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Periods of cut-and-project tiling spaces obtained from root lattices

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Abstract. Let $\mathscr{T}(E)$ be the tiling space given by the cut-and-project method in $\mathbf{R}^d = E \oplus E^{\perp}$ with a root lattice L. We will consider the dimension n of the linear space of the periods of $\mathscr{T}(E)$. We present a theorem which determines n from the root lattice L in $\mathbf{R}^d$.

1. Introduction

In 1981 de Bruijn [2], [3] introduced the cut-and-project method to construct tilings such as Penrose tilings with icosahedral symmetry. The cut-and-project method was extended to the higher dimensional hypercubic lattices [5] and to more general lattices [6]. To construct tilings and tiling spaces by the cut-and-project method, the hypercubic lattices are most frequently used. However, we are interested in tilings and tiling spaces obtained from root lattices because it is impossible for some tilings to be obtained from hypercubic lattices but possible from root lattices (cf. [1]).

First, we recall the definitions of tilings and tiling spaces by the cut-and-project method (cf. [5], [6], [7], [9]). Let L be a lattice in $\mathbf{R}^d$ with a basis $\{b_i \mid i = 1, 2, \dots, d\}$. Let E be a p-dimensional subspace of $\mathbf{R}^d$, and $E^{\perp}$ its orthogonal complement with respect to the standard inner product. Let $\pi : \mathbf{R}^d \to E$ be the orthogonal projection onto E, and $\pi^{\perp} : \mathbf{R}^d \to E^{\perp}$ the orthogonal projection onto $E^{\perp}$. Let A be a Voronoi cell of L. For any $x \in \mathbf{R}^d$ we put $W_x = \pi^{\perp}(x) + \pi^{\perp}(A) = \{\pi^{\perp}(x) + u \mid u \in \pi^{\perp}(A)\}$, which is called a window for the projection. We define $\Lambda(x)$ by $\Lambda(x) = \pi((W_x \times E) \cap L)$. Let $\mathscr{V}(x)$ denote the Voronoi tiling induced by $\Lambda(x)$, which consists of the Voronoi cells of $\Lambda(x)$. For a vertex v in $\mathscr{V}(x)$ we define $S(v)$ by $S(v) = \bigcup\{P \in \mathscr{V}(x) \mid v \in P\}$. The tiling $T(x)$ given by the cut-and-project method is defined as the collection of tiles $\mathrm{Conv}(S(v) \cap \Lambda(x))$, where $\mathrm{Conv}(B)$ denotes the convex hull of a set B. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$.

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The tiling space $\mathscr{T}(E)$ is defined as the space of the tilings $T(x)$ for all $x \in \mathbf{R}^d$. Tiling spaces are topological dynamical systems with a continuous $\mathbf{R}^p$ action, where a topology is defined by a tiling metric on tilings of $\mathbf{R}^p$ (see for example [8]). Periods of the tiling space $\mathscr{T}(E)$ are defined to be the common periods of $T(x)$ for all $x \in \mathbf{R}^d$.

For $L = \mathbf{Z}^d$, C. Hillman characterized the number of periods of the tiling space. He also constructed periods for given tiling spaces [7].

The purpose of this paper is to extend one of Hillman's results to the case that L is a root lattice.

THEOREM. *Let $\mathscr{T}(E)$ be the tiling space given by the cut-and-project method in terms of a p-dimensional subspace E of $\mathbf{R}^d$ and assume that L is a root lattice. Then,* $\operatorname{rank} \operatorname{Ker}(\pi^{\perp}|L)$ *is equal to the dimension of the linear space of the periods of $\mathscr{T}(E)$.*

For the general lattices Theorem is not true. We have the following example. Let L be a lattice in $\mathbf{R}^2$ with a basis $\{(1, \sqrt{2}), (1, -1)\}$ and E be the x-axis of $\mathbf{R}^2$. In this case it is easy to see that all tilings in $\mathbf{R}^1$ obtained by the cut-and-project method are periodic and $\operatorname{rank} \operatorname{Ker}(\pi^{\perp}|L) = 0$.

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2. Proof of Theorem

In order to prove Theorem it suffices to show that for all $x \in \mathbf{R}^d$ each element of $\operatorname{Ker}(\pi^{\perp}|L)$ is a period of the tiling $T(x)$ and $\operatorname{rank} \operatorname{Ker}(\pi^{\perp}|L)$ is equal to the dimension of the linear space of the periods of $T(x)$. For convenience we assume $0 \in W_x$. By the similar argument we can also prove it in the case that $0 \notin W_x$. We see that a basis of $\operatorname{Ker}(\pi^{\perp}|L)$ consists of linearly independent periods of $(W_x \times E) \cap L$. By the definition of the cut-and-project method, the images by π of vectors in a basis are linearly independent periods of the tiling $T(x)$. We obtain that the dimension of the linear space of the periods of $T(x)$ is greater than or equal to $\operatorname{rank} \operatorname{Ker}(\pi^{\perp}|L)$.

We first prove the following proposition to show that $\operatorname{rank} \operatorname{Ker}(\pi^{\perp}|L)$ is greater than or equal to the dimension of the linear space of the periods of $T(x)$.

PROPOSITION. *Let $\tilde{v}'$ be a vector in $(W_x \times E) \cap L$ such that $\pi(\tilde{v}')$ is a period of $T(x)$. Then, there exist a positive integer a and $\tilde{v} \in \operatorname{Ker}(\pi^{\perp}|L)$ such that $\pi(a\tilde{v}') = \pi(\tilde{v})$.*

PROOF OF PROPOSITION. If $\pi^{\perp}(\tilde{v}') = 0$, then we can take $a = 1$ and $\tilde{v} = \tilde{v}'$. So we may assume that $\pi^{\perp}(\tilde{v}') \neq 0$. Since the root lattices are integral lattices (cf. [4, Chapter 4]), due to [9, p. 52, Th. 2.3] there exist subspaces V_1 and V_2 which satisfy the following:

$$E^\perp = V_1 \oplus V_2,$$

$$\pi^\perp((V_1 \oplus E) \cap L) = V_1 \cap \pi^\perp(L) \text{ is a discrete lattice in } V_1,$$

$$\pi^\perp((V_2 \oplus E) \cap L) = V_2 \cap \pi^\perp(L) \text{ is dense in the nontrivial subspace } V_2.$$

The set $W_x \cap V_1 \cap \pi^\perp(L)$ is finite since W_x is compact. We put $W_x \cap V_1 \cap \pi^\perp(L) = \{p_1, \dots, p_t\}$ and $D_i = \{p_i + y \mid y \in V_2 \cap \pi^\perp(L)\} \cap W_x$. We see that $D_i = D_j$ if $p_i - p_j \in V_2$, and $D_i \cap D_j = \varnothing$ otherwise. By modifying the indices if necessary, we may assume that $W_x \cap \pi^\perp(L)$ is a disjoint union of $D_1, \dots D_s$ $(s \le t)$. Since W_x is compact, there exists a positive integer c such that $r\pi^\perp(\tilde{v}') \notin W_x$ for any integer $r \ge c$. Since $\pi(\tilde{v}')$ is a period of $T(x)$, for any integer $r \ge c$ there exists $w_r \in (W_x \times E) \cap L$ such that $\pi(r\tilde{v}') = \pi(w_r)$. Since $W_x \cap \pi^\perp(L)$ is a disjoint union of $D_1, \dots D_s$, there exist positive integers $r_1 < r_2$ and s_0 such that $\pi^\perp(w_{r_1})$, $\pi^\perp(w_{r_2}) \in D_{s_0}$. Then we see that $\pi^\perp(w_{r_1}) - \pi^\perp(w_{r_2}) \in V_2 \cap \pi^\perp(L)$. If $\pi^\perp(w_{r_1} - w_{r_2}) = 0$, then we can take $a = r_2 - r_1$ and $\tilde{v} = w_{r_1} - w_{r_2}$.

So we may assume that $\pi^\perp(w_{r_1} - w_{r_2}) \neq 0$. We put $u_1 = w_{r_1} - w_{r_2}$ and let h be the maximum value of length of line segments of the direction $\pi^\perp(u_1)$ in W. We take a positive integer c_0 such that $c_0\|\pi^\perp(u_1)\| > h$, where $\| \ \|$ denotes the length of a vector. Since W_x is compact, there exists a positive integer c_1 such that $r_0\pi^\perp(u_1) \notin W_x$ for any integer $r_0 \ge c_1$. Since $\pi(u_1)$ is a period of a tiling $T(x)$, for any integer $t_0 \ge c_1$ there exists $z_{t_0} \in (W_x \times E) \cap L$ such that $\pi(t_0 u_1) = \pi(z_{t_0})$. Since $W_x \cap \pi^\perp(L)$ is a disjoint union of $D_1, \dots D_s$, there exist positive integers t_1, t_2 and s_0 with $t_1 > c_1$ and $t_2 > t_1 + c_0$ which satisfy the property that $\pi^\perp(z_{t_1})$ and $\pi^\perp(z_{t_2})$ are in D_{s_0}. Then we see that $z_{t_2} - t_2 u_1 - (z_{t_1} - t_1 u_1) \in \operatorname{Ker}(\pi|L)$ is a non-zero vecor and $\pi^\perp(z_{t_2} - t_2 u_1 - (z_{t_1} - t_1 u_1)) \in V_2 \cap \pi^\perp(L)$. Because the root lattice L is integral and $\pi^\perp((V_2 \oplus E) \cap L)$ is dense, we obtain that $\pi \mid (V_2 \oplus E) \cap L$ is one to one by [9, p. 55, Prop. 2.15]. Thus we have a contradiction. We obtain that $\pi^\perp(w_{r_1} - w_{r_2}) = 0$. Hence, in any case we can take a positive integer a and $\tilde{v} \in \operatorname{Ker}(\pi^\perp|L)$ that satisfy the properties stated in Proposition. q.e.d.

We take linearly independent periods $v_1', v_2', \dots, v_n'$ of $T(x)$, where n denotes the dimension of the linear space of the periods of $T(x)$. Then we take vectors $\tilde{v}_1', \tilde{v}_2', \dots, \tilde{v}_n'$ in $(W_x \times E) \cap L$ such that $\pi(\tilde{v}_i') = v_i'$ for each i. By Proposition there exist positive integers a_i and $\tilde{v}_i \in \operatorname{Ker}(\pi^\perp|L)$ such that $\pi(a_i \tilde{v}_i') = \pi(\tilde{v}_i)$ $(i = 1, 2, \dots, n)$. Then we obtain that $\operatorname{rank} \operatorname{Ker}(\pi^\perp|L)$ is greater than or equal to the dimension of the linear space of the periods of $T(x)$. Hence the proof of Theorem is completed. q.e.d.

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