HIROSHIMA MATH. J.
32 (2002), 207–215

Minimal sets of certain annular homeomorphisms

Shigenori MATSUMOTO and Mitsuhiro SHISHIKURA

(Received June 11, 2001)
(Revised November 5, 2001)

ABSTRACT. We consider a homeomorphism of the annulus $S^1 \times \mathbf{R}$ of the form $F_{\alpha,\varphi}(x, y) = (x + \alpha, y + \varphi(x))$, where α is an irrational number and φ is a continuous function on S^1 with vanishing integral. We show that if φ is of bounded variation and if $F_{\alpha,\varphi}$ is not topologically conjugate to $F_{0,\varphi}$, then $F_{\alpha,\varphi}$ does not admit a minimal set. We also show the abundance of such homeomorphisms.

1. Introduction

In [I], T. Inaba constructed an example of a smooth flow without a minimal set on an open surface of infinite genus. This was generalized by J.-C. Beniere and G. Meigniez [BM] to show that there are always flows without minimal sets on any noncompact manifolds other than the real line and surfaces of finite genus. This made us interested in considering the same problem for homeomorphisms on open surfaces.

Let f be a homeomorphism of a metric space X. A subset $\mathscr{M}$ of X is called a *minimal set* if $\mathscr{M}$ is a nonempty closed subset invariant by the homeomorphism f, which is minimal among such subsets with respect to the inclusion. By Zorn's lemma any homeomorphism on a compact space admits a minimal set. However this is no longer the case for a noncompact space.

If a homeomorphism f does not admit a minimal set, then there is no discrete orbit, since such an orbit would be a minimal set. As a consequence either the α-limit set or the ω-limit set of any point is nonempty, and thus the nonwandering set of f is nonempty. It follows for example that any homeomorphism of the plane must have a minimal set, because any homeomorphism with nonempty nonwandering set has a fixed point by a classical result of L. E. J. Brouwer. See *e.g.* [G].

Therefore first example of surfaces to be considered is an open annulus. Here we deal only with a special type of homeomorphisms, called *skew products*, which are defined as follows. Denote by R_α the rotation by α of the

2000 *Mathematics Subject Classification*. 37E30.
Keywords and phrases. skew-product, minimal set, Diophantine number, residual set.

circle $S^1 = \mathbf{R}/\mathbf{Z}$, and let φ be a real valued continuous function on S^1. Define a homeomorphism $F_{\alpha,\varphi}$ of the annulus $S^1 \times \mathbf{R}$ by

$$F_{\alpha,\varphi}(x, y) = (R_\alpha(x), y + \varphi(x)).$$

Throughout this paper we always work under the following hypotheses, since otherwise the dynamics of a skew product $F_{\alpha,\varphi}$ is not so much interesting, as will be explained below.

(1.1) The rotation number α is irrational.

(1.2) The function φ satisfies

$$\int_{S^1} \varphi(x)dx = 0.$$

Computation shows that for any integer $n > 0$,

$$F^n_{\alpha,\varphi}(x, y) = (R^n_\alpha(x), y + \varphi_n(x)), \qquad \text{where } \varphi_n(x) = \sum_{i=0}^{n-1} \varphi(R^i_\alpha(x)).$$

Notice that the unique ergodicity of the transformation R_α implies under the assumption (1.2) that the functions $(1/n)\varphi_n$ converge to zero uniformly on S^1 as n tends to the infinity. If on the contrary the assumption (1.2) is not fulfilled, then the functions $(1/n)\varphi_n$ converge to some nonzero constant, and all the orbits of $F_{\alpha,\varphi}$ are discrete. This is why we assume (1.2) throughout.

The set of C^r-functions φ $(r = 0, 1, 2, \dots, \infty)$ satisfying (1.2) is denoted by $C^r_0(S^1)$. Also the set of continuous functions φ of bounded variation satisfying (1.2) is denoted by $C^{BV}_0(S^1)$.

A skew product $F_{\alpha,\varphi}$ is called *C^r-integrable* if there is a C^r-function h on S^1 such that

$$h \circ R_\alpha - h = \varphi.$$

Then the graphs of the functions $h + c$ $(c \in \mathbf{R})$ are invariant under $F_{\alpha,\varphi}$, and $F_{\alpha,\varphi}$ is C^r-conjugate to the horizontal rotation $F_{\alpha,0}$ by the C^r-diffeomorphism $F_{0,h}$, *i.e.* $F_{\alpha,\varphi} = F_{0,h} \circ F_{\alpha,0} \circ F^{-1}_{0,h}$.

The main result of the present paper is the following.

THEOREM 1. *For any function $\varphi \in C^{BV}_0(S^1)$, the skew product $F_{\alpha,\varphi}$ is either C^0-integrable or admits no minimal set.*

REMARK 1.1. The hypothesis that φ is of bounded variation is actually necessary. According to [B], there is an example of a skew product $F_{\alpha,\varphi}$, with $\varphi \in C^0_0(S^1)$, which admits a discrete orbit. Of course the function φ is continuous, but not of bounded variation.

The key fact for the proof of Theorem 1 is Proposition 2.1 in Sect. 2. This proposition was already proved by A. B. Krygin (Proposition 2, [Kr]) for

C^1-maps φ, in connection with other interesting descriptions of the behaviour of orbits of $F_{\alpha,\varphi}$.

Next we shall show the abundance of homeomorphisms without minimal set. The variation $\mathrm{var}(f)$ (see next section) of a function f defines a norm on $C_0^{BV}(S^1)$ and $(C_0^{BV}(S^1), \mathrm{var}(\cdot))$ becomes a Banach space since we assume (1.2). The space $C_0^r(S^1)$ is to be equipped with the C^r-topology.

For a real number α denote by $\|\alpha\|$ the distance of α to the set of integers $\mathbf{Z}$. An irrational number α is called *Diophantine of exponent* ρ $(\rho > 0)$ if there is a positive constant C such that $\|k\alpha\|\,|k|^{\rho} > C$ for any nonzero integer k, *Diophantine* if it is Diophantine of exponent ρ for some $\rho > 0$, and *Liouville* if it is not Diophantine.

Theorem 2. *For any irrational α, there exists a residual subset $\mathscr{R}$ of $C_0^{BV}(S^1)$ such that for any $\varphi \in \mathscr{R}$ the skew product $F_{\alpha,\varphi}$ is not C^0-integrable.*

As for smooth functions the following fact is well known. See for example [Ko].

Remark 1.2. Assume α is a Diophantine number. Then for any function $\varphi \in C_0^{\infty}(S^1)$, the skew product $F_{\alpha,\varphi}$ is C^{∞}-integrable.

For a Liouville number α, we obtain the following nonintegrability result.

Theorem 3. *Given a Liouville number α, there exists a residual subset $\mathscr{R}$ of $C_0^{\infty}(S^1)$ such that for any $\varphi \in \mathscr{R}$ the skew product $F_{\alpha,\varphi}$ is not C^0-integrable.*

More precisely we have:

Theorem 4. *Assume α is not a Diophantine number of exponent ρ and $\rho \geq r$ $(r = 1, 2, \ldots)$. Then there exists a residual subset $\mathscr{R}$ of $C_0^r(S^1)$ such that for any $\varphi \in \mathscr{R}$, the skew product $F_{\alpha,\varphi}$ is not C^0-integrable.*

The authors are grateful to T. Inaba for helpful comments, and to B. Fayad for information on the reference [Kr].

2. Proof of Theorem 1

Fix once and for all an irrational number α and a function $\varphi \in C_0^{BV}(S^1)$, and assume that the skew product $F_{\alpha,\varphi}$ admits a minimal set $\mathscr{M} \subset S^1 \times \mathbf{R}$. Our aim is to show that $F_{\alpha,\varphi}$ is C^0-integrable. Recall that φ is *of bounded variation* if there is a constant $C > 0$ such that for any partition $x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n = x_0$ of S^1, we have

$$\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| \leq C,$$

where $\prec$ denotes the positive circular order in S^1. The smallest such C is called the *variation* of φ and is denoted by $\mathrm{var}(\varphi)$.

For any $t \in \mathbf{R}$ denote by $V_t : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R}$ the vertical translation by t, *i.e.*

$$V_t(x, y) = (x, y + t).$$

Clearly the skew product $F_{\alpha,\varphi}$ commutes with V_t, and thus the set $V_t(\mathscr{M})$ is also a minimal set of $F_{\alpha,\varphi}$. Therefore the two sets $\mathscr{M}$ and $V_t(\mathscr{M})$ are either identical or disjoint. Denote by $\mathscr{Z}$ the set of t for which $V_t(\mathscr{M})$ and $\mathscr{M}$ coincide. Clearly $\mathscr{Z}$ is a closed subgroup of $\mathbf{R}$, and we have the following trichotomy:

CASE 1. $\mathscr{Z} = \mathbf{R}$.

CASE 2. $\mathscr{Z}$ is free cyclic.

CASE 3. $\mathscr{Z}$ is trivial.

First let us show that Cases 1 and 2 are impossible. We follow an argument in [B]. Denote by $\pi_2 : S^1 \times \mathbf{R} \to \mathbf{R}$ the projection onto the second factor. Notice that in these cases $\mathscr{M}$ is unbounded both above and below, *i.e.* the projection $\pi_2(\mathscr{M})$ is unbounded above and below. Also by the assumption of Cases 1 and 2, $\mathscr{M}$ does not consist of a single discrete orbit. This implies that $\mathscr{M}$ is a perfect set. A standard argument shows that for a generic point p in $\mathscr{M}$, both the forward and backward orbits are dense in $\mathscr{M}$.

Now there exist sequences $n_k \to \infty$ and $m_k \to -\infty$ such that

$$\pi_2 F_{\alpha,\varphi}^{n_k}(p) \to \infty, \qquad \pi_2 F_{\alpha,\varphi}^{m_k}(p) \to \infty.$$

Choose $m_k \le j_k \le n_k$ such that $\pi_2 F_{\alpha,\varphi}^{j_k}(p) \le \pi_2 F_{\alpha,\varphi}^{j}(p)$ for any $m_k \le j \le n_k$.

By the assumption of Cases 1 and 2, one can choose $t_k \in \mathscr{Z}$ so that the points $q_k = V_{t_k} F_{\alpha,\varphi}^{j_k}(p)$ satisfy the following conditions.

(a) The points q_k's lie in a compact subset of $S^1 \times \mathbf{R}$.

(b) $\pi_2 F_{\alpha,\varphi}^{j}(q_k) \ge \pi_2(q_k)$ $(m_k' \le j \le n_k')$.

(c) $n_k' \to \infty$, and $m_k' \to -\infty$.

Now let q_∞ be a point of accumulation of $\{q_k\}$. Then q_k and hence q_∞ lie in $\mathscr{M}$, but the orbit of q_∞ is bounded below. This contradicts the assumption that $\mathscr{M}$ is not bounded below.

The rest of this section is devoted to the proof of the following proposition.

PROPOSITION 2.1. *If the group $\mathscr{Z}$ is trivial, then the skew product $F_{\alpha,\varphi}$ is C^0-integrable.*

This proposition for a C^1-function φ can be found in [Kr]. Our argument is essentially the same as that of Proposition 2 in [Kr], but more elementary since we are not involved in the theory of continued fractions.

First we prepare some fundamental facts about Diophantine approximation of real numbers. Given any two points $a, b \in S^1$, not antipodal, denote by $|a, b|$ the smaller open interval in S^1 bounded by a and b.

A positive integer q is called a *closest return time* for α if $\|j\alpha\| > \|q\alpha\|$ for any integer j such that $0 < j < q$. As is well known q is a closest return time for α if and only if it is the denominator of a convergent of the continued fraction of α.

THEOREM 2.2 (Denjoy-Koksma). *If $\varphi \in C_0^{BV}(S^1)$ and q is a closest return time for α, then we have*

$$\|\varphi_q\|_\infty \leq \mathrm{var}(\varphi).$$

For more details, see Theorem 3.1 of M. R. Herman [H], Chap. VI.

LEMMA 2.3. *For integers $0 \leq n_1 < n_2 < n_3$ assume that n_3 is the smallest positive integer j such that $j\alpha$ lies in the interval $|n_1\alpha, n_2\alpha|$. Then $q = n_3 - n_2$ is a closest return time for α.*

PROOF. By the hypothesis, for j with $0 < j < n_3$, $j\alpha \notin |n_1\alpha, n_2\alpha|$. By applying $R_\alpha^{-n_1}$ and shifting j by $-n_1$, we have $j\alpha \notin |0, (n_2 - n_1)\alpha|$ for j with $0 < j < q = n_3 - n_2$ $(< n_3 - n_1)$. By a similar argument for $|n_3\alpha, n_2\alpha| \subset |n_1\alpha, n_2\alpha|$, we have $j\alpha \notin |q\alpha, 0|$ for j with $0 < j < q = n_3 - n_2$.

To fix an idea, let us assume that $n_1\alpha \prec n_3\alpha \prec n_2\alpha \prec (n_1\alpha + 1/2)$ (the other case is similar). Then it is easy to see that $-1/2 \prec q\alpha \prec 0 \prec (n_2 - n_1)\alpha \prec 1/2$. Moreover since $|n_3\alpha, n_2\alpha| \subsetneqq |n_1\alpha, n_2\alpha|$, the length of $|0, (n_2 - n_1)\alpha|$ is greater than that of $|q\alpha, 0|$, which is $\|q\alpha\|$. Therefore $|0, -q\alpha| \subset |0, (n_2 - n_1)\alpha|$.

We now conclude that $j\alpha \notin |q\alpha, 0| \cup |0, -q\alpha|$ for $0 < j < q$, which implies that q is a closest return time. $\square$

PROOF OF PROPOSITION 2.1. Notice first of all that we only need to show that the minimal set $\mathscr{M}$ is compact. In fact then Theorem 14.11 of Gottshalk and Hedlund [GH] shows that $F_{\alpha,\varphi}$ is C^0-integrable. Since the proof of this fact is very short, we shall include it for the convenience of the readers. The triviality of $\mathscr{Z}$ implies that any vertical line $\{x\} \times \mathbf{R}$ intersects at $\mathscr{M}$ at most one point. In fact, by the compactness of $\mathscr{M}$ and the minimality of the base map $R_\alpha : S^1 \to S^1$, the intersection is always one point, *i.e.* $\mathscr{M}$ is the graph of a function h. Now the closedness of $\mathscr{M}$ is equivalent to the continuity of h, and the the invariance of $\mathscr{M}$ means that h satisfies $h \circ R_\alpha - h = \varphi$. This shows the C^0-integrability of $F_{\alpha,\varphi}$.

Assume to fix the idea that the fiber $\{0\} \times \mathbf{R}$ intersects the minimal set $\mathscr{M}$ at the point $(0,0)$. The rest of this section is devoted to the proof of the following claim.

CLAIM. *There exists a neighbourhood U of* 0 *in* S^1 *such that whenever* $R_\alpha^n(0) \in U$, *we have* $|\varphi_n(0)| \le 1$.

Let us show why this claim suffices for our purpose. The claim implies that the projection $\pi_2((U \times \mathbf{R}) \cap \mathscr{M})$ is bounded. There are integers $n_1, \dots, n_k$ such that

$$R_\alpha^{n_1}(U) \cup \cdots \cup R_\alpha^{n_k}(U) = S^1.$$

Then

$$F_{\alpha,\varphi}^{n_1}((U \times \mathbf{R}) \cap \mathscr{M}) \cup \cdots \cup F_{\alpha,\varphi}^{n_k}((U \times \mathbf{R}) \cap \mathscr{M}) = \mathscr{M}$$

also has bounded π_2-projection. Since $\mathscr{M}$ is closed, it must be compact.

Now let us begin the proof of the claim. First of all since the minimal set $\mathscr{M}$ intersects the fiber $\{0\} \times \mathbf{R}$ exactly at one point $(0,0)$, one can choose a small positive number ε such that the union of two rectangles

$$Y = \{(x,y) \mid |x| \le \varepsilon, 1 \le |y| \le 1 + \mathrm{var}(\varphi)\}$$

does not intersect $\mathscr{M}$. Choose a positive integer n_- and n_+ such that

(2.1) $-\varepsilon < n_-\alpha < 0 < n_+\alpha < \varepsilon$,

(2.2) $n_\pm$ is a closest return time for α.

Let us define the interval U in the claim to be $(n_-\alpha, n_+\alpha)$. We are going to show the claim only for positive n, since the negative case can be dealt with similarly. Assume to fix the idea that $n\alpha \in (0, n_+\alpha)$. Define a sequence $n_1, n_2, \dots, n_k = n$ as follows. First of all let $n_1 = n_+$, and let n_2 be the smallest positive integer j such that $j\alpha$ lies in the interval $I_1 = (0, n_+\alpha)$. If $n_2 = n$, we are done. If not, $n_2\alpha$ divides the interval I_1 into two subintervals, one of which, say I_2, contains $n\alpha$. Let n_3 be the smallest positive integer j such that $j\alpha$ lies in the interval I_2. Proceeding in this way, we end up with some n_k eventually matching the given n.

Now by Lemma 2.3, all the integers $n_1, n_2 - n_1, \dots, n_k - n_{k-1}$ are closest return times for α. By Theorem 2.2, we have $|\varphi_{n_1}(0)| \le \mathrm{var}(\varphi)$. But since the point $F_{a,\varphi}^{n_1}(0,0) = (R_\alpha^{n_1}(0), \varphi_{n_1}(0))$ does not lie in Y, we have in fact $|\varphi_{n_1}(0)| \le 1$. Now applying Theorem 2.2 once again to $n_2 - n_1$, one obtains that

$$|\varphi_{n_2}(0)| = |\varphi_{n_2 - n_1}(\varphi_{n_1}(0)) + \varphi_{n_1}(0)| \le \mathrm{var}(\varphi) + 1.$$

Again since $F_{\alpha,\varphi}^{n_2}(0,0)$ does not lie in Y, we actually have $|\varphi_{n_2}(0)| \le 1$. In this way one obtains inductively that $|\varphi_n(0)| \le 1$, completing the proof of Proposition 2.1, and thus of Theorem 1.

3. Proof of Theorems 2, 3 and 4

To show Theorem 2, recall first of all the following classical theorem due to Fejér (Theorem 73 of [HR]) giving a criterion for the continuity of L^2 functions. For any $f \in L^2(S^1)$, define an analytic function $r_k(f)$ on S^1 by

$$r_k(f)(x) = \sum_{\nu=-k}^{k} \hat{f}_\nu e^{2\pi i \nu x},$$

where

$$\hat{f}_\nu = \int_{S^1} f(t) e^{-2\pi i \nu t}\, dt.$$

Next define a function $s_k(f)$ (the Cesàro sum) by

$$s_k(f)(x) = \frac{1}{k}(r_0(f)(x) + r_1(f)(x) + \cdots + r_{k-1}(f)(x)).$$

THEOREM 3.1 (Fejér). *If f is continuous, then $s_k(f)$ converges uniformly to f.*

Direct computation shows that given a function f, the solution h of the equation $f = h \circ R_\alpha - h$ must have Fourier coefficients

$$\hat{h}_k = \frac{\hat{f}_k}{e^{2\pi i k\alpha} - 1}.$$

Thus we make the following notations,

$$s_k^\alpha(f)(x) = \frac{1}{k}(r_0^\alpha(f)(x) + r_1^\alpha(f)(x) + \cdots + r_{k-1}^\alpha(f)(x)),$$

where

$$r_k^\alpha(f)(x) = \sum_{\nu=-k}^{k} \frac{\hat{f}_\nu}{e^{2\pi i \nu\alpha} - 1} e^{2\pi i \nu x}.$$

Thus if f admits a continuous solution h of the equation $f = h \circ R_\alpha - h$, then $s_k^\alpha(f)$ must converge uniformly to some continuous function.

Let C_0^* be either of $C_0^{BV}(S^1)$ or $C_0^r(S^1)$ $(r = 1, \ldots, \infty)$. For a positive integer l, define

$$\mathscr{B}_l^*(\alpha) = \{f \in C_0^* \mid |s_k^\alpha(f)(0)| > l,\ \exists k > l\}.$$

LEMMA 3.2. (1) *$\mathscr{B}_l^*(\alpha)$ is open in the C^0-topology, hence in the C^*-topology.*

(2) *If* φ *belongs to* $\bigcap_l \mathscr{B}_l^*(\alpha)$, *then there is no continuous function* h *such that* $\varphi = h \circ R_\alpha - h$.

(3) *If* $\bigcap_l \mathscr{B}_l^*(\alpha)$ *is nonempty, then* $\mathscr{B}_l^*(\alpha)$ *is dense in* C_0^*.

PROOF. (1) is clear from the continuity of $s_k^\alpha(f)$ in the C^0-topology. For (2), notice that if $f \in \bigcap_l \mathscr{B}_l^*(\alpha)$, then $s_k^\alpha(f)(0)$ cannot converge. To show (3), notice that $C_0^* \setminus \bigcap_l \mathscr{B}_l^*(\alpha)$ is a proper linear subspace of C_0^*, hence its complement has to be dense. □

In order to show Theorem 2, it suffices to construct a function $\varphi \in \bigcap_l \mathscr{B}_l^{BV}(\alpha)$ for any irrational α. Choose a sequence $\{k_n\}_{n=1}^\infty$ of positive integers satisfying

$$(3.1) \quad \|k_n\alpha\| < 1/|k_n|, \qquad (3.2) \quad k_{n+1} > 2k_n,$$

and define

$$\varphi(x) = \sum_{n=1}^\infty \frac{1}{n}((e^{2\pi i k_n \alpha} - 1)e^{2\pi i k_n x} + (e^{-2\pi i k_n \alpha} - 1)e^{-2\pi i k_n x}).$$

We have

$$|e^{\pm 2\pi i k_n \alpha} - 1| < 2\pi\|k_n\alpha\| < \frac{2\pi}{k_n}.$$

This, together with the assumption (3.2) shows that $r_k(\varphi)$ converges uniformly to φ. To show that φ is of bounded variation, we have;

$$\begin{aligned} \mathrm{var}(r_k(\varphi)) &= \int_{S^1} |r_k(\varphi)'(x)|dx \le \left(\int_{S^1} |r_k(\varphi)'(x)|^2 dx\right)^{1/2} \\ &\le \left(2\sum_n \frac{1}{n^2}|e^{2\pi i k_n \alpha} - 1|^2 (2\pi k_n)^2\right)^{1/2} \le 4\sqrt{2}\pi^2 \sum_n \frac{1}{n^2} < \infty. \end{aligned}$$

On the other hand the Fourier coefficient $\hat{h}_k$ of the solution h of the equation $h \circ R_\alpha - h = \varphi$ is given by $\hat{h}_k = 1/n$ if $k = \pm k_n$ and 0 otherwise. Thus $r_k^\alpha(\varphi)(0) \to \infty$ and hence $s_k^\alpha(\varphi)(0) \to \infty$ by the property of the Cesàro sum. Therefore φ belongs to $\bigcap_l \mathscr{B}_l^{BV}(\alpha)$.

In order to show Theorem 4, we construct a function $\varphi \in \bigcap_l \mathscr{B}_l^r(\alpha)$ for α not Diophantine of exponent ρ, where $\rho \ge r$.

Notice that a L^2 function f is C^r if its Fourier coefficients $\hat{f}_k$ satisfy

$$\sum_k |\hat{f}_k|\,|k|^r < \infty.$$

For α not Diophantine of exponent ρ ($\rho \ge r$), one can find an increasing sequence $\{k_n\}$ of positive integers which satisfy

$$\|k_n\alpha\| k_n^\rho \le 1/2^n, \qquad k_{n+1} > k_n^2,$$

and let

$$\varphi(x) = \sum_n ((e^{2\pi i k_n \alpha} - 1)e^{2\pi i k_n x} + (e^{-2\pi i k_n \alpha} - 1)e^{-2\pi i k_n x}).$$

The rest of the proof is analogous as before.

Finally to show Theorem 3 for a Liouville number α, choose a sequence $\{k_n\}$ such that $\|k_n\alpha\|k_n^n \leq 1$ and define a function φ by the above expression. The rest of the proof is left to the reader.

References

[BM] J.-C. Beniere and G. Meigniez, Flows without minimal set, Ergod. Th. Dyn. Sys. **19** (1999), 21–30.

[B] A. S. Besikovitch, A problem on topological transformations of the plane II, Proc. Cambridge Philos. Soc. **47** (1951), 38–45.

[GH] W. H. Gottschalk and G. A. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloquium Publication Vol. 36 (1955).

[G] L. Guillou, Théorème de translation plane de Brouwer et generalisations du théorème de Poincaré-Birkhoff, Topology **33** (1994), 331–351.

[HR] G. H. Hardy and W. W. Rogosinski, Fourier series, Cambridge Tracts in Mathematics and Mathematical Physics, No. 38 (1950).

[H] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations, Publications Math. I. H. E. S. **49** (1979), 5–234.

[I] T. Inaba, An example of a flow on a non-compact surface which has no minimal set, Ergod. Th. Dyn. Sys. **19** (1999), 31–33.

[Ko] A. N. Kolmogorov, On dynamical systems with integral invariant on the torus, Doklady A. N. SSSR **93** (1953), 763–766 In Russian.

[Kr] A. B. Krygin, ω-limit sets of smooth cylindrical cascades, Math. Zametki **23** (1978), 873–884, English Translation; Math. Notes **23** (1978), 479–485.

Shigenori Matsumoto
Department of Mathematics
College of Science and Technology
Nihon University
1-8 *Kanda-Surugadai, Chiyoda-ku*
Tokyo 101-8308 *Japan*
E-mail: matsumo@cst.nihon-u.ac.jp

Mitsuhiro Shishikura
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa-Oiwakecho, Sakyo-ku
Kyoto 606-8502 *Japan*
E-mail: mitsu@kusm.kyoto-u.ac.jp