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Oscillation of solutions of first-order neutral differential equations

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ABSTRACT. First order neutral differential equations are studied and sufficient conditions are derived for every solution to be oscillatory. Our approach is to reduce the oscillation of neutral differential equations to the nonexistence of eventually positive solutions of non-neutral differential inequalities. Our results extend and improve several known results in the literature.

1. Introduction

We shall be concerned with the oscillatory behavior of solutions of the neutral differential equation

$$\frac{d}{dt}[x(t) + h(t)x(t-\tau)] + \sigma q(t)|x(t-\rho)|^{\gamma} \operatorname{sgn} x(t-\rho) = 0, \tag{1}$$

where $\sigma = +1$ or -1, $\gamma > 0$, and the following conditions (H1) and (H2) are assumed to hold:

(H1) $h \in C[t_0, \infty)$, $h(t) > 0$ for $t \geq t_0$ and $\tau > 0$;
(H2) $q \in C[t_0, \infty)$, $q(t) > 0$ for $t \geq t_0$ and $\rho \in \mathbf{R}$.

If $\gamma = 1$, then equation (1) becomes the linear equation

$$\frac{d}{dt}[x(t) + h(t)x(t-\tau)] + \sigma q(t)x(t-\rho) = 0. \tag{2}$$

By a solution of (1), we mean a function $x(t)$ that is continuous and satisfies (1) on $[t_x, \infty)$ for some $t_x \geq t_0$. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

In recent years there has been much current interest in studying the oscillations of first-order neutral differential equations. For typical results we refer the reader to [2], [3], [4], [5], [6], [7], [8], [12], [13], [14] for oscillation criteria, and to [1], [2], [3], [7], [8], [11], [12], [13] for existence of nonoscillatory solutions. In particular, Jaroš and Kusano [8] have shown that (1) is oscillatory if and only if

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$$\int^{\infty} q(t)dt = \infty,$$

under one of the following conditions (a) and (b):

(a) $\sigma = +1,\ 0 < \gamma < 1,\ 1 < \mu \le h(t) \le \lambda,\ \rho > \tau > 0;$

(b) $\sigma = -1,\ \gamma > 1,\ 0 < h(t) \le \lambda < 1,\ \tau > 0,\ \rho < 0.$

However, very little is known about sufficient conditions for (1) to be oscillatory for the case where restrictive conditions on $h(t)$, such as $h(t) \le \lambda$, is not assumed. For equation (2) with $h(t) > 0$ and $\sigma = +1$, the sufficient conditions for all solutions to be oscillatory have been established in [5, Theorem 10], [6, Corollary 2] and [7, Theorem 6.4.4]. All of them, however, assume that $q(t)$ is τ-periodic. Györi and Ladas put forward the following question in [7, Problem 6.12.10 (b)]: find sufficient conditions for the oscillation of all solutions of equation (2) with $\sigma = +1$ in the case where $q(t)$ is not τ-periodic.

The purpose of this paper is to obtain sufficient conditions for equation (1) to be oscillatory without the restrictive condition on $h(t)$ or the periodicity of $q(t)$. In Section 2 we reduce the oscillation of the neutral differential equation (1) to the nonexistence of eventually positive solutions of non-neutral differential inequalities of the form

$$\sigma z'(t) + p(t)[z(t-\eta)]^{\gamma} \le 0, \tag{3}$$

where $\eta \in \mathbf{R}$, $p \in C[t_1, \infty)$ and $p(t) > 0$ for $t \ge t_1$. Sufficient conditions for (3) to have no eventually positive solution have been established by many authors. For example, see [2], [7], [9] and [10]. By combining these known results with the results obtained in Section 2, we derive oscillation criteria for equation (1) in Section 3. Our results are extensions and improvements of the results in [4], [5], [6] and [7].

2. Reduction to non-neutral inequalities

In this section we consider the equation

$$\frac{d}{dt}[x(t) + h(t)x(t-\tau)] + \sigma q(t) f(x(t-\rho)) = 0, \tag{4}$$

where $\sigma = +1$ or -1, and conditions (H1), (H2) and the following conditions (H3)–(H5) are assumed:

(H3) $f \in C(\mathbf{R})$, $f(u)$ is nondecreasing in $u \in \mathbf{R}$ and $uf(u) > 0$ for $u \ne 0$;

(H4) there exists a function $\varphi \in C(\mathbf{R})$ such that $\varphi(u)$ is nondecreasing in $u \in \mathbf{R}$, $u\varphi(u) > 0$ for $u \ne 0$ and

$$|\varphi(u+v)| \le |f(u) + f(v)| \qquad \text{for } uv > 0;$$

(H5) there exists a function $\omega \in C(0, \infty)$ such that $\omega(u) > 0$ for $u > 0$ and

$$|f(uv)| \le \omega(u)|f(v)| \qquad \text{for } u > 0 \text{ and } v \in \mathbf{R}.$$

Remark 2.1. For the case $f(u) = |u|^{\gamma} \operatorname{sgn} u$ $(\gamma > 0)$, we can choose

$$\varphi(u) = \min\{1, 2^{1-\gamma}\}|u|^{\gamma} \operatorname{sgn} u \qquad \text{and} \qquad \omega(u) = u^{\gamma}.$$

Theorem 2.1. *Let* $\sigma = +1$. *Suppose that* (H1)–(H5) *hold. Then* (4) *is oscillatory if there exists a function* $\lambda \in C[t_0, \infty)$ *such that* $0 < \lambda(t) < 1$ *for* $t \geq t_0$ *and the differential inequality*

$$\{z'(t) + Q(t)\varphi(z(t - \rho + \tau))\} \operatorname{sgn} z(t) \leq 0 \tag{5}$$

has no nonoscillatory solution, where

$$Q(t) = \min\left\{\lambda(t)q(t), \frac{[1 - \lambda(t - \tau)]q(t - \tau)}{\omega(h(t - \rho))}\right\}. \tag{6}$$

Proof. Suppose to the contrary that there is a nonoscillatory solution $x(t)$ of (4). We may assume that $x(t) > 0$ for all large $t \geq t_0$ since the case where $x(t)$ is eventually negative can be treated similarly. Set $y(t) = x(t) + h(t)x(t - \tau)$. Then, by (4), $y(t) > 0$ and $y(t)$ is decreasing for all large $t \geq t_0$. Integrating (4) over $[t, \infty)$, we have

$$y(t) = \lim_{t\to\infty} y(t) + \int_t^{\infty} q(s)f(x(s - \rho))ds \geq \int_t^{\infty} q(s)f(x(s - \rho))ds$$

for all large $t \geq t_0$. We observe that

$$\begin{aligned}
y(t) &\geq \int_t^{\infty} \lambda(s)q(s)f(x(s-\rho))ds + \int_t^{\infty} [1 - \lambda(s)]q(s)f(x(s-\rho))ds \\
&= \int_t^{\infty} \lambda(s)q(s)f(x(s-\rho))ds + \int_{t+\tau}^{\infty} [1 - \lambda(s-\tau)]q(s-\tau)f(x(s-\rho-\tau))ds \\
&\geq \int_t^{\infty} Q(s)f(x(s-\rho))ds + \int_{t+\tau}^{\infty} Q(s)\omega(h(s-\rho))f(x(s-\rho-\tau))ds \\
&\geq \int_{t+\tau}^{\infty} Q(s)f(x(s-\rho))ds + \int_{t+\tau}^{\infty} Q(s)f(h(s-\rho)x(s-\rho-\tau))ds \\
&= \int_{t+\tau}^{\infty} Q(s)[f(x(s-\rho)) + f(h(s-\rho)x(s-\rho-\tau))]ds \\
&\geq \int_{t+\tau}^{\infty} Q(s)\varphi(x(s-\rho) + h(s-\rho)x(s-\rho-\tau))ds \\
&= \int_{t+\tau}^{\infty} Q(s)\varphi(y(s-\rho))ds
\end{aligned}$$

for all large $t \geq t_0$. Put

$$z(t) = \int_t^\infty Q(s)\varphi(y(s-\rho))ds > 0.$$

Then $y(t) \geq z(t+\tau)$ eventually. We see that

$$z'(t) = -Q(t)\varphi(y(t-\rho)) \leq -Q(t)\varphi(z(t-\rho+\tau))$$

for all large $t \geq t_0$, so that $z(t)$ is an eventually positive solution of (5). This is a contradiction. The proof is complete.

Theorem 2.2. *Let* $\sigma = -1$. *Suppose that* (H1)–(H5) *hold. Then* (4) *is oscillatory if there exists a function* $\lambda \in C[t_0, \infty)$ *such that* $0 < \lambda(t) < 1$ *for* $t \geq t_0$ *and the differential inequality*

$$\{-z'(t) + Q(t)\varphi(z(t-\rho))\} \operatorname{sgn} z(t) \leq 0$$

has no nonoscillatory solution, where $Q(t)$ *is the function defined by* (6).

Proof. Let $x(t)$ be an eventually positive solution of (4). Put $y(t) = x(t) + h(t)x(t-\tau) > 0$. Integration of (4) over $[T, t]$ yields

$$y(t) \geq \int_T^t q(s)f(x(s-\rho))ds, \qquad t \geq T,$$

where T is sufficiently large. By the same arguments as in the proof of Theorem 2.1, we find that

$$y(t) \geq \int_{T+\tau}^t Q(s)\varphi(y(s-\rho))ds \equiv z(t) > 0$$

for all large $t > T + \tau$. Then we obtain

$$z'(t) = Q(t)\varphi(y(t-\rho)) \geq Q(t)\varphi(z(t-\rho))$$

for all large $t > T + \tau$, which is a contradiction. This completes the proof.

Applying Theorems 2.1 and 2.2 to equation (1), we have the following corollaries.

Corollary 2.1. *Let* $\sigma = +1$ *and* $\gamma > 0$. *Suppose that* (H1) *and* (H2) *hold. Then* (1) *is oscillatory if the differential inequality*

$$z'(t) + P(t)[z(t-\rho+\tau)]^\gamma \leq 0$$

has no eventually positive solution, where

$$(7) \qquad P(t) = \min\{1, 2^{1-\gamma}\} \cdot \min\left\{\frac{q(t)}{1 + [h(t-\rho+\tau)]^\gamma}, \frac{q(t-\tau)}{1 + [h(t-\rho)]^\gamma}\right\}.$$

Corollary 2.2. *Let* $\sigma = -1$ *and* $\gamma > 0$. *Suppose that* (H1) *and* (H2) *hold. Then* (1) *is oscillatory if the differential inequality*

$$z'(t) \geq P(t)[z(t-\rho)]^{\gamma}$$

has no eventually positive solution, where $P(t)$ *is the function defined by* (7).

Proof of Corollaries 2.1 and 2.2. We note that

$$\sigma z'(t) + p(t)[z(t-\eta)]^{\gamma} \leq 0 \tag{8}$$

has no eventually positive solution if and only if

$$\{\sigma z'(t) + p(t)|z(t-\eta)|^{\gamma} \operatorname{sgn} z(t-\eta)\} \operatorname{sgn} z(t) \leq 0 \tag{9}$$

has no nonoscillatory solution, where $\eta \in \mathbf{R}$ and $p \in C[t_1, \infty)$. In fact, if $w(t)$ is an eventually negative solution of (9), then $z(t) \equiv -w(t)$ is an eventually positive solution of (8). Therefore, the conclusions of Corollaries 2.1 and 2.2 follow, by applying Theorems 2.1 and 2.2 to equation (1) and by choosing $\varphi(u) = \min\{1, 2^{1-\gamma}\}|u|^{\gamma} \operatorname{sgn} u$, $\omega(u) = u^{\gamma}$ and $\lambda(t) = 1/[1 + [h(t-\rho+\tau)]^{\gamma}]$. (Recall Remark 2.1.)

3. Oscillation theorems

In this section we derive oscillation theorems for equation (1). First we consider the case $\gamma \neq 1$. We need the following lemmas. For the proofs, see Kitamura and Kusano [9].

Lemma 3.1. *Let* $0 < \gamma < 1$ *and* $\rho > 0$. *Suppose that* (H2) *holds. Then the differential inequality*

$$u'(t) + q(t)[u(t-\rho)]^{\gamma} \leq 0$$

has no eventually positive solution if

$$\int^{\infty} q(t)dt = \infty. \tag{10}$$

Lemma 3.2. *Let* $\gamma > 1$ *and* $\rho < 0$. *Suppose that* (H2) *holds. Then the differential inequality*

$$u'(t) \geq q(t)[u(t-\rho)]^{\gamma}$$

has no eventually positive solution if (10) *holds.*

Combining Corollaries 2.1 and 2.2 with Lemmas 3.1 and 3.2, we establish the following theorem.

Theorem 3.1. *Suppose that* (H1) *and* (H2) *are satisfied and that one of the following cases holds:*

(i) $\sigma = +1$, $0 < \gamma < 1$ *and* $\rho > \tau$;
(ii) $\sigma = -1$, $\gamma > 1$ *and* $\rho < 0$.

Then equation (1) *is oscillatory if*

$$\int^{\infty} \min\left\{\frac{q(s)}{1 + [h(s-\rho+\tau)]^{\gamma}}, \frac{q(s-\tau)}{1 + [h(s-\rho)]^{\gamma}}\right\} ds = \infty. \tag{11}$$

COROLLARY 3.1. *Suppose that* (H1) *and* (H2) *are satisfied and that either* (i) *or* (ii) *of Theorem* 3.1 *holds. Assume moreover that* $h(t)$ *is bounded on* $[t_0, \infty)$. *Then equation* (1) *is oscillatory if*

$$\int^{\infty} \min\{q(s), q(s-\tau)\} ds = \infty. \tag{12}$$

PROOF. If $h(t)$ is bounded on $[t_0, \infty)$, then we easily see that (12) implies (11). Hence, the conclusion follows from Theorem 3.1.

Now we consider equation (2).

The following results will be required. For the proofs, see Györi and Ladas [7, Theorems 2.3.3 and 2.3.4] or Ladas [10] or Erbe, Kong and Zhang [2, Theorem 2.1.1].

LEMMA 3.3. *Suppose that* (H2) *holds and*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} q(s) ds > \frac{1}{e}.$$

Then the differential inequality

$$u'(t) + q(t)u(t-\rho) \le 0$$

has no eventually positive solution.

LEMMA 3.4. *Suppose that* (H2) *holds and*

$$\liminf_{t\to\infty} \int_{t}^{t-\rho} q(s) ds > \frac{1}{e}.$$

Then the differential inequality

$$u'(t) \ge q(t)u(t-\rho)$$

has no eventually positive solution.

From Corollaries 2.1, 2.2, Lemmas 3.3 and 3.4, we have the following results.

THEOREM 3.2. *Let* $\sigma = +1$. *Suppose that* (H1) *and* (H2) *hold. Then equation* (2) *is oscillatory if*

$$\liminf_{t\to\infty} \int_{t-\rho+\tau}^{t} \min\left\{\frac{q(s)}{1 + h(s-\rho+\tau)}, \frac{q(s-\tau)}{1 + h(s-\rho)}\right\} ds > \frac{1}{e}.$$

Theorem 3.3. *Let* $\sigma = -1$. *Suppose that* (H1) *and* (H2) *hold. Then equation* (2) *is oscillatory if*

$$\liminf_{t\to\infty} \int_t^{t-\rho} \min\left\{\frac{q(s)}{1+h(s-\rho+\tau)}, \frac{q(s-\tau)}{1+h(s-\rho)}\right\} ds > \frac{1}{e}.$$

Remark 3.2. It should be emphasized that the restrictive conditions on $h(t)$, such as $h(t) \le \lambda$, is not assumed in Theorems 3.1–3.3. Theorem 3.2 improves Theorem 10 in [5], Corollary 2 in [6] and Theorem 6.4.4 in [7], and give an answer to Problem 6.12.10 (b) raised by Györi and Ladas in [7].

References

[1] M. P. Chen, J. S. Yu and Z. C. Wang, Nonoscillatory solutions of neutral delay differential equations, Bull. Austral. Math. Soc. **48** (1993), 475–483.

[2] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation theory for functional-differential equations, Marcel Dekker, Inc., New York, 1995.

[3] K. Gopalsamy and B. G. Zhang, Oscillation and nonoscillation in first order neutral differential equations, J. Math. Anal. Appl. **151** (1990), 42–57.

[4] J. R. Graef, P. W. Spikes and M. K. Grammatikopoulos, On the behavior of solutions of a first order nonlinear neutral delay differential equation, Appl. Anal. **40** (1991), 111–121.

[5] M. K. Grammatikopoulos, E. A. Grove and G. Ladas, Oscillations of first-order neutral delay differential equations, J. Math. Anal. Appl. **120** (1986), 510–520.

[6] M. K. Grammatikopoulos, G. Ladas and Y. G. Sficas, Oscillation and asymptotic behavior of neutral equations with variable coefficients, Rad. Mat. **2** (1986), 279–303.

[7] I. Györi and G. Ladas, Oscillation theory of delay differential equations, Oxford University Press, 1991.

[8] J. Jaroš and T. Kusano, Oscillation properties of first order nonlinear functional differential equations of neutral type, Differential Integral Equations **4** (1991), 425–436.

[9] Y. Kitamura and T. Kusano, Oscillation of first-order nonlinear differential equations with deviating arguments, Proc. Amer. Math. Soc. **78** (1980), 64–68.

[10] G. Ladas, Sharp conditions for oscillations caused by delays, Appl. Anal. **9** (1979), 93–98.

[11] S. Tanaka, Existence of positive solutions for a class of first-order neutral functional differential equations, J. Math. Anal. Appl. **229** (1999), 501–518.

[12] B. Yang and B. G. Zhang, Qualitative analysis of a class of neutral differential equations, Funkcial. Ekvac. **39** (1996), 347–362.

[13] J. S. Yu, Z. C. Wang and Q. Chuanxi, Oscillation of neutral delay differential equations, Bull. Austral. Math. Soc. **45** (1992), 195–200.

[14] Y. Zhou, The distribution of zeros of neutral differential equations, Hiroshima Math. J. **29** (1999), 361–370.

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