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Periodic solutions for nonautonomous predator-prey system with diffusion and time delay*

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ABSTRACT. By using the continuation theorem of coincidence degree theory, the existence of a positive periodic solution for the Lotka-Volterra population model, considered by Song-Chen. [1], is established.

1. Introduction

X. Song and L. Chen [1] considered the following Lotka-Volterra population model:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)(a_1(t) - b_1(t)x_1(t) - c(t)y(t)) + D_1(t)(x_2(t) - x_1(t)), \\ \dfrac{dx_2(t)}{dt} = x_2(t)(a_2(t) - b_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ \dfrac{dy(t)}{dt} = y(t)\left(- d(t) + p(t)x_1(t) - q(t)y(t) - \beta(t)\displaystyle\int_{-\tau}^{0} k(s)y(t+s)ds \right), \end{cases} \tag{1.1}$$

where x_1 and y are the population density of prey species x and predator species y in patch 1, and x_2 is the density of species x in patch 2. Predator species y is confined to patch 1, while the prey species x can diffuse between two patches. $D_i(t)$ $(i = 1, 2)$ are diffusion coefficients of species x.

In [1], they proved that system (1.1) is uniformly persistent under appropriate conditions and obtained sufficient conditions for global stability of the system (1.1). Our purpose in this paper is, by using the continuation theorem which was proposed in [2] by Gaines and Mawhin, to establish the existence of at least one positive w-periodic solution of system (1.1).

First, consider an abstract equation in a Banach space X,

$$Lx = \lambda Nx, \qquad \lambda \in (0, 1), \tag{1.2}$$

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where $L : \mathrm{Dom}\, L \cap X \to X$ is a linear operator and λ is a parameter. Let P and Q denote two projectors,

$$P : X \cap \mathrm{Dom}\, L \to \mathrm{Ker}\, L \qquad \text{and} \qquad Q : X \to X \backslash \mathrm{Im}\, L.$$

For convenience we introduce a continuation theorem [2, p. 40] as follows.

Lemma 1.1. *Let X be a Banach space and L a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore assume:*

(a) *for each $\lambda \in (0,1), x \in \partial\Omega \cap \mathrm{Dom}\, L$,*

$$Lx \neq \lambda Nx;$$

(b) *for each $x \in \partial\Omega \cap \mathrm{Ker}\, L$,*

$$QNx \neq 0;$$

(c) $deg\{QNx, \Omega \cap \mathrm{Ker}\, L, 0\} \neq 0.$

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

2. Main result

In what follows, we use the following notation:

$$\bar{f} = \frac{1}{w}\int_0^w f(t)dt, \qquad f^l = \min_{t \in [0,w]} |f(t)|, \qquad f^u = \max_{t \in [0,w]} |f(t)|,$$

where f is a periodic continuous function with period $w > 0$.

In system (1.1), we always assume the following.

Assumption (H_1). $a_i(t)$, $b_i(t)$, $D_i(t)$ $(i = 1,2)$, $c(t)$, $d(t)$, $p(t)$, $q(t)$ and $\beta(t)$ are positive periodic continuous functions with period $w > 0$.

Assumption (H_2). $k(s)$ is a continuous and nonnegative function on $[-\tau, 0]$, $0 \leq \tau < \infty$.

Now we state our fundamental theorem about the existence of a positive w-periodic solution of system (1.1).

Theorem 2.1. *In addition to Assumption (H_1) and (H_2), we assume the following:*

(i) $p^l(a_1 - D_1)^l > b_1^u d^u$;

(ii) $a_2(t) > D_2(t)$, *for* $\forall t \in R$.

Then system (1.1) has at least one positive w-periodic solution.

Proof. Consider the system

$$\begin{cases} \dfrac{dy_1}{dt} = a_1(t) - D_1(t) - b_1(t)e^{y_1(t)} - c(t)e^{y_3(t)} + D_1(t)e^{y_2(t)-y_1(t)}, \\ \dfrac{dy_2}{dt} = a_2(t) - D_2(t) - b_2(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}, \\ \dfrac{dy_3}{dt} = -d(t) + p(t)e^{y_1(t)} - q(t)e^{y_3(t)} - \beta(t)\displaystyle\int_{-\tau}^{0} k(s)e^{y_3(t+s)}\,ds, \end{cases} \tag{2.1}$$

where $a_i(t)$, $b_i(t)$, $D_i(t)$ $(i = 1, 2)$, $c(t)$, $d(t)$, $p(t)$, $q(t)$ and $\beta(t)$ are the same as those in Assumption (H_1), and τ and $k(s)$ are the same as those in Assumption (H_2), it is easy to see that if system (2.1) has an w-periodic solution $(y_1^*(t), y_2^*(t), y_3^*(t))$, then $(e^{y_1^*(t)}, e^{y_2^*(t)}, e^{y_3^*}(t))$ is a positive w-periodic solution of system (1.1). Therefore, for (1.1) to have at least one positive w-periodic solution it is sufficient that (2.1) has at least one w-periodic solution. In order to apply Lemma 1.1 to system (2.1), we take

$$X = \{(y_1(t), y_2(t), y_3(t))^T \in C(R, R^3) : y_i(t + w) = y_i(t), i = 1, 2, 3\}$$

and

$$\|(y_1(t), y_2(t), y_3(t))^T\| = \max_{t\in[0,w]} |y_1(t)| + \max_{t\in[0,w]} |y_2(t)| + \max_{t\in[0,w]} |y_3(t)|.$$

With this norm, X is a Banach space. Let

$$N\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1(t) - D_1(t) - b_1(t)e^{y_1(t)} - c(t)e^{y_3(t)} + D_1(t)e^{y_2(t)-y_1(t)} \\ a_2(t) - D_2(t) - b_2(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)} \\ -d(t) + p(t)e^{y_1(t)} - q(t)e^{y_3(t)} - \beta(t)\displaystyle\int_{-\tau}^{0} k(s)e^{y_3(t+s)}\,ds \end{bmatrix},$$

$$L\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}, \qquad P\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Q\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{w}\displaystyle\int_0^w y_1(t)dt \\ \dfrac{1}{w}\displaystyle\int_0^w y_2(t)dt \\ \dfrac{1}{w}\displaystyle\int_0^w y_3(t)dt \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in X.$$

Since $\operatorname{Ker} L = R^3$ and $\operatorname{Im} L$ is closed in X, L is a Fredholm mapping of index zero. Furthermore, we have that N is L-compact on $\overline{\Omega}$ [2]; here Ω is any open bounded set in X. Corresponding to equation (1.2), we have

$$
\begin{cases}
\dfrac{dy_1}{dt} = \lambda[a_1(t) - D_1(t) - b_1(t)e^{y_1(t)} - c(t)e^{y_3(t)} + D_1(t)e^{y_2(t)-y_1(t)}], \\
\dfrac{dy_2}{dt} = \lambda[a_2(t) - D_2(t) - b_2(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}], \\
\dfrac{dy_3}{dt} = \lambda\left[-d(t) + p(t)e^{y_1(t)} - q(t)e^{y_3(t)} - \beta(t)\displaystyle\int_{-\tau}^{0} k(s)e^{y_3(t+s)}\,ds\right].
\end{cases} \tag{2.2}
$$

Suppose that $(y_1(t), y_2(t), y_3(t))^T \in X$ is a solution of system (2.2) for a certain $\lambda \in (0,1)$. By integrating (2.2) over the interval $[0,w]$, we obtain

$$
\int_0^w D_1(t)e^{y_2(t)-y_1(t)}\,dt + \int_0^w (a_1(t) - D_1(t))dt = \int_0^w b_1(t)e^{y_1(t)}\,dt + \int_0^w c(t)e^{y_3(t)}\,dt, \tag{2.3}
$$

$$
\int_0^w D_2(t)e^{y_1(t)-y_2(t)}\,dt + \int_0^w (a_2(t) - D_2(t))dt = \int_0^w b_2(t)e^{y_2(t)}\,dt \tag{2.4}
$$

and

$$
\int_0^w p(t)e^{y_1(t)}\,dt = \int_0^w d(t)dt + \int_0^w q(t)e^{y_3(t)}\,dt + \int_0^w \beta(t)\left(\int_{-\tau}^0 k(s)e^{y_3(t+s)}\,ds\right)dt. \tag{2.5}
$$

From (2.2)–(2.5), it follows that

$$
\int_0^w |y_1'(t)|dt \le \int_0^w (a_1(t) - D_1(t))dt + \int_0^w b_1(t)e^{y_1(t)}\,dt + \int_0^w c(t)e^{y_3(t)}\,dt + \int_0^w D_1(t)e^{y_2(t)-y_1(t)}\,dt, \tag{2.6}
$$

$$
\int_0^w |y_2'(t)|dt \le \int_0^w (a_2(t) - D_2(t))dt + \int_0^w b_2(t)e^{y_2(t)}\,dt + \int_0^w D_2(t)e^{y_1(t)-y_2(t)}\,dt \tag{2.7}
$$

and

$$
\int_0^w |y_3'(t)|dt \le \int_0^w d(t)dt + \int_0^w p(t)e^{y_1(t)}\,dt + \int_0^w q(t)e^{y_3(t)}\,dt + \int_0^w \beta(t)\left(\int_{-\tau}^0 k(s)e^{y_3(t+s)}\,ds\right)dt. \tag{2.8}
$$

By (2.3)–(2.5) and (2.6)–(2.8), we obtain

$$
\int_0^w |y_1'(t)|dt \le 2\int_0^w b_1(t)e^{y_1(t)}\,dt + 2\int_0^w c(t)e^{y_3(t)}\,dt, \tag{2.9}
$$

$$\int_0^w |y_2'(t)|dt \le 2\int_0^w b_2(t)e^{y_2(t)}\, dt \tag{2.10}$$

and

$$\int_0^w |y_3'(t)|dt \le 2\int_0^w p(t)e^{y_1(t)}\, dt. \tag{2.11}$$

Choose $t_i^* \in [0, w]$, $i = 1, 2, 3$ such that

$$y_i(t_i^*) = \max_{t\in[0,w]} y_i(t), \qquad i = 1, 2, 3.$$

Then it is clear that

$$y_i'(t_i^*) = 0, \qquad i = 1, 2, 3.$$

From this and (2.2), we have

$$a_1(t_1^*) - D_1(t_1^*) - b_1(t_1^*)e^{y_1(t_1^*)} - c(t_1^*)e^{y_3(t_1^*)} + D_1(t_1^*)e^{y_2(t_1^*)-y_1(t_1^*)} = 0, \tag{2.12}$$

$$a_2(t_2^*) - D_2(t_2^*) - b_2(t_2^*)e^{y_2(t_2^*)} + D_2(t_2^*)e^{y_1(t_2^*)-y_2(t_2^*)} = 0 \tag{2.13}$$

and

$$-d(t_3^*) + p(t_3^*)e^{y_1(t_3^*)} - q(t_3^*)e^{y_3(t_3^*)} - \beta(t_3^*)\int_{-\tau}^0 k(s)e^{y_3(t_3^*+s)}\, ds = 0. \tag{2.14}$$

(2.14) implies that

$$q(t_3^*)e^{y_3(t_3^*)} < p(t_3^*)e^{y_1(t_3^*)} \tag{2.15}$$

and

$$e^{y_1(t_3^*)} > \frac{d(t_3^*)}{p(t_3^*)} > \frac{d^l}{p^u}. \tag{2.16}$$

Combining (2.16) with (2.12), we have

$$b_1^l e^{y_1(t_1^*)} < b_1(t_1^*)e^{y_1(t_1^*)} < (a_1 - D_1)^u + \frac{D_1^u p^u}{d^l} e^{y_2(t_2^*)}, \tag{2.17}$$

from which, together with (2.13), it implies that

$$d^l b_1^l b_2^l e^{2y_2(t_2^*)} < (b_1^l d^l (a_2 - D_2)^u + D_2^u D_1^u p^u)e^{y_2(t_2^*)} + D_2^u d^l (a_1 - D_1)^u.$$

Thus

$$2d^l b_1^l b_2^l e^{y_2(t_2^*)} < b_1^l d^l (a_2 - D_2)^u + D_1^u D_2^u p^u$$
$$+ \{(b_1^l a^l (a_2 - D_2)^u + D_1^u D_2^u p^u)^2 + 4d^l b_1^l b_2^l D_2^u d^l (a_1 - D_1)^u\}^{1/2},$$

from which, by using inequality

$$(a+b)^{1/2} < a^{1/2} + b^{1/2}, \qquad \text{for } a > 0, \ b > 0, \tag{2.18}$$

it implies that

$$2d^l b_1^l b_2^l e^{y_2(t_2^*)} < 2b_1^l d^l (a_2 - D_2)^u + 2D_1^u D_2^u p^u + 2\sqrt{d^l b_1^l b_2^l D_2^u d^l (a_1 - D_1)^u}.$$

That is

$$e^{y_2(t_2^*)} < \frac{(a_2 - D_2)^u}{b_2^l} + \frac{D_1^u D_2^u p^u + \sqrt{d^l b_1^l b_2^l D_2^u d^l (a_1 - D_1)^u}}{d^l b_1^l b_2^l} \overset{def}{=} A_1, \tag{2.19}$$

from which, together with (2.17), it follows that

$$e^{y_1(t_1^*)} < \frac{(a_1 - D_1)^u}{b_1^l} + \frac{D_1^u p^u}{b_1^l d^l} A_1 \overset{def}{=} A_2. \tag{2.20}$$

Combining (2.15) with (2.20), we obtain

$$e^{y_3(t_3^*)} < \frac{p^u}{q^l}\left[\frac{(a_1 - D_1)^u}{b_1^l} + \frac{D_1^u p^u}{b_1^l d^l} A_1\right] \overset{def}{=} A_3. \tag{2.21}$$

Therefore for $\forall t \in R$,

$$e^{y_1(t)} < A_2, \qquad e^{y_2(t)} < A_1, \qquad e^{y_3(t)} < A_3. \tag{2.22}$$

From (2.13), we have

$$e^{y_2(t_2^*)} > \frac{a_2(t_2^*) - D_2(t_2^*)}{b_2(t_2^*)} > \frac{(a_2 - D_2)^l}{b_2^u}. \tag{2.23}$$

By (2.3) and (2.5), it implies that there exist two points $\eta_1, \eta_2 \in (0, w)$ such that

$$b_1(\eta_1)\int_0^w e^{y_1(t)}\,dt + \int_0^w c(t)e^{y_3(t)}\,dt = \int_0^w D_1(t)e^{y_2(t)-y_1(t)}\,dt + \int_0^w (a_1(t) - D_1(t))dt \tag{2.24}$$

and

$$p(\eta_2)\int_0^w e^{y_1(t)}\,dt = \int_0^w d(t)dt + \int_0^w q(t)e^{y_3(t)}\,dt + \int_0^w \beta(t)\left(\int_{-\tau}^0 k(s)e^{y_3(t+s)}\,ds\right)dt. \tag{2.25}$$

Substituting (2.25) into (2.24) gives

$$b_1(\eta_1)\int_0^w q(t)e^{y_3(t)}\,dt + p(\eta_2)\int_0^w c(t)e^{y_3(t)}\,dt + b_1(\eta_1)\int_0^w \beta(t)\left(\int_{-\tau}^0 k(s)e^{y_3(t+s)}\,ds\right)dt$$
$$= p(\eta_2)\int_0^w D_1(t)e^{y_2(t)-y_1(t)}\,dt + p(\eta_2)\int_0^w (a_1(t)-D_1(t))dt - b_1(\eta_1)\int_0^w d(t)dt$$
$$> p^l w(a_1 - D_1)^l - b_1^u w d^u. \tag{2.26}$$

In view of

$$b_1(\eta_1)\int_0^w q(t)e^{y_3(t)}\,dt + p(\eta_2)\int_0^w c(t)e^{y_3(t)}\,dt + b_1(\eta_1)\int_0^w \beta(t)\left(\int_{-\tau}^0 k(s)e^{y_3(t+s)}\,ds\right)dt$$
$$< \left(b_1^u q^u + p^u c^u + b_1^u \beta^u \int_{-\tau}^0 k(s)ds\right) w e^{y_3(t_3^*)},$$

from this and (2.26), we have

$$e^{y_3(t_3^*)} > \frac{p^l(a_1 - D_1)^l - b_1^u d^u}{b_1^u q^u + p^u c^u + b_1^u \beta^u \int_{-\tau}^0 k(s)ds} \stackrel{def}{=} A_4. \tag{2.27}$$

By (2.9)–(2.11) and (2.22), we can get

$$\int_0^w |y_1'(t)|dt \le 2wb_1^u A_2 + 2wc^u A_3 \stackrel{def}{=} d_1, \tag{2.28}$$

$$\int_0^w |y_2'(t)|dt \le 2wb_2^u A_1 \stackrel{def}{=} d_2 \tag{2.29}$$

and

$$\int_0^w |y_3'(t)|dt \le 2p^u w A_2 \stackrel{def}{=} d_3 \tag{2.30}$$

From (2.16), (2.23) and (2.27), it follows that there exist three constants ρ_1, ρ_2 and ρ_3 such that

$$y_1(t_3^*) > -\rho_1, \qquad y_2(t_2^*) > -\rho_2, \qquad y_3(t_3^*) > -\rho_3. \tag{2.31}$$

Since for $\forall t \in [0, w]$,

$$y_1(t) = y_1(t_3^*) - \int_{t_3^*}^t y_1'(s)ds$$

$$y_2(t) = y_2(t_2^*) - \int_{t_2^*}^t y_2'(s)ds$$

and

$$y_3(t) = y_3(t_3^*) - \int_{t_3^*}^{t} y_3'(s)ds,$$

from (2.31) and (2.28)–(2.30), it implies that for $i = 1, 2, 3$,

$$y_i(t) > -\rho_i - \int_0^w |y_i'(s)|ds > -\rho_i - d_i. \tag{2.32}$$

From (2.22) and (2.32), we can obtain

$$|y_1(t)| \le \max\{|\ln A_2|, \rho_1 + d_1\} \stackrel{def}{=} R_1,$$

$$|y_2(t)| \le \max\{|\ln A_1|, \rho_2 + d_2\} \stackrel{def}{=} R_2$$

and

$$|y_3(t)| \le \max\{|\ln A_3|, \rho_3 + d_3\} \stackrel{def}{=} R_3.$$

Clearly, R_i $(i = 1, 2, 3)$ are independent of λ. Denote $M = R_1 + R_2 + R_3 + R_0$; here R_0 is taken sufficiently large such that each solution $(\alpha^*, \beta^*, \gamma^*)$ of the following system:

$$\begin{cases} \overline{a_1} - \overline{D_1} - \overline{b_1}e^{\alpha} - \bar{c}e^{\gamma} + \overline{D_1}e^{\beta-\alpha} = 0, \\ \overline{a_2} - \overline{D_2} - \overline{b_2}e^{\beta} + \overline{D_2}e^{\alpha-\beta} = 0, \\ -\bar{d} + \bar{p}e^{\alpha} - \left(\bar{q} + \bar{\beta}\displaystyle\int_{-\tau}^{0} k(s)ds\right)e^{\gamma} = 0, \end{cases} \tag{2.33}$$

satisfies $\|(\alpha^*, \beta^*, \gamma^*)\| = |\alpha^*| + |\beta^*| + |\gamma^*| < M$, provided that system (2.33) has a solution or a number of solution. Now we take $\Omega = \{(y_1(t), y_2(t), y_3(t))^T \in X : \|(y_1, y_2, y_3)^T\| < M\}$. This satisfies condition (a) of Lemma 1.1. When $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^T$ is a constant vector in R^3 with $|y_1| + |y_2| + |y_3| = M$. If system (2.33) has a solution or a number of solutions, then

$$QN\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \overline{a_1} - \overline{D_1} - \overline{b_1}e^{y_1} - \bar{c}e^{y_3} + \overline{D_1}e^{y_2-y_1} \\ \overline{a_2} - \overline{D_2} - \overline{b_2}e^{y_2} + \overline{D_2}e^{y_1-y_2} \\ -\bar{d} + \bar{p}e^{y_1} - \left(\bar{q} + \bar{\beta}\displaystyle\int_{-\tau}^{0} k(s)ds\right)e^{y_3} \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If system (2.33) does not have a solution, then naturally

$$QN\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This prove that condition (b) of Lemma 1.1 is satisfied. Finally we will prove that condition (c) of Lemma 1.1 is satisfied. To this end, we define $\phi : \mathrm{Dom}\, L \times [0,1] \to X$ by

$$\phi(y_1, y_2, y_3, \mu) = \begin{bmatrix} \overline{a_1} - \overline{D_1} - \overline{b_1}e^{y_1} - \bar{c}e^{y_3} \\ \overline{a_2} - \overline{D_2} - \overline{b_2}e^{y_2} \\ -\bar{d} + \bar{p}e^{y_1} - \left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)e^{y_3} \end{bmatrix} + \mu \begin{bmatrix} \overline{D_1}e^{y_2 - y_1} \\ \overline{D_2}e^{y_1 - y_2} \\ 0 \end{bmatrix},$$

where $\mu \in [0,1]$ is a parameter. When $(y_1, y_2, y_3)^T \in \partial\Omega \cap \mathrm{Ker}\, L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^T$ is a constant vector in R^3 with $|y_1| + |y_2| + |y_3| = M$. We will show that when $(y_1, y_2, y_3)^T \in \partial\Omega \cap \mathrm{Ker}\, L, \phi(y_1, y_2, y_3, \mu) \neq 0$. If the conclusion is not true, a.e., constant vector $(y_1, y_2, y_3)^T$ with $|y_1| + |y_2| + |y_3| = M$ satisfies $\phi(y_1, y_2, y_3, \mu) = 0$, then from

$$\begin{cases} \overline{a_1} - \overline{D_1} - \overline{b_1}e^{y_1} - \bar{c}e^{y_3} + \mu\overline{D_1}e^{y_2 - y_1} = 0, \\ \overline{a_2} - \overline{D_2} - \overline{b_2}e^{y_2} + \mu\overline{D_2}e^{y_1 - y_2} = 0, \\ -\bar{d} + \bar{p}e^{y_1} - \left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)e^{y_3} = 0, \end{cases}$$

following the argument of (2.22) and (2.31) gives

$$|y_1| < \max\{|\ln A_2|, \rho_1\},$$
$$|y_2| < \max\{|\ln A_1|, \rho_2\}$$

and

$$|y_3| < \max\{|\ln A_3|, \rho_3\}.$$

Thus

$$|y_1| + |y_2| + |y_3| < \max\{|\ln A_2|, \rho_1\} + \max\{|\ln A_1|, \rho_2\} + \max\{|\ln A_3|, \rho_3\} < M,$$

which contradicts the fact that $|y_1| + |y_2| + |y_3| = M$. Therefore

$$
\begin{aligned}
& deg(QN(y_1, y_2, y_3)^T, \Omega \cap \operatorname{Ker} L, (0,0,0)^T) \\
& \quad = deg(\phi(y_1, y_2, y_3, 1), \Omega \cap \operatorname{Ker} L, (0,0,0)^T) \\
& \quad = deg(\phi(y_1, y_2, y_3, 0), \Omega \cap \operatorname{Ker} L, (0,0,0)^T) \\
& \quad = deg\Bigg(\Bigg(\overline{a_1} - \overline{D_1} - \overline{b_1}e^{y_1} - \bar{c}e^{y_3}, \overline{a_2} - \overline{D_2} - \overline{b_2}e^{y_2}, -\bar{d} + \bar{p}e^{y_1} \\
& \qquad\qquad - \left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)e^{y_3}\Bigg)^T, \Omega \cap \operatorname{Ker} L, (0,0,0)^T\Bigg).
\end{aligned}
$$

Because of (i) of Theorem 2.1, then the system of algebraic equations

$$
\begin{cases}
\overline{a_1} - \overline{D_1} - \overline{b_1}u - \bar{c}z = 0, \\
\overline{a_2} - \overline{D_2} - \overline{b_2}v = 0, \\
-\bar{d} + \bar{p}u - \bar{q}z - \left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)z = 0
\end{cases}
$$

has a unique solution (u^*, v^*, z^*) which satisfies $u^* > 0$, $v^* > 0$ and $z^* > 0$, thus

$$
\begin{aligned}
& deg\Bigg(\Bigg(\overline{a_1} - \overline{D_1} - \overline{b_1}e^{y_1} - \bar{c}e^{y_3}, \overline{a_2} - \overline{D_2} - \overline{b_2}e^{y_2}, -\bar{d} + \bar{p}e^{y_1} \\
& \qquad - \left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)e^{y_3}\Bigg)^T, \Omega \cap \operatorname{Ker} L, (0,0,0)^T\Bigg) \\
& \quad = \operatorname{sign}\begin{vmatrix} -\overline{b_1}u^* & 0 & -\bar{c}z^* \\ 0 & -\overline{b_2}v^* & 0 \\ \bar{p}u^* & 0 & -\left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right)z^* \end{vmatrix} \\
& \quad = -\operatorname{sign}\left[\left(\overline{b_1}\,\overline{b_2}\left(\bar{q} + \bar{\beta}\int_{-\tau}^{0} k(s)ds\right) + \bar{p}\overline{b_2}\bar{c}\right)u^*v^*z^*\right] = -1.
\end{aligned}
$$

Consequently

$$
deg(QN(y_1, y_2, y_3)^T,\ \Omega \cap \operatorname{Ker} L, (0,0,0)^T) \neq 0.
$$

This completes the proof of condition (c) of Lemma 1.1. By now we have known that Ω verifies all the requirements of Lemma 1.1 and then system (2.1) has at least one w-periodic solution. This completes the proof.

References

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