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On the Cowling-Price theorem for $SU(1,1)$

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Abstract. M. G. Cowling and J. F. Price showed a kind of uncertainty principle on Fourier analysis. If v and w grow very rapidly then the finiteness of $\|vf\|_p$ and $\|w\hat{f}\|_q$ implies that $f = 0$, where $\hat{f}$ denotes the Fourier transform of f. We give an analogue of this theorem for $SU(1,1)$.

1. Introduction

The Hardy theorem asserts that if a measurable function f on $\mathbf{R}$ satisfies $|f(x)| \le Ce^{-ax^2}$ and $|\hat{f}(y)| \le Ce^{-by^2}$ and $ab > \frac{1}{4}$ then $f = 0$ (a.e.). Here we use the Fourier transform defined by $\hat{f}(y) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)e^{\sqrt{-1}xy}\,dx$. M. G. Cowling and J. F. Price [4] generalized the Hardy theorem as follows: Suppose that $1 \le p, q \le \infty$ and one of them is finite. If a measurable function f on $\mathbf{R}$ satisfies $\|\exp\{ax^2\}f(x)\|_{L^p(\mathbf{R})} < \infty$ and $\|\exp\{by^2\}\hat{f}(y)\|_{L^q(\mathbf{R})} < \infty$ and $ab \ge 1/4$ then $f = 0$ (a.e.). The case where $p = q = \infty$ and $ab > 1/4$ is covered by the Hardy theorem. S. C. Bagchi and S. K. Ray [1] showed that if $ab > 1/4$, then the Hardy theorem is equivalent to the Cowling-Price theorem.

A. Sitaram and M. Sundari [14] obtained the Hardy theorem in the case of noncompact semisimple Lie groups with one conjugacy class of Cartan subgroups, $SL(2,\mathbf{R})$ and Riemannian symmetric spaces of the noncompact type. Recently J. Sengupta [12] and M. Ebata et al. [6] obtained the Hardy theorem for all Lie groups of Harish-Chandra class and all connected semisimple Lie groups with finite center respectively. Also, M. Cowling, A. Sitaram and M. Sundari [5] gave another simple proof of the Hardy theorem for connected real semisimple Lie groups with finite center. On the other hand, S. C. Bagchi and S. K. Ray [1] obtained the Cowling-Price theorem for some Lie groups and M. Eguchi, S. Koizumi and K. Kumahara [7] also obtained the Cowling-Price theorem for motion groups. Further, J. Sengupta [13] obtained the Cowling-Price theorem on Riemannian symmetric spaces of the noncompact type.

In this paper, we prove the Cowling-Price theorem for $SU(1,1)$ under the assumption that $1 \le p, q \le \infty$ and $ab > 1/4$.

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2. Notation and preliminaries

The standard symbols $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ shall be used for the sets of the integers, the real numbers and the complex numbers respectively. For $z \in \mathbf{C}$, $\Re z$ and $\Im z$ denote its real and imaginary part, respectively. Let $\mathbf{Z}_{\geq k} = \{n \in \mathbf{Z}; n \geq k\}$ for $k \in \mathbf{Z}$. If V is a vector space over $\mathbf{R}$, $V_{\mathbf{C}}$, V^* and $V_{\mathbf{C}}^*$ denote its complexification, its real dual and its complex dual, respectively. For a Lie group L, $\hat{L}$ denotes the set of equivalence classes of irreducible unitary representations of L. As usual, we use lower case German letters to denote the corresponding Lie algebras.

If $\mathscr{H}$ is a complex separable Hilbert space, $\mathbf{B}(\mathscr{H})$ denotes the Banach space comprised of all bounded operators on $\mathscr{H}$ with operator norm $\|\cdot\|_\infty$. For $T \in \mathbf{B}(\mathscr{H})$ and $1 \leq p < \infty$, we indicate its Schatten norm by $\|T\|_p$, that is, $\|T\|_p = (\mathrm{tr}(T^*T)^{p/2})^{1/p}$, T^* being the adjoint operator of T. For a complex separable Hilbert space $\mathscr{H}$ and a σ-finite measure space (X, μ), we denote by $L^p(X, \mathbf{B}(\mathscr{H}))$ the Banach space comprised of all $\mathbf{B}(\mathscr{H})$-valued L^p functions on X. Here the L^p-norm $\|F\|_{L^p(X, \mathbf{B}(\mathscr{H}))}$ of $F \in L^p(X, \mathbf{B}(\mathscr{H}))$ is given by the following:

$$\|F\|_{L^p(X,\mathbf{B}(\mathscr{H}))} = \left(\int_X \|F(x)\|_p^p d\mu(x)\right)^{1/p}, \qquad 1 \leq p < \infty,$$

$$\|F\|_{L^\infty(X,\mathbf{B}(\mathscr{H}))} = \text{ess.} \sup_{x \in X} \|F(x)\|_\infty.$$

Throughout this paper, G denotes the matrix group $SU(1,1)$, that is,

$$G = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}; |\alpha|^2 - |\beta|^2 = 1, \alpha, \beta \in \mathbf{C} \right\}.$$

Let

$$K = \left\{ k_\theta = \begin{pmatrix} e^{\sqrt{-1}\theta/2} & 0 \\ 0 & e^{-\sqrt{-1}\theta/2} \end{pmatrix}; 0 \leq \theta < 4\pi \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}; t \in \mathbf{R} \right\},$$

$$N = \left\{ n_\eta = \begin{pmatrix} 1 + \sqrt{-1}\eta/2 & -\sqrt{-1}\eta/2 \\ \sqrt{-1}\eta/2 & 1 - \sqrt{-1}\eta/2 \end{pmatrix}; \eta \in \mathbf{R} \right\},$$

$$\overline{N} = \left\{ \bar{n}_\zeta = \begin{pmatrix} 1+\sqrt{-1}\zeta/2 & \sqrt{-1}\zeta/2 \\ -\sqrt{-1}\zeta/2 & 1-\sqrt{-1}\zeta/2 \end{pmatrix}; \zeta \in \mathbf{R} \right\},$$

$$M = \left\{ \pm I, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $G = KAN$ is an Iwasawa decomposition of G (cf. [11]). Each $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G$ can be uniquely decomposed as $g = k_{\theta(g)} a_{t(g)} n_{\eta(g)}$, where

$$\theta(g) = 2 \arg \frac{\alpha+\beta}{|\alpha+\beta|}, \qquad t(g) = 2 \log|\alpha+\beta|$$

$$\text{and } \eta(g) = \frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{\sqrt{-1}|\alpha+\beta|^2}.$$

Then we can choose a Haar measure dg so that

$$\int_G f(g)dg = \frac{1}{4\pi} \int_0^{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k_\theta a_s n_\eta) e^t \, d\theta dt d\eta.$$

Take $H = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathfrak{a}$ and identify $\mathfrak{a}^*_{\mathbf{C}}$ with $\mathbf{C}$ via the correspondence $v \mapsto v(H)$. We fix $\mathfrak{a}^+ = \{tH; t > 0\}$. Let $A^+ = \exp \mathfrak{a}^+$ and $\mathrm{Cl}(A^+)$ denote the closure of A^+ in G.

By the Cartan decomposition $G = K\,\mathrm{Cl}(A^+)K$, each $g \in G$ can be written as $g = k_1 a_t k_2$ for $k_1, k_2 \in K$ and $t \geq 0$ (cf. [15]). In relation to this decomposition, we have

(2.1) $$\int_G f(g)dg = 2\pi \int_K \int_0^\infty \int_K f(ka_t k') \sinh t \, dk dt dk',$$

where dk is the normalized Haar measure $(4\pi)^{-1} d\theta$ on K.

As is well known,

$$\hat{K} = \{\chi_n(k_\theta) = e^{\sqrt{-1}n\theta/2}; n \in \mathbf{Z}\}.$$

A function f on G is said to be (m,n)-spherical if

$$f(k_1 g k_2) = \chi_m(k_1) f(g) \chi_n(k_2)$$

for all $g \in G$ and $k_1, k_2 \in K$.

Let σ and Ξ be the spherical functions defined by Harish-Chandra. In our case, for $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G$,

$$\sigma(g) = 2\sqrt{2}\log(|\alpha| + |\beta|), \qquad \Xi(g) = \frac{1}{|\alpha|}F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{|\beta|^2}{|\alpha|^2}\right),$$

where F denotes the hypergeometric function.

Let $U(\mathfrak{g}_c)$ be the universal enveloping algebras of $\mathfrak{g}_c$. The elements of $U(\mathfrak{g}_c)$ act on $C^\infty(G)$ on both sides as differential operators. Following Harish-Chandra, we write $f(D; g; E)$ for the action of $D, E \in U(\mathfrak{g})$ on $f \in C^\infty(G)$ at $g \in G$.

3. Irreducible unitary representations and the Fourier transform on G

We give here a quick review of the Fourier transform on G. For $\varepsilon = 0, 1$ and $v \in \mathbf{R}$, let

$$\mathscr{H}_{\varepsilon, v} = \{\varphi \in L^2(K); \varphi(k(\pm I)) = (\pm 1)^\varepsilon \varphi(k), k \in K\}.$$

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathscr{H}_{\varepsilon, v}$. We define the action $\pi_{\varepsilon, v}$ on $\mathscr{H}_{\varepsilon, v}$ by

$$(\pi_{\varepsilon, v}(g)\varphi)(k) = e^{(\sqrt{-1}v - 1/2)t(g^{-1}k)}\varphi(k_{\theta(g^{-1}k)}).$$

Then $\pi_{\varepsilon, v}$ is a unitary representation on $\mathscr{H}_{\varepsilon, v}$ and is called a principal series representation. It follows from the Frobenius reciprocity theorem that

$$\pi_{\varepsilon, v}|_K = \sum_{n \in \mathbf{Z}(\varepsilon)} \chi_n,$$

where $\mathbf{Z}(\varepsilon) = \{m \in \mathbf{Z}; m \equiv \varepsilon \pmod 2\}$. We set $e_\ell(k_\theta) = e^{\sqrt{-1}\ell\theta/2}$. Then the set $\{e_\ell; \ell \in \mathbf{Z}(\varepsilon)\}$ is an orthonormal basis of $\mathscr{H}_{\varepsilon, v}$.

Let $I_{\varepsilon, v}$ be the standard intertwining operator defined by Knapp and Stein (cf. [9]). For each $\varepsilon = 0, 1$, it is satisfied that

$$I_{\varepsilon, v}\pi_{\varepsilon, v}(g) = \pi_{\varepsilon, -v}(g)I_{\varepsilon, v}$$

for all $v \in \mathbf{R}$ and $g \in G$. We take from [16] that

$$I_{\varepsilon, v}e_\ell = (-1)^\ell c_\ell(v)e_\ell,$$

where $c_\ell(v)$ is the Harish-Chandra C-function given by

$$\begin{aligned} c_\ell(v) &= \int_{\bar{N}} e^{(\sqrt{-1}v - 1/2)t(\bar{n})} e_\ell(k_{\theta(\bar{n})}) d\bar{n} \\ &= \frac{2^{-2\sqrt{-1}v+1}\Gamma(2\sqrt{-1}v)}{\Gamma(\sqrt{-1}v + 1/2 - \ell/2)\Gamma(\sqrt{-1}v + 1/2 + \ell/2)}. \end{aligned}$$

Let $\Phi^{\varepsilon,\nu}_{\ell_1\ell_2}(g)$ denote the matrix coefficient for $\pi_{\varepsilon,\nu}$ with respect to $\{e_\ell; \ell \in \mathbf{Z}(\varepsilon)\}$, that is,

$$\begin{aligned}\Phi^{\varepsilon,\nu}_{\ell_1\ell_2}(g) &= \langle \pi_{\varepsilon,\nu}(g)e_{\ell_2}, e_{\ell_1}\rangle \\ &= \int_K e^{(\sqrt{-1}\nu-1/2)t(g^{-1}k)} e_{\ell_2}(k_{\theta(g^{-1}k)})\overline{e_{\ell_1}(k)}dk.\end{aligned}$$

We also need another representation. Let $D = \{z \in \mathbf{C}; |z| < 1\}$. For $\lambda \in \mathbf{Z}_{\geq 2}$, denote by $\mathscr{H}_\lambda^+$ (resp. $\mathscr{H}_\lambda^-$) the Hilbert space of all holomorphic (resp. anti-holomorphic) functions φ on D such that

$$\|\varphi\|_\lambda^2 = \int_{|z|<1} |\varphi(z)|^2(1-|z|^2)^{\lambda-2}dz < \infty.$$

We define the action π_λ^+ (resp. π_λ^-) of G on $\mathscr{H}_\lambda^+$ (resp. $\mathscr{H}_\lambda^-$) by

$$\left(\pi_\lambda^+\left(\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}\right)\varphi\right)(z) = (-\beta z+\bar\alpha)^{-\lambda}\varphi\left(\frac{\alpha z-\bar\beta}{-\beta z+\bar\alpha}\right),$$

$$\left(\pi_\lambda^-\left(\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}\right)\varphi\right)(z) = \overline{(-\beta z+\bar\alpha)}^{-\lambda}\varphi\left(\frac{\alpha z-\bar\beta}{-\beta z+\bar\alpha}\right).$$

These representations π_λ^+ and π_λ^- are unitary and called the discrete series representations. Hereafter we write $(\pi_\lambda, \mathscr{H}_\lambda)$ instead of $(\pi_{|\lambda|+1}^{\operatorname{sgn}\lambda}, \mathscr{H}_{|\lambda|+1}^{\operatorname{sgn}\lambda})$. We denote by $(\cdot,\cdot)_\lambda$ the inner product on $\mathscr{H}_\lambda$. Then $\{z^\ell; \ell \in \mathbf{Z}_{\geq 0}\}$ and $\{\bar z^\ell; \ell \in \mathbf{Z}_{\geq 0}\}$ are orthogonal bases of $\mathscr{H}_\lambda$ for $\lambda > 0$ and $\lambda < 0$ respectively. Put

$$\mathbf{Z}_\lambda = \begin{cases}\{-\lambda-2\ell-1; \ell \in \mathbf{Z}_{\geq 0}\} & \text{if } \lambda > 0\\ \{-\lambda+2\ell+1; \ell \in \mathbf{Z}_{\geq 0}\} & \text{if } \lambda < 0\end{cases}$$

Then

$$\pi_\lambda|_K = \sum_{n\in\mathbf{Z}_\lambda} \chi_n.$$

Let

$$\psi_{-\lambda-2\ell-1}(z) = \left(\frac{2}{B(\lambda,\ell+1)}\right)^{1/2} z^\ell \qquad \text{if } \lambda > 0,$$

$$\psi_{-\lambda+2\ell+1}(z) = \left(\frac{2}{B(-\lambda,\ell+1)}\right)^{1/2} \bar z^\ell \qquad \text{if } \lambda < 0,$$

where B is the Beta function. We denote by $\Psi^\lambda_{\ell_1\ell_2}(g)$ the matrix coefficient for π_λ with respect to $\{\psi_\ell; \ell \in \mathbf{Z}_\lambda\}$, that is,

$$\Psi^{\lambda}_{\ell_1\ell_2}(g) = (\pi_\lambda(g)\psi_{\ell_2}, \psi_{\ell_1})_\lambda.$$

For $\ell_1, \ell_2 \in \mathbf{Z}$, it is known that the set of λ for which $\Psi^{\lambda}_{\ell_1\ell_2}$ does not vanish coincides with

$$L(\ell_1, \ell_2) = \begin{cases} \{\ell \in \mathbf{Z}; \max\{\ell_1, \ell_2\} \le \ell \le -1\} & \text{if } \ell_1 < 0 \text{ and } \ell_2 < 0 \\ \{\ell \in \mathbf{Z}; 1 \le \ell \le \min\{\ell_1, \ell_2\}\} & \text{if } \ell_1 > 0 \text{ and } \ell_2 > 0 \\ \varnothing & \text{otherwise} \end{cases}$$

For $f \in L^1(G)$, its Fourier transform on G is defined by

$$\mathscr{F}^c f(\varepsilon, v) = \int_G f(g)\pi_{\varepsilon, v}(g)dg, \tag{3.1}$$

$$\mathscr{F}^d f(\lambda) = \int_G f(g)\pi_\lambda(g)dg. \tag{3.2}$$

We write $\mathscr{F} = (\mathscr{F}^c, \mathscr{F}^d)$. If $f \in C_0^\infty(G)$, then the following inversion formula holds

$$\begin{aligned} f(g) = \sum_{\varepsilon=0}^{1} \int_0^\infty &\operatorname{tr}(\mathscr{F}^c f(\varepsilon, v)\pi_{\varepsilon,v}(g^{-1}))\mu(\varepsilon, v)dv \\ &+ \sum_{\lambda \in \mathbf{Z}\setminus\{0\}} d(\lambda)\{\operatorname{tr}(\mathscr{F}^d f(\lambda)\pi_\lambda(g^{-1}))\}, \end{aligned} \tag{3.3}$$

where $\mu(0, v) = \pi v \tanh \pi v$, $\mu(1, v) = \pi v \coth \pi v$ and $d(\lambda) = |\lambda|/(4\pi)$.

For convenience we write $\mathscr{L}^p_\varepsilon(\mathfrak{a}^*) = L^p(\mathfrak{a}^*, \mathbf{B}(\mathscr{H}_{\varepsilon,v}), \mu(\varepsilon, v)dv)$ and $L^p_\varepsilon(\mathfrak{a}^*) = L^p(\mathfrak{a}^*, \mu(\varepsilon, v)dv)$.

4. Schwartz space and tempered distribution

In this section we review the definitions of the Schwartz spaces $\mathscr{C}(G)$ and $\mathscr{C}(\hat{G})$ and prove the pointwise inversion formula of the (m, n)-spherical transform for the very rapidly decreasing functions. To prove the main theorem in the next section, we need this inversion formula, or Proposition 4.7. For proving Proposition 4.7, we use the isomorphism between $\mathscr{C}'(G)$ and $\mathscr{C}'(\hat{G})$. The contents of this section are almost same as the arguments of Baker [2], but he didn't give the statement as Proposition 4.7. The Schwartz space on G is defined by

$$\mathscr{C}(G) = \{\phi \in C^\infty(G); \|\phi\|_{r,D,E} < \infty \text{ for all } r \in \mathbf{Z}_{\ge 0}, D, E \in U(\mathfrak{g}_c)\},$$

$$\text{where } \|\phi\|_{r,D,E} = \sup_{g \in G} |(1 + \sigma(g))^r \Xi(g)^{-1}\phi(D; g; E)|.$$

As is well known, the system of seminorms $\|\cdot\|_{r,D,E}$ makes $\mathscr{C}(G)$ into a Fréchet space.

Let $\mathscr{C}_c(\hat{G})$ be the set of operator valued functions $F : \{0,1\} \times \mathbf{R} \to \bigoplus_{\varepsilon=0}^{1} \mathbf{B}(\mathscr{H}_{\varepsilon,v})$ such that

(i) $F(\varepsilon, v) \in \mathbf{B}(\mathscr{H}_{\varepsilon,v})$ for each $\varepsilon = 0,1,\ v \in \mathbf{R}$

(ii) $v \mapsto F(\varepsilon, v)$ is smooth on $\mathbf{R}$

(iii) $I_{\varepsilon,v}F(\varepsilon,v) = F(\varepsilon,-v)I_{\varepsilon,v}$ for each $\varepsilon = 0,1,\ v \in \mathbf{R}$

(iv) $\sup_{\substack{\varepsilon=0,1,v\in\mathbf{R}\\ \ell_1,\ell_2\in\mathbf{Z}(\varepsilon)}} \left|\left(\frac{d}{dv}\right)^r \langle F(\varepsilon,v)e_{\ell_2}, e_{\ell_1}\rangle\right|(1+|v|)^{r_1}(1+|\ell_1|)^{r_2}(1+|\ell_2|)^{r_3} < \infty$ for all $r_1, r_2, r_3, r \in \mathbf{Z}_{\geq 0}$.

The system of seminorms given by (iv) makes $\mathscr{C}_c(\hat{G})$ into a Fréchet space.

Let $\mathscr{C}_d(\hat{G})$ be the set of all $F : \mathbf{Z}\backslash\{0\} \to \bigoplus_{\lambda\in\mathbf{Z}\backslash\{0\}} \mathbf{B}(\mathscr{H}_\lambda)$ such that

(i) $F(\lambda) \in \mathbf{B}(\mathscr{H}_\lambda)$ for each $\lambda \in \mathbf{Z}\backslash\{0\}$

(ii) $\sup_{\substack{\lambda\in\mathbf{Z}\backslash\{0\}\\ \ell_1,\ell_2\in\mathbf{Z}_\lambda}} |(F(\lambda)\psi_{\ell_2}, \psi_{\ell_1})_\lambda|(1+|\lambda|)^{r_1}(1+|\ell_1|)^{r_2}(1+|\ell_2|)^{r_3} < \infty$ for all $r_1, r_2, r_3 \in \mathbf{Z}_{\geq 0}$.

The system of seminorms given by (ii) makes $\mathscr{C}_d(\hat{G})$ into a Fréchet space. Put $\mathscr{C}(\hat{G}) = \mathscr{C}_c(\hat{G}) \oplus \mathscr{C}_d(\hat{G})$. Then $\mathscr{C}(\hat{G})$ is a Fréchet space in an obvious manner. We put $\mathscr{S}^c = (\mathscr{F}^c)^{-1}$ and $\mathscr{S}^d = (\mathscr{F}^d)^{-1}$. Then they are given by

$$\mathscr{S}^c F(g) = \int_0^\infty \mathrm{tr}(F(\varepsilon,v)\pi_{\varepsilon,v}(g^{-1}))\mu(\varepsilon,v)dv \qquad \text{for } F \in \mathscr{C}_c(\hat{G}),$$

$$\mathscr{S}^d F(g) = \sum_{\lambda\in\mathbf{Z}\backslash\{0\}} d(\lambda)\mathrm{tr}(F(\lambda)\pi_\lambda(g^{-1})) \qquad \text{for } F \in \mathscr{C}_d(\hat{G}).$$

Proposition 4.1 (cf. [8]). *The Fourier transform $\mathscr{F}$ is a topological isomorphism from $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$. And its inverse transform is given by* (3.3).

Let

$$\mathscr{C}_c(G) = \{\phi \in \mathscr{C}(G); \mathscr{F}^d\phi(\lambda) = 0, \lambda \in \mathbf{Z}\backslash\{0\}\},$$

$$\mathscr{C}_d(G) = \{\phi \in \mathscr{C}(G); \mathscr{F}^c\phi(\varepsilon,v) = 0, \varepsilon = 0,1, v \in \mathbf{R}\},$$

and $\mathscr{C}_{c,mn}(G)$ (resp. $\mathscr{C}_{d,mn}(G)$) denote the subset of $\mathscr{C}_c(G)$ (resp. $\mathscr{C}_d(G)$) consisting of the (m,n)-spherical functions.

Let $m,n \in \mathbf{Z}$. If $m-n \in 2\mathbf{Z}+1$, we set $\mathscr{C}_{c,mn}(\hat{G}) = \varnothing$. If $m-n \in 2\mathbf{Z}$, we choose ε so that $m,n \in \mathbf{Z}(\varepsilon)$ and let $\mathscr{C}_{c,mn}(\hat{G})$ be the set of C^∞ functions $F : \mathbf{R} \to \mathbf{C}$ such that

(i) $F(-v) = c_n(v)^{-1}c_m(v)F(v)$ for each $v \in \mathbf{R}$,

(ii) $\sup_{v\in\mathbf{R}}\left|(1+|v|)^r\left(\frac{d}{dv}\right)^s F(v)\right| < \infty$ for all $r,s \in \mathbf{Z}_{\geq 0}$.

The system of seminorms given by (ii) makes $\mathscr{C}_{c,mn}(\hat{G})$ into a Fréchet space.

Let $\mathscr{C}_{d,mn}(\hat{G})$ be the set of all functions $F : \mathbf{Z}\backslash\{0\} \to \mathbf{C}$ such that

$$F(\lambda) = 0 \qquad \text{for all } \lambda \notin L(m,n).$$

We equip $\mathscr{C}_{d,mn}(\hat{G})$ with the topology induced by the system of seminorms $\|F\|_{\ell} = \sup_{\lambda \in L(m,n)} |F(\lambda)|(1+|\lambda|)^{\ell}$ for $\ell \in \mathbf{Z}_{\geq 0}$. Then $\mathscr{C}_{d,mn}(\hat{G})$ becomes a Fréchet space. It is also known that $\mathscr{C}(G) \subseteq L^2(G)$ and $\mathscr{C}_{c,mn}(\hat{G}) \subseteq L^p_{\varepsilon}(\mathfrak{a}^*)$ for all $p \in [1,\infty]$.

For $f \in L^1(G)$, we define its (m,n)-spherical transforms $\mathscr{F}^c_{mn} f$ and $\mathscr{F}^d_{mn} f$ by

$$(\mathscr{F}^c_{mn} f)(\varepsilon, \nu) = \int_G f(g) \Phi^{\varepsilon,\nu}_{mn}(g) dg,$$

$$(\mathscr{F}^d_{mn} f)(\lambda) = \int_G f(g) \Psi^{\lambda}_{mn}(g) dg.$$

For $\phi \in L^1_{\varepsilon}(\mathfrak{a}^*)$ and $m,n \in \mathbf{Z}(\varepsilon)$, we set

$$(\mathscr{S}^c_{mn}\phi)(g) = \int_0^{\infty} \phi(\nu) \Phi^{\varepsilon,\nu}_{nm}(g^{-1}) \mu(\varepsilon,\nu) d\nu.$$

For an arbitrary function $\phi : \mathbf{Z}\backslash\{0\} \to \mathbf{C}$, we put

$$(\mathscr{S}^d_{mn}\phi)(g) = \sum_{\lambda \in L(m,n)} d(\lambda) \phi(\lambda) \Psi^{\lambda}_{nm}(g^{-1}).$$

Proposition 4.2 (cf. [8]). *The (m,n)-spherical transform $\mathscr{F}^c_{mn}$ (resp. $\mathscr{F}^d_{mn}$) is a topological isomorphism of $\mathscr{C}_{c,mn}(G)$ (resp. $\mathscr{C}_{d,mn}(G)$) onto $\mathscr{C}_{c,mn}(\hat{G})$ (resp. $\mathscr{C}_{d,mn}(\hat{G})$). And the inverse transform of $\mathscr{F}^c_{mn}$ (resp. $\mathscr{F}^d_{mn}$) is given by $\mathscr{S}^c_{mn}$ (resp. $\mathscr{S}^d_{mn}$).*

For $\phi \in \mathscr{C}(G)$, we define the wave packets $\phi_{c,mn} \in \mathscr{C}_{c,mn}(G)$ and $\phi_{d,mn} \in \mathscr{C}_{d,mn}(G)$ by

$$\phi_{c,mn}(g) = \mathscr{S}^c_{mn}(\mathscr{F}^c_{mn}\phi)(g) = \int_0^{\infty} (\mathscr{F}^c_{mn}\phi)(\varepsilon,\nu) \Phi^{\varepsilon,\nu}_{nm}(g^{-1}) \mu(\varepsilon,\nu) d\nu,$$

$$\phi_{d,mn}(g) = \mathscr{S}^d_{mn}(\mathscr{F}^d_{mn}\phi)(g) = \sum_{\lambda \in L(m,n)} d(\lambda) (\mathscr{F}^d_{mn}\phi)(\lambda) \Psi^{\lambda}_{nm}(g^{-1}),$$

and put $\phi_{mn}(g) = \phi_{c,mn}(g) + \phi_{d,mn}(g)$. Then the following proposition is valid.

Proposition 4.3 (see [2]). *For each $\phi \in \mathscr{C}(G)$, there is a unique expansion*

$$\phi = \sum_{m,n \in \mathbf{Z}} \phi_{mn} = \sum_{m,n \in \mathbf{Z}} \phi_{c,mn} + \sum_{m,n \in \mathbf{Z}} \phi_{d,mn}.$$

The series converges absolutely to ϕ in $\mathscr{C}(G)$, and the mappings $\phi \to \phi_{c,mn}$ and $\phi \to \phi_{d,mn}$ are continuous.

Let $\mathscr{C}'(G)$ be the set of tempered distributions on G. For a tempered distribution $T \in \mathscr{C}'(G)$, we define $T_{c,mn}, T_{d,mn} \in \mathscr{C}'(G)$ by

$$T_{c,mn}[\phi] = T[\phi_{c,mn}], \qquad T_{d,mn}[\phi] = T[\phi_{d,mn}] \qquad (\phi \in \mathscr{C}(G)).$$

Similarly, we also define $T_{mn} \in \mathscr{C}'(G)$ by

$$T_{mn}[\phi] = T[\phi_{mn}].$$

Proposition 4.4 (see [2]). *Retain the notation above.*

$$T = \sum_{m,n \in \mathbf{Z}} T_{mn} = \sum_{m,n \in \mathbf{Z}} T_{c,mn} + \sum_{m,n \in \mathbf{Z}} T_{d,mn},$$

where the series converges absolutely to T in the weak topology of $\mathscr{C}'(G)$.

Let $\Phi \in \mathscr{C}(\hat{G})$ and write $\Phi = \Phi_c + \Phi_d$ for $\Phi_c \in \mathscr{C}_c(\hat{G})$ and $\Phi_d \in \mathscr{C}_d(\hat{G})$. We set

$$\Phi_{c,mn}(\nu) = \langle \Phi_c(\varepsilon, \nu) e_n, e_m \rangle,$$

$$\Phi_{d,mn}(\lambda) = (\Phi_d(\lambda)\psi_n, \psi_m)_\lambda,$$

where ε is chosen so that $m, n \in \mathbf{Z}(\varepsilon)$. Then $\Phi_{c,mn} \in \mathscr{C}_{c,mn}(\hat{G})$, $\Phi_{d,mn} \in \mathscr{C}_{d,mn}(\hat{G})$ and $\Phi = \sum_{m,n \in \mathbf{Z}} \Phi_{c,mn} + \sum_{m,n \in \mathbf{Z}} \Phi_{d,mn}$.

For $T \in \mathscr{C}'(G)$, we define its Fourier transform by

$$\mathscr{F}T[\Phi] = T[\mathscr{F}^{-1}\Phi], \qquad \Phi \in \mathscr{C}(\hat{G}).$$

And also, for $S \in \mathscr{C}'(\hat{G})$, we define its inverse transform of $\mathscr{F}$ by

$$\mathscr{F}^{-1}S[\phi] = S[\mathscr{F}\phi], \qquad \phi \in \mathscr{C}(G).$$

Let $T \in \mathscr{C}'(G)$ and we define its Fourier transforms $\mathscr{F}^c T$ and $\mathscr{F}^d T$ by

$$\mathscr{F}^c T[\Phi] = T[\mathscr{S}^c \Phi], \qquad \mathscr{F}^d T[\Phi] = T[\mathscr{S}^d \Phi]$$

for $\Phi \in \mathscr{C}(\hat{G})$. Following Barker [2], we define the (m,n)-spherical transforms $\mathscr{F}^c_{mn} T$ and $\mathscr{F}^d_{mn} T$ of T by

$$\mathscr{F}^c_{mn} T[\Phi] = T[\mathscr{S}^c_{mn} \Phi_{c,mn}],$$

$$\mathscr{F}^d_{mn} T[\Phi] = T[\mathscr{S}^d_{mn} \Phi_{d,mn}]$$

for $\Phi \in \mathscr{C}(\hat{G})$. Here we give some lemmas.

Lemma 4.5 (cf. [2]). *Let $T \in \mathscr{C}'(G)$. Then*

$$\mathscr{F}^c T_{c,mn} = \mathscr{F}^c_{mn} T, \qquad \mathscr{F}^d T_{c,mn} = 0, \qquad \mathscr{F}^c T_{d,mn} = 0, \qquad \mathscr{F}^d T_{d,mn} = \mathscr{F}^d_{mn} T.$$

If a function f satisfies $\|e^{a\sigma(g)^2} f(g)\|_{L^p(G)} \le C$ for $a > 0$ and $1 \le p \le \infty$, we call that f is very rapidly decreasing. Such functions belong to $L^1(G)$. When f is very rapidly decreasing, we define $T_f \in \mathscr{C}'(G)$ by

$$T_f[\phi] = \int_G f(g)\phi(g)dg, \qquad \phi \in \mathscr{C}(G). \tag{4.1}$$

If f is very rapidly decreasing and (m,n)-spherical, then

$$(T_f)_{c,rs} = \delta_{r,-m}\delta_{s,-n}(T_f)_{c,(-m)(-n)},$$

$$(T_f)_{d,rs} = \delta_{r,-m}\delta_{s,-n}(T_f)_{d,(-m)(-n)},$$

for $r,s \in \mathbf{Z}$. From this fact and Proposition 4.4, we see that

$$T_f = (T_f)_{c,(-m)(-n)} + (T_f)_{d,(-m)(-n)},$$

where f is very rapidly decreasing and (m,n)-spherical. Let $F \in L^p_\varepsilon(\mathfrak{a}^*)$ and fix $m,n \in \mathbf{Z}(\varepsilon)$. If we set

$$T_F[\Phi] = \int_0^\infty F(v)\Phi(v)\mu(\varepsilon,v)dv \qquad \text{for } \Phi \in \mathscr{C}_{c,mn}(\hat{G}),$$

then $T_F \in \mathscr{C}'_{c,mn}(\hat{G})$.

For an arbitrary function $F : \mathbf{Z}\backslash\{0\} \to \mathbf{C}$, we put

$$T_F[\Phi] = \sum_{\lambda \in L(m,n)} d(\lambda)F(\lambda)\Phi(\lambda) \qquad \text{for } \Phi \in \mathscr{C}_{d,mn}(\hat{G}).$$

Then $T_F \in \mathscr{C}'_{d,mn}(\hat{G})$.

LEMMA 4.6. *Let f be very rapidly decreasing and (m,n)-spherical, and $\mathscr{F}^c_{mn}f \in L^1_\varepsilon(\mathfrak{a}^*)$. Then*

$$\mathscr{F}^{-1}\mathscr{F}^c_{(-m)(-n)}T_f = T_{(\mathscr{S}^c_{(-n)(-m)}\mathscr{F}^c_{(-n)(-m)}\check{f})^\vee},$$

$$\mathscr{F}^{-1}\mathscr{F}^d_{(-m)(-n)}T_f = T_{(\mathscr{F}^d_{(-n)(-m)}\mathscr{F}^d_{(-n)(-m)}\check{f})^\vee},$$

where $\check{f}(g) = f(g^{-1})$.

PROOF. For $\Phi \in \mathscr{C}(\hat{G})$, we have

$$\begin{aligned}
\mathscr{F}^c_{(-m)(-n)}T_f[\Phi] &= T_f[(\mathscr{S}^c_{(-m)(-n)})\Phi_{c,(-m)(-n)}] \\
&= \int_G f(g)\int_0^\infty \Phi_{c,(-m)(-n)}(v)\Phi^{\varepsilon,v}_{(-n)(-m)}(g^{-1})\mu(\varepsilon,v)dvdg \\
&= \int_0^\infty \Phi_{c,(-m)(-n)}(v)\int_G \check{f}(g)\Phi^{\varepsilon,v}_{(-n)(-m)}(g)dg\mu(\varepsilon,v)dv \\
&= \int_0^\infty \Phi_{c,(-m)(-n)}(v)\mathscr{F}^c_{(-n)(-m)}\check{f}(v)\mu(\varepsilon,v)dv \\
&= T_{\mathscr{F}^c_{(-n)(-m)}\check{f}}[\Phi_{c,(-m)(-n)}].
\end{aligned}$$

Thus we have, for $\phi \in \mathscr{C}(G)$,

$$\begin{aligned}
&\mathscr{F}^{-1}\mathscr{F}^{c}_{(-m)(-n)}T_f[\phi] \\
&\quad = T_{\mathscr{F}^{c}_{(-n)(-m)}\check{f}}[(\mathscr{F}\phi)_{c,(-m)(-n)}] \\
&\quad = T_{\mathscr{F}^{c}_{(-n)(-m)}\check{f}}[\mathscr{F}_{c,(-n)(-m)}\phi_{c,(-n)(-m)}] \\
&\quad = \int_0^\infty \mathscr{F}^{c}_{(-n)(-m)}\check{f}(v)\int_G \phi_{c,(-m)(-n)}(g)\Phi^{\varepsilon,v}_{(-m)(-n)}(g)dg\mu(\varepsilon,v)dv \\
&\quad = \int_G \phi_{c,(-m)(-n)}(g^{-1})\int_0^\infty \mathscr{F}^{c}_{(-n)(-m)}\check{f}(v)\Phi^{\varepsilon,v}_{(-m)(-n)}(g^{-1})\mu(\varepsilon,v)dvdg \\
&\quad = \int_G \phi_{c,(-m)(-n)}(g^{-1})(\mathscr{S}^{c}_{(-n)(-m)}\mathscr{F}^{c}_{(-n)(-m)}\check{f})(g)dg \\
&\quad = T_{(\mathscr{S}^{c}_{(-n)(-m)}\mathscr{F}^{c}_{(-n)(-m)}\check{f})^{\vee}}[\phi_{c,(-m)(-n)}].
\end{aligned}$$

Similarly, using the definition of $T_F \in \mathscr{C}'_{d,mn}(\hat{G})$, we also have

$$\mathscr{F}^{-1}\mathscr{F}^{d}_{(-m)(-n)}T_f = T_{(\mathscr{F}^{d}_{(-n)(-m)}\mathscr{F}^{d}_{(-n)(-m)}\check{f})^{\vee}}. \quad \square$$

Finally, we conclude section with the following proposition.

Proposition 4.7. *Let f be very rapidly decreasing and (m,n)-spherical, and $\mathscr{F}^{c}_{mn}f \in L^1_\varepsilon(\mathfrak{a}^*)$. Then*

$$f(g) = (\mathscr{S}^{c}_{mn}\mathscr{F}^{c}_{mn}f)(g) + (\mathscr{S}^{d}_{mn}\mathscr{F}^{d}_{mn}f)(g) \qquad \text{(a.e.)}.$$

Proof. From the above lemmas, we have

$$\begin{aligned}
T_f &= \mathscr{F}^{-1}\mathscr{F}T_f = \mathscr{F}^{-1}\mathscr{F}\{(T_f)_{c,(-m)(-n)} + (T_f)_{d,(-m)(-n)}\} \\
&= \mathscr{F}^{-1}\mathscr{F}^{c}_{(-m)(-n)}(T_f) + \mathscr{F}^{-1}\mathscr{F}^{d}_{(-m)(-n)}(T_f) \\
&= T_{(\mathscr{S}^{c}_{(-n)(-m)}\mathscr{F}^{c}_{(-n)(-m)}\check{f})^{\vee}} + T_{(\mathscr{S}^{d}_{(-n)(-m)}\mathscr{F}^{d}_{(-n)(-m)}\check{f})^{\vee}}.
\end{aligned}$$

Thus we have

$$f = (\mathscr{S}^{c}_{(-n)(-m)}\mathscr{F}^{c}_{(-n)(-m)}\check{f})^{\vee} + (\mathscr{S}^{d}_{(-n)(-m)}\mathscr{F}^{d}_{(-n)(-m)}\check{f})^{\vee} \qquad \text{(a.e.)}.$$

Interchanging f with $\check{f}$, we conclude

$$f(g) = (\mathscr{S}^{c}_{mn}\mathscr{F}^{c}_{mn}f)(g) + (\mathscr{S}^{d}_{mn}\mathscr{F}^{d}_{mn}f)(g) \qquad \text{(a.e.)}. \quad \square$$

5. The main theorem

We need the following lemma of Cowling-Price (cf. [4]).

Lemma 5.1. *Let $1 \le p \le \infty$ and $A > 0$. Let g be an entire function such that*

$$|g(x+\sqrt{-1}y)| \le Ae^{\pi x^2},$$

$$\left(\int_{\mathbf{R}} |g(x)|^p x^2\, dx\right)^{1/p} \le A.$$

Then g is a constant function on $\mathbf{C}$. Moreover, if $p < \infty$ then $g = 0$.

Proof. The lemma can be proved as Cowling-Price [4] by a slight modification. □

Proposition 5.2. *Let $1 \le p, q \le \infty$. Let f be a (m,n)-spherical measurable function on G such that*

$$\|e^{a\sigma(g)^2} f(g)\|_{L^p(G)} \le C,$$

$$\|e^{bv^2}(\mathscr{F}_{mn}^c f)(\varepsilon, v)\|_{L_\varepsilon^q(\mathfrak{a}^*)} \le C$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ (a.e.).

Proof. We recall $\Phi_{mn}^{\varepsilon,v}(g)$ is a holomorphic function of $v \in \mathbf{C}$, and satisfies

$$|\Phi_{mn}^{\varepsilon,v}(a_t)| \le e^{(|\Im v|+1/2)|t|}. \tag{5.1}$$

So $(\mathscr{F}_{mn}^c f)(\varepsilon, v)$ is also holomorphic function. Let p' denote the conjugate exponent of p, that is, $1/p + 1/p' = 1$. Then the Hölder ineqality and the first assumption of f implies that

$$\begin{aligned}
|(\mathscr{F}_{mn}^c f)(\varepsilon, v)| &= \left|\int_G f(g)\Phi_{mn}^{\varepsilon,v}(g)dg\right| \\
&\le \left\{\int_G |e^{a\sigma(g)^2} f(g)|^p dg\right\}^{1/p} \left\{\int_G |e^{-a\sigma(g)^2}\Phi_{mn}^{\varepsilon,v}(g)|^{p'} dg\right\}^{1/p'} \\
&\le C\left\{\int_G |e^{-a\sigma(g)^2}\Phi_{mn}^{\varepsilon,v}(g)|^{p'} dg\right\}^{1/p'}.
\end{aligned}$$

Using similar arguments of [14], we have from (2.1), (5.1) and $|\sinh t| \le e^t$ $(t \in [0, \infty))$ that

$$|(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)| \le C\left\{\int_0^\infty e^{-2ap't^2+p'(|\Im\nu|+1/2)t+t}\,dt\right\}^{1/p'}$$
$$\le C\left\{\int_0^\infty e^{-ap't^2+p'(|\Im\nu|+1/2)t/\sqrt{2}+t/\sqrt{2}}\,dt\right\}^{1/p'}.$$

We choose $0 < a' < a$ so that $a'b > 1/4$ and $e^{-ap't^2+(p'/2+1)t/\sqrt{2}} \le \text{Const.}\ e^{-a'p't^2}$. Then, for a constant C_0, we obtain

$$|(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)| \le C_0 e^{(\Im\nu)^2/(8a')}.$$

Therefore we have

$$|e^{\nu^2/4a'}(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)| \le C_1 e^{(\Re\nu)^2/(4a')} \tag{5.2}$$

for $\nu\in\mathbf{C}$ and a constant $C_1 > 0$. On the other hand, there exist positive constants B_1 and B_2 such that, for $\varepsilon = 0,1$ and $\nu\in\mathbf{R}$,

$$B_1\nu^2(1+|\nu|)^{-1} \le \mu(\varepsilon,\nu) \le B_2\nu^2(1+|\nu|)^{-1},$$

and then the Hölder inequality implies

$$\|e^{\nu^2/(4a')}(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)\|_{L^1_\varepsilon(\mathfrak{a}^*,\nu^2d\nu)} \tag{5.3}$$
$$\le \|(1+|\nu|)e^{-(b-1/4a')\nu^2}\|_{L^{q'}_\varepsilon(\mathfrak{a}^*)}\|e^{b\nu^2}(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)\|_{L^q_\varepsilon(\mathfrak{a}^*)},$$

where q' denotes the conjugate exponent of q. From the second assumption of $\mathscr{F}^c_{mn}f$, $a'b > 1/4$ and (5.4), we can find a constant $C_2 > 0$ such that

$$\|e^{\nu^2/(4a')}(\mathscr{F}^c_{mn}f)(\varepsilon,\nu)\|_{L^1_\varepsilon(\mathfrak{a}^*,\nu^2d\nu)} \le C_2, \tag{5.4}$$

for $\nu\in\mathbf{R}$. Therefore Lemma 5.1 implies

$$(\mathscr{F}^c_{mn}f)(\varepsilon,\nu) = 0. \tag{5.5}$$

Interchanging f with $\check{f}$, we have from Lemma 4.6 and (5.5) that

$$\mathscr{F}^c_{(-m)(-n)}(T_f)[\Phi] = (T_{\mathscr{F}^c_{mn}\check{f}})[\Phi_{c,(-n)(-m)}] = 0$$

for all $\Phi\in\mathscr{C}(\hat{G})$. From Proposition 4.7, we have

$$f(g) = \mathscr{S}^d_{mn}\mathscr{F}^d_{mn}f(g) \tag{5.6}$$
$$= \sum_{\lambda\in L(m,n)}(\mathscr{F}^d_{mn}f)(\lambda)\Psi^\lambda_{mn}(g) \qquad \text{(a.e.)}.$$

If $f \ne 0$ (a.e.), there exists $C_4 \ne 0$ such that

$$f(a_t) = C_4 e^{-t} + \text{higher order terms} \qquad \text{(a.e.)}. \tag{5.7}$$

On the other hand, the first assumption of f implies that, when $1 \le p < \infty$, the integral

$$\int_G |e^{a\sigma(g)^2} f(g)|^p dg = \int_{K\times K}\int_0^\infty |e^{a\sigma(k_1 a_t k_2)^2} f(k_1 a_t k_2)|^p \sinh t\, dt dk_1 dk_2$$
$$= \int_0^\infty |C_4 e^{2at^2 - t} + e^{2at^2}(\text{higher order terms})|^p \sinh t\, dt$$

must be finite, and when $p = \infty$,

$$\sup_{g\in G} |e^{a\sigma(g)^2} f(g)| = \sup_{t>0} |e^{2at^2} f(a_t)| < \infty.$$

However, the function $e^{2at^2 - t}$ diverges as $t \to \infty$, so the integral does not converge, and similarly, the supremum is infinite. This leads to a contradiction. Thus we conclude $f = 0$ (a.e.). □

REMARK. By using the proof of the discrete series part in Proposition 5.2, that is asymptotic behavior of Ψ^{λ}_{mn}, we can give another proof of the Hardy theorem for G proved by A. Sitaram and M. Sundari [14].

The following theorem is an easy consequence of Proposition 5.2.

THEOREM 5.3 (the Cowling-Price theorem for $SU(1,1)$). *Let* $1 \le p$, $q \le \infty$. *Let* f *be a measurable function on* G *such that*

$$\|e^{a\sigma(g)^2} f(g)\|_{L^p(G)} \le C,$$
$$\|e^{bv^2} \mathscr{F}^c f(\varepsilon, v)\|_{\mathscr{L}^q_\varepsilon(\mathfrak{a}^*)} \le C_\varepsilon$$

for $C > 0$, $a > 0$ *and* $b > 0$. *If* $ab > 1/4$ *then* $f = 0$ *(a.e.).*

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