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The homotopy groups $\pi_*(L_2V(0) \wedge T(k))$

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Abstract. Let $V(0)$ and $T(k)$ denote the mod p Moore spectrum and the Ravenel spectrum at a prime p, respectively. We determine the homotopy groups $\pi_*(L_2V(0) \wedge T(k))$ for $k \geq 2$ and $p > 2$. This is done by determining the chromatic E_1-term $H(k)^*M_1^1$, which is obtained by using only two key lemmas: one is to define the Miller-Ravenel-Wilson elements and the other is to give a one dimensional element ζ.

1. Introduction and the statement of results

Let $T(k)$ denote the Ravenel ring spectrum at a prime p, which is characterized by the Brown-Peterson homology $BP_*(T(k)) = BP_*[t_1, t_2, \dots, t_k] \subset BP_*(BP) = BP_*[t_1, t_2, \dots]$, where $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \dots]$. Note that $T(0) = S^0$. Then the homotopy groups $\pi_*(T(k))$ are, in a sense, an approximation of the homotopy groups $\pi_*(S^0)$ of spheres. For the Bousfield localization functor L_n on the stable homotopy category with respect to $v_n^{-1}BP$, the homotopy groups $\pi_*(L_nS^0)$ are also an approximation of $\pi_*(S^0)$. Both of the homotopy groups are considerably easier to compute than the homotopy groups of spheres. In this paper we determine the homotopy groups of $L_2V(0) \wedge T(k)$ for each $k \geq 2$ at an odd prime p, where $V(0)$ denotes the mod p Moore spectrum. These groups are computed by the Adams-Novikov spectral sequence and the chromatic spectral sequence. The E_1-terms of the chromatic spectral sequence are $H(k)^*M_1^0$ and $H(k)^*M_1^1$, where $H(k)^*M = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, M \otimes_{BP_*} BP_*(T(k)))$. If the prime p is odd, then the Adams-Novikov spectral sequence collapses from the E_2-term, and so it suffices to determine the chromatic E_1-terms to obtain the module structure of $\pi_*(L_2V(0) \wedge T(k))$. Ravenel determined $H(k)^*M_1^0$ (*cf.* [4]) and we determine $H(k)^*M_1^1$ here not only for an odd prime p but also for the prime 2. We note that our computation for $H(k)^0M_1^1$ also works in the case where $k = 1$ and $p > 2$. For the case $k = 0$, it is determined in [10] and [6] if $p > 3$, in [9] if $p = 3$ and in [8] if $p = 2$, which show that the computation is very

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tough. For the case $k=1$, $H(1)^*M_1^1$ at $p=2$ is determined in [7]. If $k>1$, then it is much easier, and has a kind of good nature. This will be studied in a forthcoming paper.

Throughout this note, we fix an integer $k \ge 1$ and $k>1$ if $p=2$. In order to state our theorems, we introduce some notations. The modules $k(1)_*$, $K(1)_*$, $k(1,j)_*$ and $k[3,k+1]_*$ are given by

$$k(1)_* = \boldsymbol{Z}/p[v_1],$$
$$K(1)_* = v_1^{-1}k(1)_*,$$
$$k(1,j)_* = k(1)_*/(v_1^j),$$
$$k[3,k+1]_* = \boldsymbol{Z}/p[v_3,\dots,v_{k+1}].$$

The integers a_n and c_n for each $n \ge 0$ are defined by

$$(1.1)\qquad \begin{aligned} a_0 &= 1,\\ a_{3m+1} &= p^{3m+1} + p(p^2+1)e_m,\\ a_{3m+2} &= pa_{3m+1} + p,\\ a_{3m+3} &= pa_{3m+2},\\ c_{3m+i} &= p^i(p-1)e_m,\\ e_m &= (p^{3m}-1)/(p^3-1) \end{aligned}$$

for $m \ge 0$ and $i = 1,2,3$. Our first theorem gives the chromatic E_1-term $H(k)^*M_1^1$. For this theorem, we consider the $k(1)_*$-modules defined by

$$\bar{A}(0) = K(2)_*\{v_{k+2}^s/v_1 \mid p\nmid s>0\},$$
$$\bar{A}(n) = k(1)_*/(v_1^{a_n})\{v_{k+2}^{sp^n}/v_1^{a_n} \mid p\nmid s>0\} \quad \text{for } n>0,$$
$$\bar{A}^0 = \textstyle\sum_{n>0}\bar{A}(n),$$

$$(1.2)\qquad \begin{aligned} \bar{A}^1 = \bar{A}(1)\{\zeta_2,\zeta_3\} \oplus \textstyle\sum_{m\ge 0}(&\bar{A}(3m+2)\{v_{k+2}^{p^{3m+1}(p-1)}\zeta_{1,m},\zeta_{3,m}\}\\ &\oplus \bar{A}(3m+3)\{\zeta_{1,m+1}, v_{k+2}^{p^{3m+1}(p-1)}\zeta_{2,m}\}\\ &\oplus \bar{A}(3m+4)\{\zeta_{2,m+1}, v_{k+2}^{p^{3m+3}(p-1)}\zeta_{3,m}\}), \end{aligned}$$

$$\begin{aligned} \bar{A}^2 = \bar{A}(1)\zeta_1^* \oplus \bar{A}(2)\zeta_2^* \oplus \textstyle\sum_{m\ge 0}(&\bar{A}(3m+3)v_{k+2}^{p^{3m+1}(p^2-1)}\zeta_{3,m}^*\\ &\oplus \bar{A}(3m+4)v_{k+2}^{p^{3m+2}(p^2-1)}\zeta_{1,m}^*\\ &\oplus \bar{A}(3m+5)v_{k+2}^{p^{3m+3}(p^2-1)}\zeta_{2,m}^*). \end{aligned}$$

Here $v_{k+2}^{sp^n}/v_1^{a_n}$ denotes the homology class represented by a cocycle whose leading term is $v_{k+2}^{sp^n}/v_1^{a_n}$ in the cobar complex $\Omega^0 M_1^1$. We denote $\zeta_1 = h_{k+1,0}$, $\zeta_i = h_{k+2,i-2}$ for $i = 2, 3$,

$$\zeta_{1,m} = [v_{k+2}^{c_{3m+1}} t_{k+1}], \qquad \zeta_{2,m} = [v_{k+2}^{c_{3m+2}} t_{k+2}], \qquad \zeta_{3,m} = [v_{k+2}^{c_{3m+3}} t_{k+2}^p],$$

and

$$\zeta_{1,m}^* = \zeta_{2,m}\zeta_{3,m}, \qquad \zeta_{2,m}^* = \zeta_{1,m+1}\zeta_{3,m} \qquad \zeta_{3,m}^* = \zeta_{1,m}\zeta_{2,m}.$$

Furthermore ζ denotes a cocycle whose leading term is $v_{k+2}^{p-1} t_{k+1}^p$ given in Lemma 2.3. Put $\bar{A}^* = (K(1)_*/k(1)_* \oplus \sum_{i=0}^2 \bar{A}^i) \otimes K(2)_*$.

Theorem 1.3. *The chromatic E_1-term $H(k)^* M_1^1$ is given as follows:*

1. *$H(k)^* M_1^1$ for $k > 1$ is the tensor product of $k[3, k+1]_*$ and the direct sum of $\bar{A}^* \otimes \Lambda(\zeta)$ and $\bar{A}(0) \otimes \Lambda(\zeta_1, \zeta_2, \zeta_3)$.*
2. *$H(1)^0 M_1^1$ at $p > 2$ is the direct sum of $K(1)_*/k(1)_*$, $\bar{A}(0)$ and $\bar{A}^0$.*

Let $E(2)$ denote the second Johnson-Wilson spectrum. We use the $E(2)$-based Adams spectral sequence to show our main theorem. Its E_2-term is $\mathrm{Ext}^*_{E(2)_*(E(2))}(E(2)_*, E(2)_*(X))$, and we use the notation

$$h(k)^* M = \mathrm{Ext}^*_{E(2)_*(E(2))}(E(2)_*, M \otimes_{E(2)_*} E(2)_*[t_1, \dots, t_k]).$$

Then the spectral sequence converging to $\pi_*(L_2V(0) \wedge T(k))$ has the E_2-term $h(k)^* E(2)_*/(p)$. By the change of rings theorem [1], we have

$$h(k)^* M_1^1 \otimes E(2)_* = H(k)^* M_1^1.$$

Consider the connecting homomorphism $\delta : h(k)^s M_1^1 \otimes E(2)_* \to h(k)^{s+1} E(2)_*/(p)$, and define

$$\begin{aligned} \beta_{i/j} &= \delta(v_{k+2}^i/v_1^j), \\ \beta_{i/j}\xi_{i,m} &= \delta(v_{k+2}^i \zeta_{i,m}/v_1^j), \\ \beta_{i/j} v_{k+2}^a \xi_{i,m} &= \delta(v_{k+2}^{i+a} \zeta_{i,m}/v_1^j), \qquad (1.4) \\ \beta_{i/j}\xi_{i,m}^* &= \delta(v_{k+2}^i \zeta_{i,m}^*/v_1^j), \\ \beta_{i/j} v_{k+2}^a \xi_{i,m}^* &= \delta(v_{k+2}^{i+a} \zeta_{i,m}^*/v_1^j). \end{aligned}$$

Furthermore, we put $\xi = \delta(x\zeta/v_1^j)$ for an element $x/v_1^j \in h(k)^* M_1^1 \otimes E(2)_*$. We put the $k(1)_*$-module

$$\bar{\boldsymbol{B}}(n) = \left(\sum_{s \in \boldsymbol{N} - p\boldsymbol{N}} k(1, a_n)_* \langle \beta_{sp^n/a_n} \rangle \right) \otimes k[3, k+1]_*$$

for each $n \geq 0$, and we define the $k(1)_*$-modules

$$\tilde{\boldsymbol{B}}_0 = \bar{\boldsymbol{B}}(0) \otimes \Lambda(\xi_1, \xi_2, \xi_3),$$

$$\bar{\boldsymbol{B}}^0 = \sum_{n>0} \bar{\boldsymbol{B}}(n),$$

$$\bar{\boldsymbol{B}}^1 = \bar{\boldsymbol{B}}(1)\{\xi_2, \xi_3\} \oplus \sum_{m \geq 0} (\bar{\boldsymbol{B}}(3m+2)\{v_{k+2}^{p^{3m+1}(p-1)}\xi_{1,m}, \xi_{3,m}\}$$
$$\oplus \bar{\boldsymbol{B}}(3m+3)\{\xi_{1,m+1}, v_{k+2}^{p^{3m+1}(p-1)}\xi_{2,m}\} \oplus \bar{\boldsymbol{B}}(3m+4)\{\xi_{2,m+1}, v_{k+2}^{p^{3m+3}(p-1)}\xi_{3,m}\}),$$

$$\bar{\boldsymbol{B}}^2 = \bar{\boldsymbol{B}}(1)\xi_1^* \oplus \bar{\boldsymbol{B}}(2)\xi_2^* \oplus \sum_{m \geq 0} (\bar{\boldsymbol{B}}(3m+3)v_{k+2}^{p^{3m+1}(p^2-1)}\xi_{3,m}^*$$
$$\oplus \bar{\boldsymbol{B}}(3m+4)v_{k+2}^{p^{3m+2}(p^2-1)}\xi_{1,m}^* \oplus \bar{\boldsymbol{B}}(3m+5)v_{k+2}^{p^{3m+3}(p^2-1)}\xi_{2,m}^*).$$

Then we have our main theorem:

THEOREM 1.5. *Let p be an odd prime and $k \geq 2$. Then the homotopy groups $\pi_*(L_2 V(0) \wedge T(k))$ are isomorphic as a $k(1)_*$-module to the direct sum of $k(1)_* \otimes \Lambda(\zeta)$ and $\tilde{\boldsymbol{B}}_0 \oplus (\bar{\boldsymbol{B}}^0 \oplus \bar{\boldsymbol{B}}^1 \oplus \bar{\boldsymbol{B}}^2) \otimes \Lambda(\xi)$.*

Note that Theorem 1.5 also holds for $p = 2$, if we replace the homotopy groups by the E_2-term of the $E(2)$-based Adams spectral sequence. We prove these theorems in the next section assuming two key lemmas, Lemmas 2.3 and 2.4, which are proved in §4 and §3, respectively. The authors express their gratitude to Prof. Xiangjun Wang who pointed out that our original proof works even at the prime 2.

2. $H(k)^* M_1^1$

We consider the Hopf algebroid $(A, \Gamma(k)) = (BP_*, BP_*[t_{k+1}, t_{k+2}, \dots])$ whose structure maps, say, $\eta_R : A \to \Gamma(k)$ and $\Delta : \Gamma(k) \to \Gamma(k) \otimes \Gamma(k)$, are induced from those of the Hopf algebroid $(BP_*, BP_*(BP))$ under the projection $BP_*BP \to \Gamma(k)$. Then $H(k)^*M$ for a comodule M is the cohomology of the cobar complex $(\Omega^* M, d)$, where $\Omega^s M = M \otimes_A \Gamma(k) \otimes_A \cdots \otimes_A \Gamma(k)$ (s factors of $\Gamma(k)$) and the differential d is defined inductively by $d(m) = \eta_R(m) - m$ for $m \in M$, $d(x) = 1 \otimes x + x \otimes 1 - \Delta(x)$ for $x \in \Gamma(k)$ and $d(x \otimes y) = d(x) \otimes y + (-1)^s x \otimes d(y)$ for $x \in \Omega^s M$ and $y \in \Gamma(k)$. Note that the change of rings theorem gives an isomorphism

$$H(k)^* M = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, M \otimes_{BP_*} BP_*[t_1, \dots, t_k]).$$

The comodules M_j^i are defined inductively by $N_j^0 = A/I_j$, $M_j^i = v_{i+j}^{-1} N_j^i$ and the short exact sequence

$$0 \to N_j^i \to M_j^i \to N_j^{i+1} \to 0. \tag{2.1}$$

We start with Ravenel's results (*cf.* [4]):

$$H(k)^*M_2^0 = K(2)_*[v_3, \dots, v_{k+2}] \otimes \Lambda(h_{k+1,0}, h_{k+1,1}, h_{k+2,0}, h_{k+2,1})$$
$$= K(2)_*[v_{k+2}] \otimes \Lambda(h_{k+1,0}, h_{k+1,1}, h_{k+2,0}, h_{k+2,1}) \otimes k[3, k+1]_*,$$

where $K(2)_* = \boldsymbol{Z}/p[v_2, v_2^{-1}]$, $k[3, k+1]_*$ denotes the trivial $BP_*(BP)$-comodule $\boldsymbol{Z}/p[v_3, \dots, v_{k+1}]$ and $h_{i,j}$ denotes the cohomology class represented by $t_i^{p^j}$.

We have the exact sequence

$$H(k)^*M_2^0 \xrightarrow{\varphi} H(k)^*M_1^1 \xrightarrow{v_1} H(k)^*M_1^1 \xrightarrow{\delta} H(k)^*M_2^0 \tag{2.2}$$

of graded modules associated to the short exact sequence $0 \to M_2^0 \xrightarrow{\varphi} M_1^1 \xrightarrow{v_1} M_1^1 \to 0$, where $\varphi(x) = x/v_1$ and δ raises the dimension by 1.

Now we state our key lemmas, whose proofs will be given in the following sections.

Lemma 2.3. *If $k \geq 2$, then there exists a cocycle r_0 of $\Omega^1 v_2^{-1}BP_*/(p, v_1^j)$ for any $j > 0$ whose leading term is $v_{k+2}^{p-1}t_{k+1}^p$.*

We denote by ζ the homology class represented by r_0.

Lemma 2.4. *Let k be a positive integer with $k > 1$ if $p = 2$. Then for the connecting homomorphism δ, we have*

$$\delta(v_{k+2}^s/v_1) = sv_{k+2}^{s-p}\zeta,$$
$$\delta(v_{k+2}^{sp^n}/v_1^{a_n}) = sv_2^{b_n}v_{k+2}^{p^n(s-1)}\zeta_{i,m}$$

for $s > 0$ and $n = 3m + i > 0$ with $i = 1, 2, 3$. Here the integers b_n are defined by

$$b_{3m+i} = p^{3m+i-1}(p^{k+1} - 1) + p^{i-1}(p^{k+1} - 1)e_m$$

for $i = 1, 2, 3$, and we abbreviate $\zeta_{i,0}$ to ζ_i for $i = 1, 2, 3$.

Consider the $K(2)_*$-modules

$$A(n) = K(2)_*\{v_{k+2}^{sp^n} \mid p \nmid s\}$$

for $n \geq 0$, and put $A^* = (K(2)_* \oplus \sum_{n\geq 1} A(n)) \otimes \Lambda(h_{k+1,0}, h_{k+2,0}, h_{k+2,1}) \otimes k[3, k+1]_* \subset H(k)^*M_2^0$ if $k > 1$. Then we have the following proposition whose proof will be given after the proofs of the theorems.

Proposition 2.5. *For $k \geq 2$, the module $\bar{A}^*$ of (1.2) fits in the commutative diagram of exact sequences*

$$\begin{array}{ccccccc}
A^* & \xrightarrow{\varphi} & \bar{A}^* & \xrightarrow{v_1} & \bar{A}^* & \xrightarrow{\delta} & A^* \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
H(k)^*M_2^0 & \xrightarrow{\varphi} & H(k)^*M_1^1 & \xrightarrow{v_1} & H(k)^*M_1^1 & \xrightarrow{\delta} & H(k)^*M_2^0.
\end{array}$$

Proof of Theorem 1.3. Let B^* denote the module of Theorem 1.3. Then it suffices to show that the sequence

$$H(k)^*M_2^0 \xrightarrow{\varphi} B^* \xrightarrow{v_1} B^* \xrightarrow{\delta} H(k)^*M_2^0$$

is exact by [2, Remark 3.11]. Note that $K(2)_*[v_{k+2}] \otimes \Lambda(h_{k+1,1})$ is isomorphic to

$$(A(0) \oplus A^*) \oplus (v_2^{-p}\zeta A(0) \oplus \zeta A^*).$$

Since $\delta(\xi\zeta_i) = \delta(\xi)\zeta_i$ for $\xi \in \bar{A}^*$, we have the exact sequence

$$(2.6) \quad A(0) \otimes \Lambda(\zeta_1, \zeta_2, \zeta_3) \xrightarrow{\varphi} \bar{A}(0) \otimes \Lambda(\zeta_1, \zeta_2, \zeta_3) \xrightarrow{0} \bar{A}(0) \otimes \Lambda(\zeta_1, \zeta_2, \zeta_3)$$
$$\xrightarrow{\delta} v_{k+2}^{-p} A(0) \otimes \Lambda(\zeta_1, \zeta_2, \zeta_3)$$

by the first equation of Lemma 2.4, where φ and δ are isomorphisms.

By Proposition 2.5, we have the exact sequence

$$(2.7) \qquad A^* \otimes \Lambda(\zeta) \to \bar{A}^* \otimes \Lambda(\zeta) \to \bar{A}^* \otimes \Lambda(\zeta) \xrightarrow{\delta} A^* \otimes \Lambda(\zeta).$$

The direct sum of these exact sequences yields the desired one. q.e.d.

Proof of Theorem 1.5. By (2.1), we have the short exact sequence $0 \to \tilde{N}_1^0 \to \tilde{M}_1^0 \to \tilde{M}_1^1 \to 0$, where we put $\tilde{M} = M \otimes_{BP_*} E(2)_*$ for a $BP_*(BP)$-comodule M. Ravenel shows that $h(k)^*\tilde{M}_1^0 = K(1)_*[v_2, \dots, v_{k+1}] \otimes \Lambda(h_{k+1,0})$. The long exact sequence associated to the short sequence splits into the exact sequence

$$0 \to h(k)^0\tilde{N}_1^0 \to h(k)^0\tilde{M}_1^0 \to h(k)^0\tilde{M}_1^1 \xrightarrow{\delta} h(k)^1\tilde{N}_1^0$$
$$\to h(k)^1\tilde{M}_1^0 \to h(k)^1\tilde{M}_1^1 \xrightarrow{\delta} h(k)^2\tilde{N}_1^0 \to 0$$

and the isomorphisms

$$\delta : h(k)^i\tilde{M}_1^1 \cong h(k)^{i+1}\tilde{N}_1^0$$

for $i > 1$. By the definition (1.4), we see that $\delta(\bar{A}(n)) = \bar{\boldsymbol{B}}(n)$. Note that ζ_1/v_1^j is homologous to ζ/v_1^{j-2} by Lemmas 3.3 and 3.4. Thus we obtain the E_2-term $h(k)^*\tilde{N}_1^0$ of the Adams-Novikov spectral sequence. Since $h^s(k)\tilde{N}_1^0 = 0$ for $s > 4$, we see that the Adams-Novikov spectral sequence collapses from the E_2-term. q.e.d.

The rest of this section is devoted to showing Proposition 2.5.
First we note a property of the non-negative integers.

REMARK. *For any* $n \geq 0$,
1. *There exist integers* l *and* m *with* $l \not\equiv p^2 - p \mod(p^3)$ *such that*

$$n = p^{3m}l + c_{3m+1}.$$

2. *For any* $n \geq 0$, *there exist integers* l *and* m *with* $l \not\equiv p^2 - 1 \mod(p^3)$ *such that*

$$np = p^{3m+1}l + c_{3m+1} + c_{3m+2}.$$

Consider the modules

$$A_2(m) = A(3m+1) \oplus A(3m+2)v_{k+2}^{-p^{3m+2}} \oplus A(3m+3)v_{k+2}^{p^{3m+2}(p-1)},$$

$$A_3(m) = A(3m+2) \oplus A(3m+3)v_{k+2}^{-p^{3m+3}} \oplus A(3m+4)v_{k+2}^{p^{3m+3}(p-1)},$$

$$A_1(m) = A(3m+3) \oplus A(3m+4)v_{k+2}^{-p^{3m+4}} \oplus A(3m+5)v_{k+2}^{p^{3m+4}(p-1)},$$

$$A_3(m)^* = A(3m+1)v_{k+2}^{-p^{3m+1}} \oplus A(3m+2)v_{k+2}^{-p^{3m+1}} \oplus A(3m+3)v_{k+2}^{p^{3m+1}(p^2-1)},$$

$$A_1(m)^* = A(3m+2)v_{k+2}^{-p^{3m+2}} \oplus A(3m+3)v_{k+2}^{-p^{3m+2}} \oplus A(3m+4)v_{k+2}^{p^{3m+2}(p^2-1)},$$

$$A_2(m)^* = A(3m+3)v_{k+2}^{-p^{3m+3}} \oplus A(3m+4)v_{k+2}^{-p^{3m+3}} \oplus A(3m+5)v_{k+2}^{p^{3m+3}(p^2-1)}.$$

Then we decompose $\boldsymbol{Z}/p[v_{k+2}^p]$ into a direct sum

$$\begin{aligned}
\boldsymbol{Z}/p[v_{k+2}^p] &\cong \sum_{m\geq 0} A_2(m)v_{k+2}^{c_{3m+2}} \\
&\cong A(1) \oplus \sum_{m\geq 0} A_3(m)v_{k+2}^{c_{3m+3}} \\
&\cong A(1)v_{k+2}^{-p} \oplus A(2)v_{k+2}^{p(p-1)} \oplus \sum_{m\geq 0} A_1(m)v_{k+2}^{c_{3m+1}} \\
&\cong \sum_{m\geq 0} A_3(m)^* v_{k+2}^{c_{3m+1}+c_{3m+2}} \\
&\cong A(1) \oplus \sum_{m\geq 0} A_1(m)^* v_{k+2}^{c_{3m+2}+c_{3m+3}} \\
&\cong A(1)v_{k+2}^{-p} \oplus A(2)v_{k+2}^{p^2-p} \oplus \sum_{m\geq 0} A_2(m)^* v_{k+2}^{c_{3m+3}+c_{3m+4}}.
\end{aligned}$$

This follows immediately from the above remark. In fact, we notice that

$$A_i(m) = \{v_{k+2}^{p^{3m-1+i}l} \mid l \not\equiv p^2 - p \bmod(p^3)\} \quad \text{for } i = 2, 3,$$

$$A_1(m) = \{v_{k+2}^{p^{3m+3}l} \mid l \not\equiv p^2 - p \bmod(p^3)\},$$

$$A_3(m)^* = \{v_{k+2}^{p^{3m+1}l} \mid l \not\equiv p^2 - 1 \bmod(p^3)\},$$

$$A_i(m)^* = \{v_{k+2}^{p^{3m+1+i}l} \mid l \not\equiv p^2 - 1 \bmod(p^3)\} \quad \text{for } i = 1, 2.$$

We now reformulate the action of the connecting homomorphism $\delta: H^*M_2^0 \to H^{*+1}M_1^1$ given in Lemma 2.4.

LEMMA 2.8. *The connecting homomorphism* δ *behaves* $\delta(\bar{A}(n)) = A(n)v_{k+2}^{-p^n}\zeta_{i,m}$ *for* $n = 3m + i$ *with* $i = 1, 2, 3$, *and for* $\bar{A}(n)$ *of* (1.2).

This implies immediately

COROLLARY 2.9. *For* $m \geq 0$, *the connecting homomorphism* δ *acts as follows:*

$$\delta(\bar{A}(1)\zeta_3) = A(1)v_{k+2}^{-p}\zeta_2^*,$$

$$\delta(\bar{A}(3m)\zeta_{1,m}) = A(3m)v_{k+2}^{-p^{3m}}\zeta_{2,m-1}^*,$$

$$\delta(\bar{A}(3m+2)v_{k+2}^{p^{3m+1}(p-1)}\zeta_{1,m}) = A(3m+2)v_{k+2}^{-p^{3m+1}}\zeta_{3,m}^*,$$

$$\delta(\bar{A}(3m+1)\zeta_{2,m}) = A(3m+1)v_{k+2}^{-p^{3m+1}}\zeta_{3,m}^*,$$

$$\delta(\bar{A}(3m+3)v_{k+2}^{p^{3m+2}(p-1)}\zeta_{2,m}) = A(3m+3)v_{k+2}^{-p^{3m+2}}\zeta_{1,m}^*,$$

$$\delta(\bar{A}(3m+2)\zeta_{3,m}) = A(3m+2)v_{k+2}^{-p^{3m+2}}\zeta_{1,m}^*,$$

$$\delta(\bar{A}(3m+4)v_{k+2}^{p^{3m+3}(p-1)}\zeta_{3,m}) = A(3m+4)v_{k+2}^{-p^{3m+4}}\zeta_{2,m}^*,$$

$$\delta(\bar{A}(1)\zeta_1^*) = A(1)1^*,$$

$$\delta(\bar{A}(2)\zeta_2^*) = A(2)1^*,$$

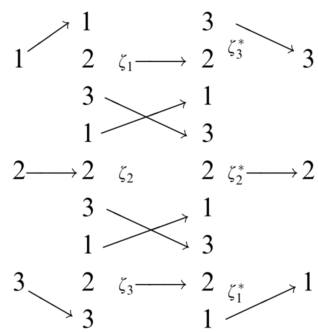
$$\delta(\bar{A}(3m+3)v_{k+2}^{p^{3m+1}(p^2-1)}\zeta_{3,m}^*) = A(3m+3)1^*,$$

$$\delta(\bar{A}(3m+4)v_{k+2}^{p^{3m+2}(p^2-1)}\zeta_{1,m}^*) = A(3m+4)1^*,$$

$$\delta(\bar{A}(3m+5)v_{k+2}^{p^{3m+3}(p^2-1)}\zeta_{2,m}^*) = A(3m+5)1^*,$$

where $1^* = v_{k+2}^{-p}\zeta_1\zeta_2\zeta_3$.

The proof of Proposition 2.5 is now an easy check as displayed in the following pattern.



For example, the picture on the left upper corner displays the diagram

$$A(3m+1) \to A(3m+1)v_{k+2}^{-p^{3m+1}}\zeta_{1,m}$$

$$A(3m+2)v_{k+2}^{p^{3m+1}(p-1)}\zeta_{1,m} \to A(3m+2)v_{k+2}^{-p^{3m+1}}\zeta_{3,m}^*$$

$$A(3m)\zeta_{1,m} \to A(3m)v_{k+2}^{-p^{3m}}\zeta_{2,m-1}^*.$$

3. The Miller-Ravenel-Wilson elements x_i

In this section we prove Lemma 2.4.

First we compute the action of $\eta_R : BP_* \to \Gamma(k) = BP_*[t_{k+1}, t_{k+2}, \dots]$ by using Hazewinkel's and Quillen's formulae

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i} \qquad \text{and} \qquad \eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i}.$$

Here $BP_* \otimes \boldsymbol{Q} = \boldsymbol{Q}[m_1, m_2, \dots]$. Since $t_i = 0$ for $0 < i \le k$, Quillen's formula turns into

$$\eta_R(m_n) = \begin{cases} m_n & \text{for } n \le k, \\ m_n + \sum_{i=k+1}^{n} m_{n-i} t_i^{p^{n-i}} & \text{for } n > k. \end{cases}$$

To describe $\eta_R(v_n)$ we introduce an element $w_{n,i}$ given by

$$pw_{n,i} = \eta_R(v_n^{p^i}) - \sum_l w_l^{p^i} \tag{3.1}$$

if we have $\eta_R(v_n) \equiv \sum_l w_l \mod(p)$ for some monomials w_l of $\Gamma(k)$.

LEMMA 3.2. *The right unit* $\eta_R : BP_* \to \Gamma(k)$ *acts on* v_n *for* $0 < n \le k+4$ *as follows:*

$$\eta_R(v_n) = v_n \qquad \text{for } n \le k,$$

$$\eta_R(v_{k+1}) \equiv v_{k+1} \qquad \mathrm{mod}(p),$$

$$\eta_R(v_{k+2}) \equiv v_{k+2} + v_1 t_{k+1}^p - v_1^{p^{k+1}} t_{k+1} \qquad \mathrm{mod}(p),$$

$$\eta_R(v_{k+3}) \equiv v_{k+3} + v_2 t_{k+1}^{p^2} + v_1 t_{k+2}^p - v_1 w_{k+2,1} - v_2^{p^{k+1}} t_{k+1} - v_1^{p^{k+2}} t_{k+2} \qquad \mathrm{mod}(p),$$

$$\begin{aligned} \eta_R(v_{k+4}) \equiv\ & v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_2 w_{k+2,2} - v_1 w_{k+3,1} \\ & - v_3^{p^{k+1}} t_{k+1} - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} \qquad \mathrm{mod}(p) \qquad \text{for } k > 1, \\ \equiv\ & v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_2 w_{k+2,2} - v_1 w_{k+3,1} \\ & - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} - t_{k+1}(v_{k+2}^{p^2} + v_1^{p^2} t_{k+1}^{p^3} - v_1^{p^{k+3}} t_{k+1}^{p^2}) \qquad \mathrm{mod}(p) \\ & \qquad \text{for } k = 1, \\ \equiv\ & v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_2 w_{k+2,2} - v_1 w_{k+3,1} \\ & - v_3^{p^{k+1}} t_{k+1} - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} \qquad \mathrm{mod}(p, v_1^{p^2}). \end{aligned}$$

PROOF. For v_n with $n \le k$, it follows immediately from the formulae. We compute $\eta_R(v_{k+1}) = \eta_R(pm_{k+1} - \sum_{i=1}^k m_i v_{1+k-i}^{p^i}) = p(m_{k+1} + t_{k+1}) - \sum_{i=1}^k m_i v_{1+k-i}^{p^i} = v_{k+1} + pt_{k+1}$. For v_{k+2}, we compute

$$\begin{aligned} \eta_R(v_{k+2}) &= \eta_R\left(pm_{k+2} - m_1 v_{k+1}^p - \sum_{i=2}^k m_i v_{2+k-i}^{p^i} - m_{k+1} v_1^{p^{k+1}}\right) \\ &= p(m_{k+2} + m_1 t_{k+1}^p + t_{k+2}) - m_1(v_{k+1}^p + p^2 w_{k+1}) \\ &\quad - \sum_{i=2}^k m_i v_{2+k-i}^{p^i} - (m_{k+1} + t_{k+1}) v_1^{p^{k+1}} \\ &= v_{k+2} + v_1 t_{k+1}^p + pt_{k+2} - pv_1 w_{k+1} - v_1^{p^{k+1}} t_{k+1}, \end{aligned}$$

where w_{k+1} is defined by $\eta_R(v_{k+1}^p) = (v_{k+1} + pt_{k+1})^p = v_{k+1}^p + p^2 w_{k+1}$.

If $k > 1$, then

$$\eta_R(v_{k+3}) = \eta_R\left(pm_{k+3} - (m_1 v_{k+2}^p + m_2 v_{k+1}^{p^2}) - \sum_{i=3}^{k} m_i v_{3+k-i}^{p^i} - (m_{k+1} v_2^{p^{k+1}} + m_{k+2} v_1^{p^{k+2}})\right)$$
$$= p(\underline{m_{k+3}}_1 + \underline{m_2 t_{k+1}^{p^2}}_2 + m_1 t_{k+2}^p + t_{k+3})$$
$$- m_1(\underline{v_{k+2}^p}_1 + \underline{v_1^p t_{k+1}^{p^2}}_2 - \underline{v_1^{p^{k+2}} t_{k+1}^p}_3 + p w_{k+2,1})$$
$$- m_2(\underline{v_{k+1}^{p^2}}_1 + p^3 w_{k+2}) - \underline{\sum_{i=3}^{k} m_i v_{3+k-i}^{p^i}}_1$$
$$- (\underline{m_{k+1}}_1 + t_{k+1}) v_2^{p^{k+1}} - (\underline{m_{k+2}}_1 + \underline{m_1 t_{k+1}^p}_3 + t_{k+2}) v_1^{p^{k+2}}$$
$$= v_{k+3} + v_2 t_{k+1}^{p^2} + v_1 t_{k+2}^p + p t_{k+3} - v_1 w_{k+2,1}$$
$$- p^3 m_2 w_{k+2} - v_2^{p^{k+1}} t_{k+1} - v_1^{p^{k+2}} t_{k+2}.$$

The underlined terms numbered 1, 2 and 3 sum up to v_{k+3}, $v_2 t_{k+1}^{p^2}$ and 0, respectively. Here note that $p^{i+1} m_i \equiv 0 \mod(p)$. If $k = 1$, then $k + 1 = 2$, and so

$$\eta_R(v_{k+3}) = \eta_R(pm_{k+3} - m_1 v_{k+2}^p - m_{k+1} v_{k+1}^{p^{k+1}} - m_{k+2} v_1^{p^{k+2}})$$
$$= p(\underline{m_{k+3}}_1 + \underline{m_2 t_{k+1}^{p^2}}_2 + m_1 t_{k+2}^p + t_{k+3})$$
$$- m_1(\underline{v_{k+2}^p}_1 + \underline{v_1^p t_{k+1}^{p^2}}_2 - \underline{v_1^{p^{k+2}} t_{k+1}^p}_3 + p w_{k+2,1})$$
$$- (\underline{m_{k+1}}_1 + t_{k+1})(v_{k+1}^{p^2} + p^3 w_{k+2}) - (\underline{m_{k+2}}_1 + \underline{m_1 t_{k+1}^p}_3 + t_{k+2}) v_1^{p^{k+2}}$$
$$\equiv v_{k+3} + v_2 t_{k+1}^{p^2} + v_1 t_{k+2}^p - v_1 w_{k+2,1} - v_2^{p^{k+1}} t_{k+1} - v_1^{p^{k+2}} t_{k+2} \qquad \mathrm{mod}(p),$$

and we see there is no difference between the cases $k > 1$ and $k = 1$. In the same way, assume $k > 2$, and

$$\eta_R(v_{k+4})$$
$$= \eta_R\left(pm_{k+4} - (m_1 v_{k+3}^p + m_2 v_{k+2}^{p^2} + m_3 v_{k+1}^{p^3}) - \sum_{i=4}^{k} m_i v_{4+k-i}^{p^i} - (m_{k+1} v_3^{p^{k+1}} + m_{k+2} v_2^{p^{k+2}} + m_{k+3} v_1^{p^{k+3}})\right)$$

$$
\begin{aligned}
&= p(\underline{m_{k+4}}_1 + \underline{m_3 t_{k+1}^{p^3}}_2 + \underline{m_2 t_{k+2}^{p^2}}_3 + m_1 t_{k+3}^p + t_{k+4}) \\
&\quad - m_1(\underline{v_{k+3}^p}_1 + \underline{v_2^p t_{k+1}^{p^3}}_2 + \underline{v_1^p t_{k+2}^{p^2}}_3 - \underline{v_1^p w_{k+2,2}}_7 - \underline{v_2^{p^{k+2}} t_{k+1}^p}_4 \\
&\quad - \underline{v_1^{p^{k+3}} t_{k+2}^p}_6 + p w_{k+3,1}) \\
&\quad - m_2(\underline{v_{k+2}^{p^2}}_1 + \underline{v_1^{p^2} t_{k+1}^{p^3}}_2 - \underline{v_1^{p^{k+3}} t_{k+1}^{p^2}}_5 + \underline{p w_{k+2,2}}_7) - m_3(\underline{v_{k+1}^{p^3}}_1 + p^4 w_{k+3}) \\
&\quad - \underline{\sum_{i=4}^{k} m_i v_{4+k-i}^{p^i}}_1 - (\underline{m_{k+1}}_1 + t_{k+1}) v_3^{p^{k+1}} - (\underline{m_{k+2}}_1 + \underline{m_1 t_{k+1}^p}_4 + t_{k+2}) v_2^{p^{k+2}} \\
&\quad - (\underline{m_{k+3}}_1 + \underline{m_2 t_{k+1}^{p^2}}_5 + \underline{m_1 t_{k+2}^p}_6 + t_{k+3}) v_1^{p^{k+3}} \\
&\equiv v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_1 w_{k+3,1} \\
&\quad - v_3^{p^{k+1}} t_{k+1} - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} - v_2 w_{k+2,2} \qquad \bmod (p).
\end{aligned}
$$

Here the underlined terms numbered 1 to 7 sum up to v_{k+4}, $v_3 t_{k+1}^{p^3}$, $v_2 t_{k+2}^{p^2}$, 0, 0, 0 and $-v_2 w_{k+2,2}$, respectively.

If $k = 2$, then

$$
\begin{aligned}
&\eta_R(v_{k+4}) \\
&\quad = \eta_R(p m_{k+4} - (m_1 v_{k+3}^p + m_2 v_{k+2}^{p^2}) - m_{k+1} v_{k+1}^{p^{k+1}} - (m_{k+2} v_2^{p^{k+2}} + m_{k+3} v_1^{p^{k+3}})) \\
&\quad = p(\underline{m_{k+4}}_1 + \underline{m_3 t_{k+1}^{p^3}}_2 + \underline{m_2 t_{k+2}^{p^2}}_3 + m_1 t_{k+3}^p + t_{k+4}) \\
&\qquad - m_1(\underline{v_{k+3}^p}_1 + \underline{v_2^p t_{k+1}^{p^3}}_2 + \underline{v_1^p t_{k+2}^{p^2}}_3 - \underline{v_1^p w_{k+2,2}}_7 - \underline{v_2^{p^{k+2}} t_{k+1}^p}_4 \\
&\qquad\quad - \underline{v_1^{p^{k+3}} t_{k+2}^p}_6 + p w_{k+3,1}) \\
&\qquad - m_2(\underline{v_{k+2}^{p^2}}_1 + \underline{v_1^{p^2} t_{k+1}^{p^3}}_2 - \underline{v_1^{p^{k+3}} t_{k+1}^{p^2}}_5 + \underline{p w_{k+2,2}}_7) \\
&\qquad - (\underline{m_{k+1}}_1 + t_{k+1})(v_{k+1}^{p^{k+1}} + p^4 w_{k+3}) \\
&\qquad - (\underline{m_{k+2}}_1 + \underline{m_1 t_{k+1}^p}_4 + t_{k+2}) v_2^{p^{k+2}} - (\underline{m_{k+3}}_1 + \underline{m_2 t_{k+1}^{p^2}}_5 + \underline{m_1 t_{k+2}^p}_6 + t_{k+3}) v_1^{p^{k+3}} \\
&\quad \equiv v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_1 w_{k+3,1} \\
&\qquad - v_3^{p^{k+1}} t_{k+1} - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} - v_2 w_{k+2,2} \qquad \bmod (p).
\end{aligned}
$$

Finally we compute the case $k = 1$,

$$\eta_R(v_{k+4})$$

$$= \eta_R(pm_{k+4} - m_1 v_{k+3}^p - m_{k+1} v_{k+2}^{p^{k+1}} - m_{k+2} v_{k+1}^{p^{k+2}} - m_{k+3} v_1^{p^{k+3}})$$

$$= p(\underline{m_{k+4}}_1 + \underline{m_3 t_{k+1}^{p^3}}_2 + \underline{m_2 t_{k+2}^{p^2}}_3 + m_1 t_{k+3}^p + t_{k+4})$$

$$- m_1(\underline{v_{k+3}^p}_1 + \underline{v_2^p t_{k+1}^{p^3}}_2 + \underline{v_1^p t_{k+2}^{p^2}}_3 - \underline{v_1^p w_{k+2,2}}_7 - \underline{v_2^{p^{k+2}} t_{k+1}^p}_4$$

$$- \underline{v_1^{p^{k+3}} t_{k+2}^p}_6 + pw_{k+3,1})$$

$$- m_{k+1}(\underline{v_{k+2}^{p^2}}_1 + \underline{v_1^{p^2} t_{k+1}^{p^3}}_2 - \underline{v_1^{p^{k+3}} t_{k+1}^{p^2}}_5 + \underline{pw_{k+2,2}}_7)$$

$$- t_{k+1}(v_{k+2}^{p^2} + v_1^{p^2} t_{k+1}^{p^3} - v_1^{p^{k+3}} t_{k+1}^{p^2} + pw_{k+2,2})$$

$$- (\underline{m_{k+2}}_1 + \underline{m_1 t_{k+1}^p}_4 + t_{k+2})(v_{k+1}^{p^{k+2}} + p^4 w_{k+3})$$

$$- (\underline{m_{k+3}}_1 + \underline{m_2 t_{k+1}^{p^2}}_5 + \underline{m_1 t_{k+2}^p}_6 + t_{k+3}) v_1^{p^{k+3}}$$

$$\equiv v_{k+4} + v_3 t_{k+1}^{p^3} + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_1 w_{k+3,1}$$

$$- \eta_R(v_{k+2}^{p^{k+1}}) t_{k+1} - v_2^{p^{k+2}} t_{k+2} - v_1^{p^{k+3}} t_{k+3} - v_2 w_{k+2,2} \qquad \mathrm{mod}(p).$$ q.e.d.

Lemma 3.3. *For the differential* $d : \Omega^0 v_2^{-1} BP_* \to \Omega^1 v_2^{-1} BP_*$, *there exist elements* x_0, x_1, w_4 *and* x_2' *such that*

$$d(x_0) = v_1 t_{k+1}^p - v_1^{p^{k+1}} t_{k+1},$$

$$d(x_1) \equiv v_1^p v_2^{p^{k+1}-1} t_{k+1} - v_1^{p+1} v_2^{-1} (t_{k+2}^p - w_{k+2,1}) \qquad \mathrm{mod}(p, v_1^{p^{k+2}}),$$

$$d(x_4) \equiv v_1^p (v_2 t_{k+2}^{p^2} - v_2^{p^{k+2}} t_{k+2}) \qquad \mathrm{mod}(p, v_1^{p+1}),$$

$$d(x_2') \equiv -v_1^{p^2+p} v_2^{-p} t_{k+2}^{p^2} - v_1^{p^2+2p} v_2^{p^{k+1}-p-1} v_{k+2}^{p^2-p} t_{k+1}$$

$$+ v_1^{p^2+2p+1} v_2^{-p-1} v_{k+2}^{p^2-p} t_{k+2}^p \qquad \mathrm{mod}(p, v_1^{p^2+2p+2}).$$

Proof. The first one follows immediately from Lemma 3.2 if we set $x_0 = v_{k+2}$. Since $d(v_{k+2}^p) \equiv v_1^p t_{k+1}^{p^2} \mod(p, v_1^{p^{k+2}})$ and $d(v_1^p v_2^{-1} v_{k+3}) \equiv v_1^p v_2^{-1} \cdot (v_2 t_{k+1}^{p^2} + v_1 t_{k+2}^p - v_1 w_{k+2,1} - v_2^{p^{k+1}} t_{k+1} - v_1^{p^{k+2}} t_{k+2}) \mod(p)$, we put $x_1 = v_{k+2}^p + v_1^p v_2^{-1} v_{k+3}$ to obtain the second.

Put $w_4 = v_1^p v_{k+4} - v_1^p v_2^{-p} v_3 v_{k+3}^p - v_1^{p-1} v_2^{p^{k+2}-p} v_3 v_{k+2} + v_2^{-p^{k+1}+1} v_3^{p^{k+1}} x_1$. Then, $\mathrm{mod}(p, v_1^{2p})$, we have

$$\begin{aligned}
d(v_1^p v_{k+4}) &\equiv v_1^p(\underline{v_3 t_{k+1}^{p^3}}_1 + v_2 t_{k+2}^{p^2} + v_1 t_{k+3}^p - v_2 w_{k+2,2} \\
&\qquad - v_1 w_{k+3,1} - \underline{v_3^{p^{k+1}} t_{k+1}}_3 - v_2^{p^{k+2}} t_{k+2}), \\
d(-v_1^p v_2^{-p} v_3 v_{k+3}^p) &\equiv -v_1^p v_2^{-p} v_3(\underline{v_2^p t_{k+1}^{p^3}}_1 - \underline{v_2^{p^{k+2}} t_{k+1}^p}_2), \\
d(-v_1^{p-1} v_2^{p^{k+2}-p} v_3 v_{k+2}) &\equiv \underline{-v_1^p v_2^{p^{k+2}-p} v_3 t_{k+1}^p}_2, \\
d(v_2^{-p^{k+1}+1} v_3^{p^{k+1}} x_1) &\equiv v_2^{-p^{k+1}+1} v_3^{p^{k+1}}(\underline{v_1^p v_2^{p^{k+1}-1} t_{k+1}}_3 - v_1^{p+1} v_2^{-1}(t_{k+2}^p - w_{k+2,1})).
\end{aligned}$$

Here $w_{k+2,2} \equiv 0 \mod(p, v_1^p)$. This holds for $k > 1$. If $k = 1$, put $w_4 = v_1^p v_{k+4} - v_1^p v_2^{-p} v_3 v_{k+3}^p - \frac{1}{2} v_1^{p-1} v_2^{p^{k+2}-p} v_{k+2}^2 + v_2^{-p^{k+1}+1} v_3^{p^{k+1}} x_1$. Then we have, $\mod(p, v_1^{p+1})$,

$$\begin{aligned}
d(v_1^p v_{k+4}) &\equiv v_1^p(\underline{v_3 t_{k+1}^{p^3}}_1 + v_2 t_{k+2}^{p^2} - v_2 w_{k+2,2} - \underline{v_3^{p^{k+1}} t_{k+1}}_3 - v_2^{p^{k+2}} t_{k+2}), \\
d(-v_1^p v_2^{-p} v_3 v_{k+3}^p) &\equiv -v_1^p v_2^{-p} v_3(\underline{v_2^p t_{k+1}^{p^3}}_1 - \underline{v_2^{p^{k+2}} t_{k+1}^p}_2), \\
d\left(-\frac{1}{2} v_1^{p-1} v_2^{p^{k+2}-p} v_{k+2}^2\right) &\equiv \underline{-v_1^p v_2^{p^{k+2}-p} v_3 t_{k+1}^p}_2, \\
d(v_2^{-p^{k+1}+1} v_3^{p^{k+1}} x_1) &\equiv v_2^{-p^{k+1}+1} v_3^{p^{k+1}} \underline{v_1^p v_2^{p^{k+1}-1} t_{k+1}}_3.
\end{aligned}$$

Here $v_{k+2} = v_3$.

Put $x_2' = x_1^p - v_1^{p^2-1} v_2^{p^{k+2}-p} v_{k+2} - v_1^{p^2+2p} v_2^{-p-1} v_{k+2}^{p^2-p} v_{k+3}$. Then

$$\begin{aligned}
d(x_1^p) &\equiv v_1^{p^2} v_2^{p^{k+2}-p} t_{k+1}^p - v_1^{p^2+p} v_2^{-p}(t_{k+2}^{p^2} - w_{k+2,1}^p) \qquad \mod(p, v_1^{p^{k+3}}) \\
&\equiv v_1^{p^2} v_2^{p^{k+2}-p} t_{k+1}^p - v_1^{p^2+p} v_2^{-p}(t_{k+2}^{p^2} - v_1^p v_{k+2}^{p^2-p} t_{k+1}^{p^2}) \qquad \mod(p, v_1^{p^2+3p}), \\
d(-v_1^{p^2-1} v_2^{p^{k+2}-p} v_{k+2}) &\equiv -v_1^{p^2} v_2^{p^{k+2}-p} t_{k+1}^p \qquad \mod(p, v^{p_1^{2+p^{k+1}-1}}), \\
d(v_1^{p^2+2p} v_2^{-p-1} v_{k+2}^{p^2-p} v_{k+3}) & \\
&\equiv v_1^{p^2+2p} v_2^{-p-1} v_{k+2}^{p^2-p}(v_2 t_{k+1}^{p^2} + v_1 t_{k+2}^p - v_2^{p^{k+1}} t_{k+1}) \qquad \mod(p, v_1^{p^2+2p+2}).
\end{aligned}$$

Thus the last congruence follows. q.e.d.

Now we define x_i:

$$\begin{aligned}
x_{3m+1} &= x_{3m}^p - v_1^{a_{3m+1}-2p} v_2^{b_{3m+1}-p^{k+1}} v_{k+2}^{c_{3m+1}-p^2+p} x_2', \\
x_{3m+2} &= x_{3m+1}^p - v_1^{a_{3m+2}-p-1} v_2^{b_{3m+2}} v_{k+2}^{c_{3m+2}+1} + v_1^{a_{3m+2}-p} v_2^{b_{3m+2}-p^{k+2}} v_{k+2}^{c_{3m+2}} w_4, \\
x_{3m+3} &= x_{3m+2}^p,
\end{aligned}$$

where a_n, b_n and c_n are the integers defined in (1.1) and Lemma 2.4.

LEMMA 3.4.

$$d(x_{3m+1}) \equiv (-1)^m v_1^{a_{3m+1}} v_2^{b_{3m+1}} v_{k+2}^{c_{3m+1}} t_{k+1} \qquad \mathrm{mod}(p, v_1^{a_{3m+1}+1}),$$

$$d(x_{3m+2}) \equiv (-1)^{m+1} v_1^{a_{3m+2}} v_2^{b_{3m+2}} v_{k+2}^{c_{3m+2}} t_{k+2} \qquad \mathrm{mod}(p, v_1^{a_{3m+2}+1}),$$

$$d(x_{3m+3}) \equiv (-1)^{m+1} v_1^{a_{3m+3}} v_2^{b_{3m+3}} v_{k+2}^{c_{3m+3}} t_{k+2}^p \qquad \mathrm{mod}(p, v_1^{a_{3m+3}+p}).$$

PROOF. Suppose that

$$d(x_{3m+1}) \equiv v_1^a v_2^b v_{k+2}^c t_{k+1} - v_1^{a+1} v_2^{b'} v_{k+2}^c t_{k+2}^p \qquad \mathrm{mod}(p, v_1^{a+2}),$$

where $b' = b - p^{k+1}$. Then noticing that $p|c$, we compute $\mathrm{mod}(p, v_1^{ap+p+1})$,

$$d(x_{3m+1}^p) \equiv v_1^{ap} v_2^{bp} v_{k+2}^{cp} t_{k+1}^p - v_1^{ap+p} v_2^{pb'} v_{k+2}^{pc} t_{k+2}^{p^2},$$

$$d(-v_1^{ap-1} v_2^{bp} v_{k+2}^{cp+1}) \equiv -v_1^{ap} v_2^{bp} v_{k+2}^{cp} t_{k+1}^p,$$

$$d(v_1^{ap} v_2^{pb'} v_{k+2}^{pc} w_4) \equiv v_1^{ap+p} v_2^{pb'} v_{k+2}^{pc} (v_2 t_{k+2}^{p^2} - v_2^{p^{k+2}} t_{k+2}),$$

and we obtain

$$d(x_{3m+2}) \equiv -v_1^{ap+p} v_2^{pb} v_{k+2}^{pc} t_{k+2} \qquad \mathrm{mod}(p, v_1^{ap+p+1}),$$

and so

$$d(x_{3m+3}) \equiv -v_1^{ap^2+p^2} v_2^{p^2 b} v_{k+2}^{p^2 c} t_{k+2}^p \qquad \mathrm{mod}(p, v_1^{ap^2+p^2+p}).$$

The p-th power shows that

$$d(x_{3m+3}^p) \equiv -v_1^{ap^3+p^3} v_2^{p^3 b} v_{k+2}^{p^3 c} t_{k+2}^{p^2} \qquad \mathrm{mod}(p, v_1^{ap^3+p^3+p^2}),$$

$$d(-v_1^{ap^3+p^3-p^2-p} v_2^{p^3 b+p} v_{k+2}^{p^3 c} x_2')$$

$$\equiv -v_1^{ap^3+p^3} v_2^{p^3 b+p} v_{k+2}^{p^3 c} (-v_2^{-p} t_{k+2}^{p^2} - v_1^p v_2^{p^{k+1}-p-1} v_{k+2}^{p^2-p} t_{k+1} + v_1^{p+1} v_2^{-p-1} v_{k+2}^{p^2-p} t_{k+2}^p)$$

$\mathrm{mod}(p, v_1^{ap^3+p^3+p+2})$. This completes the induction. q.e.d.

4. Computation of $d(r_0)$

In this section, we fix an integer $k \geq 2$, and we will prove Lemma 2.3.

LEMMA 4.1. The coproduct $\Delta : \Gamma(k) \to \Gamma(k) \otimes \Gamma(k)$ is given by

$$\Delta(t_{k+1}) = t_{k+1} \otimes 1 + 1 \otimes t_{k+1},$$

$$\Delta(t_{k+2}) = t_{k+2} \otimes 1 + v_1 b_{k+1,0} + 1 \otimes t_{k+2},$$

$$\Delta(t_{k+3}) = t_{k+3} \otimes 1 + v_1 b_{k+2,0} + v_2 b_{k+1,1} + 1 \otimes t_{k+3}.$$

Here $pb_{k+1,n-1} = -\sum_{i=1}^{p^n-1}\binom{p^n}{i} t_{k+1}^i \otimes t_{k+1}^{p^n-i}$ and

$$-pb_{k+2,0} = \Delta(t_{k+2})^p - \psi(t_{k+2})^{(1)} - v_1^p b_{k+1,1}.$$

PROOF. The coproduct is given by

$$\Delta(t_{k+n}) + \sum_{i=1}^{n-1} m_i \Delta(t_{k+n-i})^{p^i} = \psi(t_{k+n}) + \sum_{i=1}^{n-1} m_i \psi(t_{k+n-i})^{(i)},$$

where $\psi(t_{k+n})^{(l)} = 1 \otimes t_{k+n}^{p^l} + \sum_{i=1}^{n-k-1} t_{k+i}^{p^l} \otimes t_{n-i}^{p^{k+i+l}} + t_{k+n}^{p^l} \otimes 1$. Take $n = 2$, and we have

$$\Delta(t_{k+2}) + m_1 \Delta(t_{k+1})^p = \psi(t_{k+2}) + m_1 \psi(t_{k+1})^{(1)}.$$

Thus we have $\Delta(t_{k+2})$.

For $\Delta(t_{k+3})$,

$$\Delta(t_{k+3}) + m_1 \Delta(t_{k+2})^p + m_2 \Delta(t_{k+1})^{p^2} = \psi(t_{k+3}) + m_1 \psi(t_{k+2})^{(1)} + m_2 \psi(t_{k+1})^{(2)}.$$

Note that $m_2(\psi(t_{k+1})^{(2)} - \Delta(t_{k+1})^{p^2}) = pm_2 b_{k+1,1} = (v_2 + m_1 v_1^p) b_{k+1,1}$. Then

$$\Delta(t_{k+3}) = \psi(t_{k+3}) - m_1(\Delta(t_{k+2})^p - \psi(t_{k+2})^{(1)} - v_1^p b_{k+1,1}) + v_2 b_{k+1,1}. \quad \text{q.e.d.}$$

LEMMA 4.2. *Let $w_{n,i}$ be the element of* (3.1). *Then*

1. $w_{k+2,i} \equiv v_1^{p^{i-1}} v_{k+2}^{p^i - p^{i-1}} t_{k+1}^{p^i} \bmod (p, v_1^{p^{k+i}})$ and
$$d(w_{k+2,i}) = -v_1^{p^i} b_{k+1,i} + v_1^{p^{k+i+1}} b_{k+1,i-1},$$
2. $d(w_{k+3,i}) \equiv -v_2^{p^i} b_{k+1,i+1} - v_1^{p^i} b_{k+2,i} + v_2^{p^{k+i+1}} b_{k+1,i-1} + v_1^{p^{k+i+2}} b_{k+2,i-1}$ $\bmod(p)$.

PROOF. The first one follows immediately from $d(t_{k+1}^{p^{i+1}}) = pb_{k+1,i}$. For the second, note that

$$pw_{k+3,i} \equiv d(v_{k+3}^{p^i}) - v_2^{p^i} t_{k+1}^{p^{i+2}} - v_1^{p^i} t_{k+2}^{p^{i+1}} - v_1^{p^i} w_{k+2,i+1} + v_2^{p^{k+i+1}} t_{k+1}^{p^i} + v_1^{p^{k+i+2}} t_{k+2}^{p^i}$$

$\bmod(p^2)$, and $d(t_{k+2}^{p^i}) = v_1^{p^i} b_{k+1,i} + pb_{k+2,i-1}$ by the definition of $b_{k+2,i-1}$. Then we compute that

$$\begin{aligned} d(pw_{k+3,i}) = & -pv_2^{p^i} b_{k+1,i+1} - v_1^{p^{i+1}+p^i} b_{k+1,i+1} - pv_1^{p^i} b_{k+2,i-1} \\ & - v_1^{p^i}(-v_1^{p^{i+1}} b_{k+1,i+1} + v_1^{p^{k+i+2}} b_{k+1,i}) + pv_2^{p^{k+i+1}} b_{k+1,i-1} \\ & + v_1^{p^{k+i+2}}(v_1^{p^i} b_{k+1,i} + pb_{k+2,i-1}). \end{aligned}$$

Proof of Lemma 2.3. Put $H_{11} = w_{k+2,1} + v_1^p v_2^{-p^{k+3}} w_{k+3,2} - v_1^p v_2^{-p^{k+3}} t_{k+3}^{p^2} + v_1^{p^{k+4}} v_2^{-p^{k+3}} t_{k+3}^p - v_1^{p^{k+4}-p^2} v_2^{-p^{k+3}+p} t_{k+2}^{p^2} + v_1^{p^{k+2}-1} t_{k+2}$. Since $H_{11} \equiv v_1 v_{k+2}^{p-1} t_{k+1}^p$ mod (p, v_1^2), it suffices to show that $d(H_{11}) = 0$, which is verified by Lemma 4.2, as follows:

$$
\begin{aligned}
d(w_{k+2,1}) &= -\underline{v_1^p b_{k+1,1}}_1 + \underline{v_1^{p^{k+2}} b_{k+1,0}}_6, \\
d(v_1^p v_2^{-p^{k+3}} w_{k+3,2}) &= v_1^p v_2^{-p^{k+3}} (-\underline{v_2^{p^2} b_{k+1,3}}_2 - \underline{v_1^{p^2} b_{k+2,2}}_3 + \underline{v_2^{p^{k+3}} b_{k+1,1}}_1 \\
&\quad + \underline{v_1^{p^{k+4}} b_{k+2,1}}_4), \\
d(-v_1^p v_2^{-p^{k+3}} t_{k+3}^{p^2}) &= v_1^p v_2^{-p^{k+3}} (\underline{v_1^{p^2} b_{k+2,2}}_3 + \underline{v_2^{p^2} b_{k+1,3}}_2), \\
d(v_1^{p^{k+4}} v_2^{-p^{k+3}} t_{k+3}^p) &= -v_1^{p^{k+4}} v_2^{-p^{k+3}} (\underline{v_1^p b_{k+2,1}}_4 + \underline{v_2^p b_{k+1,2}}_5), \\
d(-v_1^{p^{k+4}-p^2} v_2^{-p^{k+3}+p} t_{k+2}^{p^2}) &= \underline{v_1^{p^{k+4}-p^2} v_2^{-p^{k+3}+p} (v_1^{p^2} b_{k+1,2})}_5, \\
d(v_1^{p^{k+2}-1} t_{k+2}) &= -\underline{v_1^{p^{k+2}} b_{k+1,0}}_6.
\end{aligned}
$$

Now take r_0 to be $v_1^{-1} H_{11}$. q.e.d.

Remark. This proof does not work for the case $k = 1$, since $\Delta(t_4) \neq t_4 \otimes 1 + v_1 b_{3,0} + v_2 b_{2,1} + 1 \otimes t_4$.

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