A note on nonremovable cusp singularities

Dedicated to Professors Takuo Fukuda and Shuzo Izumi on their 60-th birthdays

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Abstract. Let \( M^4 \) be a closed oriented 4-manifold such that \( \text{rank}_\mathbb{Z} \mathbb{H}_2(M^4; \mathbb{Z}) = 1. \) Then we prove that every stable map \( f : M^4 \to \mathbb{R}^3 \) always has cusp singularities.

1. Introduction

Is the Thom polynomial the only obstruction to removing the corresponding singularities of a given smooth map? This is one of the important questions in Global Singularity Theory and several affirmative answers had been given mainly on the elimination of 0-dimensional singularities (see [9, 8]). In [5, Theorem 4] and [6, Theorem 4.5] Saeki discovered a negative case that every stable map of a closed 4-dimensional manifold \( M^4 \) with \( \mathbb{H}_1(M^4; \mathbb{Z}) \cong \mathbb{H}_1(\mathbb{C}P^2; \mathbb{Z}) \) into an orientable 3-manifold has necessarily nonempty one dimensional cusp singularities. This means that the cusp singularities cannot be removed, although the Thom polynomial vanishes.

Analyzing his proof in [5], it seems that this phenomenon is caused by a Rohlin type theorem peculiar to 4-dimensional differential topology and essentially based on the condition that \( \mathbb{H}_2(M^4; \mathbb{Z}) \cong \mathbb{Z} \). The condition that \( \mathbb{H}_1(M^4; \mathbb{Z}) = 0 \) is also technically needed to use the author’s mod 4 formula in [10].

The purpose of this paper is to extend Saeki’s theorem for a wider class of 4-manifolds at least without the assumption that \( \mathbb{H}_1(M^4; \mathbb{Z}) = 0. \) Actually, we prove that the same statement holds for a closed orientable 4-manifold \( M^4 \) with \( \text{rank}_\mathbb{Z} \mathbb{H}_2(M^4; \mathbb{Z}_2) = 1. \) Our proof heavily depends on Yamada’s result in [13].

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Throughout the paper we work in the $C^\infty$ category.

2. Main theorem

Let $f : M^4 \to N^3$ be a stable map of a 4-manifold into a 3-manifold. We denote the singular set of $f$ by $S(f)$; i.e.,

$$S(f) = \{ x \in M^4 ; \text{rank}(df_x : T_xM^4 \to T_{f(x)}N^3) < 3 \},$$

where $df_x : T_xM^4 \to T_{f(x)}N^3$ is the differential of $f$. If $x \in S(f)$, then there exist local coordinates $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3)$ centered at $x$ and $f(x)$ respectively such that $f$ has one of the following normal forms ([3]):

1. $y_i = x_i$ ($i = 1, 2$), $y_3 = x_3^2 + x_4^2$ (definite fold singularity),
2. $y_i = x_i$ ($i = 1, 2$), $y_3 = x_3^2 - x_4^2$ (indefinite fold singularity),
3. $y_i = x_i$ ($i = 1, 2$), $y_3 = x_3^2 + x_1x_2 \pm x_4^2$ (cusp singularity),
4. $y_i = x_i$ ($i = 1, 2$), $y_3 = x_3^3 + x_1x_2^2 + x_1x_2 \pm x_4^2$ (swallowtail singularity)

We denote by $A_k(f)$ ($k = 1, 2, 3$) the set of the fold, cusp and swallowtail singularities respectively. Then their Thom polynomials turn out to be:

$$[A_1(f)]^2_2 = w_2, \quad [A_2(f)]^2_2 = 0, \quad [A_3(f)]^2_2 = 0,$$

where $w_2 \in H^2(M^4; \mathbb{Z}_2)$ is the second Stiefel-Whitney class of $M^4$. The first expression is due to Thom [12] and the last one is an easy exercise. The reason for the second one (the Thom polynomial of cusp singularities) is as follows: it must be expressed as $[A_2(f)]^2_2 = aw_1^3 + bw_1 \cdot w_2 + cw_3$ ($a, b, c \in \mathbb{Z}_2$) in general and $w_3 = w_1 \cdot w_2 = 0$, since $M^4$ is orientable. Moreover, by using Wu formula ([4, §11]), we have

$$w_3 = Sq^1v_2 = v_1 \cdot v_2 = w_1 \cdot v_2 = 0,$$

where $v_i$ denotes the $i$-th Wu class.

In this section we prove the following

**Theorem 2.1.** Let $M^4$ be a closed orientable 4-manifold such that $\text{rank}_{\mathbb{Z}_2} H_2(M^4; \mathbb{Z}_2) = 1$ and $N^3$ an orientable 3-manifold. Then there exists no smooth map $f : M^4 \to N^3$ with only fold singularities.

We have an immediate corollary.
Corollary 2.2. Let $M^4$ be a closed orientable 4-manifold such that \(\text{rank}_{\mathbb{Z}_2} H_2(M^4; \mathbb{Z}_2) = 1\) and $N^3$ an orientable 3-manifold. Then every stable map $f : M^4 \to N^3$ always has cusp singularities, although the Thom polynomial of cusp singularities of $f$ always vanishes.

Now we proceed to the proof of Theorem 2.1.

Suppose that there is a smooth map $f : M^4 \to N^3$ with only fold singularities for a closed oriented 4-manifold with \(\text{rank}_{\mathbb{Z}_2} H_2(M^4; \mathbb{Z}_2) = 1\). Note that the Euler characteristic $w(M^4)$ is odd by Poincaré duality. We denote by $S^+(f)$ (resp. $S^-(f)$) the set of the definite (resp. indefinite) fold singularities. If $S^-(f) = \emptyset$, then by [11] $\chi(M^4)$ must be even. So we may assume that $S^-(f) \neq \emptyset$. Then the following results play crucial roles.

Lemma 2.3 ([2], [5], [6]).

(i) $\chi(M^4) \equiv \chi(S(f)) \pmod{2}$ ([2], [6]).

(ii) Every component of $S^+(f)$ is a closed orientable surface when $N$ is orientable ([5], [6]).

(iii) For each connected component $S^- \subset S^-(f)$, the integral self-intersection number $S^- \cdot S^-$ of $S^-$ in $M^4$ is zero ([5]).

Moreover, to complete the proof we need the following theorem proved by Yamada [13], which enables us to discuss without any assumption on the first homology group of $M^4$.

Proposition 2.4 ([13]). Let $M^4$ be an oriented 4-manifold. Then a $\mathbb{Z}_4$-quadratic form

$$q : H_2(M^4; \mathbb{Z}_2) \to \mathbb{Z}_4$$

is defined by the correspondence mapping $[F^2]_2 \in H_2(M^4; \mathbb{Z}_2)$ to $e(F^2 \hookrightarrow M^4) + 2\chi(F^2) \pmod{4}$. Here $F^2$ is a closed surface embedded in $M^4$ and $e(F^2 \hookrightarrow M^4)$ denotes the normal Euler number of the embedding $F^2 \hookrightarrow M^4$ which is equal to the integral self-intersection number of $F^2$ in $M^4$.

Here $q : V \to \mathbb{Z}_4$ is called a $\mathbb{Z}_4$-quadratic form over a $\mathbb{Z}_2$-vector space $V$ with finite rank if for any $x, y \in V$ it satisfies the following:

$$q(x + y) = q(x) + q(y) + 2(x \cdot y),$$

where $x \cdot y$ is an inner product over $V$ and $2 : \mathbb{Z}_2 \to \mathbb{Z}_4$ is the module homomorphism sending 1 to 2.

By Lemma 2.3 (i) and (ii) there must exist at least one component, say $S^-_0 \subset S^-(f)$, such that $\chi(S^-_0)$ is odd. Then applying Proposition 2.4 to the component $S^-_0$ together with Lemma 2.3 (iii), we have

$$q([S^-_0]) \equiv e(S^-_0 \hookrightarrow M^4) + 2\chi(S^-_0) \equiv 2 \pmod{4}. \quad (1)$$
On the other hand, since $q$ is a $\mathbb{Z}_4$-quadratic form, it immediately follows
\[ q([0]_4) \equiv 0 \pmod{4} \]
for the zero element $[0]_4 \in H_2(M^4; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and we also have
\[ q([1]_4) \equiv \pm 1 \pmod{4} \]
for a generator $[1]_4 \in H_2(M^4; \mathbb{Z}_2) \cong \mathbb{Z}_2$ under the condition that rank $\mathbb{Z}_2 H_2(M^4; \mathbb{Z}_2) = 1$; both cases contradict (1). This completes the proof of Theorem 2.1.

**Remark 1.** The author raised the following question in [11]: *Does every stable map $f : M^4 \to \mathbb{R}^3$ have cusp singularities for any closed oriented 4-manifold $M^4$ with odd Euler characteristic?* However, Saeki has explicitly constructed a smooth map $f : \mathbb{C}P^2 \to \mathbb{C}P^2 \to \mathbb{R}^3$ with only fold singularities such that $S^+(f) \cong S^2 \cup S^2 \cup S^2$ and $S^-(f) \cong RP^2$ ([7]). Hence the author’s question is negative in the case rank $\mathbb{Z}_2 H_2(M^4; \mathbb{Z}_2) \geq 3$. There is also a smooth map $f : \mathbb{C}P^2 \to \mathbb{C}P^2 \to \mathbb{R}^3$ with only fold singularities such that $S^+(f) \cong S^2 \cup S^2$ and $S^-(f) = \emptyset$ which is called a *special generic map* ([10]). Therefore, it raises the new question whether there exists a smooth map with only fold singularities $f : \mathbb{C}P^2 \to \mathbb{C}P^2 \to \mathbb{R}^3$ or not.

**Remark 2.** Akhmetev and Sadykov [1] also have recently given another proof of Saeki’s theorem from a slightly different viewpoint.

**References**


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