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A note on nonremovable cusp singularities

Dedicated to Professors Takuo Fukuda and Shuzo Izumi on their 60-th birthdays

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Abstract. Let M^4 be a closed oriented 4-manifold such that $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4; \mathbf{Z}_2) = 1$. Then we prove that every stable map $f : M^4 \to \mathbf{R}^3$ always has cusp singularities.

1. Introduction

Is the Thom polynomial the only obstruction to removing the corresponding singularities of a given smooth map? This is one of the important questions in Global Singularity Theory and several affirmative answers had been given mainly on the elimination of 0-dimensional singularities (see [9, 8]). In [5, Theorem 4] and [6, Theorem 4.5] Saeki discovered a negative case that every stable map of a closed 4-dimensional manifold M^4 with $H_*(M^4; \mathbf{Z}) \cong H_*(\mathbf{C}P^2; \mathbf{Z})$ into an orientable 3-manifold has necessarily nonempty one dimensional cusp singularities. This means that the cusp singularities cannot be removed, although the Thom polynomial vanishes.

Analyzing his proof in [5], it seems that this phenomenon is caused by a Rohlin type theorem peculiar to 4-dimensional differential topology and essentially based on the condition that $H_2(M^4; \mathbf{Z}) \cong \mathbf{Z}$. The condition that $H_1(M^4; \mathbf{Z}) = 0$ is also technically needed to use the author's mod 4 formula in [10].

The purpose of this paper is to extend Saeki's theorem for a wider class of 4-manifolds at least without the assumption that $H_1(M^4; \mathbf{Z}) = 0$. Actually, we prove that the same statement holds for a closed orientable 4-manifold M^4 with $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4; \mathbf{Z}_2) = 1$. Our proof heavily depends on Yamada's result in [13].

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Throughout the paper we work in the C^∞ category.

2. Main theorem

Let $f : M^4 \to N^3$ be a *stable map* of a 4-manifold into a 3-manifold. We denote the singular set of f by $S(f)$; i.e.,

$$S(f) = \{x \in M^4; \mathrm{rank}(df_x : T_x M^4 \to T_{f(x)} N^3) < 3\},$$

where $df_x : T_x M^4 \to T_{f(x)} N^3$ is the differential of f. If $x \in S(f)$, then there exist local coordinates (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3) centered at x and $f(x)$ respectively such that f has one of the following normal forms ([3]):

(1) $y_i = x_i$ $(i = 1, 2)$, $y_3 = x_3^2 + x_4^2$ (definite fold singularity),

(2) $y_i = x_i$ $(i = 1, 2)$, $y_3 = x_3^2 - x_4^2$ (indefinite fold singularity),

(3) $y_i = x_i$ $(i = 1, 2)$, $y_3 = x_3^3 + x_1 x_2 \pm x_4^2$ (cusp singularity),

(4) $y_i = x_i$ $(i = 1, 2)$, $y_3 = x_3^4 + x_1 x_2^2 + x_1 x_2 \pm x_4^2$. (swallowtail singularity)

We denote by $A_k(f)$ $(k = 1, 2, 3)$ the set of the fold, cusp and swallowtail singularities respectively. Then their Thom polynomials turn out to be:

$$[\overline{A_1(f)}]_2^* = w_2, \qquad [\overline{A_2(f)}]_2^* = 0, \qquad [\overline{A_3(f)}]_2^* = 0,$$

where $w_2 \in H^2(M^4; \mathbf{Z}_2)$ is the second Stiefel-Whitney class of M^4. The first expression is due to Thom [12] and the last one is an easy exercise. The reason for the second one (the Thom polynomial of cusp singularities) is as follows: it must be expressed as $[\overline{A_2(f)}]_2^* = aw_1^3 + bw_1 \cdot w_2 + cw_3$ $(a, b, c \in \mathbf{Z}_2)$ in general and $w_1^3 = w_1 \cdot w_2 = 0$, since M^4 is orientable. Moreover, by using Wu formula ([4, §11]), we have

$$w_3 = Sq^1 v_2 = v_1 \cdot v_2 = w_1 \cdot v_2 = 0,$$

where v_i denotes the i-th Wu class.

In this section we prove the following

Theorem 2.1. *Let M^4 be a closed orientable 4-manifold such that $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4; \mathbf{Z}_2) = 1$ and N^3 an orientable 3-manifold. Then there exists no smooth map $f : M^4 \to N^3$ with only fold singularities.*

We have an immediate corollary.

Corollary 2.2. *Let* M^4 *be a closed orientable* 4*-manifold such that* $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4;\mathbf{Z}_2) = 1$ *and* N^3 *an orientable* 3*-manifold. Then every stable map* $f : M^4 \to N^3$ *always has cusp singularities, although the Thom polynomial of cusp singularities of* f *always vanishes.*

Now we proceed to the proof of Theorem 2.1.

Suppose that there is a smooth map $f : M^4 \to N^3$ with only fold singularities for a closed oriented 4-manifold with $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4;\mathbf{Z}_2) = 1$. Note that the Euler characteristic $\chi(M^4)$ is odd by Poincaré duality. We denote by $S^+(f)$ (resp. $S^-(f)$) the set of the definite (resp. indefinite) fold singularities. If $S^-(f) = \emptyset$, then by [11] $\chi(M^4)$ must be even. So we may assume that $S^-(f) \neq \emptyset$. Then the following results play crucial roles.

Lemma 2.3 ([2], [5], [6]).

(i) $\chi(M^4) \equiv \chi(S(f)) \pmod 2$ ([2], [6]).

(ii) *Every component of* $S^+(f)$ *is a closed orientable surface when* N *is orientable* ([5], [6]).

(iii) *For each connected component* $S^- \subset S^-(f)$, *the integral self-intersection number* $S^- \cdot S^-$ *of* S^- *in* M^4 *is zero* ([5]).

Moreover, to complete the proof we need the following theorem proved by Yamada [13], which enables us to discuss without any assumption on the first homology group of M^4.

Proposition 2.4 ([13]). *Let* M^4 *be an oriented* 4*-manifold. Then a* $\mathbf{Z}_4$*-quadratic form*

$$q : H_2(M^4;\mathbf{Z}_2) \to \mathbf{Z}_4$$

is defined by the correspondence mapping $[F^2]_2 \in H_2(M^4;\mathbf{Z}_2)$ *to* $e(F^2 \hookrightarrow M^4) + 2\chi(F^2) \pmod 4$. *Here* F^2 *is a closed surface embedded in* M^4 *and* $e(F^2 \hookrightarrow M^4)$ *denotes the normal Euler number of the embedding* $F^2 \hookrightarrow M^4$ *which is equal to the integral self-intersection number of* F^2 *in* M^4.

Here $q : V \to \mathbf{Z}_4$ is called a $\mathbf{Z}_4$-quadratic form over a $\mathbf{Z}_2$-vector space V with finite rank if for any $x, y \in V$ it satisfies the following:

$$q(x+y) = q(x) + q(y) + 2(x \cdot y),$$

where $x \cdot y$ is an inner product over V and $2 : \mathbf{Z}_2 \to \mathbf{Z}_4$ is the module homomorphism sending 1 to 2.

By Lemma 2.3 (i) and (ii) there must exist at least one component, say $S_0^- \subset S^-(f)$, such that $\chi(S_0^-)$ is odd. Then applying Proposition 2.4 to the component S_0^- together with Lemma 2.3 (iii), we have

$$q([S_0^-]) \equiv e(S_0^- \hookrightarrow M^4) + 2\chi(S_0^-) \equiv 2 \pmod 4. \tag{1}$$

On the other hand, since q is a $\mathbf{Z}_4$-quadratic form, it immediately follows

$$q([0]_2) \equiv 0 \pmod 4$$

for the zero element $[0]_2 \in H_2(M^4; \mathbf{Z}_2) \cong \mathbf{Z}_2$ and we also have

$$q([1]_2) \equiv \pm 1 \pmod 4$$

for a generator $[1]_2 \in H_2(M^4; \mathbf{Z}_2) \cong \mathbf{Z}_2$ under the condition that $\mathrm{rank}_{\mathbf{Z}_2} H_2 \cdot (M^4; \mathbf{Z}_2) = 1$; both cases contradict (1). This completes the proof of Theorem 2.1.

Remark 1. The author raised the following question in [11]: *Does every stable map* $f : M^4 \to \mathbf{R}^3$ *have cusp singularities for any closed oriented* 4*-manifold* M^4 *with odd Euler characteristic*? However, Saeki has explicitly constructed a smooth map $f : 2\mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2} \to \mathbf{R}^3$ with only fold singularities such that $S^+(f) \cong S^2 \cup S^2 \cup S^2$ and $S^-(f) \cong \mathbf{R}P^2$ ([7]). Hence the author's question is negative in the case $\mathrm{rank}_{\mathbf{Z}_2} H_2(M^4; \mathbf{Z}_2) \geq 3$. There is also a smooth map $f : \mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2} \to \mathbf{R}^3$ with only fold singularities such that $S^+(f) \cong S^2 \cup S^2$ and $S^-(f) = \emptyset$ which is called a *special generic map* ([10]). Therefore, it raises the new question whether there exists a smooth map with only fold singularities $f : \mathbf{C}P^2 \sharp \mathbf{C}P^2 \to \mathbf{R}^3$ or not.

Remark 2. Akhmetev and Sadykov [1] also have recently given another proof of Saeki's theorem from a slightly different viewpoint.

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