A note on the multilinear oscillatory singular integral operators

Zefu Chu, Guoen Hu and Zhibo Lu
(Received December 17, 1999)
(Revised November 13, 2000)

Abstract. In this paper, we consider the \( L^p(\mathbb{R}^n) \) boundedness for a class of multilinear oscillatory singular integral operators with polynomial phases. We show that if the polynomial phases are non-trivial and the homogeneous kernels satisfy a certain minimum size condition, then the \( L^p(\mathbb{R}^n) \) boundedness for the multilinear oscillatory singular integral operators can be deduced from the \( L^p(\mathbb{R}^n) \) boundedness for the corresponding local multilinear singular integral operators.

1. Introduction

We will work on \( \mathbb{R}^n(n \geq 2) \). Let \( P(x, y) \) be a real-valued polynomial on \( \mathbb{R}^n \times \mathbb{R}^n \), \( \Omega(x) \) be homogeneous of degree zero which has a mean value zero on the unit sphere \( S^{n-1} \). Define the oscillatory singular integral operator

\[
Tf(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.
\]

It is well-known that the operators of this type have arisen in the study of Hilbert transforms along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. A celebrated result of Ricci and Stein [9] says that if \( \Omega \in \text{Lip}_1(S^{n-1}) \), then \( T \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \), with a bound depending only on \( n \), \( p \) and \( \deg P \) (the total degree of \( P \)), not on the coefficients of the polynomial. Chanillo and Christ [2] showed that \( \Omega \in \text{Lip}_1(S^{n-1}) \) is also sufficient for \( T \) to be a bounded mapping from \( L^1 \) to weak \( L^1 \), and the bound depends only on \( n \) and \( \deg P \). Lu and Zhang [7] improved the result of Ricci and Stein, and proved that if \( \Omega \in \bigcup_{q>1} L^q(S^{n-1}) \), then \( T \) is bounded on \( L^p(\mathbb{R}^n) \) with a bound \( C(n, p, \deg P) \) for \( 1 < p < \infty \).

In this paper, we will study the multilinear operators defined by

\[
T_{A_1, \ldots, A_k} f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+\delta}} \prod_{j=1}^k R_{m_j+1}(A_j, x, y) f(y) dy,
\]

2000 Mathematics Subject Classification. 42B20

Key words and phrases. multilinear operator, oscillatory singular integral, BMO.

The research was supported by the NSF of China (19701039) and the NSF of Henan Province.
where \( k \) and \( m_j \) \((j = 1, \ldots, k)\) are positive integers, \( m = \sum_{j=1}^{k} m_j \), \( A_j \) \((j = 1, \ldots, k)\) has derivatives of order \( m_j \) in \( \text{BMO}(\mathbb{R}^n) \), \( R_{m+1}(A_j; x, y) \) denotes the \((m_j + 1)\)-th Taylor series remainder of \( A_j \) at \( x \) about \( y \), that is,

\[
R_{m+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.
\]

Operators of this type have been studied in [3], [4], [6] and many other works. It is easy to see that the operator \( T_{A_1, \ldots, A_k} \) is closely related to the oscillatory singular integral operator defined by (1) and the multilinear singular integral operator defined by

\[
T_{A_1, \ldots, A_k} f(x) = \int_{\mathbb{R}^k} \frac{\Omega(x - y)}{|x - y|^{n+m} \prod_{j=1}^{k} R_{m+1}(A_j; x, y)} f(y) dy.
\]

Using good-\( \lambda \)-inequality techniques, Cohen and Gosselin [5] showed that if \( \Omega \) satisfies a certain vanishing moment and \( \Omega \in \text{Lip}_1(S^{n-1}) \), then for \( 1 < p < \infty \),

\[
\|T_{A_1, A_2} f\|_p \leq \prod_{j=1}^{k} \left( \sum_{|\alpha| \leq m_j} \|D^\alpha A_j\|_{\text{BMO}(\mathbb{R}^n)} \right) \|f\|_p.
\]

In [3], Chen, Hu and Lu considered the \( L^p(\mathbb{R}^n) \) boundedness for the operator \( T_{A_1, A_2} \) and proved that if \( \Omega \in \bigcup_{q \geq 1} L^q(S^{n-1}) \), and the polynomial \( P(x, y) \) is non-trivial, then the \( L^p(\mathbb{R}^n) \) boundedness for \( T_{A_1, A_2} \) can be obtained from the \( L^p(\mathbb{R}^n) \) boundedness for the local multilinear singular integral operator

\[
S_{A_1, A_2} f(x) = \int_{|x-y| \leq 1} \frac{\Omega(x - y)}{|x-y|^{n+m_1+m_2}} \prod_{j=1}^{k} R_{m+1}(A_j; x, y) f(y) dy,
\]

(see [2, Theorem 2]). The purpose of this paper is to show that if \( \Omega \in L(\log L)^{k+1}(S^{n-1}) \), and \( P \) is non-trivial, then the \( L^p(\mathbb{R}^n) \) boundedness for \( T_{A_1, \ldots, A_k} \) can be obtained from the \( L^p(\mathbb{R}^n) \) boundedness for the local version of the operator \( T_{A_1, \ldots, A_k} \). Our main result in this paper can be stated as follows.

**Theorem 1.** Let \( 1 < p < \infty \), \( k \) and \( m_j \) \((j = 1, 2, \ldots, k)\) be positive integers, \( m = \sum_{j=1}^{k} m_j \), \( A_j \) \((j = 1, 2, \ldots, k)\) are functions on \( \mathbb{R}^n \) whose derivatives of order \( m_j \) are in \( \text{BMO}(\mathbb{R}^n) \). Suppose that \( \Omega \) is homogeneous of degree zero and belongs to the space \( L(\log L)^{k+1}(S^{n-1}) \), that is,

\[
\int_{S^{n-1}} |\Omega(x')| \log^{k+1}(2 + |\Omega(x')|) dx' < \infty,
\]
and the operator

\[ S_{A_1, \ldots, A_k} f(x) = \int_{|x-y| \leq 1} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^k R_{m_j+1}(A_j; x, y) f(y) \, dy \]

is bounded on \( L^p(\mathbb{R}^n) \). Then for any real-valued non-trivial polynomial \( P(x, y) \), the operator \( T_A \) defined by (2) is also bounded on \( L^p(\mathbb{R}^n) \), with a bound depending on \( n, p, m_j (j = 1, \ldots, k), \prod_{j=1}^k (\sum_{|z_j|=m_j} \|D^\gamma A_j\|_{\text{BMO}(\mathbb{R}^n)}) \) and \( \deg P \), not on the coefficients of \( P \).

2. Proof of Theorem 1

We begin with some preliminary lemmas.

**Lemma 1** (see [5]). Let \( b(x) \) be a function on \( \mathbb{R}^n \) with derivatives of order \( m \) in \( L^q(\mathbb{R}^n) \) for some \( n < q \leq \infty \). Then

\[
|R_m(b; x, y)| \leq C_{m,n}|x-y|^m \sum_{|z|=m} \left( \frac{1}{|I(x, y)|} \int_{I(x, y)} |D^\gamma b(z)|^q \, dz \right)^{1/q},
\]

where \( I(x, y) \) is the cube centered at \( x \) with diameter \( 5\sqrt{n}|x-y| \).

**Lemma 2.** Let \( 1 < p < \infty, k \) and \( m_j (j = 1, 2, \ldots, k) \) be positive integers, \( m = \sum_{j=1}^k m_j, A_j (j = 1, 2, \ldots, k) \) be functions on \( \mathbb{R}^n \) whose derivatives of order \( m_j \) are in \( \text{BMO}(\mathbb{R}^n) \). Suppose that \( \Omega \) is homogeneous of degree zero and belongs to the space \( L^\infty(S^{n-1}) \). Set

\[
\lambda_{\Omega, k} = \inf \left\{ \lambda > 0 : \frac{\|\Omega\|_{L^1}}{\lambda} \log^k \left( 2 + \frac{\|\Omega\|_{L^{\infty}}}{\lambda} \right) \leq 1 \right\}.
\]

Then for any \( r > 0 \), the operator

\[ U_{A_1, \ldots, A_k} f(x) = r^{-n-m} \int_{r/2 < |x-y| \leq r} \frac{|\Omega(x-y)|}{|I(x, y)|} \prod_{j=1}^k |R_{m_j+1}(A_j; x, y)||f(y)| \, dy \]

is bounded on \( L^p(\mathbb{R}^n) \) with a bound \( C(n, m, p) \lambda_{\Omega, k} \prod_{j=1}^k (\sum_{|z_j|=m_j} \|D^\gamma A_j\|_{\text{BMO}(\mathbb{R}^n)}) \).

**Proof.** Note that for each \( t > 0 \),

\[
\lambda_{\tilde{\Omega}, k} = \inf \left\{ \lambda > 0 : \frac{\|t\tilde{\Omega}\|_{L^1}}{\lambda} \log^k \left( 2 + \frac{\|t\tilde{\Omega}\|_{L^{\infty}}}{\lambda} \right) \leq 1 \right\} = \inf \left\{ t\tilde{\lambda} : \tilde{\lambda} > 0, \frac{t\tilde{\lambda}}{\|t\tilde{\Omega}\|_{L^1}} \log^k \left( 2 + \frac{\|t\tilde{\Omega}\|_{L^{\infty}}}{t\tilde{\lambda}} \right) \leq 1 \right\} = t\lambda_{\tilde{\Omega}, k}.
\]
Thus we may assume that $\lambda_{Q,k} = 1/2$. Therefore,

$$\|\tilde{\Omega}\|_1 \log^k (2 + \|\tilde{\Omega}\|_\infty) \leq 1.$$ 

Define the operator $E$ by

$$Eh(x) = \int_{|x-y| \leq 1} |\tilde{\Omega}(x-y)| h(y) dy.$$

Denote by $E^*$ the adjoint operator of $E$, that is,

$$E^* h(x) = \int_{|x-y| \leq 1} |\tilde{\Omega}(y-x)| h(y) dy.$$

Let $b_1, b_2, \ldots, b_k \in \text{BMO}(\mathbb{R}^n)$ and $Q$ be a cube with side length 1. Denote by $m_Q(b_j)$ the mean value of $b_j$ on $Q$. We claim that for $1 < p < \infty$, supp $h \subset 10nQ$ and non-negative integer $l \leq k$,

\begin{equation}
\int_Q |E^* h(x)|^p \prod_{j=1}^k |b_j(x) - m_Q(b_j)|^p dx 
\leq C \log^{(k+1)p}(2 + \|\tilde{\Omega}\|_\infty) \prod_{j=1}^k \|b_j\|_{\text{BMO}(\mathbb{R}^n)}^p \|h\|_p^p,
\end{equation}

with the interpretation that when $l = 0$, $\prod_{j=1}^l |b_j(x) - m_Q(b_j)| \equiv 1$. To prove (6), we can assume that $\|h\|_p = 1$. Choose $1 < r_j < \infty$ such that $\sum_{j=1}^k 1/r_j = 1$. By the well-known John-Nirenberg inequality, there is a positive constant $C_j = C(r_j, r_j, n)$ such that

$$\left( \int_Q |b_j(x) - m_Q(b_j)|^{2r_j} dx \right)^{1/(2r_j)} \leq C_j \|b_j\|_{\text{BMO}(\mathbb{R}^n)}^p.$$

We may also assume that $\|b_j\|^p_{\text{BMO}(\mathbb{R}^n)} = 1/C_j$ for all $1 \leq j \leq k$. We shall carry out our argument by induction on $l$. If $l = 0$, the Young inequality gives that

$$\int_Q |E^* h(y)|^p dy \leq C \|\tilde{\Omega}\|^p \|h\|_p^p \leq C \log^{-kp}(2 + \|\tilde{\Omega}\|_\infty).$$

Now let $d \leq k - 1$ be a non-negative integer and assume that the estimate (6) holds for $l = d$. We will show that (6) holds for $l = d + 1$. Observe that $\Phi(t) = t \log^p (2 + t)$ is a Young function and its complementary Young function is $\Psi(t) \approx \exp t^{1/p}$. By the general Hölder inequality, it follows that
\[ \int_Q |E^* h(x)|^p \prod_{j=1}^{d+1} |b_j(x) - m_Q(b_j)|^p dx \]
\[
\leq C \inf \left\{ \lambda > 0 : \int_Q \frac{|E^* h(x)|^p}{\lambda} \log \left( 2 + \frac{|E^* h(x)|^p}{\lambda} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 1 \right\} \times \inf \left\{ \lambda > 0 : \int_Q \exp \left( \frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 2 \right\},
\]

(see [1] or [8]). Applying the Young inequality again, we have
\[ \|E^* h\|_\infty \leq \|\hat{Q}\|_\infty \|h\|_1 \leq C \|\hat{Q}\|_\infty \|h\|_p \leq C \|\hat{Q}\|_\infty. \]

Our induction assumption now gives that
\[ \int_Q \frac{|E^* h(x)|^p}{\lambda} \log \left( 2 + \frac{|E^* h(x)|^p}{\lambda} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq C \log \left( 2 + \frac{C \|\hat{Q}\|_\infty}{\lambda} \right) \log^{(-k+d)p}(2 + \|\hat{Q}\|_\infty). \]

Set \( \lambda_0 = \log^{(-k+d+1)p}(2 + \|\hat{Q}\|_\infty) \). An easy computation then leads to that
\[ \int_Q \frac{|E^* h(x)|^p}{\lambda_0} \log \left( 2 + \frac{|E^* h(x)|^p}{\lambda_0} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq C \lambda_0. \]

On the other hand, by the Hölder inequality,
\[ \int_Q \exp \left( \frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \]
\[
\leq \left( \int_Q \exp \left( \frac{2|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}} \right) dx \right)^{1/2} \left( \int_Q \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^{2p} dx \right)^{1/(2r)} \]
\[
\leq \left( \int_Q \exp \left( \frac{2|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}} \right) dx \right)^{1/2} ,
\]

which together with the John-Nirenberg inequality implies that
\[ \inf \left\{ \lambda > 0 : \int_Q \exp \left( \frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}} \right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 2 \right\} \leq C, \]
Therefore,
\[
\int_Q |E^* h(x)|^p \prod_{j=1}^{d+1} |b_j(x) - m_Q(b_j)|^p \, dx \leq C \log^{(-k+d+1)p}(2 + \|\Omega\|_{\infty}).
\]

We can now prove our Lemma 2. By dilation-invariance, it suffices to consider the case \( r = 1 \). Write \( \mathbb{R}^n = \bigcup I_j \), where each \( I_j \) is a cube having side length 1 and the cubes have disjoint interiors. Let \( \chi_j \) be the characteristic function of \( I_j \). Set \( f_j = f \chi_j \). Then
\[
f(x) = \sum_j f_j(x), \quad \text{a.e. } x \in \mathbb{R}^n.
\]
Since the support of \( U_{A_1, \ldots, A_k; 1} f_j \) is contained in a fixed multiple of \( I_j \), the supports of various terms \( \{ U_{A_1, \ldots, A_k; 1} f_j \} \) have bounded overlaps, and so we have
\[
\| U_{A_1, \ldots, A_k; 1} f \|_p^p \leq C \sum_j \| U_{A_1, \ldots, A_k; 1} f_j \|_p^p.
\]
Thus we may assume that \( \text{supp } f \subseteq I \) for some cube \( I \) with side length 1. Set
\[
\tilde{A}_j(y) = A_j(y) - \sum_{|z_j| = m_j} \frac{1}{a_j} m_I(D^{z_j} A_j) y^a.
\]
A straightforward computation shows that for \( x, y \in \mathbb{R}^n \),
\[
R_{m_j+1}(A_j; x, y) = R_{m_j+1}(\tilde{A}_j; x, y).
\]
Choose \( n < q < \infty \), Lemma 1 now tells us that
\[
|R_m(\tilde{A}_j; x, y)|
\]
\[
\leq C|x - y|^{m_0} \sum_{|z_j| = m_j} \left( \frac{1}{I(x, y)} \int_{I(x, y)} |D^{z_j} A_j(z) - m_I(D^{z_j} A_j)|^q \, dz \right)^{1/q}
\]
\[
\leq C|x - y|^{m_0} \sum_{|z_j| = m_j} \left( \frac{1}{I(x, y)} \int_{I(x, y)} |D^{z_j} A_j(z) - m_{I(x,y)}(D^{z_j} A_j)|^q \, dz \right)^{1/q}
\]
\[
+ C|x - y|^{m_0} \sum_{|z_j| = m_j} \left| m_I(D^{z_j} A_j) - m_{I(x,y)}(D^{z_j} A_j) \right|
\]
\[
\leq C|x - y|^{m_0} \sum_{|z_j| = m_j} (\|D^{z_j} A_j\|_{\text{BMO}(\mathbb{R}^n)} + |m_I(D^{z_j} A_j) - m_{I(x,y)}(D^{z_j} A_j)|).
\]
Note that if \( y \in I \) and \(|x - y| \leq 1\), then \( \tilde{I}(x, y) \subset 100nI \). This in turn implies that for \( y \in I \) and \( 1/2 \leq |x - y| \leq 1\),

\[
|m_I(D^y A_j) - m_{I(y)}(D^y A_j)| \leq C\|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)}.
\]

Thus in this case, we have

\[
|R_{m_I}(A_j; x, y)| \leq C|x - y|^m \sum_{|x| = m} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} \leq C \sum_{|x| = m} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)}.
\]

Let

\[
\phi(y) = \prod_{j=1}^{k} \left( \sum_{|x| = m_j} \left( \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} + \|D^y A_j(y) - m_I(D^y A_j)\| \right) \right).
\]

We can write

\[
U_{A_1, \ldots, A_k; 1} f(x) \leq E(\|f\|)(x).
\]

A standard duality argument and the Hölder inequality then show that

\[
\|U_{A_1, \ldots, A_k; 1} f\|_p \leq \sup_{h \in 10nI, \|h\|_p \leq 1} \left| \int E(\|f\|)(x) h(x) dx \right|
\]

\[
= \sup_{h \in 10nI, \|h\|_p \leq 1} \int |E^* h(y) \phi(y) f(y)| dy
\]

\[
\leq C\|f\|_p \sup_{h \in 10nI, \|h\|_p \leq 1} \|E^* h\|_{p'},
\]

where \( p' \) is the dual exponent of \( p \), i.e. \( p' = p/(p - 1) \). Invoking the estimate (6) for \( 0 \leq l \leq k \), we finally obtain

\[
\|U_{A_1, \ldots, A_k; 1} f\|_p \leq C \prod_{j=1}^{k} \left( \sum_{|x| = m_j} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} \right) \|f\|_p.
\]

This completes the proof of Lemma 2.

**Proof of Theorem 1.** Without loss of generality, we may assume that for \( 1 \leq j \leq k \),

\[
\sum_{|x| = m_j} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} = 1.
\]

Let \( k_0 \) be a positive integer and \( P(x, y) \) be a real-valued non-trivial polynomial having degree \( k_0 \) in \( x \) and degree \( l_0 \) in \( y \). Write

\[
P(x, y) = \sum_{|\alpha| = k_0, |\beta| = l_0} a_{\alpha, \beta} x^\alpha y^\beta + R(x, y),
\]

where \( a_{\alpha, \beta} \) and \( R(x, y) \) are to be determined.

We can write

\[
UA_1, \ldots, A_k; 1 f(x) \leq E(\|f\|)(x).
\]

A standard duality argument and the Hölder inequality then show that

\[
\|UA_1, \ldots, A_k; 1 f\|_p \leq \sup_{h \in 10nI, \|h\|_p \leq 1} \left| \int E(\|f\|)(x) h(x) dx \right|
\]

\[
= \sup_{h \in 10nI, \|h\|_p \leq 1} \int |E^* h(y) \phi(y) f(y)| dy
\]

\[
\leq C\|f\|_p \sup_{h \in 10nI, \|h\|_p \leq 1} \|E^* h\|_{p'},
\]

where \( p' \) is the dual exponent of \( p \), i.e. \( p' = p/(p - 1) \). Invoking the estimate (6) for \( 0 \leq l \leq k \), we finally obtain

\[
\|UA_1, \ldots, A_k; 1 f\|_p \leq C \prod_{j=1}^{k} \left( \sum_{|x| = m_j} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} \right) \|f\|_p.
\]

This completes the proof of Lemma 2.

**Proof of Theorem 1.** Without loss of generality, we may assume that for \( 1 \leq j \leq k \),

\[
\sum_{|x| = m_j} \|D^y A_j\|_{\text{BMO}(\mathbb{R}^n)} = 1.
\]

Let \( k_0 \) be a positive integer and \( P(x, y) \) be a real-valued non-trivial polynomial having degree \( k_0 \) in \( x \) and degree \( l_0 \) in \( y \). Write

\[
P(x, y) = \sum_{|\alpha| = k_0, |\beta| = l_0} a_{\alpha, \beta} x^\alpha y^\beta + R(x, y),
\]

where \( a_{\alpha, \beta} \) and \( R(x, y) \) are to be determined.
Thus by dilation-invariance and Lemma 3, we may assume that $\sum_{|\alpha| = k_0, |\beta| = \beta_0} |a_{\alpha\beta}| = 1$. Split $T_{A_1, \ldots, A_k}$ as

$$T_{A_1, \ldots, A_k} f(x) = \int_{|x-y| \leq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+\delta}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy$$

$$+ \sum_{j=1}^{\infty} \int_{2^{j-1} < |x-y| \leq 2^j} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+\delta}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy$$

$$= T_{A_1, \ldots, A_k}^0 f(x) + \sum_{j=1}^{\infty} T_{A_1, \ldots, A_k}^j f(x).$$

We first consider the operator $T_{A_1, A_2, \ldots, A_k}^j$ for $j \geq 1$. Let $E_0 = \{ x' \in S^{n-1}, |\Omega(x')| \leq 2 \}$ and $E_l = \{ x' \in S^{n-1}, 2^l < |\Omega(x')| \leq 2^{l+1} \}$ for positive integer $l$. Let $\Omega_l$ be the restriction of $\Omega$ on $E_l$. Define the operator $T_{A_1, \ldots, A_k; l}$ by

$$T_{A_1, \ldots, A_k; l}^j f(x) = \int_{2^{l-1} < |x-y| \leq 2^l} e^{iP(x,y)} \frac{\Omega_l(x-y)}{|x-y|^{n+\delta}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy.$$ 

To estimate the $L^p(\mathbb{R}^d)$ boundedness for $T_{A_1, \ldots, A_k; l}^j$, we will use the following lemma.

**Lemma 3.** Let the polynomial $P(x, y)$, $k$, $m_u$ and $A_u$ ($u = 1, \ldots, k$) be the same as above, $\Omega$ be homogeneous of degree zero and belong to the space $L^\infty(S^{n-1})$. Define the operator

$$V_j f(x) = \int_{1 < |x-y| \leq 2} e^{iP(2x, 2y)} \frac{\tilde{\Omega}(x-y)}{|x-y|^{n+\delta}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy.$$ 

Then for $1 < p < \infty$, there exists positive constants $C$ and $\delta$ which are depending only on $n$, $p$ and $\text{deg } P$ such that

$$\| V_j f \|_p \leq C\| \tilde{\Omega} \|_\infty 2^{-j\delta} \| f \|_p.$$ 

For the case of $k = 1$, this lemma was proved essentially in [3, page 43–46]. For general positive integer $k$, Lemma 3 can be proved by induction on $k$. We omit the details.

We now estimate $T_{A_1, \ldots, A_k; l}^j$. Note that for $b \in \text{BMO}(\mathbb{R}^d)$ and $t > 0$, $b_t(x) = b(tx)$ also belongs to the space $\text{BMO}(\mathbb{R}^d)$ and $\| b_t \|_{\text{BMO}(\mathbb{R}^d)} = \| b \|_{\text{BMO}(\mathbb{R}^d)}$. Thus by dilation-invariance and Lemma 3,

$$\| T_{A_1, \ldots, A_k; l}^j f \|_p \leq C 2^{-j\delta} \| f \|_p.$$ 

(7)
On the other hand, Lemma 2 states that
\begin{equation}
\|T_{A_1, \ldots, A_k; i} f\|_p \leq C \lambda \Omega, k \|f\|_p.
\end{equation}

Set \( \lambda = l^k \|\Omega\|_1 + 2^{-i} \). A trivial computation gives that
\[
\frac{\|\Omega\|_1}{\lambda} \log \left(2 + \frac{\|\Omega\|_\infty}{\lambda}\right) \leq \frac{\|\Omega\|_1}{l^k \|\Omega\|_1} \log \left(2 + \frac{\|\Omega\|_\infty}{2^{-i}}\right) \leq C,
\]
which in turn implies
\begin{equation}
\lambda \Omega, k \leq C (l^k \|\Omega\|_1 + 2^{-i}).
\end{equation}

Our hypothesis on \( \Omega \) now says that \( \sum_{l \geq 0} l^k \|\Omega\|_1 < \infty \). Let \( N \) be a positive integer such that \( N > 2\delta^{-1} \). Combining the inequalities (7) and (8) yields that
\[
\left\| \sum_{l \geq 1} \sum_{i \geq 0} T_{A_1, \ldots, A_k; i} f \right\|_p \leq \sum_{l \geq 1} \|T_{A_1, \ldots, A_k; 0} f\|_p + \sum_{l \geq 0} \sum_{j \geq N} \|T_{A_1, \ldots, A_k; j} f\|_p
\]
\[
+ \sum_{l \geq 0} \sum_{j \geq N} \|T_{A_1, \ldots, A_k; j} f\|_p
\]
\[
\leq C \sum_{l \geq 1} 2^{-\delta l} \|f\|_p + C \sum_{l \geq 0} 2^{l} \sum_{j \geq N} 2^{-\delta j} \|f\|_p
\]
\[
+ C \sum_{l \geq 0} l \lambda \Omega, k \|f\|_p \leq C \|f\|_p.
\]

Now we turn our attention to the operator \( T_{A_1, \ldots, A_k}^0 \). The estimate for this term follows from the following lemma directly.

**Lemma 4.** Let \( 1 < p < \infty \), and \( S_{A_1, \ldots, A_k} \) be defined by (4) with \( \Omega \in L(\log L)^k (S^{n-1}) \). Suppose that \( S_{A_1, \ldots, A_k} \) is bounded on \( L^p(\mathbb{R}^n) \). Then for any real-valued polynomial \( \tilde{P}(x, y) \), the operator
\[
W_{A_1, \ldots, A_k} f(x) = \int_{|x-y| \leq 1} e^{i\tilde{P}(x, y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy,
\]
is bounded on \( L^p(\mathbb{R}^n) \) with a bound \( C(n, m, p, \deg \tilde{P}) \).

**Proof.** We follow along the same line as in the proof of Lemma 6 in [3]. We shall carry out the argument by a double induction on the degree in \( x \) and \( y \) of the polynomial. Obviously, Lemma 4 holds if the polynomial \( \tilde{P}(x, y) \) depends only on \( x \) or only on \( y \). Let \( u \) and \( v \) be two positive integers and the...
polynomial $\tilde{P}(x, y)$ have degree $u$ in $x$ and $v$ in $y$. We assume that Lemma 4 is known for all polynomials which are sums of monomials of degree less than $u$ in $x$ times monomials of any degree in $y$, together with monomials which are of degree $u$ in $x$ times monomials which are of degree less than $v$ in $y$.

We can now write

$$\tilde{P}(x, y) = \sum_{|\mu|=u, |\nu|=v} b_{\mu\nu} x^\mu y^\nu + P_0(x, y),$$

where $P_0(x, y)$ satisfies the inductive assumption. We consider the following two cases.

**Case I.** $\sum_{|\mu|=u, |\nu|=v} |b_{\mu\nu}| \leq 1$. As in the proof of Lemma 2, we may assume that $\text{supp } f \subset I$ for some cube $I$ centered at $x_0$ and having side length 1. By translation-invariance (note that our result is independent of the coefficients of the polynomial), we may assume that $\text{supp } f \subset I_0$, the cube centered at the origin and having side length 1. Set

$$P(x, y) = P_0(x, y) + \sum_{|\mu|=u, |\nu|=v} b_{\mu\nu} x^\mu y^\nu.$$

Observe that if $|x - y| \leq 1$ and $y \in I_0$, then

$$|e^{i\tilde{P}(x, y)} - e^{iP(x, y)}| \leq C|x - y|.$$

Thus,

$$|W_{A_1, \ldots, A_k} f(x)| \leq \left| \int_{|x-y| \leq 1} e^{iP(x, y)} \frac{\Omega(x - y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j; x, y) f(y) dy \right|$$

$$+ C \int_{|x-y| \leq 1} \frac{\Omega(x - y)}{|x-y|^{n+m}} \prod_{j=1}^{k} |R_{m_j+1}(A_j; x, y)||f(y)| dy$$

$$\leq \left| \int_{|x-y| \leq 1} e^{iP(x, y)} \frac{\Omega(x - y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j; x, y) f(y) dy \right|$$

$$+ C \sum_{j=0}^{\infty} 2^{-j} U_{A_1, \ldots, A_k; 2^{-j}} f(x),$$

where $U_{A_1, \ldots, A_k; 2^{-j}}$ is defined by (5). Set

$$U_{A_1, \ldots, A_k; 2^{-j}} f(x) = 2^{-j(n+m)} \int_{2^{-j-1} < |x-y| \leq 2^{-j}} \frac{\Omega_f(x - y)}{|x-y|^{n+m}} \prod_{a=1}^{k} |R_{m_a+1}(A_a; x, y)||f(y)| dy.$$
It follows from Lemma 2 and the inequality (9) that
\[ \sum_{j=0}^{\infty} 2^{-j} \| U_{A_1, \ldots, A_k; 2^{-j} f} \|_p \leq C \sum_{j=0}^{\infty} 2^{-j} \| U'_{A_1, \ldots, A_k; 2^{-j} f} \|_p \]
\[ \leq C \| f \|_p + \sum_{j=0}^{\infty} 2^{-j} \sum_{\ell \geq 1} \| F_{\ell} \|_p \| f \|_p \leq C \| f \|_p. \]

This via the induction hypothesis tells us that
\[ \| W_{A_1, \ldots, A_k f} \|_p \leq C(n, m, p, \deg \tilde{P}) \| f \|_p. \]

**Case II.** \( \sum_{|p|=m, |v|=v} |b_{pv}| > 1. \) Set \( J = (\sum_{|p|=m, |v|=v} |b_{pv}|)^{1/(u+e)}. \) Let
\[ Q(x, y) = \sum_{|p|=m, |v|=v} \frac{b_{pv}}{J^{u+e}} x^u y^v + P_0(x/J, y/J). \]

Then \( \tilde{P}(x, y) = Q(Jx, Jy). \) Define the operator
\[ \tilde{W}_{A_1, \ldots, A_k} f(x) = \int_{|x-y| \leq J} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy. \]

By dilation-invariance, it suffices to prove that
\[ \| \tilde{W}_{A_1, \ldots, A_k} f \|_p \leq C(n, m, p, \deg \tilde{P}) \| f \|_p. \]

We split the operator \( \tilde{W}_{A_1, \ldots, A_k} \) as
\[ \tilde{W}_{A_1, \ldots, A_k} f(x) = \int_{|x-y| \leq 1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy \]
\[ + \sum_{j=1}^{j_0} \int_{2^{j-1} \leq |x-y| \leq 2^j} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy \]
\[ + \int_{2^{j_0} \leq |x-y| \leq J} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy \]
\[ = \tilde{W}^1 f(x) + \tilde{W}^{II} f(x) + \tilde{W}^{III} f(x), \]

where \( j_0 \) is the positive integer such that \( 2^{j_0} < J \leq 2^{j_0+1}. \) The conclusion of Case I applies to \( \tilde{W}^1, \) so
\[ \| \tilde{W}^1 f \| \leq C(n, m, p, \deg \tilde{P}) \| f \|_p. \]
By the inequalities (7), (8) and (9) as in the estimate for \( \sum_{j \geq 1} T_{A_1, A_2, \ldots, A_k} \), we can obtain that
\[
\| \tilde{W}^{II} f \|_p \leq C(n, m, p, \deg \tilde{P}) \| f \|_p.
\]
On the other hand, it follows from Lemma 2 and the estimate (9) that
\[
\| \tilde{W}^{III} f \|_p \leq C(n, m, p, \deg \tilde{P}) \| f \|_p.
\]
This leads to the estimate (10), and completes the proof of Lemma 4.

Acknowledgement

The authors would like to thank the referee for some valuable suggestions and corrections.

References


Zefu Chu, Guoen Hu and Zhibo Lu
Department of Applied Mathematics
University of Information Engineering
P.O. Box 1001-747, Zhengzhou 450002
People’s Republic of China
Z. Chu: chuzf@163.net
G. Hu: huguoen@371.net
Z. Lu: luzhb21@sina.com