HIROSHIMA MATH. J.
31 (2000), 171–181

Stability of F-harmonic maps into pinched manifolds

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(Received June 16, 2000)
(Revised October 4, 2000)

ABSTRACT. We extend two stability theorems due to Howard and Okayasu to the case of F-harmonic maps. In fact we show that every stable F-harmonic map into sufficiently pinched simply-connected Riemannian manifold is constant.

1. Introduction

Many variational problems are to find critical points of a given functional on an infinite dimensional manifold and to study their properties. In particular, there are many studies on harmonic maps, which are critical points of the Dirichlet energy functional defined on the space of smooth maps between Riemannian manifolds. Also, the studies on p-harmonic maps and exponentially harmonic maps were started by Hardt-Lin [8] and Eells-Lemaire [4] respectively, and have been developed. We are interested in critical points of more general functionals. So the author [1] introduced the notion of F-harmonic maps which unifies p-harmonic maps and exponentially harmonic maps. This notion was suggested by Eells-Sampson [5], and provides many variational problems of differential geometric interest.

To state our main theorems, we spell out the definition of F-harmonic maps. Let $F : [0, \infty) \to [0, \infty)$ be a C^2 strictly increasing function. For a smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds (M, g) and (N, h), we define the F-energy $E_F(\phi)$ of ϕ by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) v_g,$$

where v_g is the volume element of (M, g). We call ϕ an F-harmonic map if it is a critical point of the F-energy functional.

Leung [10] showed that there exist no nonconstant stable harmonic maps from any compact Riemannian manifold into a unit sphere. A natural question is "Does the above fact hold also for the case that the target is a simply-

2000 *Mathematics Subject Classification*. 58E20.
Key words and phrases. F-harmonic maps, stability, pinched manifolds.

connected δ-pinched Riemannian manifold (i.e., the sectional curvature K_N of the target manifold satisfies $\delta \leq K_N \leq 1$)?". Howard [9] proved that there exist no nonconstant stable harmonic maps into a compact simply-connected $\delta(n)$-pinched Riemannian manifold of dimension $n > 2$ for some $\delta(n)$ with $1/4 < \delta(n) < 1$ and $\lim_{n\to\infty} \delta(n) = 1$. Okayasu [11] then replaced $\delta(n)$ by a dimension independent pinching constant $\delta = 0.83$. Takeuchi [12] showed similar theorems for p-harmonic maps. In this paper, we extend their works to the case of F-harmonic maps as Theorems 3.1 and 4.1. Note that there are many C^2 strictly increasing functions which satisfy the assumption of our theorems, for instance, $F(t) = t^p (p = 1$ or $2 \leq p < \infty)$, $(1+t)^\alpha (\alpha > 1)$, $(1+t)\log(1+t) - t$, and so on. In the case where $F(t) = t^p$, the constant c_F equals $(p/2) - 1$. Then our Theorem 3.1 becomes the following due to Takeuchi.

Theorem ([12, Theorem 3]). *Let N be a compact simply-connected δ-pinched n-dimensional Riemannian manifold. Assume that n and δ satisfy $n > p$ and*

$$\int_0^\pi \left\{ (p-1) g_2(t,\delta) \left(\frac{\sin \sqrt{\delta} t}{\sqrt{\delta}} \right)^{n-1} - (n-1)\delta \cos^2(t) \sin^{n-1}(t) \right\} dt < 0,$$

where $g_2(t,\delta) = \max\{\cos^2(t), \delta \sin^2(t) \cot^2(\sqrt{\delta} t)\}$. *Then for any compact Riemannian manifold M, every stable p-harmonic map $\phi : M \to N$ is constant.*

Thus our result contains the stability theorem previously known in the case of p-harmonic maps. Also our Theorem 4.1 contains the theorem of Takeuchi for p-harmonic maps (cf. [12, Theorem 4]).

The author wishes to express his gratitude to Professor Hajime Urakawa for his constant encouragement and valuable advice, and also to Professor Makoto Sakaki for his suggestions in improving this work. Finally, he thanks the referee who gives him valuable comments.

2. Preliminaries

Let $F : [0, \infty) \to [0, \infty)$ be a C^2 strictly increasing function. Let $\phi : M \to N$ be a smooth map from an m-dimensional Riemannian manifolds (M, g) to a Riemannian manifold (N, h). We call ϕ an F-harmonic map if it is a critical point of the F-energy functional. That is, ϕ is an F-harmonic map if and only if

$$\frac{d}{dt} E_F(\phi_t)|_{t=0} = 0$$

for any compactly supported variation $\phi_t : M \to N (-\varepsilon < t < \varepsilon)$ with $\phi_0 = \phi$.

Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\tilde{\nabla}$ be the induced connection on $\phi^{-1}TN$ defined by $\tilde{\nabla}_X W = {}^N\nabla_{\phi_* X} W$, where X is a tangent vector of M and W is a section of $\phi^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M. We define the F-tension field $\tau_F(\phi)$ of ϕ by

$$\begin{aligned}\tau_F(\phi) &= \sum_{i=1}^m \left[\tilde{\nabla}_{e_i}\left\{F'\left(\frac{|d\phi|^2}{2}\right)\phi_* e_i\right\} - F'\left(\frac{|d\phi|^2}{2}\right)\phi_*\nabla_{e_i} e_i\right] \\ &= F'\left(\frac{|d\phi|^2}{2}\right)\tau(\phi) + \phi_* \operatorname{grad}\left\{F'\left(\frac{|d\phi|^2}{2}\right)\right\},\end{aligned}$$

where $\tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i} e_i)$ is the tension field of ϕ.

Under the notation above we have the following:

Theorem 2.1 (The first variation formula).

$$\frac{d}{dt} E_F(\phi_t)|_{t=0} = -\int_M h(V, \tau_F(\phi)) v_g,$$

where $V = d\phi_t/dt|_{t=0}$.

Therefore a smooth map $\phi : M \to N$ is an F-harmonic map if and only if the F-tension field $\tau_F(\phi) = 0$.

Next we give the second variation formula for F-harmonic maps.

Theorem 2.2 (The second variation formula). *Let* $\phi : M \to N$ *be an* F*-harmonic map. Let* $\phi_{s,t} : M \to N (-\varepsilon < s, t < \varepsilon)$ *be a compactly supported two-parameter variation such that* $\phi_{0,0} = \phi$, *and set* $V = \partial\phi_{s,t}/\partial t|_{s,t=0}$, $W = \partial\phi_{s,t}/\partial s|_{s,t=0}$. *Then*

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t} E_F(\phi_{s,t})|_{s,t=0} &= \int_M F''\left(\frac{|d\phi|^2}{2}\right)\langle\tilde{\nabla} V, d\phi\rangle\langle\tilde{\nabla} W, d\phi\rangle v_g \\ &\quad + \int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot\left\{\langle\tilde{\nabla} V, \tilde{\nabla} W\rangle - \sum_{i=1}^m h(R^N(V, \phi_* e_i)\phi_* e_i, W)\right\} v_g,\end{aligned}$$

where $\langle , \rangle$ *is the inner product on* $T^*M \otimes \phi^{-1}TN$ *and* R^N *is the curvature tensor of* N.

We put

$$I(V, W) = \frac{\partial^2}{\partial s\partial t} E_F(\phi_{s,t})|_{s,t=0}.$$

An F-harmonic map ϕ is called F-stable or stable if $I(V,V) \geq 0$ for any compactly supported vector field V along ϕ.

Some geometric properties of F-harmonic maps are described in [1].

3. The Howard type theorem

In this section, we show the Howard type theorem.

Theorem 3.1. *Let $F : [0,\infty) \to [0,\infty)$ be a C^2 strictly increasing function. Assume that there exists a constant $c_F := \inf\{c \geq 0 \mid F'(t)/t^c \text{ is nonincreasing}\}$. Let N be a compact simply-connected δ-pinched n-dimensional Riemannian manifold. Assume that n and δ satisfy $n > 2(c_F + 1)$ and*

$$\Psi_{n,F}(\delta) := \int_0^\pi \left\{ (2c_F+1) g_2(t,\delta) \left(\frac{\sin \sqrt{\delta} t}{\sqrt{\delta}} \right)^{n-1} - (n-1)\delta \cos^2(t) \sin^{n-1}(t) \right\} dt < 0,$$

where $g_2(t,\delta) = \max\{\cos^2(t), \delta \sin^2(t) \cot^2(\sqrt{\delta} t)\}$. Then for any compact Riemannian manifold M, every stable F-harmonic map $\phi : M \to N$ is constant.

Remark. In the case where $F(t) = (2t)^{p/2} (p = 2 \text{ or } 4 \leq p < \infty)$, the constant c_F equals $(p/2) - 1$. So the above theorem is an extension of [9, Theorem 6.1] for harmonic maps and [12, Theorem 3] for p-harmonic maps.

Proof. The proof is a complete analogue to that due to Howard [9]. For the gradient vector field V of a smooth function f on N, we define a self-adjoint map on the tangent bundle TN of N by

$$A^V(X) = {}^N\nabla_X V \qquad \text{for } X \in T_y N, \ y \in N.$$

Then we see that

$$\langle A^V(X), Y \rangle = \operatorname{Hess} f(X,Y),$$

where Hess f denotes the hessian of f. Note that from pinching conditions of sectional curvatures, the injectivity radius of N is greater than π, due to W. Klingenberg [3]. Define a function $f : \mathbf{R} \to \mathbf{R}$ by $f(t) = -\cos(t)$ for $|t| \leq \pi$, $f(t) = 1$ for $|t| \geq \pi$. Let ρ_y denotes the geodesic distance from $y \in N$, and V^y be the vector field defined by $V^y = \nabla(f \circ \rho_y) = f'(\rho_y)\nabla\rho_y$. V^y is continuous and smooth away from the cut locus defined by $\rho_y = \pi$, and $V^y = 0$ on the set defined by $\rho_y > \pi$. So we must approximate the function f by smooth functions f_k $(k = 1, 2, \ldots)$ as follows. We choose $\varepsilon > 0$ so that $\pi + \varepsilon$ is less than the injectivity radius of N, and f_k with $f_k(t) = f_k(-t)$, f_k converging to f uniformly, f_k' converging to f' uniformly, f_k'' converging to f'' uniformly on all compact sets disjoint from $\{\pi, -\pi\}$, f_k'' uniformly bounded with respect to both

t and k, and f_k constant on the interval $[\pi + \varepsilon, \infty)$. Then for each k and each $y \in N$ the vector field $V_k^y = \nabla(f_k \circ \rho_y)$ is smooth on N and converges uniformly to V^y as $k \to \infty$. Let $\{\lambda_i\}$ be the eigenvalues of the map A^{V^y} on the tangent space $T_{y_0}N$ for a fixed point $y_0 \in N$. Noting $\rho_y(y_0) = \rho_{y_0}(y)$ for $\rho_{y_0}(y) < \pi$ and putting $\rho = \rho_{y_0}$, by the Hessian comparison theorem (cf. [6]) we have

$$\cos(\rho(y)) \leq \lambda_i \leq \sqrt{\delta}\sin(\rho(y))\cot(\sqrt{\delta}\rho(y)), \qquad 1 \leq i \leq n. \tag{3.1}$$

Set

$$\tilde{g}_1(t,\delta) = \text{middle value of } \{\cos(t), 0, \sqrt{\delta}\sin(t)\cot(\sqrt{\delta}t)\},$$

$$g_1(t,\delta) = (\tilde{g}_1(t,\delta))^2, \qquad 0 \leq t \leq \pi.$$

Then by squaring (3.1) we have

$$g_1(\rho,\delta) \leq \lambda_i^2 \leq g_2(\rho,\delta). \tag{3.2}$$

Let $U_{y_0}N$ be the unit sphere in $T_{y_0}N$. For $w \in U_{y_0}N$ we can view (ρ, w) as polar coordinates on N near y_0. Then $\nabla(\rho_y) = -\gamma'(t)$ for a geodesic $\gamma(t) = \exp(tw)$. Denote the volume density on N by v_N and the volume density on $U_{y_0}N$ by $v_{UN_{y_0}}$. Then we have

$$\sin^{n-1}(\rho)d\rho\, v_{UN_{y_0}}(w) \leq v_N(y) \leq \left\{\frac{\sin(\sqrt{\delta}\rho)}{\sqrt{\delta}}\right\}^{n-1} d\rho\, v_{UN_{y_0}}(w) \tag{3.3}$$

on the open set $N \backslash \mathrm{cut}(y_0)$, $\mathrm{cut}(y_0)$ being the cut locus of y_0. And we can find the following inequality in [9, 4–14].

$$\begin{aligned} &\int_N \langle R^N(V^y, X)X, V^y\rangle v_N(y) \\ &\qquad \geq (n-1)|X|^2\,\mathrm{Vol}(S^{n-1})\int_0^\pi \delta\cos^2(\rho)\sin^{n-1}(\rho)d\rho. \end{aligned} \tag{3.4}$$

Now we put as a one-parameter variation, $\phi_t = u_t(V_k^y) \circ \phi$, where $u_t(V_k^y) : N \to N$ is a one parameter diffeomorphism of N such that $\partial\phi_t/\partial t|_{t=0} = V_k^y$. It follows from the second variation formula and the Schwarz inequality that

$$\begin{aligned} \lim_{k\to\infty}\int_N I(V_k^y, V_k^y)v_N(y) &= \int_N I(V^y, V^y)v_N(y) \\ &= \int_N \left[\int_M F''\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{i=1}^m \langle\tilde{\nabla}_{e_i}V^y, \phi_* e_i\rangle\right)^2 v_g \right. \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\quad + \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m |\tilde{\nabla}_{e_i} V^y|^2 v_g \\
&\quad \left. - \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m \langle R^N(V^y, \phi_* e_i)\phi_* e_i, V^y \rangle v_g \right] v_N(y) \\
&\leq \int_N \left[\int_M \left\{ F''\left(\frac{|d\phi|^2}{2}\right) |d\phi|^2 + F'\left(\frac{|d\phi|^2}{2}\right) \right\} \sum_{i=1}^m |\tilde{\nabla}_{e_i} V^y|^2 v_g \right. \\
&\quad \left. - \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m \langle R^N(V^y, \phi_* e_i)\phi_* e_i, V^y \rangle v_g \right] v_N(y).
\end{aligned}
$$

Now we assume that ϕ is not constant. Since $F'(t)/t^{c_F}$ is nonincreasing, $F''(t)t \leq c_F F'(t)$ on $t \in (0, \infty)$. So we have

$$
\begin{aligned}
\text{(3.6)} \qquad & \lim_{k\to\infty} \int_N I(V_k^y, V_k^y) v_N(y) \\
&\leq \int_N \sum_{i=1}^m \int_M F'\left(\frac{|d\phi|^2}{2}\right) \{(2c_F + 1)|\tilde{\nabla}_{e_i} V^y|^2 \\
&\quad - R^N(V^y, \phi_* e_i)\phi_* e_i, V^y \rangle\} v_g v_N(y).
\end{aligned}
$$

By the definition of A^{V^y} and λ_i, we get from (3.2)

$$
\text{(3.7)} \qquad |\tilde{\nabla}_{e_i} V^y|^2 = \langle A^{V^y} A^{V^y}(\phi_* e_i), \phi_* e_i \rangle \leq g_2(\rho, \delta) |\phi_* e_i|^2.
$$

Combining (3.3), (3.4), (3.6) and (3.7), we have

$$
\lim_{k\to\infty} \int_N I(V_k^y, V_k^y) v_N \leq \mathrm{Vol}(S^{n-1}) \cdot \Psi_{n,F}(\delta) \cdot \int_M F'\left(\frac{|d\phi|^2}{2}\right) |d\phi|^2 v_g.
$$

Since $\Psi_{n,F}(\delta) < 0$ by the assumption, for some $k \geq 1$ there is a certain vector field V_k^y such that the second variation for V_k^y is negative. This means that ϕ is unstable, which is a contradiction. Thus ϕ is constant. □

4. The Okayasu type theorem

We assume that N is a compact simply-connected δ-pinched Riemannian manifold. Let E denote the Whitney sum $E = TN \oplus \varepsilon(N)$ of the tangent bundle TN and the trivial line bundle $\varepsilon(N) = N \times \mathbf{R}$ with the canonical metric.

Then E admits a natural fiber metric. Let e be a cross-section of unit length in $\varepsilon(N)$. We define a metric connection ∇'' on E as follows:

$$\nabla''_X Y = {}^N\nabla_X Y - \sqrt{\frac{1+\delta}{2}}\langle X, Y\rangle e,$$

$$\nabla''_X e = \sqrt{\frac{1+\delta}{2}} X,$$

where X and Y are vector fields on N, $\langle , \rangle$ is the Riemannian metric. We define the distance of two connections ∇', ∇'', by

$$|\nabla' - \nabla''| := \mathrm{Max}\{|\nabla'_X Y - \nabla''_X Y|; X \in TN, |X| = 1, Y \in E, |Y| = 1\},$$

and four functions as follows:

$$k_1(\delta) = \frac{4}{3}\cdot\frac{1-\delta}{\delta}\left[1 + \left(\delta^{1/2}\sin\frac{1}{2}\pi\delta^{-1/2}\right)^{-1}\right],$$

$$k_2(\delta) = \left[\frac{1}{2}(1+\delta)\right]^{-1}\cdot k_1(\delta),$$

$$k_3(\delta) = k_2(\delta)\sqrt{1 + \left(1 - \frac{1}{24}\pi^2(k_1(\delta))^2\right)^{-2}},$$

$$k_4(\delta) = \sqrt{\frac{1+\delta}{2}}\cdot k_3(\delta).$$

Then there exists (cf. [7]) a flat connection ∇' such that

$$|\nabla' - \nabla''| \le \tfrac{1}{2}k_4(\delta).$$

THEOREM 4.1. *Let $F : [0, \infty) \to [0, \infty)$ be a C^2 strictly increasing function. Assume that there exists a constant $c_F := \inf\{c \ge 0 \mid F'(t)/t^c$ is nonincreasing$\}$. Let N be a compact simply-connected δ-pinched n-dimensional Riemannian manifold. Assume that n and δ satisfy $n > 2(c_F + 1)$* and

$$\Phi_{n,F}(\delta) := (2c_F + 1)\left(\frac{\sqrt{n+1}}{2}k_4(\delta) + \sqrt{\frac{1+\delta}{2}}\right)^2 - (n-1)\delta < 0.$$

Then for any compact Riemannian manifold M, every stable F-harmonic map $\phi : M \to N$ is constant.

REMARK. In the case where $F(t) = (2t)^{p/2} (p = 2 \text{ or } 4 \le p < \infty)$, the constant c_F equals $(p/2) - 1$. So the above theorem is an extension of [11, Theorem 1] for harmonic maps and [12, Theorem 4] for p-harmonic maps.

PROOF. The proof is a complete analogue to that due to Okayasu [11]. Suppose that ϕ is not constant. Let $\mathscr{W} = \{V \in \Gamma(E); \nabla' V = 0\}$, and for $V \in \mathscr{W}$ we denote by V^T and V^ε the TN-component and $\varepsilon(N)$-component of V respectively. Then we obtain

$$\begin{aligned}
I(V^T, V^T) &= \int_M F''\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{i=1}^m \langle \tilde{\nabla}_{e_i} V^T, \phi_* e_i\rangle\right)^2 v_g \\
&\quad + \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m |\tilde{\nabla}_{e_i} V^T|^2 v_g \\
&\quad - \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m \langle R^N(V^T, \phi_* e_i)\phi_* e_i, V^T\rangle v_g \\
&\le \int_M \left\{F''\left(\frac{|d\phi|^2}{2}\right)|d\phi|^2 + F'\left(\frac{|d\phi|^2}{2}\right)\right\} \sum_{i=1}^m |\tilde{\nabla}_{e_i} V^T|^2 v_g \\
&\quad - \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m \langle R^N(V^T, \phi_* e_i)\phi_* e_i, V^T\rangle v_g,
\end{aligned}$$

where we have used Schwarz's inequality. Since $F'(t)/t^{c_F}$ is nonincreasing, $F''(t)t \le c_F F'(t)$ on $t \in (0, \infty)$. So we have

$$\begin{aligned}
I(V^T, V^T) \le \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m &\{(2c_F + 1)|\tilde{\nabla}_{e_i} V^T|^2 \\
&- \langle R^N(V^T, \phi_* e_i)\phi_* e_i, V^T\rangle\} v_g.
\end{aligned} \tag{4.1}$$

On the other hand, we observe that

$$\begin{aligned}
\tilde{\nabla}_{e_i} V^T &= \{\nabla''_{\phi_* e_i}(V - V^\varepsilon)\}^T \\
&= (\nabla''_{\phi_* e_i} V)^T - (\nabla''_{\phi_* e_i}(\langle V^\varepsilon, e\rangle e))^T \\
&= (\nabla''_{\phi_* e_i} V)^T - \left(\frac{1+\delta}{2}\right)^{1/2} \langle V, e\rangle \phi_* e_i.
\end{aligned}$$

This leads to the inequality

$$(4.2)\quad |\tilde{\nabla}_{e_i} V^T|^2 \le (1+k)|\nabla''_{\phi_* e_i} V|^2 + \left(1+\frac{1}{k}\right)\frac{1+\delta}{2}\langle V, e\rangle^2 |\phi_* e_i|^2$$

$$\le \frac{1+k}{4}(k_4(\delta))^2 \cdot |V|^2 \cdot |\phi_* e_i|^2 + \left(1+\frac{1}{k}\right)\frac{1+\delta}{2}\langle V, e\rangle^2 |\phi_* e_i|^2,$$

where k is a positive constant which will be fixed below. Since N is a δ-pinched manifold, we have

$$(4.3)\qquad \langle R^N(V^T, \phi_* e_i)\phi_* e_i, V^T\rangle \ge \delta\{|V^T|^2 \cdot |\phi_* e_i|^2 - \langle V^T, \phi_* e_i\rangle^2\}.$$

Define a quadratic form Q on $\mathscr{W}$ by

$$Q(V) = \int_M F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^m \left\{(2c_F+1)\left[\frac{1+k}{4}(k_4(\delta))^2 \cdot |V|^2 \cdot |\phi_* e_i|^2 \right.\right.$$

$$\left.\left. + \left(1+\frac{1}{k}\right)\frac{1+\delta}{2}\langle V, e\rangle^2 |\phi_* e_i|^2\right] - \delta(|V^T|^2 \cdot |\phi_* e_i|^2 - \langle V^T, \phi_* e_i\rangle^2)\right\} v_g.$$

Substituting (4.2) and (4.3) into (4.1), we obtain

$$I(V^T, V^T) \le Q(V).$$

Taking an orthonormal basis $\{W_1, W_2, \ldots, W_n, W_{n+1}\}$ of $\mathscr{W}$ with respect to a natural inner product, we obtain

$$(4.4)\qquad \sum_{j=1}^{n+1} I(W_j^T, W_j^T) \le \sum_{j=1}^{n+1} Q(W_j)$$

$$= \int_M |d\phi|^2 F'\left(\frac{|d\phi|^2}{2}\right)\left\{(2c_F+1)\left[\frac{(1+k)(n+1)}{4}(k_4(\delta))^2 \right.\right.$$

$$\left.\left. + \left(1+\frac{1}{k}\right)\frac{1+\delta}{2}\right] - (n-1)\delta\right\} v_g.$$

We take $k = \sqrt{2(\delta+1)/(n+1)}(k_4(\delta))^{-1}$. From (4.4) we have

$$\sum_{j=1}^{n+1} I(W_j^T, W_j^T)$$

$$\le \int_M |d\phi|^2 F'\left(\frac{|d\phi|^2}{2}\right)\left\{(2c_F+1)\left(\frac{\sqrt{n+1}}{2}k_4(\delta) + \sqrt{\frac{1+\delta}{2}}\right)^2 - (n-1)\delta\right\} v_g,$$

which is negative by the assumption. Therefore, we obtain $I(W_j^T, W_j^T) < 0$ for some j, ϕ is unstable, which is a contradiction. Thus ϕ is constant. □

We assume that $n > 2(c_F + 1)$, and put

$$\delta_F(n) := \inf\{\delta \mid 1/4 < \delta \le 1 \text{ and } \Psi_{n,F}(\delta) < 0\},$$
$$\delta_F'(n) := \inf\{\delta \mid 1/4 < \delta \le 1 \text{ and } \Phi_{n,F}(\delta) < 0\}.$$

Remark. The functions $\Psi_{n,F}(\delta)$ and $\Phi_{n,F}(\delta)$ are continuous on $(1/4, 1]$, and $\Psi_{n,F}(1) < 0$, $\Phi_{n,F}(1) < 0$. So note that $\delta_F(n) < 1$ and $\delta_F'(n) < 1$.

Combining Theorems 3.1 and 4.1, we obtain the following:

Theorem 4.2. *Let $F : [0, \infty) \to [0, \infty)$ be a C^2 strictly increasing function. Assume that there exists a constant $c_F := \inf\{c \ge 0 \mid F'(t)/t^c$ is nonincreasing$\}$. Let N be a compact simply-connected δ_F-pinched Riemannian manifold of dimension $n > 2(c_F + 1)$ and $\delta_F = \sup_{n > 2(c_F+1)}(\min\{\delta_F(n), \delta_F'(n)\}) < 1$. Then for any compact Riemannian manifold M, every stable F-harmonic map $\phi : M \to N$ is constant.*

Proof. We prove $\delta_F < 1$. From

$$\Phi_{n,F}(\delta) = (2c_F + 1)\left(\frac{\sqrt{n+1}}{2} k_4(\delta) + \sqrt{\frac{1+\delta}{2}}\right)^2 - (n-1)\delta < 0,$$

we have

$$k_4(\delta) + \sqrt{\frac{1+\delta}{2}} \frac{2}{\sqrt{n+1}} < \frac{2\sqrt{\delta}}{\sqrt{2c_F+1}} \frac{\sqrt{n-1}}{\sqrt{n+1}}.$$

So we obtain

$$k_4(\delta) < \frac{2\sqrt{\delta}}{\sqrt{2c_F+1}} \qquad \text{as } n \to \infty.$$

Let $\delta_0 (< 1)$ be the value satisfying

$$k_4(\delta_0) = \frac{2\sqrt{\delta_0}}{\sqrt{2c_F+1}}.$$

Then $\delta_F'(n) \to \delta_0$ as $n \to \infty$. Therefore, for a sufficiently small $\varepsilon > 0$ there exists a constant n_0 such that $\delta_F'(n) \le \delta_0 + \varepsilon < 1$ for all $n \ge n_0$. □

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