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Asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity

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Abstract. We give asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations. We obtain further the uniqueness of positive decaying solutions.

1. Introduction

We consider second-order quasilinear ordinary differential equations of the form

$$(|u'|^{\alpha-1}u')' = p(t)|u|^{\lambda-1}u, \tag{1.1}$$

on which we impose the following assumptions throughout this paper:

(H1) α and λ are positive constants satisfying the *sub-homogeneity* condition $0 < \lambda < \alpha$;

(H2) $p : [t_0, \infty) \to (0, \infty)$ is a continuous function such that $p(t) \sim t^\sigma$ as $t \to \infty$.

Henceforth the notation "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$.

By a solution of (1.1) we mean a function u such that u and $|u'|^{\alpha-1}u'$ are of class C^1, and u satisfies (1.1) near $+\infty$. Throughout this paper we shall confine ourselves to the study of those solutions that do not vanish identically near $+\infty$.

When $\alpha = 1$, equation (1.1) reduces to the well-known Emden-Fowler equation of sublinear type:

$$u'' = p(t)|u|^{\lambda-1}u, \qquad 0 < \lambda < 1. \tag{1.2}$$

Qualitative theory for (1.2) has been studied in great detail by many authors; see, for example [2, 5, 7, 8]. Therefore it is an interesting problem to show

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how one can extend asymptotic theory obtained for (1.2) to the *quasilinear* equation (1.1). It was shown in [6] that some of known results can be extended surely to equation (1.1). In the earlier paper [4] the authors have considered (1.1) under the *super-homogeneity* condition $0 < \alpha < \lambda$, and have shown that asymptotic forms for positive solutions of (1.1) with $\alpha = 1$ are still valid for those of (1.1) with some obvious modifications. However, it seems to the authors that only a few of known results for (1.2) have been extended to (1.1) (under the sub-homogeneity assumption). We therefore in the present paper intend to study further asymptotic theory of positive solutions of (1.1). In particular, we aim to obtain asymptotic forms of positive solutions of (1.1).

To make a survey for possible asymptotic behavior of positive solutions of equation (1.1) we consider the equation

$$(|u'|^{\alpha-1}u')' = q(t)|u|^{\lambda-1}u, \tag{1.3}$$

with $0 < \lambda < \alpha$ and $q \in C([t_0, \infty); [0, \infty))$, which has slight generality than (1.1).

Let us give a rough classification of positive solutions of (1.3), as a first step, according to their asymptotic behavior. It is known that for every positive solution u, exactly one of the following four asymptotic behavior is possible:

(D) (*decaying* solution)

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u(t) = 0;$$

(AC) (*asymptotically constant* solution)

$$\lim_{t\to\infty} u'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} u(t) = \text{const} \in (0, \infty);$$

(AL) (*asymptotically linear* solution)

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} \frac{u(t)}{t} = \text{const} \in (0, \infty);$$

(ASL) (*asymptotically superlinear* solution)

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} \frac{u(t)}{t} = +\infty.$$

Necessary and/or sufficient conditions for the existence and nonexistence of solutions of types (D), (AC), (AL), and (ASL), respectively, have been obtained in [6]: Equation (1.3) has solutions of

(i) type (D) if $\int^{\infty}\left(\int_t^{\infty} q(s)ds\right)^{1/\alpha} dt < \infty$;

(ii) type (AC) if and only if $\int^{\infty}\left(\int_t^{\infty} q(s)ds\right)^{1/\alpha} dt < \infty$;

(iii) type (AL) if and only if $\int^{\infty} t^{\lambda} q(t)dt < \infty$;

(iv) type (ASL) if and only if $\int^{\infty} t^{\lambda} q(t)dt = \infty$;

and has no solutions of type (D) if $\liminf_{t\to\infty} t^{1+\alpha} q(t) > 0$.

These results yield necessary and sufficient conditions for (1.1) to have solutions of type (D), (AC), (AL) and (ASL), respectively: Equation (1.1) has a positive solution of

(I) type (D) if and only if $\sigma + \alpha + 1 < 0$;

(II) type (AC) if and only if $\sigma + \alpha + 1 < 0$;

(III) type (AL) if and only if $\sigma + \lambda + 1 < 0$;

(IV) type (ASL) if and only if $\sigma + \lambda + 1 \geq 0$.

It is not known how positive solutions of type (D) and type (ASL) behave near $+\infty$. In order to give asymptotic forms of all possible positive solutions it is essential to determine asymptotic forms near $+\infty$ of positive solutions of those types. We can settle this problem completely in this paper.

It is of another interest to see whether or not positive solutions of each of above-mentioned types are unique. We will show that equation (1.1) has at most one positive solution of type (D).

This paper is organized as follows: In §2 we prepare auxiliary lemmas which will be employed later. In §3 we give two theorems concerning asymptotic properties of positive solutions of type (ASL) and of type (D). These theorems play crucial role in establishing our results. Asymptotic forms for all positive solutions of equation (1.1) are given in §4, which is the main object of this paper. In §5 we give uniqueness theorems of solutions of type (D) as an application of our asymptotic theory. Lastly in §6 we will mention the duality between the results of super-homogeneous case obtained in [4] and those of sub-homogeneous case obtained in this paper.

2. Preliminary lemmas

In this section we collect preliminary lemmas which will be employed in the sequel.

Consider two differential equations of the same kind

$$(|y'|^{\alpha-1}y')' = \phi(t)|y|^{\beta-1}y, \tag{2.1}$$

$$(|Y'|^{\alpha-1}Y')' = \Phi(t)|Y|^{\beta-1}Y, \tag{2.2}$$

where $\beta > 0$ is a constant.

Lemma 2.1. *Suppose that ϕ and Φ are nonnegative and continuous and $\phi(t) \le \Phi(t)$ for $a \le t \le b$, and y and Y are positive solutions on $[a,b]$ of* (2.1) *and* (2.2), *respectively. If $y(a) \le Y(a)$ and $y'(a) < Y'(a)$ then $y(t) < Y(t)$ for $a < t \le b$.*

A proof of Lemma 2.1 is found in [4].

Lemma 2.2. *Let $f \in C^1[T,\infty)$ be such that f' is bounded, and $\int^\infty |f(t)|^\gamma dt < \infty$ for some $\gamma > 1$. Then, $\lim_{t\to\infty} f(t) = 0$.*

A proof of Lemma 2.2 is found in [3]. The following is a variant of l'Hospital's rule:

Lemma 2.3. *Let $f(t)$ and $g(t)$ be continuously differentiable functions defined near ∞ and $g'(t) \ne 0$. Then, we have*

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}$$

if either $\lim_{t\to\infty} g(t) = \infty$ or $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} f(t) = 0$ holds.

Lastly we consider the two dimensional first-order differential system

$$\frac{dw}{dt} = (A + B(t))w, \tag{2.3}$$

where A is a constant matrix with simple characteristic roots and $B \in C[T,\infty)$ satisfies $\lim_{t\to\infty} B(t) = 0$.

Lemma 2.4. *Let κ be an arbitrary characteristic root of A. Then we can find a solution w of* (2.3) *satisfying the inequalities*

$$c_1 \exp\left[(\mathrm{Re}\,\kappa)t - d_1 \int_T^t \|B(s)\| ds\right] \le \|w(t)\| \le c_2 \exp\left[(\mathrm{Re}\,\kappa)t + d_2 \int_T^t \|B(s)\| ds\right],$$

for some positive constants c_1, c_2, d_1 and d_2.

A proof of Lemma 2.4 is found in [1, Chapter 2].

3. Asymptotic equivalence theorems

Let us consider the following two equations

$$(3.1)\qquad (|x'|^{\alpha-1}x')' = a(t)|x|^{\lambda-1}x,$$

$$(3.2)\qquad (|y'|^{\alpha-1}y')' = b(t)|y|^{\lambda-1}y,$$

where $0 < \lambda < \alpha$, a and b are positive continuous functions near $+\infty$. The next theorem asserts that, if a and b have the same asymptotic behavior, then so do positive solutions of type (ASL) of equation (3.1) and of equation (3.2).

Theorem 3.1. *Suppose that*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = 1$$

holds. Let x and y be positive solutions of type (ASL) *of* (3.1) *and* (3.2), *respectively. Then $x(t) \sim y(t)$ as $t \to \infty$.*

Proof. For fixed $t_0 \gg 1$, we can find a sufficiently small number $m > 0$ and a sufficiently large number $M > 0$ such that

$$mx(t_0) \le y(t_0) \le Mx(t_0), \qquad mx'(t_0) \le y'(t_0) \le Mx'(t_0).$$

Furthermore there exists $t_1 > t_0$ such that $m^{\alpha-\lambda}a(t) \le b(t) \le M^{\alpha-\lambda}a(t)$ for $t \ge t_1$. Since the function $\bar{x}(t) = Mx(t)$ is a positive solution of type (ASL) of the equation

$$(|\bar{x}'|^{\alpha-1}\bar{x}')' = \frac{a(t)}{M^{\lambda-\alpha}}|\bar{x}|^{\lambda-1}\bar{x},$$

we see that $\bar{x}(t) \equiv Mx(t) \ge y(t)$, $t \ge t_1$ by Lemma 2.1. Similarly we see that $mx(t) \le y(t)$, $t \ge t_2$. Put $l = \liminf_{t\to\infty} x(t)/y(t)$. We know from the above observation that $0 < l < \infty$. Employing Lemma 2.3, we obtain

$$l = \liminf_{t\to\infty} \frac{x(t)}{y(t)} \ge \liminf_{t\to\infty} \frac{x'(t)}{y'(t)} = \liminf_{t\to\infty}\left(\frac{x'(t)^{\alpha}}{y'(t)^{\alpha}}\right)^{1/\alpha}$$

$$\ge \liminf_{t\to\infty}\left(\frac{[x'(t)^{\alpha}]'}{[y'(t)^{\alpha}]'}\right)^{1/\alpha} = \liminf_{t\to\infty}\left(\frac{a(t)x(t)^{\lambda}}{b(t)y(t)^{\lambda}}\right)^{1/\alpha} = l^{\lambda/\alpha},$$

hence $l \ge 1$. Similarly we obtain $\limsup_{t\to\infty} x(t)/y(t) \le 1$. This implies that $\lim_{t\to\infty} x(t)/y(t) = 1$; that is $x(t) \sim y(t)$. This completes the proof.

The next theorem asserts that if a in equation (3.1) and b in equation (3.2) have the same asymptotic behavior in some sence, then positive solutions of type (D) of equation (3.1) and of equation (3.2) also have the same asymptotic behavior.

Theorem 3.2. *Let* $\sigma + \alpha + 1 < 0$. *Suppose that*

$$0 < \liminf_{t\to\infty} \frac{a(t)}{t^\sigma} \le \limsup_{t\to\infty} \frac{a(t)}{t^\sigma} < \infty$$

and

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = 1.$$

If x *and* y *are positive solutions of* (3.1) *and of* (3.2), *respectively, of type* (D), *then* $x(t) \sim y(t)$.

Proof. First we will show that there exists positive constants c_2, c_3, c_4 and c_5 satisfying

$$c_2 t^\nu \le x \le c_3 t^\nu, \qquad c_4 t^{\nu-1} \le -x' \le c_5 t^{\nu-1} \quad \text{for large } t,$$

where $\nu = (\sigma + \alpha + 1)/(\lambda - \alpha) < 0$. Integrating (3.1) from t to $+\infty$, we obtain

$$[-x'(t)]^\alpha \le c_1 \int_t^\infty s^\sigma x^\lambda \, ds \le \frac{c_1}{-\sigma - 1} t^{\sigma+1} x(t)^\lambda, \tag{3.3}$$

that is

$$-x' x^{-\lambda/\alpha} \le c_6 t^{(\sigma+1)/\alpha} \qquad \text{for large } t,$$

where $c_6 > 0$ is a constant. Again integrating the above inequality from t to $+\infty$, we obtain $x \le c_7 t^\nu$ near $+\infty$, where $c_7 > 0$ is a constant. Substituting this estimate into (3.3), we obtain $-x'(t) \le c_8 t^{\nu-1}$, where c_8 is a positive constant.

Next we will show that there exist positive constants c_9 and c_{10} satisfying

$$x \ge c_9 t^\nu, \qquad -x' \ge c_{10} t^{\nu-1} \quad \text{for large } t.$$

To this end let $z = x(-x')^\alpha$. Then z satisfies the identity

$$-z' = z\left[\frac{(-x')}{x} + \frac{a(t)x^\lambda}{(-x')^\alpha}\right].$$

Invoking Young's inequality, we have

$$\begin{aligned} -z' &\ge c_{11} z \left[\frac{(-x')}{x}\right]^{(\lambda\alpha+\alpha)/(\lambda\alpha+2\alpha+1)} \left[\frac{a(t)x^\lambda}{(-x')^\alpha}\right]^{(\alpha+1)/(\lambda\alpha+2\alpha+1)} \\ &= c_{11} z a(t)^{(\alpha+1)/(\lambda\alpha+2\alpha+1)} x^{(\lambda-\alpha)/(\lambda\alpha+2\alpha+1)} (-x')^{(\lambda\alpha-\alpha^2)/(\lambda\alpha+2\alpha+1)} \\ &= c_{11} z^{(\lambda\alpha+\alpha+\lambda+1)/(\lambda\alpha+2\alpha+1)} a(t)^{(\alpha+1)/(\lambda\alpha+2\alpha+1)}, \end{aligned}$$

that is,

$$-z'z^{-(\lambda\alpha+\alpha+\lambda+1)/(\lambda\alpha+2\alpha+1)} \geq c_{11}a(t)^{(\alpha+1)/(\lambda\alpha+2\alpha+1)} \qquad \text{for large } t,$$

where c_{11} is a positive constant. Integrating this inequality from t to ∞, we obtain

$$z^{(\alpha-\lambda)/(\lambda\alpha+2\alpha+1)} \geq c_{12}t^{(\sigma(\alpha+1)+\lambda\alpha+2\alpha+1)/(\lambda\alpha+2\alpha+1)},$$

that is,

$$x^{\alpha-\lambda}(-x')^{\alpha(\alpha-\lambda)} \geq c_{13}t^{\sigma(\alpha+1)+\lambda\alpha+2\alpha+1} \qquad \text{for large } t,$$

where c_{12} and c_{13} are positive constants. From the result $-x' \leq c_5 t^{\nu-1}$ for large t as shown above, we see

$$x^{\alpha-\lambda}t^{\alpha(\alpha-\lambda)(\nu-1)} \geq c_{14}t^{\sigma(\alpha+1)+\lambda\alpha+2\alpha+1},$$

that is,

$$x \geq c_{15}t^{\nu} \qquad \text{for large } t,$$

where c_{14} and c_{15} are positive constants. Moreover some simple computation shows that $-x' \geq c_{16}t^{\nu-1}$, where c_{16} is a positive constant. Similarly, we find that

$$\bar{c}_2 t^{\nu} \leq y(t) \leq \bar{c}_3 t^{\nu} \qquad \text{and} \qquad \bar{c}_4 t^{\nu-1} \leq -y'(t) \leq \bar{c}_5 t^{\nu-1}$$

hold near $+\infty$ for some constants $\bar{c}_i > 0$, $i = 2, 3, 4, 5$.

Put $l = \liminf_{t\to\infty} x(t)/y(t)$. We know from the above consideration that $0 < l < \infty$. Invoking Lemma 2.3, we obtain

$$l = \liminf_{t\to\infty} \frac{x(t)}{y(t)} \geq \liminf_{t\to\infty} \frac{-x'(t)}{-y'(t)} = \liminf_{t\to\infty} \left(\frac{(-x'(t))^{\alpha}}{(-y'(t))^{\alpha}}\right)^{1/\alpha}$$

$$\geq \liminf_{t\to\infty} \left(\frac{-[(-x'(t))^{\alpha}]'}{-[(-y'(t))^{\alpha}]'}\right)^{1/\alpha} = \liminf_{t\to\infty} \left(\frac{a(t)x(t)^{\lambda}}{b(t)y(t)^{\lambda}}\right)^{1/\alpha} = l^{\lambda/\alpha}.$$

This implies that $l \geq 1$. Similarly we obtain $\limsup_{t\to\infty} x(t)/y(t) \leq 1$, and hence $\lim_{t\to\infty} x(t)/y(t) = 1$. This completes the proof.

4. Asymptotic forms of positive solutions

Possible asymptotic formulae for positive solutions of (1.1) are given here. To get an insight into our problem, let us consider the typical case that $p(t) \equiv t^{\sigma}$:

$$(|u'|^{\alpha-1}u')' = t^{\sigma}|u|^{\lambda-1}u. \tag{4.1}$$

As was pointed out in [4], (4.1) may possess an exact positive solution of the form ct^{ℓ}, $c > 0$, $\ell \in R$. In fact we can see that the function

$$u_0(t) = \hat{c}t^k \qquad \text{with } k = \frac{\sigma + \alpha + 1}{\alpha - \lambda} \quad \text{and} \quad \hat{c}^{\lambda-\alpha} = \alpha k(k-1)|k|^{\alpha-1} \tag{4.2}$$

is a positive solution of (4.1) if $k < 0$ or $k > 1$. We further emphasize that u_0 is of type (D) if $k < 0$, and of type (ASL) if $k > 1$. This observation enables us to expect that positive solutions of type (D) and of type (ASL) behave like u_0 in case of $k < 0$ and $k > 0$, respectively. We will find that this conjecture is true. It is worth noting that the following equivalences hold:

$$\begin{aligned} k < 0 \quad &\Leftrightarrow \sigma + \alpha + 1 < 0; \\ k = 0 \quad &\Leftrightarrow \sigma + \alpha + 1 = 0; \\ 0 < k < 1 &\Leftrightarrow \sigma + \lambda + 1 < 0 < \sigma + \alpha + 1; \\ k = 1 \quad &\Leftrightarrow \sigma + \lambda + 1 = 0; \\ k > 1 \quad &\Leftrightarrow \sigma + \lambda + 1 > 0. \end{aligned}$$

It will be found that asymptotic forms of positive solutions are affected by the order relations of the three numbers 0, 1 and k.

Theorem 4.1. *Let* $\sigma + \lambda + 1 > 0$ $(k > 1)$. *Then every positive solution* u *of equation* (1.1) *has the asymptotic form*

$$u(t) \sim u_0(t) \equiv \hat{c}t^k,$$

where u_0 *is the function given by* (4.2).

We can see this theorem by applying Theorem 3.1 to equations (1.1) and (4.1).

Theorem 4.2. *Let* $\sigma + \lambda + 1 = 0$ $(k = 1)$. *Then every positive solution* u *of equation* (1.1) *has the asymptotic form*

$$u(t) \sim \left(\frac{\alpha - \lambda}{\alpha}\right)^{1/(\alpha-\lambda)} t(\log t)^{1/(\alpha-\lambda)}.$$

To see this theorem, it suffices to notice that the equation

$$(|v'|^{\alpha-1}v')' = \frac{1}{t^{\lambda+1}}\left(1 + \frac{1}{(\alpha-\lambda)\log t}\right)^{\alpha-1}\left(1 + \frac{1+\lambda-\alpha}{(\alpha-\lambda)\log t}\right)|v|^{\lambda-1}v$$

has a positive increasing solution v explicitly given by

$$v(t) = \left(\frac{\alpha - \lambda}{\alpha}\right)^{1/(\alpha-\lambda)} t(\log t)^{1/(\alpha-\lambda)}.$$

This simple observation and Theorem 3.1 give the validity of Theorem 5.2.

Theorem 4.3. *Let* $\sigma + \lambda + 1 < 0 \le \sigma + \alpha + 1$ $(0 \le k < 1)$. *Then every positive solution* u *of equation* (1.1) *has the asymptotic form*

$$u \sim c_1 t,$$

where c_1 *is a positive constant.*

This is a simple consequence of the observation given in the Introduction.

Theorem 4.4. *Let* $\sigma + \alpha + 1 < 0$ $(k < 0)$. *Then every positive solution* u *of equation* (1.1) *has one of the following asymptotic forms*

$$u \sim u_0(t) \equiv \hat{c}t^k,$$

$$u(t) \sim c_1,$$

$$u(t) \sim c_2 t,$$

where c_1 *and* c_2 *are positive constants, and* u_0 *is given by* (4.2).

To see this theorem we apply Theorem 3.2 to equations (1.1) and (4.1).

5. Uniqueness of positive decaying solutions

We now turn our attention to the problem of uniqueness of positive solutions of type (D).

Theorem 5.1. *Under the assumption of Theorem* 4.4, *equation* (1.1) *does not have more than one positive solutions of type* (D).

By this theorem we find, for example, that the function $u_0(t)$ is the only positive solution of type (D) of equation (4.1) if $k < 0$.

Proof of Theorem 5.1. Let $x(t)$ and $y(t)$ be positive solutions of (1.1) of type (D). By Theorem 4.4, we know that

$$x(t), y(t) \sim \hat{c}t^k \quad \text{as } t \to \infty. \tag{5.1}$$

Furthermore, it is easy to see that

$$x'(t), y'(t) \sim \hat{c}kt^{k-1} \quad \text{as } t \to \infty, \tag{5.2}$$

where $\hat{c}$ and k are given by (4.1).

Put $z(t) = x(t) - y(t)$. It suffices to show that $z \equiv 0$. By using the mean value theorem, we know that there exist continuous functions $\tilde{\xi}_1(t)$ and $\tilde{\xi}_2(t)$ such that

$$[\alpha \tilde{\xi}_1(t)^{\alpha-1} z']' = \lambda p(t) \tilde{\xi}_2(t)^{\lambda-1} z, \tag{5.3}$$

where $\tilde{\xi}_1(t)$, $\tilde{\xi}_2(t)$ lie between $x(t)$ and $y(t)$, $x'(t)$ and $y'(t)$, respectively. By (5.1) and (5.2) we see that $\tilde{\xi}_1(t) \sim \hat{c}kt^k$, $\tilde{\xi}_2(t) \sim \hat{c}t^k$. For the simplicity, we rewrite (5.3) in the form

$$[\xi_1(t) z']' = \lambda k(k-1) \xi_2(t) z, \tag{5.4}$$

where $\xi_1(t) \sim t^{(k-1)(\alpha-1)}$ and $\xi_2(t) \sim t^{\sigma+k(\lambda-1)}$.

The proof is divided into three cases according to the exponent $(k-1)(\alpha-1)$. Here we will consider only the case $(k-1)(\alpha-1) < 1$, i.e., $\int^\infty \xi_1(r)^{-1} dr = \infty$ because parallel arguments hold in the other cases. The change of variable $s = \int^t \xi_1(r)^{-1} dr$ transforms (5.4) into

$$z_{ss} = \lambda k(k-1) \xi_1(t) \xi_2(t) z, \tag{5.5}$$

where $\xi_1(t)\xi_2(t) \sim 1/[1-(k-1)(\alpha-1)]s^2$ as $s \to \infty$. Hence we can rewrite (5.5) as

$$z_{ss} = \frac{\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2} \frac{1+\tilde{\delta}(s)}{s^2} z, \tag{5.6}$$

where $\tilde{\delta}(s)$ is a continuous function satisfying $\lim_{s\to\infty} \tilde{\delta}(s) = 0$. Moreover the change of variable $s = e^r$ transforms (5.6) into

$$\ddot{z} - \dot{z} = \frac{\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2} (1+\delta(r)) z, \tag{5.7}$$

where $\dot{} = d/dr$ and $\delta(r)$ is a continuous function satisfying $\lim_{r\to\infty} \delta(r) = 0$. Now we reduce (5.7) to a first-order system by introducing the new variables $w_1 = z$, $w_2 = \dot{z}$:

$$\frac{dw}{dr} = (A + B(r)) w, \tag{5.8}$$

where

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ \dfrac{\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2} & 1 \end{pmatrix},$$

$$B(r) = \begin{pmatrix} 0 & 0 \\ \dfrac{\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2}\,\delta(r) & 0 \end{pmatrix}.$$

The eigenvalues of A are

$$\frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2}}\right).$$

For simplicity, we put

$$\lambda_1 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2}}\right) > 0,$$

$$\lambda_2 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{4\lambda k(k-1)}{[1-(k-1)(\alpha-1)]^2}}\right) < 0.$$

By Lemma 2.4, there exist solutions $(\overline{w}_1, \overline{w}_2)$ and $(\underline{w}_1, \underline{w}_2)$ of (5.8) satisfying

$$\begin{aligned} &c_1 \exp[(\lambda_1 + o(1))r] \le |\overline{w}_1| + |\overline{w}_2| \le c_2 \exp[(\lambda_1 + o(1))r], \\ &c_3 \exp[(\lambda_2 + o(1))r] \le |\underline{w}_1| + |\underline{w}_2| \le c_4 \exp[(\lambda_2 + o(1))r], \end{aligned} \tag{5.9}$$

where c_1, c_2, c_3 and c_4 are positive constants. It is easily seen from (5.9) that $\{(\overline{w}_1, \overline{w}_2), (\underline{w}_1, \underline{w}_2)\}$ consists of a base of solutions of (5.8). Hence z is represented as

$$z = c_5 \overline{w}_1 + c_6 \underline{w}_1, \tag{5.10}$$

$$\dot{z} = c_5 \overline{w}_2 + c_6 \underline{w}_2, \tag{5.11}$$

where c_5 and c_6 are constants. We notice from (5.1) and (5.2) that

$$z(r) = O\left(\exp\left(\frac{kr}{1-(k-1)(\alpha-1)}\right)\right),$$

$$\dot{z}(r) = O\left(\exp\left(\frac{kr}{1-(k-1)(\alpha-1)}\right)\right).$$

Below we will show that $c_5 = c_6 = 0$. If $c_5 \neq 0$, then we find from (5.10) and (5.11) that

$$|\overline{w}_1| + |\overline{w}_2| \le \frac{|z| + |\dot{z}|}{|c_5|} + \left|\frac{c_6}{c_5}\right|(|\underline{w}_1| + |\underline{w}_2|). \tag{5.12}$$

By (5.9) we know that the left-hand side of (5.12) tends to ∞ as $r \to \infty$;

however, the right-hand side remains bounded. Hence we must have $c_5 = 0$; that is

$$z = c_6\underline{w}_1, \qquad \dot{z} = c_6\underline{w}_2.$$

It follows therefore that

$$(|z| + |\dot{z}|)e^{-kr/(1-(k-1)(\alpha-1))} = |c_6|(|\underline{w}_1| + |\underline{w}_2|)e^{-kr/(1-(k-1)(\alpha-1))}. \tag{5.13}$$

By the estimates for z and $\dot{z}$, the left-hand side of (5.13) is bounded, while the right-hand side of (5.13) is estimated as

$$\begin{aligned} &|c_6|(|\underline{w}_1| + |\underline{w}_2|) \exp\left(-\frac{kr}{1-(k-1)(\alpha-1)}\right) \\ &\qquad \geq c_3|c_6| \exp\left[\left(\frac{-k}{1-(k-1)(\alpha-1)} + \lambda_2 + o(1)\right)r\right]. \end{aligned}$$

Since $0 < \lambda < \alpha$, if $c_6 \neq 0$, then the right-hand side of (5.13) diverges as $r \to \infty$ which is a contradiction. Hence $c_6 = 0$, i.e., $z \equiv 0$. This completes the proof.

If $\alpha = 1$, the condition for the uniqueness of positive decaying solutions can be relaxed. We consider the equation

$$u'' = g(t)|u|^{\lambda-1}u, \qquad 0 < \lambda < 1, \tag{5.14}$$

where g is a positive function of class C^1 near $+\infty$.

Theorem 5.2. *Let $0 < \lambda < 1$ and $\sigma < -2$. Suppose that $g(t)$ satisfies*

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t^\sigma} \leq \limsup_{t\to\infty} \frac{g(t)}{t^\sigma} < \infty, \tag{5.15}$$

and

$$0 < \liminf_{t\to\infty} \frac{-g'(t)}{t^{\sigma-1}} \leq \limsup_{t\to\infty} \frac{-g'(t)}{t^{\sigma-1}} < \infty. \tag{5.16}$$

Then (5.14) *has at most one positive solution of type* (D).

Proof. We find from the proof of Theorem 3.2 that, for each positive solution u of (5.14) of type (D), there exist positive constants c_1, c_2, c_3 and c_4 satisfying

$$c_1 t^k \leq u \leq c_2 t^k, \qquad c_3 t^{k-1} \leq -u' \leq c_4 t^{k-1} \quad \text{for large } t, \tag{5.17}$$

$$k = \frac{\sigma+2}{1-\lambda}.$$

Let $x(t)$ and $y(t)$ be positive decaying solutions of (5.14). Introduce the new functions $w(t) = x(t)/y(t)$ and $\tau = \int_{t_0}^{t} dr/y(r)^2$. A computation gives

$$w_{\tau\tau} = g(t)y(t)^{\lambda+3}(w^{\lambda} - w), \qquad \tau \geq 0.$$

Moreover the change of variable

$$s = \int_0^{\tau} g(t(\xi))^{1/2} y(t(\xi))^{(\lambda+3)/2} d\xi$$

transforms this equation into

$$\ddot{w} - f(s)\dot{w} = w^{\lambda} - w, \qquad s \geq 0, \tag{5.18}$$

where $\dot{} = d/ds$, and

$$f(s) = -\tfrac{1}{2} g(t)^{-3/2} y(t)^{-(\lambda+1)/2}[g'(t)y(t) + (\lambda+3)g(t)y'(t)].$$

From condition (5.15) and (5.16) we find that

$$0 < \liminf_{s\to\infty} f(s) \leq \limsup_{s\to\infty} f(s) < \infty. \tag{5.19}$$

While from (5.17) and

$$\dot{w} = w'(t)\frac{dt}{d\tau}\frac{d\tau}{ds} = (x'y - xy')g^{-1/2}y^{-(\lambda+3)/2},$$

we know that

$$0 < \liminf_{s\to\infty} w(s) \leq \limsup_{s\to\infty} w(s) < \infty \qquad \text{and} \qquad \dot{w}(s) = O(1). \tag{5.20}$$

Multiplying (5.18) by $\dot{w}$ and integrating the resulting equation on $[s_0, s]$, we obtain

$$\left[\frac{\dot{w}^2}{2}\right]_{s_0}^{s} - \int_{s_0}^{s} f(r)\dot{w}^2 dr = \left[\frac{w^{\lambda+1}}{\lambda+1} - \frac{w^2}{2}\right]_{s_0}^{s}. \tag{5.21}$$

By (5.19) and (5.20) this implies that $\dot{w} \in L^2[s_0, \infty)$. Since obviously $\ddot{w} = O(1)$ as $s \to \infty$, Lemma 2.2 shows that $\lim_{s\to\infty} \dot{w}(s) = 0$. From Theorem 3.2 we see that $\lim_{s\to\infty} w(s) = 1$.

Multiplying (5.18) by $\dot{w}$ again and integrating the resulting equation on $[s, \infty)$, we obtain

$$-\frac{\dot{w}^2}{2} - \int_s^{\infty} f(r)\dot{w}^2\, dr = \frac{1}{\lambda+1} - \frac{1}{2} - \frac{w^{\lambda+1}}{\lambda+1} + \frac{w^2}{2}.$$

This formula shows that if $\dot{w} \not\equiv 0$, then

$$\frac{1}{\lambda+1}-\frac{1}{2}<\frac{w^{\lambda+1}}{\lambda+1}-\frac{w^2}{2} \qquad \text{for all } s \text{ near } \infty.$$

Since $w > 0$, this inequality can not hold. Hence $\dot{w} \equiv 0$, i.e., $w \equiv 1$. Hence $x \equiv y$. This completes the proof.

6. Duality between super-homogeneous case and sub-homogeneous case

It is well known [7] that if $\alpha = 1$, i.e., for Emden-Fowler equations, there is a duality between the superlinear case $(\lambda > 1)$ and the sublinear case $(0 < \lambda < 1)$. It is natural to expect that for equation (1.1) there exists similar duality between the super-homogeneous case $(0 < \alpha < \lambda)$ and the sub-homogeneous case $(0 < \lambda < \alpha)$. Hence we compare the results obtained in this paper and those obtained in [4]. To bring out the duality in full relief, we summarize our results in the following table:

	Super-homogeneous case: $\alpha < \lambda$		Sub-homogeneous case: $\lambda < \alpha$	
	Relation of parameters	Possible asymptotic forms	Relation of parameters	Possible asymptotic forms
$k > 1$	$\sigma+\lambda+1<0$	$u_0(t), c_1 t, c_2$	$\sigma+\lambda+1>0$	$u_0(t)$
$k = 1$	$\sigma+\lambda+1=0$	c_1	$\sigma+\lambda+1=0$	Logarithmic growth
$0 < k < 1$	$\sigma+\alpha+1<0<\sigma+\lambda+1$	c_1	$\sigma+\lambda+1<0<\sigma+\alpha+1$	$c_1 t$
$k = 0$	$\sigma+\alpha+1=0$	Logarithmic decay	$\sigma+\alpha+1=0$	$c_1 t$
$k < 0$	$\sigma+\alpha+1>0$	$u_0(t)$	$\sigma+\alpha+1<0$	$u_0(t), c_1 t, c_2$

In this table c_1 and c_2 are positive constants. When $\alpha < \lambda$ and $\sigma+\alpha+1=0$, every positive solution u of (1.1) has a logarithmic decay in the sense that

$$u(t) \sim \alpha^{1/(\lambda-\alpha)}\left(\frac{\alpha}{\lambda-\alpha}\right)^{\alpha/(\lambda-\alpha)}(\log t)^{-\alpha/(\lambda-\alpha)}.$$

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