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On extremal elliptic surfaces in characteristic 2 and 3

Dedicated to Professor M. Miyanishi for his 60th birthday

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Abstract. We show that all extremal elliptic surfaces in characteristic 2 and 3 are obtained from rational extremal elliptic surfaces as purely inseparable base extensions. As a corollary, we can show that the automorphism group of every supersingular elliptic $K3$ surface has an element of infinite order which acts trivially on the global sections of the sheaf of differential forms of degree 2. We also determine the structures of Mordell-Weil groups for extremal rational elliptic surfaces in these characteristics.

1. Introduction

Throughout this paper, We work over an algebraically closed field in positive characteristic. We call an algebraic surface over an algebraically closed field supersingular if its Picard number is equal to the second betti number, and call an elliptic surface extremal if it is supersingular and it has a finite Mordell-Weil group.

In the paper [8], we showed that every extremal elliptic surfaces are obtained from rational extremal elliptic surfaces by desingularization and purely inseparable base extension provided that the characteristic of the base field is greater than or equal to 5. And we gave a question for the validity of the same results in characteristic 2 and 3. But one cannot apply the same method as in [8] for both characteristic 2 and 3 cases because we used the theory of Deligne and Rapoport [4] in that paper.

On the other hand, A. Schweizer and Gekeler have studied a generic fiber of an extremal elliptic surface as a curve over the rational function field whose coductor is minimal from the Drinfel'd modular theoretic point of view ([5], [6], [17], [18]). And recently, Schweizer proved the same but weaker results in characteristic 2 and 3 using explicit calculations [19]. Namely, extremal elliptic surfaces over an algebraically closed field in characteristic 2 and 3 which are Frobenius minimal are rational surfaces. Here, a Frobenius minimal elliptic surface is an elliptic surface whose J-function is separable.

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As a corollary, he got the unirationality of extremal elliptic surfaces in characteristic 2 and 3.

In this paper, we show the following theorem using Schweizer's results.

Theorem 1.1. *All extremal elliptic surfaces in characteristic* 2 *and* 3 *arise from extremal rational elliptic surfaces via purely inseparable base extensions and its desingularization.*

By combining this result and main theorem in [8], we can give affirmative answer to Problem 2.7 in [8] which asks whether all extremal ellptic surfaces arise from rational elliptic surfaces by Frobenius base extension or not.

Furthermore, we can classify all supersingular elliptic $K3$ surfaces with finite sections in characteristic 2 and 3, and, as a corollary, we can show the same results on the automorphism groups of supersingular $K3$ surfaces in these characteristics as in [7] (Corollary 2.5).

For motivations to treat extremal elliptic surfaces, see [8], [9], [10] and [2].

Here is a plan of the paper. We state a main theorem and its corollaries in section 2, and prove them in section 4 after recalling some results on the rational case in Section 3.

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2. Results

Let k be an algebraically closed field in characteristic 2 or 3 and $f : X \to C$ be an elliptic surface with a section O where X (resp. C) be a nonsingular projective algebraic surface (resp. curve) over k.

Definition 2.1. An elliptic surface $f : X \to C$ is called *extremal* if its Picard number $\rho(X)$ is equal to the second betti number $b_2(X)$ and its Mordell-Weil group $MW(X/C)$ is finite.

Here we note that C is always isomorphic to $\mathbf{P}^1$ for an extremal elliptic surface $f : X \to C$ ([8] Prop. 4.2).

Apart from Theorem 1.1, we can say more about $K3$ surfaces.

Theorem 2.2. *There are only five (resp. three) types of extremal elliptic* $K3$ *surfaces in characteristic* 2 *(resp.* 3*) as in Table* 1 *(resp.* 2*).*

Corollary 2.3. *These three extremal elliptic* $K3$ *surfaces in characteristic* 3 *in Table* 2 *are all Kummer surfaces.*

Table 1. Extremal elliptic $K3$ surfaces in $p = 2$

type	deg J	$MW(X/C)$	equation of X
(I_1^*, I_{16})	16	$\mathbf{Z}/4\mathbf{Z}$	$y^2 + t^2xy + t^2y = x^3 + x^2$
(IV^*, I_4, I_{12})	16	$\mathbf{Z}/6\mathbf{Z}$	$y^2 + t^2xy + t^2y = x^3$
(I_{18}, I_2, I_2, I_2)	24	$\mathbf{Z}/6\mathbf{Z}$	$y^2 + t^2xy + y = x^3$
$(I_{10}, I_{10}, I_2, I_2)$	24	$\mathbf{Z}/10\mathbf{Z}$	$y^2 + t^2xy + y = x^3 + x^2 + t^2$
(I_6, I_6, I_6, I_6)	24	$\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}$	$y^2 + t^2xy + y = x^3 + 1 + t^6$

Table 2. Extremal elliptic $K3$ surfaces in $p = 3$

type	deg J	$MW(X/C)$	equation of X
(I_3^*, I_3, I_{12})	18	$\mathbf{Z}/4\mathbf{Z}$	$y^2 = x^3 + t(t^3+1)x^2 + t^2x$
(I_6^*, I_6, I_6)	18	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$y^2 = x^3 + t(t^3+1)x^2 - t^8x - t^9(t^3+1)$
(I_{12}^*, I_3, I_3)	18	$\mathbf{Z}/2\mathbf{Z}$	$y^2 = x^3 + t(t^3+1)x^2 + t^8x$

Proposition 2.4. *Every supersingular elliptic $K3$ surface with at least one section in charcteristic 2 has a structure of elliptic fibration which has infinitely many sections.*

Corollary 2.5. *Let X be a supersingular $K3$ surface which has an elliptic fibration with a section. Then $\mathrm{Aut}(X)$ contains an element σ of infinite order such that σ preserves the elliptic fibration and acts trivially on $H^0(X, \Omega_X^2)$. Furthermore, X contains infinitely many nonsingular rational curves.*

We will prove them in section 4.

3. Rational extremal elliptic surfaces

For the reference, we exhibit the classifications by W. Lang of extremal rational elliptic surfaces in characteristic 2 and 3 with some corrections from [19] of misprints in [9]. We also calculate these Mordell-Weil groups and exhibit them.

Remark 3.1. For an elliptic surface X/C with finite sections, there is an isomorphism between Néron-Severi group divided by the trivial lattice and Mordell-Weil group. Especially, we have a relation between these orders,

$$|\det NS(X)|/|\det T| = 1/|MW(X/C)|^2. \tag{3.1}$$

Since Néron-Severi group of a rational surface is unimodular, we can calculate the order of Mordell-Weil group by the type of singular fibers

$$|MW(X/C)|^2 = |\det T|. \tag{3.2}$$

Table 3. Rational extremal elliptic surfaces in characteristic 2

Notation	type	deg J	$MW(X/C)$	equation of X
I	(I_4^*)	0	$\mathbf{Z}/2\mathbf{Z}$	$y^2+txy=x^3+tx^2+at^6,\ a\neq 0$
II	(II^*)	0	$\{0\}$	$y^2+t^3y=x^3+t^5$
III	(III, I_8)	8	$\mathbf{Z}/4\mathbf{Z}$	$y^2+txy+ty=x^3+x^2$
IV	(I_1^*, I_4)	4	$\mathbf{Z}/4\mathbf{Z}$	$y^2+txy=x^3+t^2x$
V	(III^*, I_2)	2	$\mathbf{Z}/2\mathbf{Z}$	$y^2+txy=x^3+t^4$
VI	(II^*, I_1)	1	$\{0\}$	$y^2+txy=x^3+t^5$
VII	(IV, IV^*)	0	$\mathbf{Z}/3\mathbf{Z}$	$y^2+t^2y=x^3$
VIII	(IV, I_2, I_6)	8	$\mathbf{Z}/6\mathbf{Z}$	$y^2+txy+ty=x^3$
IX	(IV^*, I_1, I_3)	4	$\mathbf{Z}/3\mathbf{Z}$	$y^2+txy+t^2y=x^3$
*S*I	(I_9, I_1, I_1, I_1)	12	$\mathbf{Z}/3\mathbf{Z}$	$y^2+txy+y=x^3$
*S*II	(I_5, I_5, I_1, I_1)	12	$\mathbf{Z}/5\mathbf{Z}$	$y^2+txy+y=x^3+x^2+t$
*S*III	(I_3, I_3, I_3, I_3)	12	$(\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$	$y^2+txy+y=x^3+(t^3+1)$

Table 4. Rational extremal elliptic surfaces in characteristic 3

Notation	type	deg J	$MW(X/C)$	equation of X
I	(II^*)	0	$\{0\}$	$y^2=x^3+t^4x+t^5$
II	(II, I_9)	9	$\mathbf{Z}/3\mathbf{Z}$	$y^2=x^3+t^2x^2+t(t+1)x+t(t+2)$
III	(IV^*, I_3)	3	$\mathbf{Z}/3\mathbf{Z}$	$y^2=x^3+t^2x^2+t^3x+t^4$
IV	(II^*, I_1)	1	$\{0\}$	$y^2=x^3+t^2x^2+t^5$
V	(III^*, III)	0	$\mathbf{Z}/2\mathbf{Z}$	$y^2=x^3+t^3x$
VI	(I_0^*, I_0^*)	0	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$y^2=x^3+tx^2+bt^3, b\neq 0$
VI_{bis}	(I_0^*, I_0^*)	0	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$y^2=x^3+t^2x$
VII	(III, I_3, I_6)	9	$\mathbf{Z}/6\mathbf{Z}$	$y^2=x^3+t^2x^2+tx$
VIII	(I_1^*, I_1, I_4)	6	$\mathbf{Z}/4\mathbf{Z}$	$y^2=x^3+t(t+1)x^2+t^2x$
IX	(I_2^*, I_2, I_2)	6	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$y^2=x^3+t(t+1)x^2-t^4x-t^5(t+1)$
X	(I_4^*, I_1, I_1)	6	$\mathbf{Z}/2\mathbf{Z}$	$y^2=x^3+t(t+1)x^2+t^4x$
XI	(III^*, I_1, I_2)	3	$\mathbf{Z}/2\mathbf{Z}$	$y^2=x^3+t^2x^2+t^3x$
*S*I	(I_8, I_2, I_1, I_1)	12	$\mathbf{Z}/4\mathbf{Z}$	$y^2=x^3+(t^2+1)x^2+x$
*S*II	(I_5, I_5, I_1, I_1)	12	$\mathbf{Z}/5\mathbf{Z}$	$y^2=x^3+(t^2+1)x^2+(t-t^2)x+t^2$
*S*III	(I_4, I_4, I_2, I_2)	12	$\mathbf{Z}/4\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$	$y^2=x^3+(t^2+1)x^2+t^2x$

Using this Remark, the structures of Mordell-Weil groups for cases I, II, V, VI, VII, VIII, IX, *S*I, *S*II in Table 3 and cases I, II, III, IV, V, VII, X, XI, *S*II in Table 4 are determined straightforward because the group structures are determined by these orders uniquely.

To determine the structure of Mordell-Weil groups for other cases, we need some calculations. We treat the characteristic 2 cases first, that is, the cases in Table 3. For types III (resp. IV) in Table 3, one can easily check that a rational point $(x, y) = (0, 0)$ (resp. $(t, 0)$) has order four. For type *S*III in Table 3, let $P = (t+1, 1)$ and $Q = (\alpha(t+\alpha), 1)$ be both rational points where

α satisfies $\alpha^2 + \alpha + 1 = 0$. Then it is not so hard to compute $2P = -P = (t+1, t(t+1))$ and $2Q = -Q = (\alpha(t+\alpha), \alpha t(t+\alpha))$. (See [20] and [11] for the method of explicit calculation.)

Next, we go into the characteristic 3 cases which is in Table 4. For types VI, let β_i $(i = 1,2,3)$ be three distinct roots of the equation $\xi^3 + \xi^2 + b = 0$ over k. Then the points $P_i = (\beta_i t, 0)$ $(i = 1,2,3)$ are all rational points and satisfy the relation $2P_i = O$ and $P_1 + P_2 = P_3$, thus P_1 and P_2 generate the group $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Similary, let $Q_\pm$ (resp. R) be the rational points $(\pm\sqrt{-1}t, 0)$ (resp. $(0,0)$) for the type VI_{bis}. Then these points satisfy $Q_+ + Q_- = R$ and $2Q_\pm = 2R = O$, and get the structure. For the type VIII, it is easy to check that the point (t, t^2) has order four, and for the type IX, the points $(\pm t^2, 0)$ and $(-t(t+1), 0)$ have all order two and any two of them generate the Mordell-Weil group as $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

Finally, to determine the group structure of Mordell-Weil groups for remaining cases *S*I and *S*III in Table 4, we need more observations.

Lemma 3.2. (1) *The elliptic surface of type* *S*I *in Table* 4 *is obtained from* VIII *by base change of degree* 2 *induced from ramified double covering between base curves* $\mathbf{P}^1$*'s, whose ramification points are just the points of the base curve* $\mathbf{P}^1$ *for the surface of type* VIII *over which the singular fibers are of type* I_1^* *and* I_4.

(2) *The elliptic surface of type* *S*III *in Table* 4 *is obtained from* VIII *by base change of degree* 2 *induced from ramified double covering between* $\mathbf{P}^1$*'s, whose ramification points are just the points of the base* $\mathbf{P}^1$ *for the surface of type* VIII *over which the singular fibers are of type* I_1 *and* I_1^*.

(3) *Moreover, the elliptic surface of type* *S*III *is obtained from* IX *also by base extension of degree* 2 *induced from ramified covering of base curves whose ramification points are just the points over which the singular fibers are of type* I_2 *and* I_2^*.

This lemma is so elementary that we omit the proof. Now, using the following lemma which is a folklore we have the structure of Mordell-Weil groups of these remaining two types.

Lemma 3.3. *Let* $f : X \to C$ *be an elliptic surface and* $\pi : C' \to C$ *be a finite morphism. Then the Mordell-Weil group of* X/C *injects into the Mordell-Weil group of* $X \times_C C'/C'$.

We know the order of the group for the type *S*I (resp. *S*III) by Remark 3.1 and the group has to include the group isomorphic to $\mathbf{Z}/4\mathbf{Z}$ (resp. both $\mathbf{Z}/4\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$) by these lemmas, we get the results.

Remark 3.4. From the above tables, one can see easily that there are some sequences of extremal elliptic surfaces by Frobenius base changes. For characteristic 2 case, there are two sequences:

$$\mathrm{III} \xrightarrow{F} \mathrm{IV} \xrightarrow{F} \mathrm{V} \xrightarrow{F} \mathrm{VI}$$

$$\mathrm{VIII} \xrightarrow{F} \mathrm{IX},$$

and for characteristic 3 case, there are also two sequences:

$$\mathrm{II} \xrightarrow{F} \mathrm{III} \xrightarrow{F} \mathrm{IV}$$

$$\mathrm{VII} \xrightarrow{F} \mathrm{XI},$$

where F is the Frobenius base extension and desingularization. Moreover, cases I, II, VII in characteristic 2 and I, V, VI, VI_{bis} in characteristic 3 are *Frobenius closed*, that is, the minimal models of Frobenius base extensions of these surfaces are isomorphic to these surfaces themselves.

Here is a precise statement of the theorem by Schweizer which we will use later.

THEOREM 3.5 ([19]). *Let $f : X \to \mathbf{P}^1$ be an extremal elliptic surface and assume that it is Frobenius minimal.*

(1) *Suppose it has a constant J-function, then it is of type* I, II *or* VII *in Table* 3 *for characteristic* 2 *and of type* I, V, VI *or* VI_{bis} *in Table* 4 *for characteristic* 3.

(2) *Suppose its J-function is not constant, then* (i) *it is of type* IX *or* VI *for non-semistable case and S*I, *S*II *or S*III *for semistable case in Table* 3 *for characteristic* 2, *and* (ii) *it is of type* IV, VIII, IX, X *or* XI *for non-semistable case and S*I, *S*II *or S*III *for semistable case in Table* 4 *for characteristic* 3.

Note that these Frobenius minimal extremal elliptic surfaces are all rational surfaces.

4. Proofs of theorems and corollaries

PROOF OF THEOREM 1.1. Let $f : X \to \mathbf{P}^1$ be an extremal elliptic surface. If the J-function of its generic fiber X_η is separable then X is one of the list in Tables 3 and 4 by Schweizer's theorem (Theorem 3.5) and we get the result.

Now suppose that J-function of X is inseparable and it decomposes into the purely inseparable part J_{insep} and the separable part J_{sep}. Consider J-function of X as the j-invariant of the generic fiber X_η which is an elliptic curve over the rational function field $k(t) = k(\mathbf{P}^1)$. Since X_η is not Frobenius minimal, j-invariant of X_η is a p-th power in $k(t)$ and X_η can be obtained from another elliptic curve E over $k(t)$ by composite of Frobenius isogenies. We may suppose that this elliptic curve E over $k(t)$ is Frobenius minimal, that is, its j-function is not a p-th power in $k(t)$.

Let $g : Y \to \mathbf{P}^1$ be the minimal nonsingular model of E over $\mathbf{P}^1$, then it is a rational surface and this is in Tabels 3 and 4 by Theorem 3.5.

Thus we have the following diagram:

$$\begin{array}{ccccc} X & & Y & & \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \\ \mathbf{P}^1 & \xrightarrow{J_{insep}} & \mathbf{P}^1 & \xrightarrow{J_{sep}} & \mathbf{P}^1 \end{array}$$

furthermore, we have a rational map from X to Y given by the composite of Frebenius isogenies between generic fibers which commutes with this diagram.

Now taking the fiber product of $J_{insep} : \mathbf{P}^1 \to \mathbf{P}^1$ and $g : Y \to \mathbf{P}^1$, we get the elliptic surface $Y \times_{\mathbf{P}^1} \mathbf{P}^1$ birational to X whose generic fiber coincides with the generic fiber of X by the above consideration.

Then from the theory of (Kodaira-Néron) minimal model (the existence and uniqueness, cf. [3] for example), the minimal desingularization of $Y \times_{\mathbf{P}^1} \mathbf{P}^1$ coincides with X. □

Proof of Theorem 2.2. By Theorem 1.1 and Remark 3.4 the only possibilities for extremal elliptic surfaces with $p_g(X) \geq 1$ are those surfaces which are obtained from surfaces of type III, VIII, *S*I, *S*II or *S*III in characteristic 2 and surfaces of type II, VII, VIII, IX, X, *S*I, *S*II or *S*III in characteristic 3 by Frobenius base extensions.

For $K3$ surfaces which have $p_g = 1$ the only possibilities are exhibited in Tables 1 and 2, and these surfaces actually exist by Frobenius base extension. For the structures of Mordell-Weil groups of them, one need more precise considerations. Since the determinant of Néron-Severi groups of supersingular $K3$ surfaces with respect to the intersection pairing is equal to $-p^{2\sigma_0}$ with $1 \leq \sigma_0 \leq 10$, where σ_0 is Artin invariant, thus $|\det NS(X)|$ has to be divisible by p^2.

Combining this fact and Lemma 3.3, one can easily determine the structure of Mordell-Weil groups for non-semistable cases in characteristic 2 and 3. For example, $|\det T|$ is 2^6 for the surface of type $(\mathrm{I}_1^*, \mathrm{I}_{16})$ in characteristic 2 which is obtained by the rational surface whose Mordell-Weil group is isomorphic to $\mathbf{Z}/4\mathbf{Z}$, so the order of Mordell-Weil group is divided by 4, and 2^6 must be divisible by $2^{2\sigma_0}4^2$ from (3.1), thus we obtain $\sigma_0 = 1$ and the Mordell-Weil group is isomorphic to $\mathbf{Z}/4\mathbf{Z}$. The structures of Mordell-Weil groups for other surfaces which have non-semi-stable fibers in both characteristics are similarly determined. For the remaining cases, that is, the cases for semi-stable elliptic surfaces in characteristic 2 in Table 1, one can check that the point $(x, y) = (t, 1)$ has order six for the surface of type $(\mathrm{I}_{18}, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_2)$, and that the point $\left(\frac{1}{t^2}, \frac{t^4+t+1}{t^3}\right)$ (resp. $\left(\frac{1}{t^2}, \frac{t^6+t^3+1}{t^3}\right)$) is 2-torsion for the surface of type $(\mathrm{I}_{10}, \mathrm{I}_{10}, \mathrm{I}_2, \mathrm{I}_2)$ (resp. $(\mathrm{I}_6, \mathrm{I}_6, \mathrm{I}_6, \mathrm{I}_6)$). □

Proof of Corollary 2.3. From Table 2, we can conclude that all supersingular $K3$ surfaces in the list have Artin invariant 1 using (3.1) (See [1] more about Artin invariant). And by the result by Ogus ([12]), these surfaces are all Kummer surfaces. □

Proof of Proposition 2.4. First of all, note that all surfaces in Table 1 has its Artin invariant 1. Thus all these surfaces are isomorphic to each other (cf. [14]). So it suffices to show the proposition for the case $(\mathrm{I}_1^*, \mathrm{I}_{16})$. This will be done by giving another structure of elliptic fibration on X using the following lemma.

Lemma 4.1. *Let D be an effective divisor on a $K3$ surface X which has the same type as a singular fiber of an elliptic surface. Then there is a unique pencil $f : X \to \mathbf{P}^1$ of arithmetic genus 1 of which D is a singular fiber. Moreover, any irreducible curve C on X with $(C \cdot D) = 1$ defines a section of f.*

This lemma follows immediately from Theorem 1 in [13] §3.

Now we take an effective divisor as in this lemma for the case $(\mathrm{I}_1^*, \mathrm{I}_{16})$ as follows.

Let us take D in the lemma to be I_3^* which was indicated as bold lines in Figure 1 which is a configuration of the zero section and the singular fibers of type I_1^* and I_{16}.

Since I_3^* does not occur as a singular fiber of a quasi-elliptic fibration, this pencil is elliptic. If an elliptic $K3$ surface has a singular fiber of type I_3^*

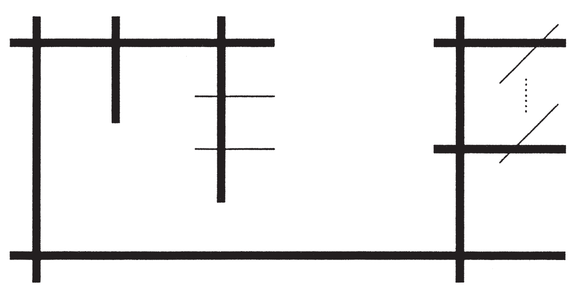

Fig. 1

then its Mordell-Weil group is infinite group by Table 2 which does not have a surface having the singular fiber of type I_3^*. Thus we are done. □

Corollary 2.5 is followed by Proposition 2.4 in characteristic 2 and Ueno's result in [21] in characteristic 3 because these are all Kummer surfaces (Corollary 2.3) (cf. [7]).

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