HIROSHIMA MATH. J.
32 (2002), 229–308

On space-time decay properties of solutions to hyperbolic-elliptic coupled systems

Dedicated to Professor Takaaki Nishida and Professor Masayasu Mimura for their 60th birthdays

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(Received August 31, 2001)

ABSTRACT. We consider the asymptotic behavior of solutions to the initial value problem for a certain class of hyperbolic-elliptic coupled systems. It will be proved that the solution is time asymptotically approximated by the superposition of diffusion waves constructed in terms of the self-similar solutions of generalized Burgers equations. We will give space-time decay estimates for the residual term through a pointwise estimate for the Green's function of the linearized system.

1. Introduction

We are concerned with large-time behavior of solutions to the initial value problem for a certain class of hyperbolic-elliptic coupled systems in one space variable. The system is written in the form

$$\begin{cases} w_t + F(w,q)_x = 0, \\ -q_{xx} + Rq + v(w,q)G(w,q)_x = 0. \end{cases} \tag{1.1}$$

Here $w = w(x,t)$ and $q = q(x,t)$ are unknown functions taking values in a domain $\Omega \subseteq \mathbf{R}^m$ and $\mathbf{R}^n$, respectively, where $x \in \mathbf{R}^1$ and $t \geq 0$, while $F = F(w,q)$, $G = G(w,q)$ and $v = v(w,q)$ are given smooth mappings from $\Omega \times \mathbf{R}^n$ into $\mathbf{R}^m$, $\mathbf{R}^n$ and $\mathbf{R}^1_+ = \{x \in \mathbf{R}^1; x > 0\}$, respectively, and R is a positive definite $n \times n$ matrix of real constant entries. We impose the initial condition at $t = 0$ in the form

$$w(x,0) = w_0(x). \tag{1.2}$$

We shall assume that the eigenvalues of the Jacobian $D_wF(w,q)$ are all real. Therefore, the first system of equations in (1.1) is a hyperbolic system of

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2000 *Mathematics Subject Classification*. 35B40, 35E05, 35M10, 35Q35

Key words and phrases. Asymptotic behavior of solutions, Hyperbolic-elliptic coupled systems, Pointwise decay estimates, L^p decay estimates, Green's function

conservation laws for w, while the second one is a elliptic system for q. Hence, we can call (1.1) a hyperbolic-elliptic coupled system.

This type of hyperbolic-elliptic coupled system was first studied by S. Kawashima, Y. Nikkuni & S. Nishibata [4]. They considered large-time behavior of solutions to the initial value problem in the case where the mappings F and G have forms $F(w,q) = f(w) + L^T q$ and $G(w,q) = g(w)$ with smooth mappings f and g and $n \times m$ matrix L of real constant entries. L^T denotes the transpose of L. Under suitable assumptions on f and g, it was proved that a unique smooth solution of the initial value problem exists for all $t \geq 0$ and converges to a given constant state $(\overline{w}, 0) \in \Omega \times \mathbf{R}^n$ as $t \to \infty$ if the initial datum $w_0 = w_0(x)$ is sufficiently close to $\overline{w}$ in a Sobolev space. They also obtained decay rates of the convergence in a Sobolev space, then in L^p for $2 \leq p \leq \infty$ by interpolation. The solution approaches the constant state at the rate $t^{-(1/2)(1-1/p)}$ as $t \to \infty$ in L^p for $2 \leq p \leq \infty$. Moreover, it was shown that the solution is well approximated by a solution of a hyperbolic-parabolic coupled system at the rate $t^{-(1/2)(2-1/p)+\varepsilon}$ as $t \to \infty$ in L^p for $2 \leq p \leq \infty$, where ε is an arbitrary positive constant.

On the other hand, T.-P. Liu & Y. Zeng [6] considered large-time behavior of solutions to the initial value problem for general hyperbolic-parabolic coupled systems and showed that the solution is time asymptotically approximated by the superposition of diffusion waves constructed in terms of the self-similar solutions of generalized Burgers equations. They proved that the solution approaches the superposition at the rate $t^{-(1/2)(1-1/p)-1/4}$ as $t \to \infty$ in L^p for $1 < p \leq \infty$ by integrating a space-time pointwise estimate. Such an estimate was obtained through a pointwise estimate for the Green's function of the linearized system around a constant state and the analysis of coupling of nonlinear diffusion waves. Since the rate $t^{-(1/2)(1-1/p)-1/4}$ is faster than that of the superposition, they got the optimal decay rate of the solution itself in L^p for $1 < p \leq \infty$, and $t^{-(1/2)(1-1/p)}$ is the rate.

The main purpose of this paper is to extend the results of Liu and Zeng to the initial value problem for the hyperbolic-elliptic coupled system (1.1) and (1.2) with extension of their results. One of our main tasks is to make a space-time pointwise estimate for the Green's function of the linearized system. The difficulty of this lies in the fact that the Fourier transform of the Green's function has essential singularities. Such singularities do not appear for hyperbolic-parabolic coupled systems and make our estimate for the Green's function to be weaker than that of Liu and Zeng's. Nevertheless, such an estimate is sufficient for the analysis of pointwise behavior of the solutions to the nonlinear systems and our estimates for the solutions are also valid for those to hyperbolic-parabolic coupled systems.

Another purpose is to weaken the hypothesis concerning the regularity of the initial data. To this end, we shall adopt the weighted energy method due to A. Matsumura [7]. He showed the existence of unique smooth solution globally in time to the initial value problem for a system of equations governing the motion of compressible, viscous and heat-conductive Newtonian fluid in three space dimensions and obtained the decay rate of the solution as $t \to \infty$. However, we can not directly apply his technique to our problem because the decay rate of the solution is not enough compared with the case of three space dimensions. Making use of the L^p estimate of the Green's function obtained by integrating the pointwise estimate, we shall evaluate the L^p norms of the solution and it's spatial derivatives in terms of the weighted energy norm. Combining the estimates for the weighted energy and L^p norms, we shall obtain decay estimates of the solution, which are valid for less regular initial data.

The contents of this paper are as follows. In section 2 we give structural conditions (existence of entropy function and stability condition) which are imposed on the hyperbolic-elliptic coupled system (1.1), preliminary propositions and statements of our results. In section 3 we prove the existence of solution locally in time to the initial value problem (1.1) and (1.2) under the assumption that the system admits an entropy function. We explain how to reduce it to the initial value problem for a symmetric hyperbolic system. In section 4, following Matsumura we evaluate the weighted energy norm by assuming the stability condition. At the same time, we obtain a priori estimate for the energy norm and then the existence of solution globally in time. In section 5 we give pointwise estimates for the Green's function and its spatial derivatives by making use of the Fourier transform technique due to Zeng [9]. The Green's function contains the Dirac δ-function because of the hyperbolicity of the system. In section 6, following Liu and Zeng we study the coupling of nonlinear diffusion waves. Although our calculation bears a resemblance to theirs, our estimates do not directly follow from theirs. In section 7 we first evaluate the L^p norm of the solution in terms of the weighted energy norm by using the L^p estimate for the Green's function. Then, combining the estimates of L^p and weighted energy norms together, we obtain a priori estimates of their norms. In section 8 we study the space-time pointwise behavior of the solution by using the pointwise estimate of the Green's function and L^p decay estimate of the solution. Finally, in section 9 we show that the order of time decay in our estimates is optimal in general by considering a particular system.

Notation. Let $F = F(w, q)$ be a smooth mapping from $\Omega \times \mathbf{R}^n$ to $\mathbf{R}^k$, where Ω is a open set in $\mathbf{R}^m$. We denote by $D_w F(w, q)$ and $D_q F(w, q)$ the Jacobians of $F(w, q)$ with respect to w and q, respectively, so that $D_w F(w, q)$ and $D_q F(w, q)$ are $k \times m$ and $k \times n$ matrices, respectively. For a matrix L, we denote by L^T the transpose of L.

For a non-negative integer l, we denote by ∂_t^l and ∂_x^l the derivatives $(\partial/\partial t)^l$ and $(\partial/\partial x)^l$, respectively. For simplicity, we write $\partial_t = \partial_t^1$ and $\partial_x = \partial_x^1$. For a non-negative integer s, we denote by H^s the usual Sobolev space on $\mathbf{R}^1$ equipped with the norm $\|u\|_s = (\sum_{l=0}^s \int_{\mathbf{R}^1} |\partial_x^l u(x)|^2 dx)^{1/2}$. For $1 \le p \le \infty$, we denote by $|\cdot|_p$ the norm of the Lebesgue space $L^p = L^p(\mathbf{R}^1)$. We use the abbreviation $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$. For $1 \le p \le \infty$, a non-negative integer s and a real number β, we denote by $W_\beta^{l,p}$ the space of all functions $u = u(x)$ on $\mathbf{R}^1$ such that $(1+|x|)^\beta \partial_x^l u \in L^p$ for $0 \le l \le s$ with the norm $\|u\|_{s,p,\beta} = \sum_{l=0}^s |(1+|x|)^\beta \partial_x^l u|_p$. For simplicity, we write $W^{s,p} = W_0^{s,p}$, $\|\cdot\|_{s,p} = \|\cdot\|_{s,p,0}$ and $|\cdot|_{(\beta)} = \|\cdot\|_{0,\infty,\beta}$. Let j be a non-negative integer and I be an interval contained in $[0,\infty)$. We say that $u \in C^j(I; H^s)$ if u is a function of C^j-class on I with values in H^s.

For functions $u = u(x)$ and $v = v(x)$ on $\mathbf{R}^1$, we denote by $u * v$ the convolution of u and v: $(u*v)(x) = \int_{\mathbf{R}^1} u(x-y)v(y)dy$. The usual inner product in $\mathbf{R}^m$ or $\mathbf{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. Throughout this paper, we denote inessential constants by the same symbol C and $C = C(a, b, \dots)$ means that C depends only on $a, b, \dots$.

2. Statement and results

First of all, we introduce a new dependent variable u by a diffeomorphism $w = w(u)$ from an open set O_u onto an open convex set $O_w \subseteq \Omega$. Putting $w = w(u)$ into system (1.1), we rewrite it as

$$
(2.1) \qquad \begin{cases} A^0(u)u_t + A(u,q)u_x + M(u,q)q_x = 0, \\ -q_{xx} + Rq + v(w(u),q)(L(u,q)u_x + J(u,q)q_x) = 0, \end{cases}
$$

where

$$
(2.2) \qquad \begin{cases} A^0(u) = D_u w(u), \\ A(u,q) = D_u(F(w(u),q)) = D_w F(w(u),q) D_u w(u), \\ M(u,q) = D_q F(w(u),q), \\ L(u,q) = D_u(G(w(u),q)) = D_w G(w(u),q) D_u w(u), \\ J(u,q) = D_q G(w(u),q). \end{cases}
$$

DEFINITION 2.1. Let O_u and O_w be an open set in $\mathbf{R}^m$ and an open convex set in Ω, respectively. We say that system (2.1) is *symmetric* on $O_u \times \mathbf{R}^n$ if the coefficient matrices verify the properties:

1. $A^0(u)$ is positive definite for $u \in O_u$;
2. $A(u,q)$ and $J(u,q)$ are symmetric and $M(u,q) = L(u,q)^T$ for $(u,q) \in O_u \times \mathbf{R}^n$.

We say that system (1.1) is symmetrizable on $O_w \times \mathbf{R}^n$ if there exists a diffeomorphism $w = w(u)$ from O_u onto O_w such that the system is reduced to a symmetric system (2.1).

The property that the system is symmetrizable can be characterized in terms of an entropy function, which is defined as follows.

Definition 2.2. Let O_w be an open convex set in Ω and $\eta = \eta(w)$ be a real-valued smooth function defined on O_w. We say that η is an *entropy function* for system (1.1) if the following properties hold:

1. η is a strictly convex function defined on O_w in the sense that the Hessian $D_w^2\eta(w)$ is positive definite for $w \in O_w$;

2. There exists a real-valued smooth function $\zeta = \zeta(w,q)$ defined on $O_w \times \mathbf{R}^n$ such that the relations

$$\begin{cases} D_w\zeta(w,q) = D_w\eta(w)D_wF(w,q), \\ D_q\zeta(w,q) = D_w\eta(w)D_qF(w,q) - G(w,q)^T \end{cases} \tag{2.3}$$

hold for $(w,q) \in O_w \times \mathbf{R}^n$. ζ is called an entropy flux corresponding to η.

Proposition 2.1 ([4]). *Let O_w be an open convex set in Ω. Then, system* (1.1) *is symmetrizable on $O_w \times \mathbf{R}^n$ if and only if the system admits an entropy function on O_w.*

We now state an existence theorem of solutions locally in time to the initial value problem (2.1) and

$$u(x,0) = u_0(x). \tag{2.4}$$

Theorem 2.1. *We assume that system* (2.1) *is symmetric on $O_u \times \mathbf{R}^n$. Let $\bar{u}$ be a constant state in O_u. There exist positive constants $c_0 = c_0(\bar{u})$ and $c_1 = c_1(\bar{u})$ such that if $u_0 - \bar{u} \in H^s$ for an integer $s \geq 2$ and $\|u_0 - \bar{u}\|_1 \leq c_0$, then the initial value problem* (2.1) *and* (2.4) *has a solution (u,q) on some time interval $[0,T]$ satisfying*

$$\begin{cases} u - \bar{u} \in C^0([0,T];H^s) \cap C^1([0,T];H^{s-1}), \qquad q \in C^0([0,T];H^{s+1}), \\ \|u(t) - \bar{u}\|_s + \|q(t)\|_{s+1} \leq C\|u_0 - \bar{u}\|_s \qquad \text{for } 0 \leq t \leq T, \end{cases}$$

where $T = T(\|u_0 - \bar{u}\|_2) > 0$ and $C = C(\|u_0 - \bar{u}\|_2, s) > 0$. Moreover, the solution is unique in the class

$$\begin{cases} u - \bar{u} \in C^0([0,T];H^2) \cap C^1([0,T];H^1), \qquad q \in C^0([0,T];H^3), \\ \|u(t) - \bar{u}\|_1 + \|q(t)\|_1 \leq c_1 \qquad \text{for } 0 \leq t \leq T. \end{cases}$$

The proof of this theorem is carried out in the next section. This theorem and Proposition 2.1 guarantee the existence of a solution locally in time to the initial value problem (1.1) and (1.2) under suitable conditions on the initial datum w_0 if the system admits an entropy function. If $L(u,q)$ and $v(w(u),q)$ in (2.1) do not depend on q and $J(u,q)$ is identically zero, then we do not have to impose the smallness condition on the initial data and the uniqueness of the solution always holds.

We proceed to state the global behavior in time of solutions to the initial value problem (1.1) and (1.2). To this end, we formulate the stability condition.

Definition 2.3. Let $\bar{u}$ be a constant state in O_u. We say that the symmetric system (2.1) satisfies the *stability condition* at $(\bar{u},0)$ if the coefficient matrices verify the property:

For $\mu \in \mathbf{R}^1$ and $\varphi \in \mathbf{R}^m$, $\mu A^0(\bar{u})\varphi + A(\bar{u},0)\varphi = 0$ and $L\varphi = 0$ imply that $\varphi = 0$.

In order to characterize the stability condition, we consider the linearization of the symmetric system (2.1) around a constant state $(\bar{u},0)$:

$$\begin{cases} A^0 u_t + A u_x + L^T q_x = 0, \\ -q_{xx} + Rq + v(Lu_x + Jq_x) = 0, \end{cases} \tag{2.5}$$

where

$$A^0 = A^0(\bar{u}), \quad A = A(\bar{u},0), \quad L = L(\bar{u},0), \quad J = J(\bar{u},0), \quad v = v(w(\bar{u}),0), \tag{2.6}$$

and the eigenvalue problem associated with this system:

$$\begin{cases} \lambda A^0 \varphi + i\xi A\varphi + i\xi L^T \psi = 0, \\ (\xi^2 I + R + i\xi vJ)\psi + i\xi vL\varphi = 0, \end{cases} \tag{2.7}$$

with a real parameter ξ, where $\lambda \in \mathbf{C}^1$, $\varphi \in \mathbf{C}^m$ and $\psi \in \mathbf{C}^n$. The admissible value of λ, which admits a non-trivial solution (φ,ψ) of (2.7), is denoted by $\lambda = \lambda(i\xi)$. We note that system (2.7) is equivalent to the system

$$\begin{cases} \lambda A^0 \varphi + \{i\xi A + \xi^2 vL^T(\xi^2 I + R + i\xi vJ)^{-1} L\}\varphi = 0, \\ \psi = -i\xi v(\xi^2 I + R + i\xi vJ)^{-1} L\varphi. \end{cases}$$

Therefore, the admissible values of λ are the roots of the algebraic equation

$$\det\{\lambda A^0 + i\xi A + \xi^2 vL^T(\xi^2 I + R + i\xi vJ)^{-1} L\} = 0.$$

Definition 2.4. Let K be an $m \times m$ matrix of real constant entries. We say that K is a *compensating matrix* for (2.5) if the following properties are satisfied:

1. KA^0 is a real skew-symmetric matrix;
2. $[KA]' + L^T L$ is positive definite, where $[X]'$ denotes the symmetric part of X.

Proposition 2.2 ([4]). *Let $\bar{u}$ be a constant state in O_u, $\lambda(i\xi)$ the value of λ corresponding to a non-trivial solution (φ, ψ) of* (2.7) *and* $(u, q) \in C^0([0, \infty); L^2)$ *a solution to* (2.5). *Then the following five conditions are equivalent to each other:*

1. *The symmetric system* (2.1) *satisfies the stability condition at* $(\bar{u}, 0)$;
2. *There exists a compensating matrix K for* (2.5);
3. $\mathrm{Re}\,\lambda(i\xi) < 0$ *for* $\xi \in \mathbf{R}^1$;
4. *There exists a positive constant* δ *such that* $\mathrm{Re}\,\lambda(i\xi) \leq -\delta\xi^2/(1+\xi^2)$ *for* $\xi \in \mathbf{R}^1$;
5. $\|u(t)\| + \|q(t)\| \to 0$ *as* $t \to \infty$.

The proof of this proposition is almost the same as that in [4], where the case $J = 0$ is investigated.

Theorem 2.2. *Suppose that system* (1.1) *admits an entropy function so that the system is put into the symmetric system* (2.1). *Let $\overline{w}$ be a constant state in O_w and $\bar{u}$ the corresponding constant state in O_u $(\overline{w} = w(\bar{u}))$. We also suppose that the symmetric system* (2.1) *satisfies the stability condition at* $(\bar{u}, 0)$. *Let $s \geq 2$ be an integer. There exists a positive constant $c_2 = c_2(\overline{w}, s)$ such that if $w_0 - \overline{w} \in H^s$ and $\|w_0 - \overline{w}\|_2 \leq c_2$, then the initial value problem* (1.1) *and* (1.2) *has a solution (w, q) satisfying*

$$\begin{cases} w - \overline{w} \in C^0([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1}), \\ q \in C^0([0,\infty); H^{s+1}). \end{cases}$$

The solution verifies the uniform estimate

$$\|w(t) - \overline{w}\|_s^2 + \|q(t)\|_{s+1}^2 + \int_0^t (\|w_x(\tau)\|_{s-1}^2 + \|q(\tau)\|_{s+1}^2)d\tau \leq C\|w_0 - \overline{w}\|_s^2$$

for $t \geq 0$, where $C = C(\overline{w}, s) > 0$.

This theorem is proved by standard continuation argument based on the local existence of solution obtained in Theorem 2.1 and a priori estimates of solutions given in section 4.

Theorem 2.3. *Assume the same conditions in Theorem 2.2. Let $s \geq 3$ be an integer. There exists a positive constant $c_3 = c_3(\overline{w}, s)$ such that if $w_0 - \overline{w} \in H^s \cap L^1$ and $E_3 \equiv \|w_0 - \overline{w}\|_3 + |w_0 - \overline{w}|_1 \leq c_3$, then the solution (w, q) obtained in Theorem 2.2 verifies the decay estimates*

$$\begin{cases} \|\partial_x^l (w(t) - \overline{w})\|_{s-l} \leq C_1 E_s (1+t)^{-l/2} & \text{if } 0 \leq l \leq s, \\ \|\partial_x^l q(t)\|_{s-l+1} \leq C_1 E_s (1+t)^{-(1/2)(l+1)} & \text{if } 0 \leq l \leq s-1 \end{cases}$$

for $t \geq 0$, *where* $C_1 = C_1(\overline{w}, s) > 0$ *and* $E_s = \|w_0 - \overline{w}\|_s + |w_0 - \overline{w}|_1$. *Moreover, for* $1 \leq p \leq \infty$ *it holds that*

$$\begin{cases} |\partial_x^l(w(t) - \overline{w})|_p \leq C_2(E_s + \|w_0 - \overline{w}\|_{l,p})(1+t)^{-(1/2)(l+1-1/p)} & \text{if } 0 \leq l \leq s-2, \\ |\partial_x^l q(t)|_p \leq C_2(E_s + \|w_0 - \overline{w}\|_{\max(l-1,0),p})(1+t)^{-(1/2)(l+2-1/p)} & \text{if } 0 \leq l \leq s-3 \end{cases}$$

for $t \geq 0$, *where* $C_2 = C_2(\overline{w}, s, p) > 0$.

This theorem is proved in section 7.

Next, we consider the asymptotic profile of the solution to the initial value problem (1.1) and (1.2). We fix $\overline{w}$ a constant state in O_w and introduce a new dependent variable v by the relation

$$w - \overline{w} = D_u w(\bar{u})v, \qquad \overline{w} = w(\bar{u}), \tag{2.8}$$

where $w = w(u)$ is the diffeomorphism used in the derivation of system (2.1) from (1.1). Under the hypothesis that system (1.1) admits an entropy function, the system can be rewritten as

$$\begin{cases} A^0(\bar{u})v_t + A(\bar{u},0)v_x + L(\bar{u},0)^T q_x = H_{3x} - Q(v,v)_x = H_{1x}, \\ -q_{xx} + Rq + v(\overline{w},0)(L(\bar{u},0)v_x + J(\bar{u},0)q_x) = H_2, \end{cases} \tag{2.9}$$

where the coefficient matrices are defined in (2.2) and

$$\begin{cases} H_1 = -(F(w,q) - F(\overline{w},0) - D_w F(\overline{w},0)(w - \overline{w}) - D_q F(\overline{w},0)q), \\ H_2 = -(v(w,q)D_w G(w,q) - v(\overline{w},0)D_w G(\overline{w},0))w_x \\ \qquad - (v(w,q)D_q G(w,q) - v(\overline{w},0)D_q G(\overline{w},0))q_x, \\ H_3 = -(F(w,q) - F(\overline{w},0) - D_w F(\overline{w},0)(w - \overline{w}) \\ \qquad - D_q F(\overline{w},0)q - \dfrac{1}{2}D_w^2 F(\overline{w},0)(w - \overline{w}, w - \overline{w})), \\ Q(v,\tilde{v}) = \dfrac{1}{2}D_w^2 F(\overline{w},0)(D_u w(\bar{u})v, D_u w(\bar{u})\tilde{v}). \end{cases} \tag{2.10}$$

Changing the variable v with $\tilde{v}$ by the formula $\tilde{v} = (A^0(\bar{u}))^{1/2}v$, we can reduce the problem to that in the case $A^0(\bar{u}) = I$. Therefore, we assume that $A^0(\bar{u}) = I$ in the following. Since $A(\bar{u},0)$ is symmetric, the eigenvalues of $A(\bar{u},0)$ are all real. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_\sigma$ be the distinct eigenvalues of $A(\bar{u},0)$ with multiplicity $m_1, m_2, \ldots, m_\sigma$; $m_1 + m_2 + \cdots + m_\sigma = m$. Let the left and the right eigenvectors associated with λ_i be l_{ij} and r_{ij}, $j = 1, \ldots, m_i$:

$$A(\bar{u},0)r_{ij} = \lambda_i r_{ij}, \qquad l_{ij}A(\bar{u},0) = \lambda_i l_{ij}, \qquad l_{ij}r_{i'j'} = \delta_{ii'}\delta_{jj'} \tag{2.11}$$

for $i, i' = 1, \dots, \sigma$, $j = 1, \dots, m_i$ and $j' = 1, \dots, m_{i'}$, where δ_{ij} is Kronecker's delta. Put

$$\left\{\begin{aligned} & l_i = \begin{pmatrix} l_{i1} \\ \vdots \\ l_{im_i} \end{pmatrix}, \qquad r_i = (r_{i1}, \dots, r_{im_i}), \quad P_i = r_i l_i \qquad \text{for } i = 1, \dots, \sigma, \\ & \tilde{L} = \begin{pmatrix} l_1 \\ \vdots \\ l_\sigma \end{pmatrix}, \qquad \tilde{R} = (r_1, \dots, r_\sigma). \end{aligned}\right. \tag{2.12}$$

Then, P_i is the projection onto the eigenspace associated with λ_i and it holds that $\tilde{L}\tilde{R} = \tilde{R}\tilde{L} = I$. Moreover, we have the spectral decomposition

$$A(\bar{u}, 0) = \sum_{j=1}^{\sigma} \lambda_i P_i. \tag{2.13}$$

Now, we derive an approximate system of equations to (2.9). Neglecting higher order nonlinear terms and derivatives of q in the elliptic system for q, we obtain

$$\left\{\begin{array}{l} v_t + Av_x + Q(v, v)_x + L^T q_x = 0, \\ Rq + \nu L v_x = 0, \end{array}\right.$$

which yields the hyperbolic-parabolic coupled system

$$v_t + Av_x + Q(v, v)_x = \nu L^T R^{-1} L v_{xx}, \tag{2.14}$$

where we used the notation in (2.6). We decompose v in the directions of the right eigenvectors as

$$v = \sum_{i=1}^{\sigma} r_i v_i, \qquad v_i = l_i v \qquad \text{for } i = 1, \dots, \sigma.$$

Then, (2.14) is equivalent to the system

$$v_{it} + \lambda_i v_{ix} + \sum_{j,k=1}^{\sigma} l_i Q(r_j v_j, r_k v_k)_x = \sum_{j=1}^{\sigma} \nu l_i L^T R^{-1} L r_j v_{jxx}$$

for $i = 1, \dots, \sigma$. Neglecting the effects of the other families than the i-field, we finally obtain

$$v_{it} + \lambda_i v_{ix} + l_i Q(r_i v_i, r_i v_i)_x = \nu l_i L^T R^{-1} L r_i v_{ixx} \tag{2.15}$$

for $i = 1, \dots, \sigma$. These are the desired approximate equations and called generalized Burgers equations. We seek a self-similar solution to (2.15) satisfying the constraint

$$\int_{\mathbf{R}^1} v_i(x,t)dx = \widetilde{\delta_i} \tag{2.16}$$

for $i = 1, \dots, \sigma$, where $\widetilde{\delta_i}$ is a given constant vector in $\mathbf{R}^{m_i}$. Under the hypothesis that the symmetric system (2.1) satisfies the stability condition at $(\bar{u}, 0)$, we see that the hyperbolic-parabolic coupled system (2.14) satisfies the stability condition at $\bar{u}$ in the sense of [2], that is, for $\lambda \in \mathbf{R}^1$ and $\varphi \in \mathbf{R}^m$, $\lambda\varphi + A\varphi = 0$ and $\nu L^T R^{-1} L\varphi = 0$ imply that $\varphi = 0$. Therefore, by Lemma 2.1 in [6] each matrix $\nu l_i L^T R^{-1} L r_i$ is positive definite and we obtain the existence theorem for the self-similar solution.

Proposition 2.3 ([1, 6]). *Assume the same conditions in Theorem 2.2. There exists a positive constant $c_4 = c_4(\bar{u})$ such that if $\widetilde{\delta_i} \in \mathbf{R}^{m_i}$ satisfies the condition $|\widetilde{\delta_i}| \le c_4$, then (2.15) and (2.16) has a unique self-similar solution of the form $\frac{1}{\sqrt{t}}\chi_i\left(\frac{x-\lambda_i t}{\sqrt{t}}\right)$ with the condition $\lim_{y\to-\infty} y\chi_i(y) = \lim_{y\to-\infty} \chi_i'(y) = 0$. Moreover, χ_i has the property*

$$\chi_i(y) = e^{-y^2/(4\mu_i)}\tilde{\chi}_i(y), \tag{2.17}$$

where μ_i is the maximum eigenvalue of $\nu l_i L^T R^{-1} L r_i$ and $\tilde{\chi}_i$ and all its derivatives are uniformly bounded by $C|\widetilde{\delta_i}|$ with a positive constant C.

We choose $\widetilde{\delta_i}$ in (2.16) as

$$\widetilde{\delta_i} = \int_{\mathbf{R}^1} v_i(x,0)dx = l_i(D_u w(\bar{u}))^{-1} \int_{\mathbf{R}^1} (w_0(x) - \overline{w})dx,$$

where w_0 is the initial datum in (1.2). If $|w_0 - \overline{w}|_1$ is sufficiently small, then Proposition 2.3 guarantees the existence of the self-similar solution. Using this we define a function θ by

$$\theta(x,t) = \sum_{i=1}^{\sigma} \theta_i(x,t), \tag{2.18}$$

$$\theta_i(x,t) = \frac{1}{\sqrt{1+t}}\chi_i\left(\frac{x-\lambda_i(1+t)}{\sqrt{1+t}}\right) \qquad \text{for } i = 1, \dots, \sigma,$$

and a function ω by

$$\omega(x,t) = (D_u w(\bar{u}))^{-1}(w(x,t) - \overline{w} - D_u w(\bar{u})\theta(x,t)), \tag{2.19}$$

where w is the solution to the initial value problem (1.1) and (1.2) obtained in Theorem 2.2. We shall show that $\overline{w} + D_u w(\bar{u})\theta(x,t)$ is an asymptotic profile of the solution $w(x,t)$. To this end, we prove that $\omega(\cdot\,,t)$ decays faster than $\theta(\cdot\,,t)$ as $t \to \infty$.

Theorem 2.4. *Assume the same conditions in Theorem* 2.2. *Let* $s \ge 3$ *be an integer and define a function* $W_0 = W_0(x)$ *by*

$$W_0(x) = \begin{cases} \displaystyle\int_x^{\infty} (w_0(y) - \overline{w})dy & \text{for } x \ge 0, \\ \displaystyle\int_{-\infty}^{x} (w_0(y) - \overline{w})dy & \text{for } x < 0. \end{cases} \tag{2.20}$$

There exists a positive constant $c_5 = c_5(\overline{w}, s)$ *such that if* $E_3 \le c_5$ *and* $0 \le l \le s-3$, *then the function* ω *defined by* (2.19) *verifies the decay estimates*

$$|\partial_x^l \omega(t)|_p \le \begin{cases} C_2((1+E_s)|W_0|_2 + E_s)(1+t)^{-(1/2)(l+1-1/p)-1/4} & \text{if } 2 \le p \le \infty, \\ C_2((1+E_s)|W_0|_1 + E_s + |w_0 - \overline{w}|_{l,p}) & \\ \qquad \times (1+t)^{-(1/2)(l+1-1/p)-1/4} & \text{if } 1 \le p \le \infty \end{cases}$$

for $t \ge 0$, *where we used the notation in Theorem* 2.3.

The decay rates in this theorem are optimal, which is explained in section 9. However, if we impose an additional condition for nonlinear terms, then we can obtain faster decay rates than the above one.

Theorem 2.5. *Assume the same conditions in Theorem* 2.2 *and the condition*

$$P_j Q(P_i u, P_i u) = 0 \qquad \text{for } u \in \mathbf{R}^m, i, j = 1, \dots, \sigma, i \ne j, \tag{2.21}$$

where Q *is the quadratic nonlinear term defined in* (2.10) *and* P_i *is the projection defined in* (2.12). *Let* $s \ge 3$ *be an integer,* $\gamma < 1/2$ *and* W_0 *the function defined in* (2.20). *There exists a positive constant* $c_6 = c_6(\overline{w}, s, \gamma)$ *such that if* $E_3 \le c_6$, $0 \le l \le s-3$ *and* $1 \le p \le \infty$, *then the function* ω *defined by* (2.19) *verifies the decay estimates*

$$|\partial_x^l \omega(t)|_p \le C_3((1+E_s)|W_0|_1 + |w_0 - \overline{w}|_{l,p} + E_s)(1+t)^{-(1/2)(l+1-1/p)-\gamma}$$

for $t \ge 0$, *where* $C_3 = C_3(\overline{w}, s, p, \gamma) > 0$ *and we used the notation in Theorem* 2.3.

Theorems 2.4 and 2.5 are also proved in section 7.

We proceed to state space-time decay estimates for the solution $(w(x,t), q(x,t))$ and the residual term $\omega(x,t)$. Those are the main results in the present paper. For $\alpha \in \mathbf{R}^1$ and $\beta = (\beta_1, \beta_2) \in \mathbf{R}^2$ we define functions by

$$
(2.22)\qquad \begin{cases} \varphi_\alpha(x,t;\lambda) = \left(1 + \dfrac{(x-\lambda(1+t))^2}{1+t}\right)^{-\alpha/2}, \\ \psi_\alpha(x,t;\lambda) = \left(1 + \dfrac{|x-\lambda(1+t)|}{1+t}\right)^{-\alpha}, \\ \Phi_\beta(x,t) = \displaystyle\sum_{i=1}^{\sigma}(\varphi_{\beta_1}(x,t;\lambda_i) + \varphi_i(x,t;\lambda_i)\psi_{\beta_2}(x,t;\lambda_i)), \end{cases}
$$

where $\lambda_1,\dots,\lambda_\sigma$ are the eigenvalues of the matrix $A(\bar{u},0)$ (cf. (2.11)).

THEOREM 2.6. *Assume the same conditions in Theorem* 2.2. *Let* $s \geq 3$ *be an integer,* $\beta_1 \geq 1$, $\beta_2 \geq 0$ *and* $\beta = (\beta_1,\beta_2)$. *There exists a positive constant* $c_7 = c_7(\overline{w},s,\beta)$ *such that if* $E_3 \leq c_7$, *then the solution* (w,q) *obtained in Theorem* 2.2 *verifies the pointwise estimates*

$$
\begin{cases} |\partial_x^l(w(x,t)-\overline{w})| + |\partial_x^l q(x,t)| \\ \qquad \leq C_4(1+E_s)^l E_{l,\beta_1}(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t) \qquad \text{if } 0 \leq l \leq s-2, \\ |\partial_x^l q(x,t)| \leq C_4(1+E_s)^{l+1}E_{\max(l,1),\beta_1}(1+t)^{-(1/2)(l+2)}\Phi_\beta(x,t) \quad \text{if } 0 \leq l \leq s-3 \end{cases}
$$

for $x \in \mathbf{R}^1$ *and* $t \geq 0$, *where* $C_4 = C_4(\overline{w},s,\beta)$, $E_{l,\beta_1} = |w_0-\overline{w}|_1 + \|w_0-\overline{w}\|_{l,\infty,\beta_1}$ *and we used the notation in Theorem* 2.3.

THEOREM 2.7. *Under the same conditions in Theorem* 2.6, *there exists a positive constant* $c_8 = c_8(\overline{w},s,\beta_1)$ *such that if* $E_3 \leq c_8$ *and* $0 \leq l \leq s-3$, *then the function* ω *defined by* (2.19) *verifies the pointwise estimate*

$$
|\partial_x^l \omega(x,t)| \leq C_4(1+E_s)^{l+1}(E_3 + \tilde{E}^{(1)}_{l,\beta_1})(1+t)^{-(1/2)(l+1)-1/4}\Phi_\beta(x,t)
$$

for $x \in \mathbf{R}^1$ *and* $t \geq 0$, *where*

$$
\tilde{E}^{(1)}_{l,\beta_1} = \begin{cases} |W_0|_2 + \|W_0\|_{0,\infty,\beta_1} + \|w_0-\overline{w}\|_{l,\infty,\beta_1} & \text{when } 1 \leq \beta_1 < 3/2, \\ |W_0|_2 + \|W_0\|_{0,\infty,1} + \|w_0-\overline{w}\|_{l,\infty,\beta_1} & \text{when } \beta_1 \geq 3/2 \end{cases}
$$

with the function W_0 *defined by* (2.20) *and we used the notation in Theorems* 2.3 *and* 2.6.

As we mentioned before, the decay rates in time are optimal. However, by imposing a condition on the quadratic nonlinear term Q we can improve them.

THEOREM 2.8. *Assume the same conditions in Theorem* 2.6 *and* (2.21). *Let* $\gamma < 1/2$. *There exists a positive constant* $c_9 = c_9(\overline{w},s,\beta,\gamma)$ *such that if* $E_3 \leq c_9$ *and* $0 \leq l \leq s-3$, *then the function* ω *defined by* (2.19) *verifies the pointwise estimate*

$$|\partial_x^l \omega(x,t)| \le C_5(1+E_s)^{l+1}(E_3+\tilde{E}_{l,\beta_1}^{(2)})(1+t)^{-(1/2)(l+1)-\gamma}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t \ge 0$, *where* $C_5 = C_5(\overline{w}, s, \beta, \gamma) > 0$,

$$\tilde{E}_{l,\beta_1}^{(2)} = \begin{cases} |W_0|_1 + \|W_0\|_{0,\infty,\beta_1} + \|w_0 - \overline{w}\|_{l,\infty,\beta_1} & \text{when } 1 \le \beta_1 < 2, \\ |W_0|_1 + \|W_0\|_{0,\infty,1} + \|w_0 - \overline{w}\|_{l,\infty,\beta_1} & \text{when } \beta_1 \ge 2 \end{cases}$$

with the function W_0 *defined by* (2.20), *and we used the notation in Theorem* 2.3.

Theorems 2.6, 2.7 and 2.8 are proved in section 8.

Remark. If the matrix $A(\bar{u},0)$ has only one eigenvalue λ with multiplicity m (in this case $A(\bar{u},0)$ must be equal to λI and the condition (2.21) is automatically satisfied), we can replace the function $\Phi_\beta(x,t)$ in Theorems 2.6 and 2.8 by $\varphi_{\beta_1}(x,t;\lambda)$.

3. Local existence

To begin with, we consider a linear elliptic system in the form

$$-q_{xx} + Rq + Jq_x = f. \tag{3.1}$$

Taking the Fourier transform of this system, we obtain

$$(-(i\xi)^2 I + R + i\xi J)\hat{q}(\xi) = \hat{f}(\xi),$$

where the hat ˆ means the Fourier transform:

$$\hat{q}(\xi) = \int_{-\infty}^{\infty} q(x)e^{-i\xi x}\,dx.$$

Lemma 3.1. *Let R and J be symmetric* $n \times n$ *matrices of constant entries and assume that R is positive definite. Then, there exists positive constants* α *and C such that for any* $z \in D_\alpha \equiv \mathbf{C}^1 \backslash \{z \in \mathbf{C}^1; |\mathrm{Im}\, z| < \alpha < |\mathrm{Re}\, z| < \alpha^{-1}\}$ *the matrix* $-z^2 I + R + zJ$ *is invertible and verifies the estimate*

$$|(-z^2 I + R + zJ)^{-1}| \le C(1+|z|^2)^{-1} \qquad \text{for } z \in D_\alpha.$$

Proof. Suppose that

$$\det(-z^2 I + R + zJ) = 0, \qquad z = x + iy, \qquad x, y \in \mathbf{R}^1.$$

Then there exists $q \in \mathbf{C}^n$ such that $(-z^2 I + R + zJ)q = 0$ and $|q| = 1$. Taking the inner product of this equation with q, we obtain

$$\begin{cases} (Rq,q) + vx(Jq,q) + (y^2 - x^2)|q|^2 = 0, \\ y(v(Jq,q) - 2x|q|^2) = 0, \end{cases}$$

where $(\cdot\,,\cdot)$ denotes the inner product in $\mathbf{C}^n$. If $y \neq 0$, then it follows from these equations that $(Rq, q) + |z|^2|q|^2 = 0$, which contradicts the positivity of R. Therefore, it holds that

$$\det(-z^2 I + R + zJ) \neq 0 \qquad \text{for } z \in \mathbf{C}^1,\ \operatorname{Im} z \neq 0.$$

On the other hand, it is easy to see that

$$\det(-z^2 I + R + zJ) = z^{2n} \det(-I + z^{-2}R + z^{-1}J) \qquad \text{for } z \in \mathbf{C}^1,\ z \neq 0.$$

Hence, if we take α sufficiently small, then there exists a positive constant c such that

$$|\det(-z^2 I + R + zJ)| \geq c(1 + |z|^2)^n \qquad \text{for } z \in D_\alpha,$$

which implies the results. □

This lemma gives directly the following one.

LEMMA 3.2. *Let s be a non-negative integer. Assume that R and J are symmetric $n \times n$ matrices of constant entries and that R is positive definite. Then, for any $f \in H^s$ the linear elliptic system* (3.1) *has a unique solution $q \in H^{s+2}$, which verifies the estimate*

$$\|q\|_{s+2} \leq C\|f\|_s,$$

where $C = C(s, R, J) > 0$.

Now, we consider the elliptic part in the symmetric system (2.1):

$$-q_{xx} + Rq + v(w(u), q)(L(u, q)u_x + J(u, q)q_x) = 0. \tag{3.2}$$

Let $\bar{u}$ be a constant state in O_u. For $c > 0$, $M > 0$ and an integer $s \geq 1$, we define a function space $X^s_{c,M}$ by

$$X^s_{c,M} = \{u; u - \bar{u} \in H^s, \|u - \bar{u}\|_1 \leq c, \|u - \bar{u}\|_s \leq M\}.$$

PROPOSITION 3.1. *There exists a positive constant $c = c(\bar{u})$ such that for any $M > 0$ and integer $s \geq 1$ we have a mapping Q from $X^s_{c,M}$ to H^{s+1} which verifies the properties:*

1. *If $u \in X^s_{c,M}$, then $q = Q(u)$ solves the elliptic system* (3.2)*;*
2. *If $u \in X^s_{c,M}$, $q \in H^2$ solves* (3.2) *and $\|q\|_1 < c$, then $q = Q(u)$;*
3. *For any $u, v \in X^s_{c,M}$, we have the estimates*

$$\begin{cases} \|Q(u)\|_2 \leq C_6\|u - \bar{u}\|_1, & \|Q(u) - Q(v)\|_2 \leq C_6\|u - v\|_1, \\ \|Q(u)\|_{s+1} \leq C_7\|u - \bar{u}\|_s, & \|Q(u) - Q(v)\|_{s+1} \leq C_7\|u - v\|_s, \end{cases}$$

where $C_6 = C_6(\bar{u}) > 0$ and $C_7 = C_7(\bar{u}, s, M) > 0$.

PROOF. We rewrite (3.2) as

$$-q_{xx} + Rq + vJq_x = \Psi(u, q),$$

where we used the notation in (2.6) and

$$\Psi(u, q) = -v(w(u), q)L(u, q)u_x - (v(w(u), q)J(u, q) - v(w(\bar{u}), 0)J(\bar{u}, 0))q_x.$$

We take a positive constant $c_{10} = c_{10}(\bar{u})$ so small that $\|u - \bar{u}\|_1 \le c_{10}$ implies that $u(x) \in O_u$ for $x \in \mathbf{R}^1$. For fixed u satisfying the condition $\|u - \bar{u}\|_1 \le c_{10}$, we define a mapping $\Phi = \Phi(q)$ by

$$\Phi(q) = (-I\partial_x^2 + R + vJ\partial_x)^{-1}\Psi(u, q).$$

By Lemma 3.2, for $q, \tilde{q} \in H^1$ satisfying $\|q\|_1, \|\tilde{q}\|_1 \le c_{10}$, we have

$$\begin{cases} \|\Phi(q)\|_2 \le C_8(\|u - \bar{u}\|_1 + \|q\|_1^2), \\ \|\Phi(q) - \Phi(\tilde{q})\|_2 \le C_8(\|u - \bar{u}\|_1 + \|q\|_1 + \|\tilde{q}\|_1)\|q - \tilde{q}\|_1, \end{cases}$$

where $C_8 = C_8(\bar{u}) \ge 1$. Therefore, if u satisfies the additional conditions $10C_8^2\|u - \bar{u}\|_1 \le 1$ and $2C_8\|u - \bar{u}\|_1 \le c_{10}$, then Φ becomes a contraction mapping from

$$S = \left\{ q \in H^2; \|q\|_2 \le \frac{2C_8\|u - \bar{u}\|_1}{1 + \sqrt{1 - 4C_8^2\|u - \bar{u}\|_1}} \right\}$$

to itself. Hence, the mapping Φ has a unique fixed point in S.

Now, we define the mapping Q such that $Q(u)$ is the fixed point. Then $Q(u)$ solves (3.2) and satisfies the estimate $\|Q(u)\|_2 \le 2C_8\|u - \bar{u}\|_1$. Moreover, if v satisfies the same conditions imposed on u, then

$$\begin{aligned} \|Q(u) - Q(v)\|_2 &= \|(-I\partial_x^2 + R + vJ\partial_x)^{-1}(\Psi(u, Q(u)) - \Psi(v, Q(v)))\|_2 \\ &\le C_9(\|u - v\|_1 + (\|u - \bar{u}\|_1 + \|v - \bar{u}\|_1)\|Q(u) - Q(v)\|_1), \end{aligned}$$

where $C_9 = C_9(\bar{u}) > 0$. Therefore, if $\|u - \bar{u}\|_1, \|v - \bar{u}\|_1 \le (4C_9)^{-1}$, then we have

$$\|Q(u) - Q(v)\|_2 \le 2C_9\|u - v\|_1.$$

It is sufficient to use the equation $Q(u)_{xx} = RQ(u) + vJQ(u)_x - \Psi(u, Q(u))$ and the induction on s to obtain higher order estimates. Finally, we define the constant c by $c = \min\{(10C_8^2)^{-1}, (2C_8)^{-1}c_{10}, (4C_9)^{-1}\}$. Then we see that the mapping Q satisfies all the properties stated in the proposition. □

Remark. If $L(u,q)$ and $v(w(u),q)$ are independent of q and $J(u,q)$ is identically zero, then we can directly define the mapping Q by $Q(u) = -(-I\partial_x^2 + R)^{-1}(v(w(u))L(u)u_x)$. In this case we do not have to impose the smallness condition $\|u - \bar{u}\|_1 \le c$ on u.

Thanks to Proposition 3.1, if we restrict our attention to small solutions which satisfy the conditions $\|u(t) - \bar{u}\|_1 \le c$ and $\|q(t)\|_1 \le c$ for $0 \le t \le T$, then the symmetric system (2.1) is equivalent to the system

$$A^0(u)u_t + A(u, Q(u))u_x + L(u, Q(u))^T Q(u)_x = 0. \tag{3.3}$$

We can regard this as a symmetric hyperbolic system because the last term behaves like lower order. Therefore, applying the standard iteration arguments we can prove the local existence theorem to the initial value problem (3.3) and (2.4), and then Theorem 2.1.

4. Estimates of weighted energy norms

We first prepare a fundamental lemma, which shall be used frequently in the following of this paper without any comments.

Lemma 4.1. *Assume that $N \ge 2$ is an integer, $l_1, l_2, \dots, l_N$ are non-negative integers, $1 \le p, q, r \le \infty$ and $1/p = 1/q + 1/r$. Put $l = l_1 + l_2 + \cdots + l_N$. Then there exists a positive constant $C = C(N, p, q, r, l)$ such that the inequality*

$$\left| \prod_{j=1}^{N} (\partial_x^{l_j} u_j) \right|_p \le C|u|_\infty^{N-2} |u|_q |\partial_x^l u|_r$$

holds for any $u = (u_1, u_2, \dots, u_N)$.

Proof. Define p_j by

$$\frac{1}{p_j} = \left(1 - \frac{l_j}{l}\right)\frac{1}{(N-1)q} + \frac{l_j}{l}\frac{1}{r}$$

for $j = 1, 2, \dots, N$. Then it holds that $p \le p_j \le \infty$ for $j = 1, 2, \dots, N$ and $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_N$. Therefore, by Hölder's inequality we obtain

$$\left| \prod_{j=1}^{N} (\partial_x^{l_j} u_j) \right|_p \le \prod_{j=1}^{N} |\partial_x^{l_j} u_j|_{p_j}. \tag{4.1}$$

Here Gagliardo-Nirenberg's and interpolation inequalities imply that

$$\begin{aligned}|\partial_x^{l_j} u_j|_{p_j} &\le C|u_j|_{(N-1)q}^{1-l_j/l}|\partial_x^l u_j|_r^{l_j/l}\\ &\le C(|u_j|_\infty^{1-1/(N-1)}|u_j|_q^{1/(N-1)})^{1-l_j/l}|\partial_x^l u_j|_r^{l_j/l}\\ &\le C|u|_\infty^{(1-1/(N-1))(1-l_j/l)}|u|_q^{(1/(N-1))(1-l_j/l)}|\partial_x^l u|_r^{l_j/l}.\end{aligned}$$

Putting this into (4.1) yields the desired inequality. □

Now, we assume the same conditions in Theorem 2.2. Let $s \ge 2$ be an integer, (w, q) a solution to (1.1) and (1.2) satisfying

$$\begin{cases} w - \overline{w} \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1}), \\ q \in C^0([0,T]; H^{s+1}) \end{cases} \tag{4.2}$$

for some $T > 0$ and (u, q) the corresponding solution to the symmetric system (2.1) and (2.4). We take a positive constant $c_{12} = c_{12}(\overline{w})$ so small that $|w - \overline{w}| \le c_{12}$ implies that $w \in O_w$ and assume, throughout this section, that the solution verifies

$$\|w(t) - \overline{w}\|_1 + \|q(t)\|_2 \le c_{12} \qquad \text{for } 0 \le t \le T. \tag{4.3}$$

For non-negative integers l and k and $1 \le p \le \infty$, we put

(4.4)

$$N_{l,k}^{(1)}(t) = \begin{cases} \sup\limits_{0\le\tau\le t}(\|w(\tau)-\overline{w}\|_k + \|q(\tau)\|_{k+1}) + \left(\int_0^t \|q(\tau)\|_{k+1}^2 d\tau\right)^{1/2} & \text{for } l = 0, \\ \sup\limits_{0\le\tau\le t}(1+\tau)^{l/2}(\|\partial_x^l w(\tau)\|_k + \|\partial_x^{l-1} q(\tau)\|_{k+2}) & \\ \qquad + \left(\int_0^t (1+\tau)^l \|\partial_x^l q(\tau)\|_{k+1}^2 d\tau\right)^{1/2} & \text{for } l \ge 1, \end{cases}$$

$$N_{l,k}^{(2)}(t) = \left(\int_0^t (1+\tau)^l \|\partial_x^{l+1} w(\tau)\|_{k-1}^2 d\tau\right)^{1/2} \qquad \text{for } k \ge 1, \tag{4.5}$$

$$N_s(t) = \sum_{l=0}^{s} N_{l,s-l}^{(1)}(t) + \sum_{l=1}^{s-1} N_{l,s-l}^{(2)}(t) \tag{4.6}$$

and

$$M_{l,p}(t) = \sup_{0\le\tau\le t}(1+\tau)^{(1/2)(l+1-1/p)}|\partial_x^l(w(\tau)-\overline{w})|_p. \tag{4.7}$$

Proposition 4.1. *Assume the same conditions in Theorem 2.2. Let $s \ge 2$ be an integer and (w, q) a solution to the initial value problem* (1.1) *and* (1.2) *satisfying* (4.2) *and* (4.3). *Then the solution verifies the estimates*

$$\begin{cases} (N^{(1)}_{0,s}(t) + N^{(2)}_{0,s}(t))^2 \le C\{\|w_0 - \overline{w}\|_s^2 + N^{(1)}_{0,2}(t)(N^{(1)}_{0,s}(t) + N^{(2)}_{0,s}(t))^2\}, \\ N_s(t)^2 \le C\{\|w_0 - \overline{w}\|_s^2 + (M_{1,\infty}(t) + N_2(t))N_s(t)^2\} \end{cases}$$

for $0 \le t \le T$, *where* $C = C(\overline{w}, s) > 0$.

PROOF. To begin with, we derive the L^2 estimate of the solution by making use of an entropy function. Let η be an entropy function for system (1.1) and ζ an entropy flux corresponding to η. Then it holds that (cf. (2.15) in [4])

$$\eta(w)_t + \{\zeta(w,q) + \langle G(w,q), q\rangle\}_x = \left(\frac{\langle q, q_x\rangle}{v(w,q)}\right)_x - \frac{\tilde{Q}}{v(w,q)}, \tag{4.8}$$

where

$$\tilde{Q} = \langle Rq, q\rangle + |q_x|^2 - \frac{v(w,q)_x}{v(w,q)}\langle q, q_x\rangle.$$

Noting that $u = (D_w\eta(w))^T$, we introduce $E[w,\overline{w}]$ and $Z[w,\overline{w},q]$ by

$$\begin{cases} E[w,\overline{w}] = \eta(w) - \eta(\overline{w}) - \langle \bar{u}, w - \overline{w}\rangle, \\ Z[w,\overline{w},q] = \zeta(w,q) - \zeta(\overline{w},0) - \langle F(w,q) - F(\overline{w},0), \bar{u}\rangle + \langle G(w,q), q\rangle. \end{cases}$$

By the convexity of η and (2.3), there exists a constant $C = C(\overline{w}) > 1$ such that

$$\begin{cases} C^{-1}|w - \overline{w}|^2 \le E[w,\overline{w}] \le C|w - \overline{w}|^2, \\ |Z[w,\overline{w},q]| \le C(|w - \overline{w}|^2 + |q|^2) \end{cases}$$

for $|w - \overline{w}| + |q| \le c_{12}$. Moreover, by (4.8) we see that

$$E[w,\overline{w}]_t + Z[w,\overline{w},q]_x + \frac{\tilde{Q}}{v(w,q)} = \left(\frac{\langle q, q_x\rangle}{v(w,q)}\right)_x.$$

Integrate this over $\mathbf{R}^1 \times [0,t]$, we obtain

$$\begin{aligned} &\|w(t) - \overline{w}\|^2 + \int_0^t \|q(\tau)\|_1^2 d\tau \\ &\qquad \le C\{\|w_0 - \overline{w}\|^2 + N^{(1)}_{0,0}(t)(N^{(1)}_{0,0}(t) + N^{(2)}_{0,1}(t))^2\} \end{aligned} \tag{4.9}$$

for $0 \le t \le T$.

We proceed to drive the L^2 estimate for the derivatives of the solution by making use of the symmetric form (2.1). Let l and k be non-negative integers such that $1 \le l + k \le s$. Differentiating (2.1) we obtain

$$
(4.10) \quad \begin{cases} A^0(u)\partial_x^{l+k}u_t + A(u,q)\partial_x^{l+k}u_x + L(u,q)^T\partial_x^{l+k}q_x = f_{l+k}, \\ \dfrac{1}{v(w(u),q)}(-\partial_x^{l+k}q_{xx} + R\partial_x^{l+k}q) + (L(u,q)\partial_x^{l+k}u)_x + J(u,q)\partial_x^{l+k}q_x = g_{l+k}, \end{cases}
$$

where

$$
\begin{cases} f_{l+k} = -A^0(u)\{[\partial_x^{l+k}, A^0(u)^{-1}A(u,q)]u_x + [\partial_x^{l+k}, A^0(u)^{-1}L(u,q)^T]q_x\}, \\ g_{l+k} = [\partial_x, L(u,q)]\partial_x^{l+k}u \\ \qquad - \dfrac{1}{v(w(u),q)}\{[\partial_x^{l+k}, v(w(u),q)L(u,q)]u_x + [\partial_x^{l+k}, v(w(u),q)J(u,q)]q_x\} \end{cases}
$$

and $[\cdot,\cdot]$ denotes the commutator. We take the inner product of the first and the second equations in (4.10) with $(1+t)^l\partial_x^{l+k}u$ and $(1+t)^l\partial_x^{l+k}q$, respectively, and add the resulting two equations to obtain

$$
\begin{aligned}
&\frac{1}{2}\langle(1+t)^l A^0(u)\partial_x^{l+k}u, \partial_x^{l+k}u\rangle_t + \frac{(1+t)^l}{v(w(u),q)}(\langle\partial_x^{l+k}q_x, \partial_x^{l+k}q_x\rangle + \langle R\partial_x^{l+k}q, \partial_x^{l+k}q\rangle) \\
&\quad + (1+t)^l\bigg\{\frac{1}{2}\langle A(u,q)\partial_x^{l+k}u, \partial_x^{l+k}u\rangle + \frac{1}{2}\langle J(u,q)\partial_x^{l+k}q, \partial_x^{l+k}q\rangle \\
&\qquad - \frac{1}{v(w(u),q)}\langle\partial_x^{l+k}q_x, \partial_x^{l+k}q\rangle + \langle L(u,q)\partial_x^{l+k}u, \partial_x^{l+k}q\rangle\bigg\}_x \\
&= \frac{l}{2}(1+t)^{l-1}\langle A^0(u)\partial_x^{l+k}u, \partial_x^{l+k}u\rangle \\
&\quad + (1+t)^l\bigg\{\frac{1}{2}\langle(A^0(u)_t + A(u,q)_x)\partial_x^{l+k}u, \partial_x^{l+k}u\rangle \\
&\qquad + \frac{1}{2}\langle J(u,q)_x\partial_x^{l+k}q, \partial_x^{l+k}q\rangle + \langle f_{l+k}, \partial_x^{l+k}u\rangle + \langle g_{l+k}, \partial_x^{l+k}q\rangle\bigg\}.
\end{aligned}
$$

Integrating this over $\mathbf{R}^1 \times [0,t]$, we get

$$
\begin{aligned}
(4.11) \qquad &(1+t)^l\|\partial_x^{l+k}u(t)\|^2 + \int_0^t (1+\tau)^l\|\partial_x^{l+k}q(\tau)\|_1^2 d\tau \\
&\leq C\bigg\{\|\partial_x^{l+k}u_0\|^2 + l\int_0^t (1+\tau)^{l-1}\|\partial_x^{l+k}u(\tau)\|^2 d\tau \\
&\qquad + \int_0^t (1+\tau)^l(|u_x(\tau)|_\infty + |q_x(\tau)|_\infty) \\
&\qquad \times (\|\partial_x^{l+k}u(\tau)\|^2 + \|\partial_x^{l+k}q(\tau)\|_1^2)d\tau\bigg\}.
\end{aligned}
$$

Note that $\|\partial_x^l(w(t)-\overline{w})\|$ is equivalent to $\|\partial_x^l(u(t)-\bar{u})\|$ for $0 \le l \le s$ under the hypothesis (4.3). In the case $l = 0$, we add (4.11) for $1 \le k \le s$ and (4.9) together to obtain

$$\|w(t)-\overline{w}\|_s^2 + \int_0^t \|q(\tau)\|_{s+1}^2 d\tau \le C\{\|w_0-\overline{w}\|_s + N_{0,2}^{(1)}(t)(N_{0,s}^{(1)}(t)+N_{0,s}^{(2)}(t))^2\} \tag{4.12}$$

for $0 \le t \le T$. In the case $1 \le l \le s$, we add (4.11) for $0 \le k \le s-l$ together to obtain

$$\begin{aligned}(1+t)^l\|\partial_x^l w(t)\|_{s-l}^2 &+ \int_0^t (1+\tau)^l \|\partial_x^l q(\tau)\|_{s-l+1}^2 d\tau \\ &\le C\{\|\partial_x^l w_0\|_{s-l}^2 + (N_{l-1,s-(l-1)}^{(2)}(t))^2 \\ &\quad + (M_{1,\infty}(t)+N_{2,0}^{(2)}(t))(N_{l-1,s-(l-1)}^{(1)}(t)+N_{l-1,s-(l-1)}^{(2)}(t))^2\}\end{aligned} \tag{4.13}$$

for $0 \le t \le T$.

We continue to estimate the solution by making use of the elliptic system in (2.1). Let k be an integer such that $0 \le k \le s-1$. Differentiating the system yields that

$$L\partial_x^{k+1}u = \partial_x^k\left\{\frac{1}{v(w(u),q)}(q_{xx}-Rq) - J(u,q)q_x - (L(u,q)-L(\bar{u},0))u_x\right\} \tag{4.14}$$

and

$$\begin{aligned}-(\partial_x^k q)_{xx} + R\partial_x^k q + vJ(\partial_x^k q)_x \\ = -\partial_x^k\{v(w(u),q)L(u,q)u_x + (v(w(u),q)J(u,q) - v(w(\bar{u}),0)J(\bar{u},0))q_x\},\end{aligned} \tag{4.15}$$

where we used the notation in (2.6). We evaluate the right hand side of (4.14) directly and apply the elliptic estimate in Lemma 3.2 to (4.15) to obtain

$$\begin{aligned}\|L\partial_x^{k+1}u(t)\| \le C\{&\|\partial_x^k q(t)\|_2 + k|q|_\infty\|\partial_x^k(w(t)-\overline{w})\| \\ &+ (|w(t)-\overline{w}|_\infty + |q(t)|_\infty + |q_x(t)|_\infty)(\|\partial_x^{k+1}w(t)\| + \|\partial_x^k q(t)\|_2)\}\end{aligned}$$

and

$$\|\partial_x^k q(t)\|_2 \le C\{\|\partial_x^{k+1}w(t)\| + (|w(t)-\overline{w}|_\infty + |q(t)|_\infty)\|\partial_x^{k+1}q(t)\|\}.$$

Therefore, by Sobolev's inequality $|q|_\infty \le \sqrt{2}\|q\|^{1/2}\|q_x\|^{1/2}$ we obtain

$$
(4.16)\quad \begin{cases} \displaystyle\int_0^t \|Lu_x(\tau)\|_{s-1}^2 d\tau \le C\Big\{\int_0^t \|q(\tau)\|_{s+1}^2 d\tau + N_{0,1}^{(1)}(t)(N_{0,s}^{(1)}(t) + N_{0,s}^{(2)}(t))^2\Big\}, \\ \displaystyle\int_0^t (1+\tau)^l \|L\partial_x^{l+1}u(\tau)\|_{s-l-1} d\tau \le C\Big\{\int_0^t (1+\tau)^l \|\partial_x^l q(\tau)\|_{s-l+1}^2 d\tau \\ \qquad + N_{2,0}^{(1)}(t)(N_{l-1,s-(l-1)}^{(2)}(t))^2 + N_{0,1}^{(1)}(t)(N_{l,s-l}^{(1)}(t) + N_{l,s-l}^{(2)}(t))^2\Big\} \end{cases}
$$

for $0 \le t \le T$ and $1 \le l \le s-1$ and

(4.17)

$$
\begin{cases} \|q(t)\|_{s+1}^2 \le C\{\|w_x(t)\|_{s-1}^2 + N_{0,1}^{(1)}(t)(N_{0,s}^{(1)}(t))^2\}, \\ (1+t)^l \|\partial_x^{l-1} q(t)\|_{s-l+2}^2 \le C\{(1+t)^l \|\partial_x^l w(t)\|_{s-l}^2 + N_{0,1}^{(1)}(t)(N_{l,s-l}^{(1)}(t))^2\} \end{cases}
$$

for $0 \le t \le T$ and $1 \le l \le s$.

Finally, we make use of a compensating matrix K for (2.5) whose existence is guaranteed by the stability condition and Proposition 2.2. Let l and k be non-negative integers such that $0 \le l+k \le s-1$. Differentiating (2.1) we obtain, in place of (4.10), that

$$
(4.18)\qquad A^0 \partial_x^{l+k} u_t + A \partial_x^{l+k} u_x + L^T \partial_x^{l+k} q_x = \partial_x^{l+k} h,
$$

where we used the notation in (2.6) and

$$
h = -A^0(\bar{u})\{(A^0(u)^{-1}A(u,q) - A^0(\bar{u})^{-1}A(\bar{u},0))u_x \\ + (A^0(u)^{-1}L(u,q)^T - A^0(\bar{u})^{-1}L(\bar{u},0)^T)q_x\}.
$$

We multiply (4.18) by K and take the inner product with $\partial_x^{l+k}u_x$ to obtain

$$
(4.19)\qquad \langle KA^0 \partial_x^{l+k} u_t, \partial_x^{l+k} u_x\rangle + \langle KA \partial_x^{l+k} u_x, \partial_x^{l+k} u_x\rangle \\ + \langle KL^T \partial_x^{l+k} q_x, \partial_x^{l+k} u_x\rangle = \langle K\partial_x^{l+k} h, \partial_x^{l+k} u_x\rangle.
$$

We compute each term in this equation as follows. Since KA^0 is real skew-symmetric, the first term can be written as

$$
\langle KA^0 \partial_x^{l+k} u_t, \partial_x^{l+k} u_x\rangle \\ = \Big\{\frac{1}{2}\langle KA^0 \partial_x^{l+k}(u-\bar{u}), \partial_x^{l+k} u_x\rangle\Big\}_t - \Big\{\frac{1}{2}\langle KA^0 \partial_x^{l+k}(u-\bar{u}), \partial_x^{l+k} u_t\rangle\Big\}_x.
$$

Since $[KA]' + L^T L$ is positive definite, the second term is estimated as

$$
\langle KA \partial_x^{l+k} u_x, \partial_x^{l+k} u_x\rangle \ge c|\partial_x^{l+1+k} u|^2 - |L\partial_x^{l+1+k} u|^2,
$$

where $c = c(\overline{w}) > 0$ is the minimum eigenvalue of $[KA]' + L^T L$. For the other terms we have

$$|\langle KL^T \partial_x^{l+k} q_x, \partial_x^{l+k} u_x \rangle| + |\langle K\partial_x^{l+k} h, \partial_x^{l+k} u_x \rangle| \\ \leq \frac{c}{2} |\partial_x^{l+1+k} u|^2 + C(|\partial_x^{l+1+k} q| + |\partial_x^{l+k} h|).$$

Therefore, multiplying (4.19) by $(1+t)^l$ and integrating the resulting equation over $\mathbf{R}^1 \times [0, t]$, we obtain

$$\begin{aligned} &\frac{c}{2}\int_0^t (1+\tau)^l \|\partial_x^{l+1+k} u(\tau)\|^2 d\tau \\ &\quad \leq \frac{1}{2}(KA^0 \partial_x^{l+k}(u_0 - \bar{u}), \partial_x^{l+1+k} u_0) - \frac{1}{2}(1+t)^l (KA^0 \partial_x^{l+k}(u(t) - \bar{u}), \partial_x^{l+1+k} u(t)) \\ &\qquad + \frac{l}{2}\int_0^t (1+\tau)^{l-1} (KA^0 \partial_x^{l+k}(u(\tau) - \bar{u}), \partial_x^{l+1+k} u(\tau)) d\tau \\ &\qquad + \int_0^t (1+\tau)^l \|L\partial_x^{l+1+k} u(\tau)\|^2 d\tau \\ &\qquad + C\int_0^t (1+\tau)^l (\|\partial_x^{l+1+k} q(\tau)\|^2 + \|\partial_x^{l+k} h(\tau)\|^2) d\tau, \end{aligned}$$

where $(\cdot,\cdot)$ denotes the usual L^2 inner product. Adding this for $0 \leq k \leq s - l - 1$ yields that

$$\begin{aligned} (4.20) \qquad &\int_0^t (1+\tau)^l \|\partial_x^{l+1} w(\tau)\|_{s-l-1}^2 d\tau \\ &\quad \leq C\Big\{ \|\partial_x^l (w_0 - \overline{w})\|_{s-l}^2 + \|\partial_x^l (w(t) - \overline{w})\|_{s-l}^2 \\ &\qquad + l\int_0^t (1+\tau)^{l-1} \|\partial_x^l (w(\tau) - \overline{w})\|_{s-l}^2 d\tau \\ &\qquad + N_{0,1}^{(1)}(t)(N_{0,s}^{(1)}(t) + N_{0,s}^{(2)}(t))^2 \\ &\qquad + \int_0^t (1+\tau)^l (\|L\partial_x^{l+1} u(\tau)\|_{s-l-1}^2 + \|\partial_x^{l+1} q(\tau)\|_{s-l-1}^2) d\tau \Big\} \end{aligned}$$

for $0 \leq t \leq T$ and $0 \leq l \leq s-1$. Combining (4.12), (4.13), (4.16), (4.17) and (4.20), we get the desired estimates. □

As a consequence of Proposition 4.1, we obtain a priori estimate which is stated as follows.

Proposition 4.2. *Assume the same conditions in Theorem* 2.2. *Let* $s \geq 2$ *be an integer and* (w, q) *a solution to the initial value problem* (1.1) *and* (1.2) *satisfying* (4.2) *and* (4.3). *There exist positive constants* $c_{13} = c_{13}(\overline{w}, s)$ *and* $C_{10} = C_{10}(\overline{w}, s)$ *which are independent of* T *such that if* $N_{0,2}^{(1)}(T) \leq c_{13}$, *then the solution verifies the estimates*

$$\|w(t) - \overline{w}\|_s^2 + \|q(t)\|_{s+1}^2 + \int_0^t (\|w_x(\tau)\|_{s-1}^2 + \|q(\tau)\|_{s+1}^2)d\tau \leq C_{10}\|w_0 - \overline{w}\|_s^2$$

for $0 \leq t \leq T$.

Theorem 2.1 and Proposition 4.2 prove Theorem 2.2.

5. Estimates of Green's functions

In view of (2.9) and noting that we have already reduced the problem to that in the case $A^0(\bar{u}) = I$, we consider a linear hyperbolic-elliptic coupled system of the form

$$\begin{cases} v_t + Av_x + L^T q_x = h_1, \\ -q_{xx} + Rq + v(Lu_x + Jq_x) = h_2, \end{cases} \tag{5.1}$$

where A, R and J are symmetric matrices of constant entries, R is positive definite and v is a positive constant. Taking the Fourier transform of this system, we obtain

$$\begin{cases} \hat{v}_t + i\xi A\hat{v} + i\xi L^T\hat{q} = \hat{h}_1, \\ (\xi^2 I + R + i\xi vJ)\hat{q} + i\xi vL\hat{v} = \hat{h}_2. \end{cases}$$

It follows from these ordinary differential and algebraic equations that

$$\begin{cases} \hat{v}(\xi, t) = e^{\Phi(\xi)t}\hat{v}(\xi, 0) + \displaystyle\int_0^t e^{\Phi(\xi)(t-\tau)}(\hat{h}_1(\xi, \tau) + i\xi L^T\Psi(\xi)\hat{h}_2(\xi, \tau))d\tau, \\ \hat{q}(\xi, t) = i\xi v\Psi(\xi)L\hat{v}(\xi, t) - \Psi(\xi)\hat{h}_2(\xi, t), \end{cases}$$

where

$$\begin{cases} \Phi(\xi) = -\{i\xi A + v\xi^2 L^T(\xi^2 I + R + i\xi vJ)^{-1}L\}, \\ \Psi(\xi) = -(\xi^2 L + R + i\xi vJ)^{-1}. \end{cases} \tag{5.2}$$

Therefore, introducing Green's functions $G_1(x, t)$ and $G_2(x)$ by

$$G_1(x, t) = \mathscr{F}^{-1}[e^{\Phi(\cdot)t}](x), \qquad G_2(x) = \mathscr{F}^{-1}[\Psi(\cdot)](x), \tag{5.3}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform, we can represent the solutions to (5.1) as

$$\left\{\begin{aligned} &v(t) = G_1(t) * v(0) + \int_0^t G_1(t-\tau) * (h_1(\tau) + L^T G_{2x} * h_2(\tau))d\tau, \\ &q(t) = \nu G_{2x} * (Lv(t)) - G_2 * h_2(t). \end{aligned}\right. \tag{5.4}$$

We remark that $\Phi(\xi)$ is not entire function. This is one of the main difference between the hyperbolic-elliptic and hyperbolic-parabolic coupled systems.

We now approximate the corresponding homogeneous system to (5.1). To this end, we assume the stability condition for the homogeneous system and use the notation in (2.11) and (2.12). As we mentioned in section 2, each matrix $\nu l_i L^T R^{-1} L r_i$ is positive definite for $i = 1, \ldots, \sigma$ so that the matrix B defined by

$$B = \sum_{i=1}^{\sigma} P_i (\nu L^T R^{-1} L) P_i \tag{5.5}$$

is also positive definite because we have the relation

$$B = \tilde{R}\,\mathrm{diag}(\nu l_1 L^T R^{-1} L r_1, \ldots, \nu l_\sigma L^T R^{-1} L r_\sigma)\tilde{L}.$$

The approximate system for v is of the form

$$v_t + Av_x = Bv_{xx}, \tag{5.6}$$

which corresponds to the linearization of (2.15). We denote by $G_1^*(x,t)$ the Green's function for this uniformly parabolic system:

$$G_1^*(x,t) = \mathscr{F}^{-1}[\mathrm{e}^{\Phi^*(\cdot)t}](x), \qquad \Phi^*(\xi) = -i\xi A + (i\xi)^2 B. \tag{5.7}$$

The aim in this section is to give pointwise behavior for these Green's functions G_1, G_2 and G_1^*, which are stated as follows.

PROPOSITION 5.1. *Suppose that A, R and J are symmetric matrices of constant entries, R is positive definite, ν is a positive constant and that the corresponding homogeneous system to* (5.1) *satisfies the stability condition. There exists a positive constant δ_1 such that the Green's function G_1 defined in* (5.3) *has the properties*

$$\begin{aligned} \partial_x^l G_1(x,t) &= \sum_{j=1}^{\sigma'} \sum_{k=0}^{l} \mathrm{e}^{c_{j,1}t} \delta^{(l-k)}(x + c_{j,0}t) Q_{j,k}(t) + \partial_x^l G_1^*(x,t) + R_1^{(l)}(x,t) \\ &= \sum_{j=1}^{\sigma'} \sum_{k=0}^{l} \mathrm{e}^{c_{j,1}t} \delta^{(l-k)}(x + c_{j,0}t) Q_{j,k}(t) + R_0^{(l)}(x,t) \end{aligned} \tag{5.8}$$

for $x \in \mathbf{R}^1$, $t \geq 0$ *and* $l = 0, 1, 2, \dots,$ *where* $c_{j,1}$ *and* $c_{j,0}$ *are constants such that* $c_{j,1} \leq -\delta_1$ *for* $j = 1, \dots, \sigma'$ *and* $\{c_{j,0}; j = 1, \dots, \sigma'\} = \{-\lambda_i; i = 1, \dots, \sigma\}$, λ_i *is an eigenvalue of A,* $Q_{j,k}(t)$ *is an polynomial matrix in t with degree not more than k,* δ *is the Dirac* δ*-function,* G_1^* *is the Green's function defined in* (5.7), *and* $R_1^{(l)}$ *and* $R_0^{(l)}$ *verify the pointwise estimates*

$$(5.9) \qquad \begin{cases} |R_1^{(l)}(x,t)| \leq Ct^{-(1/2)(l+1)}(1+t)^{-1/2} \sum_{i=1}^{\sigma} \mathrm{e}^{-\delta_1 |x - \lambda_i t|/\sqrt{t}} + C\mathrm{e}^{-t-\delta_1 |x|}, \\ |R_0^{(l)}(x,t)| \leq C(1+t)^{-(1/2)(l+1)} \sum_{i=1}^{\sigma} \mathrm{e}^{-\delta_1 |x - \lambda_i t|/\sqrt{t}} + C\mathrm{e}^{-t-\delta_1 |x|} \end{cases}$$

for $x \in \mathbf{R}^1$ *and* $t \geq 0$ *with a positive constant C. Particularly, we have the* L^p *estimates*

$$(5.10) \qquad \begin{cases} |R_1^{(l)}(t)|_p \leq C(1+t)^{-1/2} t^{-(1/2)(l+1-1/p)}, \\ |R_0^{(l)}(t)|_p \leq C(1+t)^{-(1/2)(l+1-1/p)} \end{cases}$$

for $t \geq 0$ *and* $1 \leq p \leq \infty$.

Proposition 5.2. *Suppose that J and R are symmetric matrices of constant entries, R is positive definite and that v is a constant. There exists a positive constant* δ_1 *such that the Green's function* G_2 *defined in* (5.3) *has the properties*

$$(5.11) \qquad \partial_x^l G_2(x) = \sum_{k=0}^{l-2} \delta^{(l-2-k)}(x) Q_k + R_2^{(l)}(x)$$

for $x \in \mathbf{R}^1$ *and* $l = 2, 3, 4, \dots,$ *where* δ *is the Dirac* δ*-function,* Q_k *is an symmetric matrix of constant entries and we have the pointwise estimate*

$$(5.12) \qquad |G_2(x)| + |G_{2x}(x)| + |R_2^{(l)}(x)| \leq C\mathrm{e}^{-\delta_1 |x|}$$

for $x \in \mathbf{R}^1$ *with a positive constant C. Particularly, we have the* L^p *bound*

$$(5.13) \qquad |G_2|_p + |G_{2x}|_p + |R_2^{(l)}|_p \leq C$$

for $1 \leq p \leq \infty$.

We proceed to prove these propositions. Put

$$(5.14) \qquad E(z) = -A + vzL^T(-z^2 I + R + vzJ)^{-1} L.$$

By Lemma 3.1, we see that there exists a positive constant α such that $E(z)$ is holomorphic in $D_\alpha \cup \{\infty\}$ and that $E(z)$ is real symmetric for $z \in D_\alpha \cap \mathbf{R}^1$.

Therefore, by similar arguments to those in [6] (see pages 66–69) there exist a finite number of exceptional points $z_1, z_2, \dots, z_k$ and a positive integer σ' such that for any $z \in D_\alpha \backslash \{z_1, z_2, \dots, z_k\}$, $E(z)$ has distinct eigenvalues $\tilde{\lambda}_1(z), \tilde{\lambda}_2(z), \dots, \tilde{\lambda}_{\sigma'}(z)$. We denote by $\tilde{P}_j(z)$ the eigenprojection for $\tilde{\lambda}_j(z)$, which is given by

$$\tilde{P}_j(z) = -\frac{1}{2\pi i}\int_{\Gamma_j(z)} (E(z) - \zeta I)^{-1} d\zeta,$$

where $\Gamma_j(z)$ is a closed positively-oriented curve enclosing $\tilde{\lambda}_j(z)$ but excluding all other $\tilde{\lambda}_k(z)$. Then it holds that

$$E(z) = \sum_{j=1}^{\sigma'} \tilde{\lambda}_j(z)\tilde{P}_j(z), \qquad \sum_{j=1}^{\sigma'} \tilde{P}_j(z) = I, \qquad \tilde{P}_j(z)\tilde{P}_k(z) = \delta_{jk}\tilde{P}_j(z), \tag{5.15}$$

for $z \in D_\alpha \backslash \{z_1, z_2, \dots, z_k\}$ and $j, k = 1, 2, \dots, \sigma'$, where δ_{jk} is Kronecker's delta. Moreover, $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at $z = 0$ and $z = \infty$.

Since $\Phi(\xi) = i\xi E(i\xi)$, we have

$$\hat{G}_1(\xi, t) = \sum_{j=1}^{\sigma'} \mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi), \tag{5.16}$$

where $\hat{G}_1(\xi, t)$ is the Fourier transform of the Green's function $G_1(x, t)$ with respect to x. Taking the limit of (5.15) as $z \to 0$, we obtain

$$-A = \sum_{j=1}^{\sigma'} \tilde{\lambda}_j(0)\tilde{P}_j(0), \tag{5.17}$$

which implies that $-\tilde{\lambda}_j(0)$ is an eigenvalue of A for $j = 1, \dots, \sigma'$. Considering this fact, we classify $\tilde{\lambda}_j(z)$ as $\tilde{\lambda}_{jk}(z)$ such that

$$\begin{cases} \{\tilde{\lambda}_{jk}(z); k = 1, \dots, n_j, j = 1, \dots, \sigma\} = \{\tilde{\lambda}_j(z); j = 1, \dots, \sigma'\}, \\ \tilde{\lambda}_{jk}(0) = -\lambda_j \qquad \text{for } k = 1, \dots, n_j, \ \ j = 1, \dots, \sigma, \\ n_1 + \cdots + n_\sigma = \sigma'. \end{cases}$$

Let $\tilde{P}_{jk}(z)$ be the eigenprojection for $\tilde{\lambda}_{jk}(z)$. Then we have

$$\sum_{k=1}^{n_j} \tilde{P}_{jk}(0) = P_j \qquad \text{for } j = 1, \dots, \sigma. \tag{5.18}$$

It follows from (5.14) and (5.15) that

$$\left(\sum_{k=1}^{n_j}\tilde{P}_{jk}(z)\right)(-A+\nu zL^T(-z^2I+\nu zJ+R)^{-1}L)\left(\sum_{k=1}^{n_j}\tilde{P}_{jk}(z)\right)$$
$$=\left(\sum_{k=1}^{n_j}\tilde{P}_{jk}(z)\right)\left(\sum_{k=1}^{n_j}\tilde{\lambda}_{jk}(z)\tilde{P}_{jk}(z)\right).$$

Expanding this around $z=0$ and comparing the coefficients for z on both side, we obtain

$$P_j(\nu L^TR^{-1}L)P_j=\sum_{k=1}^{n_j}\tilde{\lambda}'_{jk}(0)\tilde{P}_{jk}(0),$$

where we used the relations $AP_j=P_jA=\lambda_jP_j$ and (5.18). Adding this for $1\le j\le\sigma$ yields that

$$B=\sum_{j=1}^{\sigma'}\tilde{\lambda}'_j(0)\tilde{P}_j(0), \tag{5.19}$$

which implies that $\tilde{\lambda}'_j(0)$ is an eigenvalue of B and that it is positive for $j=1,\dots,\sigma'$. By (5.7), (5.17) and (5.19) we have

$$\hat{G}_1^*(\xi,t)=\sum_{j=1}^{\sigma'}\mathrm{e}^{(i\xi\tilde{\lambda}_j(0)-\xi^2\tilde{\lambda}'_j(0))t}\tilde{P}_j(0). \tag{5.20}$$

Particularly, we obtain the following proposition.

Proposition 5.3. *Assume the same conditions in Proposition* 5.1. *Then the Green's function* G_1^* *defined in* (5.7) *can be written in the form*

$$G_1^*(x,t)=\sum_{j=1}^{\sigma'}\frac{1}{(4\pi\tilde{\lambda}'_j(0)t)^{1/2}}\mathrm{e}^{-(x+\tilde{\lambda}_j(0)t)^2/(4\tilde{\lambda}'_j(0)t)}\tilde{P}_j(0) \tag{5.21}$$

for $x\in\mathbf{R}^1$ *and* $t>0$, *where* $\tilde{\lambda}_j(0)$ *is an eigenvalue of* $-A$ *and* $\tilde{\lambda}'_j(0)$ *is a positive constant for* $j=1,\dots,\sigma'$. *Particularly, we have the pointwise estimate*

$$|\partial_x^lG_1^*(x,t)|\le Ct^{-(1/2)(l+1)}\sum_{i=1}^{\sigma}\mathrm{e}^{-(x-\lambda_it)^2/(\mu t)} \tag{5.22}$$

for $x\in\mathbf{R}^1$, $t>0$ *and* $l=0,1,2,\dots$ *with a positive constant* μ *and the* L^p *estimate*

$$|\partial_x^lG_1^*(t)|_p\le Ct^{-(1/2)(l+1-1/p)} \tag{5.23}$$

for $t>0$ *and* $l=0,1,2,\dots,$ *where* $C=C(l)>0$.

Since $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at $z = 0$, by expanding them around $z = 0$, we obtain the following lemma.

LEMMA 5.1. *Assume the same conditions in Proposition* 5.1. *There exist positive constants* δ_2 *and* C_0 *such that the estimate*

$$|e^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - e^{(i\xi\tilde{\lambda}_j(0)-\xi^2\tilde{\lambda}_j'(0))t}\tilde{P}_j(0)| \le C_0|e^{(i\xi\tilde{\lambda}_j(0)-\xi^2\tilde{\lambda}_j'(0))t}|(|\xi| + |\xi|^3 t e^{C_0|\xi|^3 t})$$

holds for $\xi \in \mathbf{C}^1$, $|\xi| \le 2\delta_2$, $t \ge 0$ *and* $j = 1, \dots, \sigma'$.

Next, we consider the behavior of $\hat{G}_1(\xi, t)$ as $\xi \to \infty$. Since $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at $z = \infty$, we have the expansions

$$\tilde{\lambda}_j(z) = \sum_{k=0}^{\infty} c_{j,k}\frac{1}{z^k}, \qquad \tilde{P}_j(z) = \sum_{k=0}^{\infty} P_{j,k}\frac{1}{z^k} \tag{5.24}$$

as $z \to \infty$, where all the coefficients are real (matrices) and $c_{j,0}$ is an eigenvalue of $-A$ for $j = 1, \dots, \sigma'$ because $E(\infty) = -A$. Noting that $i\xi\tilde{\lambda}_j(i\xi)$ is an eigenvalue of $\Phi(\xi)$ we apply Proposition 2.2 to obtain

$$\mathrm{Re}(i\xi\tilde{\lambda}_j(i\xi)) \le -\delta\frac{\xi^2}{1+\xi^2} \qquad \text{for } \xi \in \mathbf{R}^1,$$

where δ is a positive constant. This and (5.24) imply that $c_{j,1} \le -\delta$ for $j = 1, \dots, \sigma'$. Using (5.24) we can show the following lemma.

LEMMA 5.2. *Assume the same conditions in Proposition* 5.1. *There exist positive constants* N *and* δ_3 *such that the relation*

$$(i\xi)^l e^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) = e^{(i\xi c_{j,0}+c_{j,1})t}\left(\sum_{k=0}^{l+1} Q_{j,k}(t)(i\xi)^{l-k} + R_{j,l}(\xi, t)\right)$$

holds for $l = 0, 1, 2, \dots$ *and* $j = 1, \dots, \sigma'$, *where* $c_{j,0}$, $c_{j,1}$ *and* $Q_{j,k}(t)$ *have the properties stated in Proposition* 5.1 *with* δ_1 *replaced by* δ_3 *and* $R_{j,l}$ *verifies the estimate*

$$|R_{j,l}(\xi, t)| \le C|\xi|^{-2}(1+t)^{l+2}e^{(1/4)\delta_3 t}$$

for $\xi \in \mathbf{C}^1$, $|\xi| \ge N$, $t \ge 0$, $l = 0, 1, 2, \dots$ *and* $j = 1, \dots, \sigma'$, *where* $C = C(l) > 0$.

We proceed to investigate the behavior of $\hat{G}_1(\xi, t)$ for ξ away from 0 and ∞ by making use of a compensating matrix K for the corresponding homogeneous system to (5.1).

LEMMA 5.3. *Assume the same conditions in Proposition* 5.1. *Let* δ_2 *be a positive constant and* $\Phi(\xi)$ *the matrix defined in* (5.2). *There exists a positive constant* δ_4 *such that the estimate*

$$|e^{\Phi(\xi+i\eta)t}| \le Ce^{-\delta_4\xi^2(1+\xi^2)^{-1}t}$$

holds for $\xi, \eta \in \mathbf{R}^1$, $|\xi| \ge \delta_2$, $|\eta| \le 2\delta_4$ *and* $t \ge 0$.

Proof. For $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbf{R}^1$, and $\hat{h} \in \mathbf{C}^m$, we put

$$\begin{cases} \hat{v}(\zeta, t) = e^{\Phi(\zeta)t}\hat{h}, \\ \hat{q}(\zeta, t) = -i\zeta\nu(\zeta^2 I + R + i\zeta\nu J)^{-1}L\hat{v}(\zeta, t). \end{cases}$$

Then $(\hat{v}, \hat{q})$ satisfy the system

$$\begin{cases} \hat{v}_t + i\zeta A\hat{v} + i\zeta L^T\hat{q} = 0, \\ (\zeta^2 I + R + i\zeta\nu J)\hat{q} + i\zeta\nu L\hat{v} = 0. \end{cases} \tag{5.25}$$

We take the inner product of the first and the second equations in (5.25) with $\hat{v}$ and $\frac{1}{\nu}\hat{q}$, respectively, and add the real parts of the resulting two equations to obtain

$$\frac{1}{2}\frac{d}{dt}|\hat{v}|^2 + \eta(2\,\mathrm{Re}(L\hat{v}, \hat{q}) - (A\hat{v}, \hat{v}) - (J\hat{q}, \hat{q})) + \frac{1}{\nu}((\xi^2 - \eta^2)|\hat{q}|^2 + (R\hat{q}, \hat{q})) = 0,$$

where $(\cdot\,,\cdot)$ denotes the inner product in $\mathbf{C}^m$ or $\mathbf{C}^n$. This implies that

$$\frac{d}{dt}|\hat{v}|^2 + c(1 + \xi^2)|\hat{q}|^2 \le C|\eta|(|\hat{v}|^2 + (1 + |\eta|)|\hat{q}|^2), \tag{5.26}$$

where c and C are positive constants. Let K be the compensating matrix. We multiply the first equation in (5.25) by $-i\xi K$, take the inner product with $\hat{v}$ and take the real part of the resulting equation to obtain

$$\begin{aligned} -\frac{\xi}{2}\frac{d}{dt}(iK\hat{v}, \hat{v}) &+ \xi^2\,\mathrm{Re}((KA\hat{v}, \hat{v}) + (KL^T\hat{q}, \hat{v})) \\ &- \eta\xi\,\mathrm{Im}((KA\hat{v}, \hat{v}) + (KL^T\hat{q}, \hat{v})) = 0. \end{aligned}$$

Since $[KA]' + L^TL$ is positive definite, it holds that

$$\mathrm{Re}(KA\hat{v}, \hat{v}) \ge c|\hat{v}|^2 - |L\hat{v}|^2.$$

By the second equation in (5.25), we have

$$|\xi|\,|L\hat{v}| \le C((1 + \eta^2 + \xi^2)|\hat{q}| + |\eta|\,|\hat{v}|).$$

Therefore, we obtain

$$-\xi\frac{d}{dt}(iK\hat{v}, \hat{v}) + c\xi^2|\hat{v}|^2 \le C((1 + \eta^2 + \xi^2)^2|\hat{q}|^2 + \eta^2|\hat{v}|^2). \tag{5.27}$$

Now, we put

$$E_\beta(\zeta, t) = |\hat{v}(\zeta, t)|^2 - \frac{\beta\xi}{1 + \xi^2}(iK\hat{v}(\zeta, t), \hat{v}(\zeta, t)),$$

which is equivalent to $|\hat{v}(\zeta,t)|^2$ for small positive β. We add (5.26) to (5.27) multiplied by $\beta(1+\xi^2)^{-1}$ to obtain

$$\frac{d}{dt}E_\beta(\zeta,t)+\left(c\beta\frac{\xi^2}{1+\xi^2}-C|\eta|\right)|\hat{v}(\zeta,t)|^2$$
$$+(c(1+\xi^2)-C(|\eta|+\beta(1+\xi^2)))|\hat{q}(\zeta,t)|^2\le 0$$

if β and η is small. Therefore, for any $\delta_2>0$ there exists a positive constant δ_4 such that if $|\eta|,|\beta|\le\delta_4$ and $|\xi|\ge\delta_2$, then we have

$$\frac{d}{dt}E_\beta(\zeta,t)+\delta_4\frac{\xi^2}{1+\xi^2}E_\beta(\zeta,t)\le 0$$

for $t\ge 0$, which yields the desired estimate. □

Using the constants $c_{j,0}$ and $c_{j,1}$ and the polynomial matrix $Q_{j,k}(t)$ in Lemma 5.2, we define $R_1^{(l)}(x,t)$ and $R_0^{(l)}(x,t)$ by the relation (5.8). Our task is to show that these functions satisfy the pointwise estimate (5.9). Taking the Fourier transform of (5.8), we obtain

$$\begin{cases}\hat{R}_0^{(l)}(\xi,t)=(i\xi)^l e^{\Phi(\xi)t}-\displaystyle\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}e^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k},\\ \hat{R}_1^{(l)}(\xi,t)=\hat{R}_0^{(l)}(\xi,t)-(i\xi)^l e^{\Phi^*(\xi)t}.\end{cases}\tag{5.28}$$

By Lemma 3.1, there exists a positive constant δ_5 such that $\hat{R}_1^{(l)}(\xi,t)$ and $\hat{R}_0^{(l)}(\xi,t)$ are holomorphic in $\{\xi\in\mathbf{C}^1;\,|\operatorname{Im}\xi|<\delta_5\}$. Let δ_2, C_0, N and δ_4 be the constants in Lemmas 5.1, 5.1, 5.2 and 5.3, respectively, and put

$$\delta_1=\min\left\{\delta_2,\delta_4,\delta_5,\frac{\tilde{\lambda}_1'(0)}{4C_0},\dots,\frac{\tilde{\lambda}_{\sigma'}'(0)}{4C_0}\right\}.$$

Without loss of generality, we can assume that $0<\delta_1<N$. We first evaluate $R_1^{(l)}(x,t)$ in the case $t\ge 1$. To this end, we consider the following three cases according to (x,t).

Case 1. $x-\lambda_i t\ge 0$ for all $i=1,\dots,\sigma$. By (5.16), (5.20), Lemma 5.2 and Cauchy's integral theorem, we see that

$$2\pi R_1^{(l)}(x,t)=\int_{-\infty}^{\infty}\hat{R}_1^{(l)}(\xi,t)e^{ix\xi}\,d\xi=\int_{-\infty+i\delta_1/\sqrt{t}}^{\infty+i\delta_1/\sqrt{t}}\hat{R}_1^{(l)}(\xi,t)e^{ix\xi}\,d\xi\tag{5.29}$$
$$=R_{1,1}^{(l)}(x,t)+R_{1,2}^{(l)}(x,t)+R_{1,3}^{(l)}(x,t)+R_{1,4}^{(l)}(x,t),$$

where

$$R_{1,1}^{(l)}(x,t)=\sum_{j=1}^{\sigma'}\int_{-\delta_1+i\delta_1/\sqrt{t}}^{\delta_1+i\delta_1/\sqrt{t}}(i\xi)^l(\mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi)-\mathrm{e}^{i\xi(\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0))t}\tilde{P}_j(0))\mathrm{e}^{ix\xi}\,d\xi,$$

$$R_{1,2}^{(l)}(x,t)=\sum_{j=1}^{\sigma'}\left(\int_{-\infty+i\delta_1/\sqrt{t}}^{-N+i\delta_1/\sqrt{t}}+\int_{N+i\delta_1/\sqrt{t}}^{\infty+i\delta_1/\sqrt{t}}\right)$$

$$\left((i\xi)^l\mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi)-\sum_{k=0}^{l+1}\mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\right)\mathrm{e}^{ix\xi}\,d\xi,$$

$$R_{1,3}^{(l)}(x,t)=\sum_{j=1}^{\sigma'}\left(\int_{-\infty+i\delta_1/\sqrt{t}}^{-N+i\delta_1/\sqrt{t}}+\int_{N+i\delta_1/\sqrt{t}}^{\infty+i\delta_1/\sqrt{t}}\right)\mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,l+1}(t)(i\xi)^{-1}\mathrm{e}^{ix\xi}\,d\xi$$

and

$$R_{1,4}^{(l)}(x,t)=-\sum_{j=1}^{\sigma'}\left(\int_{-\infty+i\delta_1/\sqrt{t}}^{-\delta_1+i\delta_1/\sqrt{t}}+\int_{\delta_1+i\delta_1/\sqrt{t}}^{\infty+i\delta_1/\sqrt{t}}\right)(i\xi)^l\mathrm{e}^{i\xi(\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0))t}\tilde{P}_j(0)\mathrm{e}^{ix\xi}\,d\xi$$

$$-\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}\int_{-N+i\delta_1/\sqrt{t}}^{N+i\delta_1/\sqrt{t}}\mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\mathrm{e}^{ix\xi}\,d\xi$$

$$+\left(\int_{-N+i\delta_1/\sqrt{t}}^{-\delta_1+i\delta_1/\sqrt{t}}+\int_{\delta_1+i\delta_1/\sqrt{t}}^{N+i\delta_1/\sqrt{t}}\right)(i\xi)^l\mathrm{e}^{\Phi(\xi)t}\mathrm{e}^{ix\xi}\,d\xi.$$

We compute each term in (5.29) as follows. By Lemma 5.1, $R_{1,1}^{(l)}(x,t)$ is estimated as

(5.30) $|R_{1,1}^{(l)}(x,t)|$

$$\le C_0\sum_{j=1}^{\sigma'}\int_{-\delta_1+i\delta_1/\sqrt{t}}^{\delta_1+i\delta_1/\sqrt{t}}|\xi|^l|\mathrm{e}^{i\xi(x+\tilde{\lambda}_j(0)t)-\tilde{\lambda}_j'(0)\xi^2t}|(|\xi|+|\xi|^3t\mathrm{e}^{C_0|\xi|^3t})|d\xi|$$

$$\le C\sum_{j=1}^{\sigma'}\mathrm{e}^{-\delta_1(x+\tilde{\lambda}_j(0)t)/\sqrt{t}}\int_{-\delta_1}^{\delta_1}\mathrm{e}^{-(\tilde{\lambda}_j'(0)-2C_0|\xi|)\xi^2t}(|\xi|+1/\sqrt{t})^{l+1}(1+|\xi|^2t)d\xi$$

$$\le Ct^{-(1/2)(l+1)}\sum_{i=1}^{\sigma}\mathrm{e}^{-\delta_1|x-\lambda_it|/\sqrt{t}}.$$

By Lemma 5.2, $R_{1,2}^{(l)}(x,t)$ is estimated as

$$(5.31)\quad |R_{1,2}^{(l)}(x,t)| \le \sum_{j=1}^{\sigma'}\left(\int_{-\infty+i\delta_1/\sqrt{t}}^{-N+i\delta_1/\sqrt{t}} + \int_{N+i\delta_1/\sqrt{t}}^{\infty+i\delta_1/\sqrt{t}}\right)|\mathrm{e}^{c_{j,1}t+i\xi(x+c_{j,0}t)}R_{j,l}(\xi,t)|\,|d\xi|$$

$$\le C\sum_{j=1}^{\sigma'}\mathrm{e}^{c_{j,1}t-\delta_1(x+c_{j,0}t)/\sqrt{t}}\int_{|\xi|\ge N}|\xi|^{-2}d\xi(1+t)^{l+2}\mathrm{e}^{(1/4)\delta_3 t}$$

$$\le C\mathrm{e}^{-(1/2)\delta_3 t}\sum_{i=1}^{\sigma}\mathrm{e}^{-\delta_1|x-\lambda_i t|/\sqrt{t}}.$$

We can express $R_{1,3}^{(l)}(x,t)$ as

$$(5.32)\quad R_{1,3}^{(l)}(x,t) = \sum_{j=1}^{\sigma'}\mathrm{e}^{c_{j,1}t-\delta_1(x+c_{j,0}t)/\sqrt{t}}Q_{j,l+1}(t)$$

$$\times\left\{\int_{|x+c_{j,0}t|N}^{\infty}\frac{\sin\xi}{\xi}d\xi + \left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)\frac{\delta_1/\sqrt{t}}{i\xi(i\xi-\delta_1/\sqrt{t})}\mathrm{e}^{i(x+c_{j,0}t)\xi}\,d\xi\right\}.$$

Therefore, we get

$$(5.33)\quad |R_{1,3}^{(l)}(x,t)| \le C\mathrm{e}^{-(1/2)\delta_3 t}\sum_{i=1}^{\sigma}\mathrm{e}^{-\delta_1|x-\lambda_i t|/\sqrt{t}}.$$

By Lemma 5.3, $R_{1,4}^{(l)}(x,t)$ is estimated as

$$(5.34)\quad |R_{1,4}^{(l)}(x,t)| \le C\sum_{j=1}^{\sigma'}\mathrm{e}^{-\delta_1(x+\tilde{\lambda}_j(0)t)/\sqrt{t}}\int_{|\xi|\ge\delta_1}(|\xi|+1/\sqrt{t})^l\mathrm{e}^{-(1/2)\tilde{\lambda}_j'(0)\xi^2 t}\,d\xi$$

$$+ C\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}(1+t)^k\mathrm{e}^{c_{j,1}t-\delta_1(x+\tilde{\lambda}_j(0)t)/\sqrt{t}}\int_{|\xi|\le N}(|\xi|+1/\sqrt{t})^{l-k}d\xi$$

$$+ C\mathrm{e}^{-\delta_1 x/\sqrt{t}}\int_{\delta_1\le|\xi|\le N}(|\xi|+1/\sqrt{t})^l\mathrm{e}^{-\delta_4\xi^2(1+\xi^2)^{-1}t}\,d\xi$$

$$\le C\mathrm{e}^{-\delta_6 t}\sum_{i=1}^{\sigma}\mathrm{e}^{-\delta_1|x-\lambda_i t|/\sqrt{t}}$$

with a positive constant δ_6. Collecting the above estimates, we obtain

$$(5.35)\quad |R_1^{(l)}(x,t)| \le Ct^{-(1/2)(l+2)}\sum_{i=1}^{\sigma}\mathrm{e}^{-\delta_1|x-\lambda_i t|/\sqrt{t}}.$$

Case 2. $x-\lambda_i t\le 0$ for all $i=1,\dots,\sigma$. Instead of (5.29), we take the integral pass as

$$2\pi R_1^{(l)}(x,t) = \int_{-\infty - i\delta_1/\sqrt{t}}^{\infty - i\delta_1/\sqrt{t}} \hat{R}_1^{(l)}(\xi,t)\mathrm{e}^{ix\xi}\,d\xi.$$

Similar calculation to that in Case 1 shows that the estimate (5.35) is valid in this case, too.

Case 3. $x - \lambda_{i_1}t \le 0 \le x - \lambda_{i_2}t$ for some i_1 and i_2. By (5.16) and (5.20), we decompose $R_1^{(l)}(x,t)$ as

$$(5.36)\qquad 2\pi R_1^{(l)}(x,t) = \sum_{j=1}^{\sigma'}(R_{1,1,j}^{(l)}(x,t) + R_{1,2,j}^{(l)}(x,t)) + R_{1,3}^{(l)}(x,t),$$

where

$$R_{1,1,j}^{(l)}(x,t) = \int_{-\delta_1}^{\delta_1}(i\xi)^l(\mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - \mathrm{e}^{i\xi(\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0))t}\tilde{P}_j(0))\mathrm{e}^{ix\xi}\,d\xi,$$

$$R_{1,2,j}^{(l)}(x,t) = \left(\int_{-\infty}^{-N} + \int_N^{\infty}\right) \times \left((i\xi)^l\mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - \sum_{k=0}^{l}\mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\right)\mathrm{e}^{ix\xi}\,d\xi$$

and

$$\begin{aligned} R_{1,3}^{(l)}(x,t) = &-\sum_{j=1}^{\sigma'}\left(\int_{-\infty}^{-\delta_1} + \int_{\delta_1}^{\infty}\right)(i\xi)^l\mathrm{e}^{i\xi(\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0))t}\tilde{P}_j(0)\mathrm{e}^{ix\xi}\,d\xi \\ &-\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}\int_{-N}^{N}\mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\mathrm{e}^{ix\xi}\,d\xi \\ &+\left(\int_{-N}^{-\delta_1} + \int_{\delta_1}^{N}\right)(i\xi)^l\mathrm{e}^{\Phi(\xi)t}\mathrm{e}^{ix\xi}\,d\xi. \end{aligned}$$

We compute each term in (5.36) as follows. By Cauchy's integral theorem, we further decompose $R_{1,1,j}^{(l)}(x,t)$ as

$$\begin{aligned} R_{1,1,j}^{(l)}(x,t) = &\left(\int_{-\delta_1 \pm i\delta_1/\sqrt{t}}^{\delta_1 \pm i\delta_1/\sqrt{t}} + \int_{-\delta_1}^{-\delta_1 \pm i\delta_1/\sqrt{t}} + \int_{\delta_1 \pm i\delta_1/\sqrt{t}}^{\delta_1}\right) \\ &(i\xi)^l(\mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - \mathrm{e}^{i\xi(\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0))t}\tilde{P}_j(0))\mathrm{e}^{ix\xi}\,d\xi \\ =: &\, I_{1,j}(x,t) + I_{2,j}(x,t) + I_{3,j}(x,t). \end{aligned}$$

We can evaluate $I_{1,j}(x,t)$ in the same way as in (5.30) to obtain

$$|I_{1,j}(x,t)| \le Ct^{-(1/2)(l+2)} \mathrm{e}^{\mp\delta_1(x+\tilde{\lambda}_j(0)t)/\sqrt{t}}. \tag{5.37}$$

By Lemma 5.1, $I_{2,j}(x,t)$ is estimated as

$$\begin{aligned} |I_{2,j}(x,t)| &\le C_0 \int_{-\delta_1}^{-\delta_1 \pm i\delta_1/\sqrt{t}} |\xi|^l |\mathrm{e}^{i\xi(x+\tilde{\lambda}_j(0)t)-\tilde{\lambda}_j'(0)\xi^2 t}|(|\xi| + |\xi|^3 t\mathrm{e}^{C_0|\xi|^3 t})|d\xi| \\ &\le C(1+t)\mathrm{e}^{-\delta_1^2(\tilde{\lambda}_j'(0)-2C_0\delta_1)t} \int_0^{\pm\delta_1/\sqrt{t}} \mathrm{e}^{-(x+\tilde{\lambda}_j(0)t)\eta+\tilde{\lambda}_j'(0)\eta^2 t}|d\eta| \\ &\le C(1+t)\mathrm{e}^{-\delta_1^2(\tilde{\lambda}_j'(0)-2C_0\delta_1)t}\mathrm{e}^{\delta_1|x+\tilde{\lambda}_j(0)t|/\sqrt{t}}. \end{aligned} \tag{5.38}$$

Here, we have $|x| \le (|\lambda_{i_1}| + |\lambda_{i_2}|)t$ and $2 \le \mathrm{e}^{-\delta_1|x-\lambda_{i_1}t|/\sqrt{t}}\mathrm{e}^{\delta_1\lambda_{i_1}\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_2}t|/\sqrt{t}}\mathrm{e}^{-\delta_1\lambda_{i_2}\sqrt{t}}$. Therefore, we obtain

$$|I_{2,j}(x,t)| \le C\mathrm{e}^{-\delta_7 t}(\mathrm{e}^{-\delta_1|x-\lambda_{i_1}t|/\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_2}t|/\sqrt{t}}) \tag{5.39}$$

with a positive constant δ_7. Similarly, we see that $I_{3,j}(x,t)$ also satisfies the estimate (5.39). Hence, we get

$$|R_{1,1,j}^{(l)}(x,t)| \le Ct^{-(1/2)(l+2)}(\mathrm{e}^{-\delta_1|x+\tilde{\lambda}_j(0)t|/\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_1}t|/\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_2}t|/\sqrt{t}}).$$

By Lemma 5.2 and Cauchy's integral theorem again, we decompose $R_{1,2,j}^{(l)}(x,t)$ as

$$\begin{aligned} R_{1,2,j}^{(l)}(x,t) &= \left(\left(\int_{-\infty \pm i\delta_1/\sqrt{t}}^{-N \pm i\delta_1/\sqrt{t}} + \int_{N \pm i\delta_1/\sqrt{t}}^{\infty \pm i\delta_1/\sqrt{t}}\right) + \left(\int_{-N \pm i\delta_1/\sqrt{t}}^{-N} + \int_N^{N \pm i\delta_1/\sqrt{t}}\right)\right) \\ &\qquad \left((i\xi)^l \mathrm{e}^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - \sum_{k=0}^{l} \mathrm{e}^{(i\xi c_{j,0}+c_{j,1})t} Q_{j,k}(t)(i\xi)^{l-k}\right)\mathrm{e}^{ix\xi}\, d\xi \\ &=: J_{1,j}(x,t) + J_{2,j}(x,t). \end{aligned}$$

Since $J_{1,j}(x,t)$ and $J_{2,j}(x,t)$ are estimated in the same way as in (5.31)–(5.33) and (5.38), respectively, we obtain

$$|R_{1,2,j}^{(l)}(x,t)| \le C\mathrm{e}^{-\delta_8 t}(\mathrm{e}^{-\delta_1|x+c_{j,0}t|/\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_1}t|/\sqrt{t}} + \mathrm{e}^{-\delta_1|x-\lambda_{i_2}t|/\sqrt{t}})$$

with a positive constant δ_8. By Lemma 5.3, $R_{1,3}^{(l)}(x,t)$ is estimated as

$$
\begin{aligned}
|R_{1,3}^{(l)}(x,t)| &\le C\sum_{j=1}^{\sigma'}\int_{|\xi|\ge\delta_1}|\xi|^l e^{-\tilde{\lambda}_j'(0)\xi^2 t}\,d\xi \\
&\quad + C\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}(1+t)^k e^{c_{j,1}t}\int_{|\xi|\le N}|\xi|^{l-k}\,d\xi \\
&\quad + C\int_{\delta_1\le|\xi|\le N}|\xi|^l e^{-\delta_4\xi^2(1+\xi^2)^{-1}t}\,d\xi \\
&\le Ce^{-\delta_9 t}\le Ce^{-\delta_{10}t}(e^{-\delta_1|x-\lambda_{i_1}t|/\sqrt{t}}+e^{-\delta_1|x-\lambda_{i_2}t|/\sqrt{t}}),
\end{aligned}
$$

where δ_9 and δ_{10} are positive constants. By collecting the above estimates, we see that the estimate (5.35) is valid in this case, too. Therefore, (5.35) holds for $x\in\mathbf{R}^1$ and $t\ge 1$.

We proceed to evaluate $R_0^{(l)}(x,t)$ in the case $0<t\le 1$. By (5.16), Lemma 5.2 and Cauchy's integral theorem, we see that

$$
\begin{aligned}
(5.40)\qquad 2\pi R_0^{(l)}(x,t) &= \int_{-\infty}^{\infty}\hat{R}_0^{(l)}(\xi,t)e^{ix\xi}\,d\xi=\int_{-\infty\pm i\delta_1}^{\infty\pm i\delta_1}\hat{R}_0^{(l)}(\xi,t)e^{ix\xi}\,d\xi \\
&= R_{0,1}^{(l)}(x,t)+R_{0,2}^{(l)}(x,t)+R_{0,3}^{(l)}(x,t),
\end{aligned}
$$

where

$$
\begin{aligned}
R_{0,1}^{(l)}(x,t) &= \sum_{j=1}^{\sigma'}\left(\int_{-\infty\pm i\delta_1}^{-N\pm i\delta_1}+\int_{N\pm i\delta_1}^{\infty\pm i\delta_1}\right) \\
&\qquad \left((i\xi)^l e^{i\xi\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi)-\sum_{k=0}^{l+1}e^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\right)e^{ix\xi}\,d\xi, \\
R_{0,2}^{(l)}(x,t) &= \sum_{j=1}^{\sigma'}\left(\int_{-\infty\pm i\delta_1}^{-N\pm i\delta_1}+\int_{N\pm i\delta_1}^{\infty\pm i\delta_1}\right)e^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,l+1}(t)(i\xi)^{-1}e^{ix\xi}\,d\xi
\end{aligned}
$$

and

$$
R_{0,3}^{(l)}(x,t)=\int_{-N\pm i\delta_1}^{N\pm i\delta_1}\left(e^{\Phi(\xi)t}-\sum_{j=1}^{\sigma'}\sum_{k=0}^{l}e^{(i\xi c_{j,0}+c_{j,1})t}Q_{j,k}(t)(i\xi)^{l-k}\right)e^{ix\xi}\,d\xi.
$$

We compute each term in (5.40) as follows. By Lemma 5.2, $R_{0,1}^{(l)}(x,t)$ is estimated as

$$|R_{0,1}^{(l)}(x,t)| \le \sum_{j=1}^{\sigma'} \left(\int_{-\infty \pm i\delta_1}^{-N \pm i\delta_1} + \int_{N \pm i\delta_1}^{\infty \pm i\delta_1} \right) |\mathrm{e}^{c_{j,1}t + i\xi(x + c_{j,1}t)} R_{j,l}(\xi,t)|\,|d\xi|$$

$$\le C \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t \mp \delta_1(x + c_{j,1}t)} \int_{|\xi| \ge N} |\xi|^{-2} d\xi \le C\mathrm{e}^{\mp\delta_1 x}.$$

In view of the identity

$$R_{0,2}^{(l)}(x,t) = \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t \mp \delta_1(x + c_{j,0}t)} Q_{j,l+1}(t) \left\{ \operatorname{sgn}(x + c_{j,0}t) \int_{|x + c_{j,0}t|N}^{\infty} \frac{\sin \xi}{\xi} d\xi \right.$$

$$\left. + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\delta_1}{i\xi(i\xi \mp \delta_1)} \mathrm{e}^{i(x + c_{j,0}t)\xi}\, d\xi \right\},$$

where $\operatorname{sgn} x$ is the sign function of x, we have

$$|R_{0,2}^{(l)}(x,t)| \le C\mathrm{e}^{\mp\delta_1 x}.$$

Lemmas 5.1 and 5.3 imply that $|\mathrm{e}^{\Phi(\xi)t}|$ is bounded in $\{\xi \in \mathbf{C}^1; |\mathrm{Im}\,\xi| \le \delta_1\}$. Therefore, we see that $R_{0,3}^{(l)}(x,t)$ also satisfies the same estimate as above. Hence, the estimate

$$|R_0^{(l)}(x,t)| \le C\mathrm{e}^{-\delta_1|x|} \tag{5.41}$$

holds for $x \in \mathbf{R}^1$ and $0 < t \le 1$.

To summarize, we obtain the estimates

$$\begin{cases} |\partial_x^l G_1^*(x,t)| \le Ct^{-(1/2)(l+1)} \sum_{i=1}^{\sigma} \mathrm{e}^{-\delta_1|x - \lambda_i t|/\sqrt{t}} & \text{for } x \in \mathbf{R}^1,\ t > 0, \\ |R_1^{(l)}(x,t)| \le Ct^{-(1/2)(l+2)} \sum_{i=1}^{\sigma} \mathrm{e}^{-\delta_1|x - \lambda_i t|/\sqrt{t}} & \text{for } x \in \mathbf{R}^1,\ t \ge 1, \\ |R_0^{(l)}(x,t)| \le C\mathrm{e}^{-\delta_1|x|} & \text{for } x \in \mathbf{R}^1, 0 < t \le 1. \end{cases}$$

These together with the relation

$$R_0^{(l)}(x,t) = R_1^{(l)}(x,t) + \partial_x^l G_1^*(x,t)$$

imply the desired estimate (5.9). We complete the proof of Proposition 5.1. □

We proceed to prove Proposition 5.2. Put

$$\tilde{E}(z) = -(-z^2 I + R + vzJ)^{-1}.$$

By Lemma 3.1, there exists a positive constant δ_1 such that $\tilde{E}(z)$ is holomorphic in $D_{2\delta_1} \cap \{\infty\}$. Moreover, we have the expansion

$$\tilde{E}(z) = \sum_{k=0}^{\infty} Q_k \frac{1}{z^{k+2}}$$

as $z \to \infty$, where Q_k is a real symmetric matrix because so is $\tilde{E}(z)$ for $z \in D_{2\delta_1} \cap \mathbf{R}^1$. Therefore, we obtain the following lemma.

Lemma 5.4. *Assume the same conditions in Proposition* 5.2. *There exists a positive constant* N_1 *such that the estimate*

$$\left| (i\xi)^l \boldsymbol{\Psi}(\xi) - \sum_{k=0}^{l-1} (i\xi)^{l-2-k} Q_k \right| \le C|\xi|^{-2}$$

holds for $\xi \in \mathbf{C}$, $|\xi| \ge N_1$ *and* $l = 1, 2, 3, \ldots$ *with a positive constant C, where* $\boldsymbol{\Psi}(\xi)$ *is the matrix defined in* (5.2) *and* Q_k *is a real symmetric matrix for* $k = 0, 1, 2, \ldots$.

Let $l \ge 2$ be an integer and Q_k the matrix in Lemma 5.4, and define $R_2^l(x)$ by the relation (5.11). We show that $R_2^l(x)$ verifies the pointwise estimate (5.12). Taking the Fourier transform of (5.11), we obtain

$$\hat{R}_2^{(l)}(\xi) = (i\xi)^l \boldsymbol{\Psi}(\xi) - \sum_{k=0}^{l-2} (i\xi)^{l-2-k} Q_k.$$

By Lemma 5.4 and Cauchy's integral theorem, we see that

$$2\pi R_2^{(l)}(x) = \int_{-\infty}^{\infty} \hat{R}_2^{(l)}(\xi) e^{ix\xi}\, d\xi = \int_{-\infty \pm i\delta_1}^{\infty \pm i\delta_1} \hat{R}_2^{(l)}(\xi) e^{ix\xi}\, d\xi \tag{5.42}$$
$$= R_{2,1}^{(l)}(x) + R_{2,2}^{(l)}(x) + R_{2,3}^{(l)}(x),$$

where

$$R_{2,1}^{(l)}(x) = \left(\int_{-\infty \pm i\delta_1}^{-N_1 \pm i\delta_1} + \int_{N_1 \pm i\delta_1}^{\infty \pm i\delta_1} \right) \left((i\xi)^l \boldsymbol{\Psi}(\xi) - \sum_{k=0}^{l-1} (i\xi)^{l-2-k} Q_k \right) e^{ix\xi}\, d\xi,$$

$$R_{2,2}^{(l)}(x) = \left(\int_{-\infty \pm i\delta_1}^{-N_1 \pm i\delta_1} + \int_{N_1 \pm i\delta_1}^{\infty \pm i\delta_1} \right) (i\xi)^{-1} e^{ix\xi}\, d\xi Q_{l-1}$$

and

$$R_{2,3}^{(l)}(x) = \int_{-N_1 \pm i\delta_1}^{N_1 \pm i\delta_1} \left((i\xi)^l \boldsymbol{\Psi}(\xi) - \sum_{k=0}^{l-2} (i\xi)^{l-2-k} Q_k \right) e^{ix\xi}\, d\xi.$$

We compute each term in (5.42) as follows. By Lemma 5.4, $R_{2,1}^{(l)}(x)$ is estimated as

$$|R_{2,1}^{(l)}(x)| \le C\left(\int_{-\infty\pm i\delta_1}^{-N_1\pm i\delta_1} + \int_{N_1\pm i\delta_1}^{\infty\pm i\delta_1}\right)|\xi^{-2}\mathrm{e}^{ix\xi}|\,|d\xi|$$

$$\le C\mathrm{e}^{\mp\delta_1 x}\int_{|\xi|\ge N_1}|\xi|^{-2}d\xi \le C\mathrm{e}^{\mp\delta_1 x}.$$

In view of the identity

$$R_{2,2}^{(l)}(x) = \mathrm{e}^{\mp\delta_1 x}\left\{\mathrm{sgn}\, x\int_{|x|N_1}^{\infty}\frac{\sin\xi}{\xi}d\xi + \left(\int_{-\infty}^{-N_1} + \int_{N_1}^{\infty}\right)\frac{\delta_1}{i\xi(i\xi\mp\delta_1)}\mathrm{e}^{ix\xi}\,d\xi\right\},$$

we obtain

$$|R_{2,2}^{(l)}(x)| \le C\mathrm{e}^{\mp\delta_1 x}.$$

Since $\Psi(\xi)$ is bounded in $\{\xi\in\mathbf{C}^1; |\mathrm{Im}\,\xi|\le\delta_1\}$, $R_{2,3}^{(l)}(x)$ also satisfies the above estimate. Therefore, the estimate

$$|R_2^{(l)}(x)| \le C\mathrm{e}^{-\delta_1|x|}$$

holds for $x\in\mathbf{R}^1$. Similarly, we can show that $G_2(x)$ and $G_{2x}(x)$ verify the above estimate. We complete the proof of Proposition 5.2. □

6. Estimates of coupling of diffusion waves

The aim in this section is to show the following propositions. Although the proofs given below are essentially due to [6], our results do not directly follow from theirs.

Proposition 6.1. *Let $l\ge 0$ be an integer, $G_1^*(x,t)$ the Green's function defined in* (5.7)*, $Q(u,v)$ the quadratic form defined in* (2.10) *and $\theta_i(x,t)$ the self-similar solution defined in* (2.18) *for $i=1,\dots,\sigma$. We put*

$$(6.1)\quad I_i^{(l)}(x,t) = \int_0^t\int_{\mathbf{R}^1}\partial_x^{l+1}G_1^*(x-y,t-\tau)(I-P_i)Q(r_i\theta_i(y,\tau), r_i\theta_i(y,\tau))dyd\tau.$$

Here and in what follows, we use the notation in (2.11), (2.12) *and* (2.22)*. There exists a positive constant μ_0 such that $I_i^{(l)}(x,t)$ verifies the pointwise estimate*

$$(6.2)\quad |I_i^{(l)}(x,t)| \le C|\widetilde{\delta}_i|^2(1+t)^{-(1/2)(l+1)-1/4}$$

$$\times\sum_{k=1}^{\sigma}(\mathrm{e}^{-(x-\lambda_k(1+t))^2/(\mu_0(1+t))} + \varphi_{3/2}(x,t;\lambda_k)\mathrm{e}^{-|x-\lambda_k(1+t)|/(1+t)})$$

for $x \in \mathbf{R}^1$, $t > 0$ *and* $i = 1, \dots, \sigma$. *Particularly, we have the* L^p *estimate*

$$|I_i^{(l)}(t)|_p \le C|\widetilde{\delta}_i|^2(1+t)^{-(1/2)(l+1-1/p)-1/4} \tag{6.3}$$

for $t > 0$, $1 \le p \le \infty$ *and* $i = 1, \dots, \sigma$, *where* C *is a positive constant.*

Proposition 6.2. *Using the notation in Proposition* 6.1, *we put*

$$I_{i,j}^{(l)}(x,t) = \int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y, t-\tau) Q(r_i\theta_i(y,\tau), r_j\theta_j(y,\tau))dyd\tau. \tag{6.4}$$

There exists a positive constant μ_0 *such that* $I_{i,j}^{(l)}(x,t)$ *verifies the pointwise estimate*

$$|I_{i,j}^{(l)}(x,t)| \le C|\widetilde{\delta}_i|\,|\tilde{\delta}_j|(1+t)^{-(1/2)(l+2)} \sum_{k=1}^{\sigma} \mathrm{e}^{-(x-\lambda_k(1+t))^2/(\mu_0(1+t))} \tag{6.5}$$

for $x \in \mathbf{R}^1$, $t > 0$, $i, j = 1, \dots, \sigma$ *and* $i \ne j$. *Particularly, we have the* L^p *estimate*

$$|I_{i,j}^{(l)}(t)|_p \le C|\widetilde{\delta}_i|\,|\tilde{\delta}_j|(1+t)^{-(1/2)(l+2-1/p)} \tag{6.6}$$

for $t > 0$, $1 \le p \le \infty$, $i, j = 1, \dots, \sigma$ *and* $i \ne j$, *where* C *is a positive constant.*

In order to prove these propositions, we first prepare fundamental lemmas, which shall be used frequently in the following of this paper without any comments.

Lemma 6.1. *Suppose that* $\alpha_1 < 1$ *and* $\alpha_2 \in \mathbf{R}^1$. *Then we have*

$$\int_0^t \tau^{-\alpha_1}(1+\tau)^{-\alpha_2} d\tau \le Ct^{1-\alpha_1}(1+t)^{-\min(\alpha_2, 1-\alpha_1)}(1+\delta_{\alpha_1+\alpha_2,1}\log(1+t))$$

for $t > 0$, *where* δ *is Kronecker's delta and* $C = C(\alpha_1, \alpha_2) > 0$.

Proof. Since $\alpha_1 < 1$, it holds that

$$\lim_{t \to 0} \frac{1}{t^{1-\alpha_1}} \int_0^t \tau^{-\alpha_1}(1+\tau)^{-\alpha_2} d\tau = \frac{1}{1-\alpha_1}.$$

For $t > 1$, we see that

$$\begin{aligned}\int_0^t \tau^{-\alpha_1}(1+\tau)^{-\alpha_2} d\tau &= C + \int_1^t \tau^{-\alpha_1}(1+\tau)^{-\alpha_2} d\tau \\ &\le C + C(1+t)^{1-\alpha_1-\min(\alpha_2, 1-\alpha_1)}(1+\delta_{\alpha_1+\alpha_2,1}\log(1+t)).\end{aligned}$$

These imply the desired inequality. □

This lemma and standard technique yield the following one.

Lemma 6.2. *Suppose that* $\alpha_1 < 1$, $\beta_1 < 1$ *and* $\alpha_2, \beta_2 \in \mathbf{R}^1$. *Then we have*

$$\int_0^t (t-\tau)^{-\alpha_1}(1+t-\tau)^{-\alpha_2}\tau^{-\beta_1}(1+\tau)^{-\beta_2}d\tau$$
$$\le Ct^{1-(\alpha_1+\beta_1)}\{(1+t)^{-(\alpha_2+\min(\beta_2,1-\beta_1))}(1+\delta_{\beta_1+\beta_2,1}\log(1+t))$$
$$+(1+t)^{-(\beta_2+\min(\alpha_2,1-\alpha_1))}(1+\delta_{\alpha_1+\alpha_2,1}\log(1+t))\}$$

for $t > 0$, *where* δ *is Kronecker's delta and* $C = C(\alpha_1, \alpha_2, \beta_1, \beta_2) > 0$.

Lemma 6.3. *Suppose that* $\alpha_1 > 0$, $\alpha_2 < 1$ *and* $\alpha_3 \in \mathbf{R}^1$. *Then we have*

$$\int_0^t \mathrm{e}^{-\alpha_1(t-\tau)}\tau^{-\alpha_2}(1+\tau)^{-\alpha_3}d\tau \le Ct^{1-\alpha_2}(1+t)^{-(\alpha_3+1)}$$

for $t > 0$, *where* $C = C(\alpha_1, \alpha_2, \alpha_3) > 0$.

We proceed to prove Proposition 6.1. In view of (5.18) and (5.21), we can express $I_i^{(l)}(x,t)$ in the form

$$I_i^{(l)}(x,t) = \sum_{j\ne i}\sum_{k=1}^{n_j}\int_0^t\int_{\mathbf{R}^1}\frac{1}{(4\pi\tilde{\lambda}'_{jk}(0)(t-\tau))^{1/2}}\mathrm{e}^{-(x-y-\lambda_j(t-\tau))^2/(4\tilde{\lambda}'_{jk}(0)(t-\tau))}$$
$$\tilde{P}_{jk}(0)\partial_y^{l+1}Q(r_i\theta_i(y,\tau), r_i\theta_i(y,\tau))dyd\tau. \tag{6.7}$$

Moreover, by (2.18) and Proposition 2.3 we have

$$|\partial_y^{l_1}(\partial_\tau + \lambda_i\partial_y)^{l_2}Q(r_i\theta_i(y,\tau), r_i\theta_i(y,\tau))|$$
$$\le C|\tilde{\delta}_i|^2(1+\tau)^{-(1/2)(l_1+2l_2+2)}\mathrm{e}^{-(y-\lambda_i(1+\tau))^2/(4\mu_i(1+\tau))}$$

for $y \in \mathbf{R}^1$, $\tau \ge 0$ and $l_1, l_2 = 0, 1, 2, \ldots$. Therefore, Proposition 6.1 follows from the following lemma.

Lemma 6.4. *Let* l *be non-negative integer,* μ *and* ε *positive constants, and* λ *and* λ' *real constants such that* $\lambda \ne \lambda'$. *Suppose that a function* $h(y,\tau)$ *satisfies the estimates*

$$\begin{cases} |h(y,\tau)| \le \Lambda(1+\tau)^{-1}\mathrm{e}^{-(y-\lambda'(1+\tau))^2/(4\mu(1+\tau))}, \\ |\partial_y^l h(y,\tau)| \le \Lambda(1+\tau)^{-(1/2)(l+2)}\mathrm{e}^{-(y-\lambda'(1+\tau))^2/(4\mu(1+\tau))}, \\ |(\partial_\tau + \lambda'\partial_y - \mu\partial_y^2)h(y,\tau)| \le \Lambda(1+\tau)^{-2}\mathrm{e}^{-(y-\lambda'(1+\tau))^2/(4\mu(1+\tau))}, \\ |\partial_y^l(\partial_\tau + \lambda'\partial_y - \mu\partial_y^2)h(y,\tau)| \le \Lambda(1+\tau)^{-(1/2)(l+4)}\mathrm{e}^{-(y-\lambda'(1+\tau))^2/(4\mu(1+\tau))} \end{cases}$$

for $y \in \mathbf{R}^1$ *and* $\tau > 0$ *with a positive constant* Λ. *Then we have*

$$\left|\int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-1/2} e^{-(x-y-\lambda(t-\tau))^2/(4\mu(t-\tau))} \partial_y^{l+1} h(y,\tau)\, dy d\tau\right| \tag{6.8}$$
$$\leq C\Lambda(1+t)^{-(1/2)(l+1)-1/4}\big(e^{-(x-\lambda(1+t))^2/((4\mu+\varepsilon)(1+t))}$$
$$+ e^{-(x-\lambda'(1+t))^2/((4\mu+\varepsilon)(1+t))} + \varphi_{3/2}(x,t;\lambda)\ \mathrm{char}\ S_{\lambda,\lambda'}(x,t)\big)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* $\mathrm{char}\, S_{\lambda,\lambda'}$ *is the characteristic function of the set* $S_{\lambda,\lambda'} = \{(x,t); \min(\lambda,\lambda')(1+t) + |\lambda-\lambda'|\sqrt{t} \leq x \leq \max(\lambda,\lambda')(1+t) - |\lambda-\lambda'|\sqrt{t}\}$ *and* $C = C(l,\mu,\varepsilon,\lambda,\lambda') > 0$.

Proof. By changing the variables x, y and μ into $\tilde{x}$, $\tilde{y}$ and $\tilde{\mu}$ by the relation

$$\tilde{x} = \frac{x-\lambda(1+t)}{\lambda'-\lambda}, \qquad \tilde{y} = \frac{y-\lambda(1+\tau)}{\lambda'-\lambda}, \qquad \tilde{\mu} = \frac{\mu}{(\lambda'-\lambda)^2}, \tag{6.9}$$

we can reduce the problem to that in the case $\lambda = 0$ and $\lambda' = 1$. Therefore, we assume that $\lambda = 0$ and $\lambda' = 1$ in the following.

We first consider the case $t \geq 4$. We split the integral with respect to τ in (6.8) over $(0,\sqrt{t})$, $(\sqrt{t}, t-\sqrt{t})$ and $(t-\sqrt{t}, t)$ and write the respective integrals as $I_1(x,t)$, $I_2(x,t)$ and $I_3(x,t)$. By integration by parts with respect to y and the identity

$$\frac{(x-y)^2}{t-\tau} + \frac{(y-(1+\tau))^2}{1+\tau} \tag{6.10}$$
$$= \frac{1+t}{(t-\tau)(1+\tau)}\left(y + \frac{(1+\tau)(x+(t-\tau))}{1+t}\right)^2$$
$$+ \frac{(x-(1+\tau))^2}{1+t},$$

we have

$$|I_1(x,t)| \leq C\Lambda \int_0^{\sqrt{t}} \int_{\mathbf{R}^1} (t-\tau)^{-(1/2)(l+2)}(1+\tau)^{-1}$$
$$\times e^{-(x-y)^2/((4\mu+\varepsilon/2)(t-\tau))} e^{-(y-(1+\tau))^2/(4\mu(1+\tau))}\, dy d\tau$$
$$\leq C\Lambda t^{-(1/2)(l+2)} \int_0^{\sqrt{t}} (1+\tau)^{-1/2} e^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))}\, d\tau.$$

Here, for any $K > 1$ it holds that

$$\sup_{0 \le \tau \le \sqrt{t}} e^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))}$$
$$\le \begin{cases} e^{-x^2/((4\mu+\varepsilon/2)(1+t))} & \text{for } x \le 0, \\ e^{-(1-1/K)^2 x^2/((4\mu+\varepsilon/2)(1+t))} & \text{for } x \ge 2K\sqrt{1+t}, \\ e^{K^2/\mu} e^{-x^2/(4\mu(1+t))} & \text{for } 0 \le x \le 2K\sqrt{1+t}. \end{cases}$$

Therefore, we obtain

$$|I_1(x,t)| \le C\Lambda t^{-(1/2)(l+1)-1/4} e^{-x^2/((4\mu+\varepsilon)(1+t))}$$

for $x \in \mathbf{R}^1$ and $t > 0$. Similarly, we can show that

$$|I_3(x,t)| \le C\Lambda t^{-(1/2)(l+1)-1/4} e^{-(x-(1+t))^2/((4\mu+\varepsilon)(1+t))}$$

for $x \in \mathbf{R}^1$ and $t > 0$. By the identity

$$\partial_y = (\partial_\tau + \partial_y - \mu\partial_y^2) - (\partial_\tau - \mu\partial_y^2)$$

and integration by parts with respect to y and τ, $I_2(x,t)$ is decomposed as

$$\begin{aligned} I_2(x,t) = & -t^{-1/4} \int_{\mathbf{R}^1} e^{-(x-y)^2/4\mu\sqrt{t}} \partial_y^l h(y, t-\sqrt{t}) dy \\ & + (t-\sqrt{t})^{-1/2} \int_{\mathbf{R}^1} ((-\partial_y)^l e^{-(x-y)^2/(4\mu(t-\sqrt{t}))}) h(y,\sqrt{t}) dy \\ & + \int_{\sqrt{t}}^{t/2} \int_{\mathbf{R}^1} (t-\tau)^{-1/2} ((-\partial_y)^l e^{-(x-y)^2/(4\mu(t-\tau))}) \\ & \times (\partial_\tau + \partial_y - \mu\partial_y^2) h(y,\tau) dy d\tau \\ & + \int_{t/2}^{t-\sqrt{t}} \int_{\mathbf{R}^1} (t-\tau)^{-1/2} e^{-(x-y)^2/(4\mu(t-\tau))} \partial_y^l (\partial_\tau + \partial_y - \mu\partial_y^2) h(y,\tau) dy d\tau \\ =: & \, I_4(x,t) + I_5(x,t) + I_6(x,t) + I_7(x,t), \end{aligned}$$

where we used the equality

$$(\partial_\tau + \mu\partial_y^2)((t-\tau)^{-1/2} e^{-(x-y)^2/(4\mu(t-\tau))}) = 0.$$

By the identity (6.10), $I_4(x,t)$ is estimated as

$$|I_4(x,t)| \le C\Lambda(t-\tau)^{-1/2}(1+\tau)^{-1}\int_{\mathbf{R}^1} \mathrm{e}^{-(x-y)^2/(4\mu(t-\tau))}\mathrm{e}^{-(y-(1+\tau))^2/(4\mu(1+\tau))}\,dy\bigg|_{\tau=t-\sqrt{t}}$$
$$\le C\Lambda(1+t)^{-1/2}(1+t-\sqrt{t})^{-(1/2)(l+1)}\mathrm{e}^{-(x-(1+t)+\sqrt{t})^2/(4\mu(1+t))}$$
$$\le C\Lambda t^{-(1/2)(l+2)}\mathrm{e}^{-(x-(1+t))^2/((4\mu+\varepsilon)(1+t))}.$$

Similarly, we have

$$|I_5(x,t)| \le C\Lambda t^{-(1/2)(l+1)-1/4}\mathrm{e}^{-x^2/((4\mu+\varepsilon)(1+t))}.$$

$I_6(x,t)$ and $I_7(x,t)$ are estimated as

$$|I_6(x,t)| + |I_7(x,t)|$$
$$\le C\Lambda\int_{\sqrt{t}}^{t/2}\int_{\mathbf{R}^1}(t-\tau)^{-(1/2)(l+1)}(1+\tau)^{-2}$$
$$\times\, \mathrm{e}^{-(x-y)^2/((4\mu+\varepsilon/2)(t-\tau))}\mathrm{e}^{-(y-(1+\tau))^2/(4\mu(1+\tau))}\,dyd\tau$$
$$+\, C\Lambda\int_{t/2}^{t-\sqrt{t}}\int_{\mathbf{R}^1}(t-\tau)^{-1/2}(1+\tau)^{-(1/2)(l+4)}$$
$$\times\, \mathrm{e}^{-(x-y)^2/(4\mu(t-\tau))}\mathrm{e}^{-(y-(1+\tau))^2/(4\mu(1+\tau))}\,dyd\tau$$
$$\le C\Lambda t^{-(1/2)(l+1)}\int_{\sqrt{t}}^{t-\sqrt{t}}(1+\tau)^{-3/2}\mathrm{e}^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))}\,d\tau$$
$$=: C\Lambda t^{-(1/2)(l+1)}I_8(x,t).$$

We evaluate the last integral $I_8(x,t)$ in the following way.

Case 1. $x \le \sqrt{t}$.

$$I_8(x,t) \le \int_{\sqrt{t}}^{t-\sqrt{t}}(1+\tau)^{-3/2}d\tau \mathrm{e}^{-(x-(1+\sqrt{t}))^2/((4\mu+\varepsilon/2)(1+t))}$$
$$\le Ct^{-1/4}\mathrm{e}^{-x^2/((4\mu+\varepsilon)/(1+t))}.$$

Case 2. $x \ge 1+t-\sqrt{t}$.

$$I_8(x,t) \le \int_{\sqrt{t}}^{t-\sqrt{t}}(1+\tau)^{-3/2}d\tau \mathrm{e}^{-(x-(1+t)+\sqrt{t})^2/((4\mu+\varepsilon/2)(1+t))}$$
$$\le Ct^{-1/4}\mathrm{e}^{-(x-(1+t))^2/((4\mu+\varepsilon)(1+t))}.$$

Case 3. $\sqrt{t} \le x \le 1+t-\sqrt{t}$. For any $K > 1$,

$$I_8(x,t) = \int_{\sqrt{t}}^{t-\sqrt{t}} (1+\tau)^{-3/2} \mathrm{e}^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))} \operatorname{char}\{\tau; K(1+\tau) \le x\} d\tau$$
$$+ \int_{\sqrt{t}}^{t-\sqrt{t}} (1+\tau)^{-3/2} \mathrm{e}^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))} \operatorname{char}\{\tau; K(1+\tau) \ge x\} d\tau$$
$$\le \int_{\sqrt{t}}^{t-\sqrt{t}} (1+\tau)^{-3/2} d\tau \mathrm{e}^{-(1-1/K)^2 x^2/((4\mu+\varepsilon/2)(1+t))}$$
$$+ K^{3/2} x^{-3/2} \int_{\mathbf{R}^1} \mathrm{e}^{-\tau^2/((4\mu+\varepsilon/2)(1+t))}\, d\tau$$
$$\le Ct^{-1/4} \mathrm{e}^{-(1-1/K)^2 x^2/((4\mu+\varepsilon/2)(1+t))} + CK^{3/2}(1+t)^{1/2}(1+t+x^2)^{-3/4},$$

which yields that

$$I_8(x,t) \le Ct^{-1/4}(\mathrm{e}^{-x^2/((4\mu+\varepsilon)(1+t))} + \varphi_{3/2}(x,t;0) \operatorname{char} S_{0,1}(x,t)).$$

Therefore, (6.7) holds for $x \in \mathbf{R}^1$ and $t \ge 4$.

Next, we consider the case $0 < t \le 4$. We denotes the left hand side of (6.8) by $I(x,t)$. By integration by parts with respect to y and (6.10), we have

$$I(x,t) \le C\Lambda \int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-1}(1+\tau)^{-(1/2)(l+2)}$$
$$\times \mathrm{e}^{-(x-y)^2/((4\mu+\varepsilon/2)(t-\tau))} \mathrm{e}^{-(y-(1+\tau))^2/(4\mu(1+\tau))}\, dy d\tau$$
$$\le C\Lambda(1+t)^{-1/2} \int_0^t (t-\tau)^{-1/2}(1+\tau)^{-(1/2)(l+1)} \mathrm{e}^{-(x-(1+\tau))^2/((4\mu+\varepsilon/2)(1+t))}\, d\tau$$
$$\le C\Lambda \mathrm{e}^{-x^2/((4\mu+\varepsilon)(1+t))}.$$

Therefore, we complete the proof of Lemma 6.4. □

By (2.18), Proposition 2.3 and the relation

$$\frac{(y-\lambda_i(1+\tau))^2}{1+\tau} + \frac{(y-\lambda_j(1+\tau))^2}{1+\tau}$$
$$= \frac{2}{1+\tau}\left(y - \frac{\lambda_i+\lambda_j}{2}(1+\tau)\right)^2 + \frac{(\lambda_i-\lambda_j)^2}{2}(1+\tau),$$

we see that if $i \ne j$, then there exists a positive constant α such that

$$|\partial_y^l Q(r_i\theta_i(y,\tau), r_j\theta_j(y,\tau))|$$
$$\le C|\widetilde{\delta_i}|\, |\tilde{\delta}_j| \mathrm{e}^{-\alpha\tau} \mathrm{e}^{-(y-\lambda_i(1+\tau))^2/(8\mu_i(1+\tau))} \mathrm{e}^{-(y-\lambda_j(1+\tau))^2/(8\mu_j(1+\tau))}$$

holds for $y \in \mathbf{R}^1$, $\tau \geq 0$ and $l = 0, 1, 2, \ldots$. Therefore, in order to prove Proposition 6.2, it is sufficient to show the following lemma.

Lemma 6.5. *Let l be non-negative integer, α and μ positive constants and λ and λ' real constants. There exists a positive constant μ_0 such that if a function $h(y, \tau)$ satisfies the estimate*

$$|h(y,\tau)| + |\partial_y^l h(y,\tau)| \leq \Lambda e^{-\alpha\tau} e^{-(y-\lambda'(1+\tau))^2/(\mu(1+\tau))}$$

for $y \in \mathbf{R}^1$ and $\tau > 0$ with a positive constant Λ, then we have

$$\left| \int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-1/2} e^{-(x-y-\lambda(t-\tau))^2/(\mu(t-\tau))} \partial_y^{l+1} h(y,\tau) dy d\tau \right| \leq C\Lambda (1+t)^{-(1/2)(l+2)} \left(e^{-(x-\lambda(1+t))^2/(\mu_0(1+t))} + e^{-(x-\lambda'(1+t))^2/(\mu_0(1+t))} \right) \tag{6.11}$$

for $x \in \mathbf{R}^1$ and $t > 0$, where $C = C(l, \alpha, \mu, \lambda, \lambda') > 0$.

Proof. We give the proof only in the case $\lambda \neq \lambda'$. The proof in the case $\lambda = \lambda'$ is simpler than that of the previous case. Moreover, by changing the variables in accordance with (6.9), we can assume that $\lambda = 0$ and $\lambda' = 1$.

We denotes the left hand side of (6.11) by $I(x,t)$. By integration by parts with respect to y and (6.10), we see that

$$\begin{aligned} I(x,t) &\leq C\Lambda \int_0^{t/2} \int_{\mathbf{R}^1} (t-\tau)^{-(1/2)(l+2)} e^{-\alpha\tau} e^{-(x-y)^2/(2\mu(t-\tau))} e^{-(y-(1+\tau))^2/(\mu(1+\tau))}\, dy d\tau \\ &\quad + C\Lambda \int_{t/2}^{t} \int_{\mathbf{R}^1} (t-\tau)^{-1} e^{-\alpha\tau} e^{-(x-y)^2/(2\mu(t-\tau))} e^{-(y-(1+\tau))^2/(\mu(1+\tau))}\, dy d\tau \\ &\leq C\Lambda t^{-(1/2)(l+1)} \int_0^t e^{-(\alpha/2)\tau} (t-\tau)^{-1/2} e^{-(x-(1+\tau))^2/(2\mu(1+t))}\, d\tau \\ &=: C\Lambda t^{-(1/2)(l+1)} I_1(x,t) \end{aligned}$$

and that

$$\begin{aligned} I(x,t) &\leq C\Lambda \int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-1} e^{-\alpha\tau} e^{-(x-y)^2/(2\mu(t-\tau))} e^{-(y-(1+\tau))^2/(\mu(1+\tau))}\, dy d\tau \\ &\leq C\Lambda I_1(x,t). \end{aligned}$$

We evaluate the integral $I_1(x,t)$ in the following way.

Case 1. $x \leq 0$.

$$I_1(x,t) \le \int_0^t e^{-(\alpha/2)\tau}(t-\tau)^{-1/2} d\tau e^{-x^2/(2\mu(1+t))}$$
$$\le C(1+t)^{-1/2} e^{-x^2/(2\mu(1+t))}.$$

Case 2. $x \ge 1+t$.

$$I_1(x,t) \le \int_0^t e^{-(\alpha/2)\tau}(t-\tau)^{-1/2} d\tau e^{-(x-(1+t))^2/(2\mu(1+t))}$$
$$\le C(1+t)^{-1/2} e^{-(x-(1+t))^2/(2\mu(1+t))}.$$

Case 3. $0 \le x \le 1+t$.

$$I_1(x,t) = \int_0^t e^{-(\alpha/2)\tau}(t-\tau)^{-1/2} e^{-(x-(1+\tau))^2/(2\mu(1+t))} \operatorname{char}\{\tau; 2(1+\tau) \le x\} d\tau$$
$$+ \int_0^t e^{-(\alpha/2)\tau}(t-\tau)^{-1/2} e^{-(x-(1+\tau))^2/(2\mu(1+t))} \operatorname{char}\{\tau; 2(1+\tau) \ge x\} d\tau$$
$$\le \int_0^t e^{-(\alpha/2)\tau}(t-\tau)^{-1/2} d\tau e^{-x^2/(8\mu(1+t))} + e^{-(\alpha/4)(x/2-1)} \int_0^t e^{-(\alpha/4)\tau}(t-\tau)^{-1/2} d\tau$$
$$\le C(1+t)^{-1/2}(e^{-x^2/(8\mu(1+t))} + e^{-\alpha x^2/(8(1+t))}).$$

We now complete the proof of Lemma 6.5. □

7. Decay estimates of energy and L^p norms

We shall show the decay estimates stated in Theorems 2.3, 2.4 and 2.5. Throughout this section, we assume that system (1.1) admits an entropy function and that the symmetric system (2.1) satisfies the stability condition at $(\bar{u}, 0)$. Let $G_1(x,t)$ and $G_1^*(x,t)$ be the corresponding Green's functions defined in (5.3) and (5.7). The following lemmas are simple consequences of Propositions 5.1 and 5.3.

Lemma 7.1. *Let l be a non-negative integer, $1 \le p \le \infty$ and $u \in W^{l,p} \cap L^1$. Then we have the estimate*

$$|\partial_x^l G_1(t) * u|_p \le C(e^{-\delta t}\|u\|_{l,p} + (1+t)^{-(1/2)(l+1-1/p)}|u|_1)$$

for $t > 0$, where δ is a positive constant and $C = C(l,p) > 0$.

Proof. By Proposition 5.1, we can express $\partial_x^l G_1(t) * u$ as

$$\partial_x^l G_1(t) * u = \sum_{k=0}^{l} \sum_{j=1}^{\sigma'} e^{c_{j,1}t} Q_{j,k}(t) \partial_x^{l-k} u(\cdot + c_{j,0}t) + R_0^{(l)}(t) * u.$$

This and Young's inequality imply that

$$
\begin{aligned}
|\partial_x^l G_1(t) * u|_p &\le \sum_{k=0}^{l} \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1} t} |Q_{j,k}(t)|\, |\partial_x^{l-k} u|_p + |R_0^{(l)}(t)|_p |u|_1 \\
&\le C \sum_{k=0}^{l} \mathrm{e}^{-\delta_1 t} (1+t)^k |\partial_x^{l-k} u|_p + C(1+t)^{-(1/2)(l+1-1/p)} |u|_1,
\end{aligned}
$$

which gives the desired estimate. □

Lemma 7.2. *Let l be a non-negative integer, $1 \le p \le \infty$ and $\alpha \le 1/2$. Suppose that a function $H(x,t)$ satisfies the estimates*

$$
\begin{cases}
|\partial_x^k H(\tau)|_p \le \Lambda(t), \qquad k = 0, 1, \ldots, l+1, \\
|\partial_x^l H(\tau)|_p \le \Lambda(t)(1+t)^{-\alpha-(1/2)(l+2-1/p)}, \\
|\partial_x^{l+1} H(\tau)|_p \le \Lambda(t)(1+t)^{-\alpha-(1/2)(l+1-1/p)}, \\
|H(\tau)|_1 \le \Lambda(t)(1+t)^{-\alpha-1/2}
\end{cases}
$$

for $0 < \tau < t$ with a non-negative valued function $\Lambda(t)$. Then we have the estimate

$$
\left| \int_0^t \partial_x^{l+1} G_1(t-\tau) * H(\tau) d\tau \right|_p \le C\Lambda(t)(1+t)^{-\alpha-(1/2)(l+1-1/p)} (1 + \delta_{\alpha,1/2} \log(1+t))
$$

for $t > 0$, where δ is Kronecker's delta and $C = C(l,p,\alpha) > 0$.

Proof. By Proposition 5.1, we have

$$
\begin{aligned}
&\left| \int_0^t \partial_x^{l+1} G_1(t-\tau) * H(\tau) d\tau \right|_p \\
&\quad \le \sum_{k=0}^{l+1} \sum_{j=1}^{\sigma'} \int_0^{t/2} \mathrm{e}^{c_{j,1}(t-\tau)} |Q_{j,k}(t-\tau)|\, |\partial_x^{l+1-k} H(\tau)|_p d\tau \\
&\qquad + \sum_{k=0}^{1} \sum_{j=1}^{\sigma'} \int_{t/2}^{t} \mathrm{e}^{c_{j,1}(t-\tau)} |Q_{j,k}(t-\tau)|\, |\partial_x^{l+1-k} H(\tau)|_p d\tau \\
&\qquad + \int_0^{t/2} |R_0^{(l+1)}(t-\tau)|_p |H(\tau)|_1 d\tau + \int_{t/2}^{t} |R_0^{(1)}(t-\tau)|_1 |\partial_x^l H(\tau)|_p d\tau.
\end{aligned}
$$

This together with (5.10) and Lemmas 6.2 and 6.3 gives the desired estimate. □

Similarly, we can show the following lemmas.

Lemma 7.3. *Let l be a non-negative integer, $1 \le p \le \infty$ and $u \in W^{l,p} \cap L^1$. Then we have the estimate*

$$|\partial_x^l (G_1 - G_1^*)(t) * u|_p \le C(e^{-\delta t}\|u\|_{l,p} + t^{-(1/2)(l+1-1/p)}(1+t)^{-1/2}|u|_1)$$

for $t > 0$, where δ is a positive constant and $C = C(l,p) > 0$.

Lemma 7.4. *Let l be a non-negative integer, $1 \le p \le \infty$ and $\alpha \le 1/2$. Suppose that a function $H(x,t)$ satisfies the estimates*

$$\begin{cases} |\partial_x^k H(\tau)|_p \le \Lambda(t) \qquad k = 0, 1, \dots, l+1, \\ |\partial_x^l H(\tau)|_p + |\partial_x^{l+1} H(\tau)|_p \le \Lambda(t)(1+\tau)^{-\alpha-(1/2)(l+2-1/p)}, \\ |H(\tau)|_1 \le \Lambda(t)(1+\tau)^{-\alpha-1/2} \end{cases}$$

for $0 < \tau < t$ with a non-negative valued function $\Lambda(t)$. Then we have the estimate

$$\left|\int_0^t \partial_x^{l+1}(G_1 - G_1^*)(t-\tau) * H(\tau)d\tau\right|_p$$
$$\le C\Lambda(t)t^{-(1/2)(l+1-1/p)}(1+t)^{-\alpha-1/2}(1+\log(1+t))$$

for $t > 0$, where $C = C(l,p,\alpha) > 0$.

Lemma 7.5. *Let l be a non-negative integer, $1 \le p, r, q \le \infty$, $1/q + 1/r - 1 = 1/p$ and $u \in L^q$. Then we have the estimate*

$$|\partial_x^l G_1^*(t) * u|_p \le Ct^{-(1/2)(l+1-1/r)}|u|_q$$

for $t > 0$, where $C = C(l,p,q,r) > 0$.

Lemma 7.6. *Let l be a non-negative integer, $1 \le p \le \infty$ and $\alpha < 1/2$. Suppose that a function $H(x,t)$ satisfies the estimates*

$$\begin{cases} |H(\tau)|_1 \le \Lambda(t)(1+\tau)^{-\alpha-1/2}, \\ |\partial_x^l H(\tau)|_p \le \Lambda(t)(1+\tau)^{-\alpha-(1/2)(l+2-1/p)} \end{cases}$$

for $0 < \tau < t$ with a non-negative valued function $\Lambda(t)$. Then we have the estimate

$$\left|\int_0^t \partial_x^{l+1} G_1^*(t-\tau) * H(\tau)d\tau\right|_p \le C\Lambda(t)t^{-(1/2)(l+1-1/p)}(1+t)^{-\alpha}$$

for $t > 0$, where $C = C(l,p,\alpha) > 0$.

Let (w,q) be a solution to (1.1) and (1.2) satisfying (4.2) and (4.3), and (v,q) the corresponding solution to (2.9). By (5.4), derivatives of (v,q) can be expressed as

$$(7.1)\qquad \begin{cases} \partial_x^l v(t) = \partial_x^l G_1(t) * v(0) \\ \qquad\qquad + \displaystyle\int_0^t \partial_x^{l+1} G_1(t-\tau) * (H_1(\tau) + L^T G_2 * H_2(\tau)) d\tau, \\ \partial_x^l q(t) = \nu G_2 * (L\partial_x^{l+1} v(t)) - \partial_x^l G_2 * H_2(t). \end{cases}$$

In addition to (4.4)–(4.7), we put

$$M_l(t) = \sum_{k=0}^{l} (M_{k,2}(t) + M_{k,\infty}(t)).$$

Proposition 7.1. *Assume the same conditions in Theorem* 2.2. *Let* $s \ge 3$ *be an integer,* $1 \le p \le \infty$ *and* (w,q) *a solution to the initial value problem* (1.1) *and* (1.2) *satisfying* (4.2) *and* (4.3). *There exist positive constants* $c_{14} = c_{14}(\overline{w}, s)$, $C_{11} = C_{11}(\overline{w}, s)$ *and* $C_{12} = C_{12}(\overline{w}, s, p)$ *such that if* $N_3(T) + M_1(T) \le c_{14}$, *then the solution verifies the estimates*

$$\begin{cases} N_s(T) + M_{s-2}(T) \le C_{11}(\|w_0 - \overline{w}\|_s + |w_0 - \overline{w}|_1), \\ M_{l,p}(T) \le C_{12}(\|w_0 - \overline{w}\|_{l,p} + \|w_0 - \overline{w}\|_s + |w_0 - \overline{w}|_1) \end{cases}$$

for $l = 0, 1, \ldots, s-2$.

Proof. In view of Proposition 4.1 and (2.8), it is sufficient to show that the estimate

$$(7.2)\qquad |\partial_x^l v(t)|_p \le C(1+t)^{-(1/2)(l+1-1/p)} \{\|v(0)\|_{l,p} + |v(0)|_1 + (M_0(t) + N_1(t))(M_l(t) + N_{l+2}(t))\}$$

holds for $0 \le t \le T$ and $l = 0, 1, \ldots, s-2$. To this end, we evaluate the right hand side of the first equation in (7.1). By (2.10), Lemma 4.1, Proposition 5.2 and Sobolev's inequality, we see that

$$(7.3)\qquad \begin{aligned} &|\partial_x^k H_1(\tau)|_p \\ &\le C(|v(\tau)|_{2p} + \|q(\tau)\|_1)(|\partial_x^k v(\tau)|_{2p} + \|\partial_x^k q(\tau)\|_1) \\ &\le C(1+\tau)^{-(1/2)(k+2-1/p)} (M_{0,2p}(\tau) + N_{1,0}^{(1)}(\tau))(M_{k,2p}(\tau) + N_{k+1,0}^{(1)}(\tau)), \end{aligned}$$

$$(7.4)\quad |\partial_x^k G_2 * H_2(\tau)|_p + |\partial_x^{k+1} G_2 * H_2(\tau)|_p \le C|\partial_x^k H_2(\tau)|_p$$
$$\le C(|v(\tau)|_{2p} + \|q(\tau)\|_1)(|\partial_x^{k+1} v(\tau)|_{2p} + \|\partial_x^{k+1} q(\tau)\|_1)$$
$$\le C(1+\tau)^{-(1/2)(k+3-1/p)}(M_{0,2p}(\tau) + N_{1,0}^{(1)}(\tau))(M_{k+1,2p}(\tau) + N_{k+2,0}^{(1)}(\tau))$$

and that

$$(7.5)\quad |\partial_x^{k+1} H_1(\tau)|_p + |\partial_x^{k+1} G_2 * H_2(\tau)|_p$$
$$\le C(|v(\tau)|_{2p} + \|q(\tau)\|_1)(\|\partial_x^{k+1} v(\tau)\|_1 + \|\partial_x^{k+2} q(\tau)\|_2)$$
$$\le C(1+\tau)^{-(1/2)(k+2-1/2p)}(M_{0,2p}(\tau) + N_{1,0}^{(1)}(\tau))N_{k+1,1}^{(1)}(\tau).$$

By these estimates and the interpolation inequality $M_{k,2p}(\tau) \le M_{k,2}(\tau) + M_{k,\infty}(\tau)$, we see that all the hypotheses of Lemma 7.2 are fulfilled with $\alpha = 0$, $H = H_1 + L^T G_2 * H_2$ and $\Lambda(t) = C(M_0(t) + N_1(t))(M_l(t) + N_{l+2}(t))$. Therefore, the desired estimate (7.2) follows from Lemmas 7.1 and 7.2. The proof is complete. □

By this Proposition 7.1, we obtain the decay estimates for w stated in Theorem 2.3.

Next, we investigate decay properties for q. Note that we have the decay estimate for the energy norm of q because of the above argument. In view of Proposition 5.1, we define a function $V_{l+1}(x,t)$ by

$$V_{l+1}(t) = R_0^{(l+1)}(t) * v(0) + \int_0^t \partial_x^{l+2} G_1(t-\tau) * (H_1(\tau) + L^T G_2 * H_2(\tau))d\tau.$$

Then, by the first equation in (7.1), we have

$$\partial_x^{l+1} v(x,t) = \sum_{k=0}^{l} \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t} Q_{j,k}(t) \partial_x^{l+1-k} v(x + c_{j,0}t, 0) + V_{l+1}(x,t).$$

Therefore, by the second equation in (7.1) and Proposition 5.2, q and its derivatives can be expressed as

$$(7.6)\quad q(t) = \nu G_2 * (LV_1(t)) - G_2 * H_2(t)$$
$$+ \nu \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t}(G_2 * (LQ_{j,1}(t)v(\cdot + c_{j,0}t, 0)$$
$$+ G_{2x} * (LQ_{j,0}v(\cdot + c_{j,0}t, 0)))$$

and

$$\begin{aligned}(7.7)\qquad \partial_x^l q(t) &= \nu G_2 * (LV_{l+1}(t)) - \partial_x^l G_2 * H_2(t)\\ &\quad + \nu \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t}(G_2 * (LQ_{j,l+1}(t)v(\cdot + c_{j,0}t, 0))\\ &\qquad\qquad + G_{2x} * (LQ_{j,l}(t)v(\cdot + c_{j,0}t, 0)))\\ &\quad + \nu \sum_{k=0}^{l-1}\sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t}(Q_0 LQ_{j,k}(t)\partial_x^{l-1-k} v(\cdot + c_{j,0}t, 0)\\ &\qquad\qquad + R_2^{(2)} * (LQ_{j,k}(t)\partial_x^{l-1-k} v(\cdot + c_{j,0}t, 0)))\end{aligned}$$

for $l = 1, 2, \ldots.$ Hence, we obtain

$$\begin{cases} |q(t)|_p \le C(\mathrm{e}^{-\delta_1 t}(1+t)|v(0)|_p + |V_1(t)|_p + |G_2 * H_2(t)|_p),\\ |\partial_x^l q(t)|_p \le C(\mathrm{e}^{-\delta_1 t}(1+t)^{l+1}\|v(0)\|_{l,p} + |V_{l+1}(t)|_p + |\partial_x^l G_2 * H_2(t)|_p).\end{cases}$$

Moreover, by the argument used in the proof of Proposition 7.1 and the decay estimates obtained above, we see that

$$|V_{l+1}(t)|_p \le CE_s(1+t)^{-(1/2)(l+2-1/p)}$$

for $1 \le p \le \infty$ and $l = 0, 1, \ldots, s-3$, where $E_s = \|w_0 - \overline{w}\|_s + |w_0 - \overline{w}|_1$. These estimates, (7.4) and the decay estimates obtained above yield the L^p decay estimate for q stated in Theorem 2.3. We complete the proof of Theorem 2.3. □

We proceed to give the L^p decay estimate for the function ω defined by (2.19). We first note that the approximate function $\theta(x,t)$ defined in (2.18) satisfies the parabolic system

$$\theta_t + A\theta_x + \sum_{i=1}^{\sigma} P_i Q(r_i\theta_i, r_i\theta_i)_x = B\theta_{xx},$$

because θ_i is a self-similar solution of (2.15) (see also (2.13) and (5.5)). Therefore, by using the Green's function $G_1^*(x,t)$ defined in (5.7), θ and its derivatives can be expressed as

$$\partial_x^l \theta(t) = \partial_x^l G_1^*(t) * \theta(0) - \sum_{i=1}^{\sigma}\int_0^t \partial_x^{l+1} G_1^*(t-\tau) * P_i Q(r_i\theta(\tau), r_i\theta(\tau))d\tau.$$

By this expression, (7.1) and the relations $\omega(x,t) = v(x,t) - \theta(x,t)$ and $H_1 = H_3 - Q(v,v)$, we obtain

$$\begin{aligned}(7.8)\qquad \partial_x^l\omega(t) &= \partial_x^l G_1^*(t)*\omega(0) + \partial_x^l(G_1 - G_1^*)(t)*v(0)\\ &\quad - \int_0^t \partial_x^{l+1} G_1^*(t-\tau)*\Big(Q(v(\tau),v(\tau)) - \sum_{i=1}^{\sigma} P_i Q(r_i\theta_i(\tau), r_i\theta_i(\tau))\Big)d\tau\\ &\quad + \int_0^t \partial_x^{l+1}(G_1 - G_1^*)(t-\tau)*Q(v(\tau),v(\tau))d\tau\\ &\quad + \int_0^t \partial_x^{l+1} G_1(t-\tau)*(H_3(\tau) + L^T G_2 * H_2(\tau))d\tau\\ &=: I_1(t) + I_2(t) - I_3(t) + I_4(t) + I_5(t).\end{aligned}$$

Note that $\widetilde{\delta_i}$ in (2.16) was chosen so that

$$\int_{\mathbf{R}^1} \omega(x,0)dx = 0.$$

In view of this, we define a function $\tilde{\omega}(x)$ by

$$(7.9)\qquad \tilde{\omega}(x) = \int_{-\infty}^{x} \omega(y,0)dy.$$

Then we have

$$(7.10)\qquad \tilde{\omega}_x(x) = \omega(x,0)$$

and

$$(7.11)\qquad \tilde{\omega}(x) = \int_{-\infty}^{x} v(y,0)dy - \int_{-\infty}^{x} \theta(y,0)dy = \int_{x}^{\infty} \theta(y,0)dy - \int_{x}^{\infty} v(y,0)dy.$$

Now, we put

$$(7.12)\qquad M_{l,p,\gamma}(t) = \sup_{0\le\tau\le t}(1+\tau)^{(1/2)(l+1-1/p)+\gamma}|\partial_x^l\omega(\cdot,\tau)|_p.$$

LEMMA 7.7. *Assume the same conditions in Theorem* 2.4. *Then the function* $\omega(x,t)$ *defined by* (2.19) *verifies the estimates*

$$\begin{aligned}(7.13)\qquad |\partial_x^l\omega(t)|_p &\le Ct^{-(1/2)(l+1-1/p)-1/4}|\tilde{\omega}|_2 + CE_s t^{-(1/2)(l+1-1/p)-1/4}\\ &\quad + Ct^{-(1/2)(l+1-1/p)-\gamma}\\ &\quad \times \{E_3(M_{l,2,\gamma}(t) + M_{l,\infty,\gamma}(t)) + E_s(M_{0,2,\gamma}(t) + M_{0,\infty,\gamma}(t))\}\end{aligned}$$

for $t > 0$, $2 \le p \le \infty$, $\gamma < 1/2$ *and* $l = 0,1,\dots,s-3$, *and*

(7.14) $$|\partial_x^l \omega(t)|_p \le Ct^{-(1/2)(l+1-1/p)-1/2}(|\tilde{\omega}|_1 + \|v(0)\|_{l,p}) + CE_s t^{-(1/2)(l+1-1/p)-1/4} + Ct^{-(1/2)(l+1-1/p)-\gamma} \times \{E_3(M_{l,2,\gamma}(t) + M_{l,\infty,\gamma}(t)) + E_s(M_{0,2,\gamma}(t) + M_{0,\infty,\gamma}(t))\}$$

for $t > 0$, $1 \le p \le \infty$, $\gamma < 1/2$ *and* $l = 0, 1, \dots, s-3$.

Proof. We evaluate each term in the right hand side of (7.8) as follows. By (7.10) and Lemma 7.5, $I_1(x,t)$ is estimated as

$$|I_1(t)|_p = |\partial_x^{l+1} G_1^*(t) * \tilde{\omega}|_p \le Ct^{-(1/2)(l+1-1/p)-1/4}|\tilde{\omega}|_2$$

for $t > 0$ and $2 \le p \le \infty$, and

$$|I_1(t)|_p \le Ct^{-(1/2)(l+1-1/p)-1/2}|\tilde{\omega}|_1$$

for $t > 0$ and $1 \le p \le \infty$. By Lemma 7.3, $I_2(x,t)$ is estimated as

$$|I_2(t)|_p \le t^{-(1/2)(l+2-1/p)}(|v(0)|_1 + \|v(0)\|_{l,p})$$

for $t > 0$ and $1 \le p \le \infty$. Since $Q(v,v)$ has quadratic non-linearity, by Lemma 4.1 and the results of Theorem 2.3, we have

$$|\partial_x^k Q(v(\tau), v(\tau))|_p \le C|v(\tau)|_{2p}|\partial_x^k v(\tau)|_{2p} \le CE_s(1+\tau)^{-(1/2)(k+2-1/p)}$$

for $\tau \ge 0$ and $k = 0, 1, \dots, s-2$. This implies that all the hypotheses of Lemma 7.4 are fulfilled with $\alpha = 0$, $H = Q(v,v)$ and $\Lambda(t) = CE_s$. Therefore, we obtain

$$|I_4(t)|_p \le CE_s t^{-(1/2)(l+2-1/p)}(1 + \log(1+t))$$

for $t > 0$ and $1 \le p \le \infty$. By (2.10), Lemma 4.1 and Sobolev's inequality, we see that

$$\begin{aligned} |\partial_x^k H_3(\tau)|_p &\le C|q(\tau)|_{2p}\|\partial_x^k q(\tau)\|_1 \\ &\quad + C(|v(\tau)|_\infty + |q(\tau)|_\infty)(|v(\tau)|_{2p} + |q(\tau)|_{2p}) \\ &\quad \times (|\partial_x^k v(\tau)|_{2p} + \|\partial_x^k q(\tau)\|_1) \\ &\le CE_s(1+\tau)^{-(1/2)(k+3-1/p)} \end{aligned}$$

for $\tau > 0$, $1 \le p \le \infty$ and $k = 0, 1, \dots, s-2$. Moreover, by (7.4) and the results of Theorem 2.3, we have

$$|\partial_x^k G_2 * H_2(\tau)|_p + |\partial_x^{k+1} G_2 * H_2(\tau)|_p \le CE_s(1+\tau)^{-(1/2)(k+3-1/p)}$$

for $\tau > 0$, $1 \le p \le \infty$ and $k = 0, 1, \dots, s-3$. These estimates imply that all the hypotheses of Lemma 7.2 are fulfilled with $\alpha = 1/2$, $H = H_3 + L^T G_2 * H_2$ and $\Lambda(t) = CE_s$. Therefore, we obtain

$$|I_5(t)|_p \le CE_s t^{-(1/2)(l+2-1/p)}(1 + \log(1+t))$$

for $t > 0$ and $1 \le p \le \infty$. It remains to estimate $I_3(x,t)$. We decompose it as

$$\begin{aligned} (7.15) \qquad I_3(t) &= \sum_{i=1}^{\sigma} \int_0^t \partial_x^{l+1} G_1^*(t-\tau) * ((I - P_i) Q(r_i \theta_i(\tau), r_i \theta_i(\tau))) d\tau \\ &\quad + \sum_{i \ne j} \int_0^t \partial_x^{l+1} G_1^*(t-\tau) * Q(r_i \theta_i(\tau), r_j \theta_j(\tau)) d\tau \\ &\quad + \int_0^t \partial_x^{l+1} G_1^*(t-\tau) * Q(\theta(\tau) + v(\tau), \omega(\tau)) d\tau \\ &=: I_{3,1}(t) + I_{3,2}(t) + I_{3,3}(t). \end{aligned}$$

By Propositions 6.1 and 6.2, we have

$$\begin{cases} |I_{3,1}(t)|_p \le CE_3(1+t)^{-(1/2)(l+1-1/p)-1/4}, \\ |I_{3,2}(t)|_p \le CE_3(1+t)^{-(1/2)(l+2-1/p)} \end{cases}$$

for $t > 0$ and $1 \le p \le \infty$. By Lemma 4.1, (7.12), the results of Theorem 2.3 and the estimate

$$(7.16) \qquad |\partial_x^l \theta(t)|_p \le CE_3(1+t)^{-(1/2)(l+1-1/p)}$$

for $t \ge 0$, $1 \le p \le \infty$ and $l = 0, 1, 2, \dots$, which comes from (2.18) and Proposition 2.3, we see that

$$\begin{aligned} &|\partial_x^k Q(\theta(\tau) + v(\tau), \omega(\tau))|_p \\ &\quad \le C\{(|\theta(\tau)|_{2p} + |v(\tau)|_{2p}) |\partial_x^k \omega(\tau)|_{2p} + (|\partial_x^k \theta(\tau)|_{2p} + |\partial_x^k v(\tau)|_{2p}) |\omega(\tau)|_{2p}\} \\ &\quad \le C(1+\tau)^{-(1/2)(k+2-1/p)-\gamma}(E_3 M_{k,2p,\gamma}(\tau) + E_s M_{0,2p,\gamma}(\tau)) \end{aligned}$$

for $\tau \ge 0$, $1 \le p \le \infty$ and $k = 0, 1, \dots, s-2$. Therefore, by Lemma 7.6 we obtain

$$|I_{3,3}(t)|_p \le Ct^{-(1/2)(l+1-1/p)-\gamma}(E_3 M_{l,2p,\gamma}(\tau) + E_s M_{0,2p,\gamma}(\tau))$$

for $t > 0$ and $1 \le p \le \infty$. Collecting the above estimates and using the interpolation inequality $M_{k,2p,\gamma}(t) \le M_{k,2,\gamma}(t) + M_{k,\infty,\gamma}(t)$, we obtain the desired estimates (7.13) and (7.14). We complete the proof of Lemma 7.7. □

Here, by Proposition 2.3 it holds that

$$(7.17)\qquad \begin{cases} \left|\int_{-\infty}^{x} \theta(y,0)dy\right| \le CE_3 e^{-x^2/(4\mu)} & \text{for } x \le 0, \\ \left|\int_{x}^{\infty} \theta(y,0)dy\right| \le CE_3 e^{-x^2/(4\mu)} & \text{for } x \ge 0, \end{cases}$$

where μ is a positive constant. By (7.11), (7.17) and (2.8), the L^p norm of $\tilde{\omega}$ is estimated as

$$(7.18)\qquad |\tilde{\omega}|_p \le C(|W_0|_p + E_3)$$

for $1 \le p \le \infty$, where W_0 is the function defined by (2.20). Moreover, Theorem 2.3 and (7.16) imply that the estimate

$$(7.19)\qquad \begin{aligned} |\partial_x^l \omega(t)|_p &\le |\partial_x^l v(t)|_p + |\partial_x^l \theta(t)|_p \\ &\le C(E_s + \|w_0 - \overline{w}\|_{l,p})(1+t)^{-(1/2)(l+1-1/p)} \end{aligned}$$

holds for $t \ge 0$, $1 \le p \le \infty$ and $l = 0, 1, \dots, s-2$. Using this, the estimates in Lemma 7.7 with $\gamma = 1/4$ and (7.18), we obtain the decay estimates for ω stated in Theorem 2.4, provided that E_3 is suitable small. This complete the proof of Theorem 2.4. □

Finally, we also assume the condition (2.21). From (2.21), (5.15), (5.18) and the relation $P_i r_i = r_i$ for $i = 1, \dots, \sigma$, it follows that

$$\tilde{P}_{jk}(0)Q(r_i u_i, r_i u_i) = 0 \qquad \text{for } u_i \in \mathbf{R}^{m_i},$$

where $k = 1, \dots, n_j$, $i, j = 1, \dots, \sigma$ and $i \ne j$. Therefore, by (6.1) and (6.7) we see that the integral $I_{3,1}(x,t)$ is identically zero. Hence, in place of (7.13) and (7.14) we obtain the estimate

$$\begin{aligned} |\partial_x^l \omega(t)|_p &\le Ct^{-(1/2)(l+1-1/p)-1/2}(|\tilde{\omega}|_1 + \|v(0)\|_{l,p}) \\ &\quad + CE_s t^{-(1/2)(l+1-1/p)-1/2}(1 + \log(1+t)) + Ct^{-(1/2)(l+1-1/p)-\gamma} \\ &\quad \times \{E_3(M_{l,2,\gamma}(t) + M_{l,\infty,\gamma}(t)) + E_s(M_{0,2,\gamma}(t) + M_{0,\infty,\gamma}(t))\} \end{aligned}$$

for $t > 0$, $1 \le p \le \infty$, $\gamma < 1/2$ and $l = 0, 1, \dots, s-3$, which together with (7.18) and (7.19) yields the decay estimate stated in Theorem 2.5, provided that E_3 is suitable small. This complete the proof of Theorem 2.5. □

8. Space-time decay estimates

In this section we shall show the pointwise decay estimates stated in Theorems 2.6, 2.7 and 2.8. To this end, we first establish a series of lemmas. In addition to (2.22), we use the function

$$\tilde{\varphi}_\alpha(x,t,\tau;\lambda) = \left(1 + \frac{(x-\lambda(1+\tau))^2}{t-\tau}\right)^{-\alpha/2}. \tag{8.1}$$

LEMMA 8.1. *Suppose that* $\beta \ge 1$, $\delta > 0$, $1 \le p \le \infty$, $\lambda \in \mathbf{R}^1$ *and* $u \in L^1 \cap W_\beta^{0,\infty}$. *Then we have*

$$\begin{aligned} \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y-\lambda t|/\sqrt{1+t}} |u(y)|dy &\le C|u|_p (1+t)^{(1/2)(1-1/p)} \mathrm{e}^{-\delta|x-\lambda(1+t)|/4\sqrt{1+t}} \\ &\quad + C|u|_{(\beta)}(1+t)^{-(1/2)(\beta-1)} \varphi_\beta(x,t;\lambda) \end{aligned} \tag{8.2}$$

for $x \in \mathbf{R}^1$ *and* $t \ge 0$. *Particularly, we have*

$$\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y-\lambda t|/\sqrt{1+t}} |u(y)|dy \le C(|u|_1 + |u|_{(\beta)})\varphi_\beta(x,t;\lambda) \tag{8.3}$$

for $x \in \mathbf{R}^1$ *and* $t \ge 0$, *where* C *is a positive constant.*

PROOF. We denote by $I(x,t)$ the left hand side of the inequality (8.2). By splitting the integral over $S_1 = \{y \in \mathbf{R}^1; 2|y| < \sqrt{1+t}\varphi_1(x,t;\lambda)\}$ and $S_2 = \{y \in \mathbf{R}^1; 2|y| > \sqrt{1+t}\varphi_1(x,t;\lambda)\}$, we see that

$$\begin{aligned} I(x,t) &\le \sup_{y \in S_1} \mathrm{e}^{-\delta|x-y-\lambda t|/2\sqrt{1+t}} \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y-\lambda t|/2\sqrt{1+t}} |u(y)|dy \\ &\quad + \sup_{y\in S_2} |u(y)| \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|y|/\sqrt{1+t}}\, dy \\ &\le C\Big\{ |u|_p |\mathrm{e}^{-\delta|\cdot|/2\sqrt{1+t}}|_{p/(p-1)} \mathrm{e}^{-\delta|x-\lambda(1+t)|/4\sqrt{1+t}} \\ &\quad + |u|_{(\beta)}(1+t)^{1/2} \sup_{y\in S_2}(1+y^2)^{-\beta/2} \Big\}. \end{aligned}$$

This yields the desired inequality (8.2). The proof is complete. □

LEMMA 8.2. *Suppose that* $\beta_1, \beta_2, \beta_3 \ge 0$, $\delta > 0$, $1 \le p, q \le \infty$, $1/p + 1/q = 1$ *and* $\beta_1 q > 1$. *Then we have*

$$\begin{aligned} &\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;0)\psi_{\beta_2}(y,\tau;0)dy \\ &\quad \le C\{(t-\tau)^{1/(2p)}(1+\tau)^{1/(2q)}\tilde{\varphi}_{\beta_3}(x,t,\tau;0) + (t-\tau)^{1/2}\varphi_{\beta_1}(x,\tau;0)\psi_{\beta_2}(x,\tau;0)\} \end{aligned}$$

for $x \in \mathbf{R}^1$ *and* $0 \le \tau < t$, *where* C *is a positive constant.*

PROOF. We denote by $I(x,t)$ the left hand side of the above inequality. By splitting the integral over $\{y \in \mathbf{R}^1; 2|y| < |x|\}$ and $\{y \in \mathbf{R}^1; 2|y| > |x|\}$, we see that

$$\begin{aligned}
I(x,t) &\le e^{-\delta|x|/4\sqrt{t-\tau}} \int_{|y|<|x|/2} e^{-\delta|x-y|/2\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;0)dy \\
&\quad + C\int_{|y|>|x|/2} e^{-\delta|x-y|/\sqrt{t-\tau}}\, dy \varphi_{\beta_1}(x,\tau;0)\psi_{\beta_2}(x,\tau;0) \\
&\le e^{-\delta|x|/4\sqrt{t-\tau}} |e^{-\delta|\cdot|/2\sqrt{t-\tau}}|_p |\varphi_{\beta_1}(\cdot,\tau;0)|_q \\
&\quad + C|e^{-\delta|\cdot|/\sqrt{t-\tau}}|_1 \varphi_{\beta_1}(x,\tau;0)\psi_{\beta_2}(x,\tau;0).
\end{aligned}$$

This yields the desired inequality. The proof is complete. □

Similarly, we can show the following lemma.

LEMMA 8.3. *Suppose that $\beta_1, \beta_2 \ge 0$ and $\delta > 0$. Then we have*

$$\int_{\mathbf{R}^1} e^{-\delta|x-y|} \varphi_{\beta_1}(y,\tau;0)\psi_{\beta_2}(y,\tau;0)dy \le C\varphi_{\beta_1}(x,\tau;0)\psi_{\beta_2}(x,\tau;0)$$

for $x \in \mathbf{R}^1$ and $\tau \ge 0$, where C is a positive constant.

LEMMA 8.4. *Suppose that $\beta_1, \beta_2, \beta_3 \ge 0$, $\alpha_1 - \beta_1/2 < 1$, $\alpha_2 - \beta_2/2 < 1$ and $c > 0$. Then we have*

$$\begin{aligned}
&\int_0^t (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2} \tilde{\varphi}_{\beta_1}(x,t,\tau;0)\varphi_{\beta_2}(x,\tau;0)\psi_{\beta_3}(x,\tau;0)d\tau \\
&\qquad \le C(1+t)^{-(\alpha_1+\alpha_2-1)} \varphi_{\beta_1+\beta_2}(x,t;0)\psi_{\beta_3}(x,t;0)
\end{aligned}$$

for $x \in \mathbf{R}^1$ and $t > 0$ satisfying $|x| \ge c\sqrt{1+t}$, where C is a positive constant.

PROOF. We denote by $I(x,t)$ the left hand side of the above inequality. In view of $\psi_{\beta_3}(x,\tau;0) \le \psi_{\beta_3}(x,t;0)$ for $0 \le \tau \le t$, we see that

$$\begin{aligned}
I(x,t) &\le \int_0^t (t-\tau)^{-(\alpha_1-\beta_1/2)}(1+\tau)^{-(\alpha_2-\beta_2/2)} d\tau |x|^{-(\beta_1+\beta_2)} \psi_{\beta_3}(x,t;0) \\
&\le C(1+t)^{-(\alpha_1+\alpha_2-1)+(\beta_1+\beta_2)/2}(1+t+x^2)^{-(\beta_1+\beta_2)/2}\psi_{\beta_3}(x,t;0) \\
&= C(1+t)^{-(\alpha_1+\alpha_2-1)}\varphi_{\beta_1+\beta_2}(x,t;0)\psi_{\beta_3}(x,t;0).
\end{aligned}$$

This completes the proof. □

LEMMA 8.5. *Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \ge 0$, $\alpha_1 + \alpha_2 \ge 1/2$, $\alpha_1 - \beta_1/2 < 1$, $\alpha_2 - \beta_2/2 < 1$, $r \le \min(\alpha_1, \alpha_2, \alpha_1 + \alpha_2 - 1/2)$ and $c > 0$. Then we have*

(8.4)
$$\int_0^t (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2}\tilde{\varphi}_{\beta_1}(x,t,\tau;1)\varphi_{\beta_2}(x,\tau;1)\psi_{\beta_3}(x,\tau;1)d\tau$$
$$\leq C\sum_{\lambda=0,1}\{(1+t)^{-(\alpha_1+\alpha_2-1)}\varphi_{\beta_1+\beta_2}(x,t;\lambda)\psi_{\beta_3}(x,t;\lambda)$$
$$+(1+t)^{-(1/2)(\alpha_1+\alpha_2+r-1/2)}\varphi_{\alpha_1+\alpha_2-r-1/2}(x,t;\lambda)\psi_{\beta_4}(x,t;\lambda)\}$$

for $x\in\mathbf{R}^1$ *and* $t>0$ *satisfying* $|x|\geq c\sqrt{1+t}$ *and* $|x-(1+t)|\geq c\sqrt{1+t}$, *where* C *is a positive constant.*

If, in addition, $\alpha_1,\alpha_2\leq 3/2$, *then we have*

(8.5)
$$\int_0^t (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2}\tilde{\varphi}_{\beta_1}(x,t,\tau;1)\varphi_{\beta_2}(x,\tau;1)\psi_{\beta_3}(x,\tau;1)d\tau$$
$$\leq C(1+t)^{-(\alpha_1+\alpha_2-1)}$$
$$\times\sum_{\lambda=0,1}\{\varphi_{\beta_1+\beta_2}(x,t;\lambda)\psi_{\beta_3}(x,t;\lambda)+\varphi_1(x,t;\lambda)\psi_{\beta_4}(x,t;\lambda)\}$$

for $x\in\mathbf{R}^1$ *and* $t>0$ *satisfying* $|x|\geq c\sqrt{1+t}$ *and* $|x-(1+t)|\geq c\sqrt{1+t}$.

PROOF. We denote by $I(x,t)$ the left hand side of the inequalities (8.4). Then we see that

(8.6)
$$I(x,t)\leq C\int_0^t (t-\tau)^{-(\alpha_1-\beta_1/2)}(1+\tau)^{-(\alpha_2-\beta_2/2)}$$
$$\times(1+t-\tau+(x-(1+\tau))^2)^{-\beta_1/2}$$
$$\times(1+\tau+(x-(1+\tau))^2)^{-\beta_2/2}\psi_{\beta_3}(x,\tau;1)d\tau.$$

We evaluate this integral by considering the following three cases.

Case 1. $x\leq -c\sqrt{1+t}$. Since $|x-(1+\tau)|\geq|x|\geq c\sqrt{1+t}$, we see that

$$I(x,t)\leq C\int_0^t (t-\tau)^{-(\alpha_1-\beta_1/2)}(1+\tau)^{-(\alpha_2-\beta_2/2)}d\tau|x|^{\beta_1+\beta_2}\psi_{\beta_3}(x,t;0)$$
$$\leq C(1+t)^{-(\alpha_1+\alpha_2-1)}\varphi_{\beta_1+\beta_2}(x,t;0)\psi_{\beta_3}(x,t;0).$$

Case 2. $x\geq 1+t+c\sqrt{1+t}$. Since $|x-(1+\tau)|\geq|x-(1+t)|\geq c\sqrt{1+t}$, we see that

$$I(x,t)\leq C\int_0^t (t-\tau)^{-(\alpha_1-\beta_1/2)}(1+\tau)^{-(\alpha_2-\beta_2/2)}d\tau|x-(1+t)|^{\beta_1+\beta_2}\psi_{\beta_3}(x,t;1)$$
$$\leq C(1+t)^{-(\alpha_1+\alpha_2-1)}\varphi_{\beta_1+\beta_2}(x,t;1)\psi_{\beta_3}(x,t;1).$$

Case 3. $c\sqrt{1+t} \le x \le 1+t-c\sqrt{1+t}$. We split the integral in (8.4) over $\{\tau \in (0,t); 2|x-(1+\tau)| > \min(x, 1+t-x)\}$, $\{\tau \in (0,t/2); 2|x-(1+\tau)| < \min(x, 1+t-x)\}$ and $\{\tau \in (t/2,t); 2|x-(1+\tau)| < \min(x, 1+t-x)\}$, and write the respective integrals as $I_1(x,t)$, $I_2(x,t)$ and $I_3(x,t)$. By the same way as in Cases 1 and 2, $I_1(x,t)$ is estimated as

$$I_1(x,t) \le C(1+t)^{-(\alpha_1+\alpha_2-1)} \sum_{\lambda=0,1} \varphi_{\beta_1+\beta_2}(x,t;\lambda)\psi_{\beta_3}(x,t;\lambda).$$

Note that if $2|x-(1+\tau)| \le \min(x, 1+t-x)$, then we have $\frac{1}{2}x \le 1+\tau \le \frac{3}{2}x$ and $\frac{1}{2}(1+t-x) \le t-\tau \le \frac{3}{2}(1+t-x)$. Therefore, $I_2(x,t)$ is estimated as

$$\begin{aligned} I_2(x,t) &\le C(1+t)^{-r}(1+t-x)^{-(\alpha_1-\beta_1/2-r)}x^{-(\alpha_2-\beta_2/2)} \\ &\quad \times \int_0^{(1/2)\min(x,1+t-x)} (1+t-x+y^2)^{-\beta_1/2}(x+y^2)^{-\beta_2/2}dy. \end{aligned}$$

Since $2(\alpha_1-r) \ge 0$, $2\alpha_2 \ge 0$ and $2(\alpha_1-r)+2\alpha_2 \ge 1$, there exist $p,q \in [1,\infty]$ such that $2(\alpha_1-r)-1/p \ge 0$, $2\alpha_2-1/q \ge 0$ and $1/p+1/q = 1$. Using these p and q, we further evaluate $I_2(x,t)$ as

$$\begin{aligned} I_2(x,t) &\le C(1+t)^{-r}(1+t-x)^{-(\alpha_1-\beta_1/2-r)}x^{-(\alpha_2-\beta_2/2)} \\ &\quad \times \left(\int_0^{(1/2)(1+t-x)} (1+t-x+y^2)^{-(1/2)p\beta_1}dy\right)^{1/p} \\ &\quad \times \left(\int_0^{(1/2)x} (x+y^2)^{-(1/2)q\beta_2}dy\right)^{1/q} \\ &\le C(1+t)^{-r}(1+t-x)^{-(\alpha_1-r)+1/(2p)}x^{-\alpha_2+1/(2q)} \\ &\le C(1+t)^{-r}(x^{-(\alpha_1+\alpha_2-r-1/2)} + (1+t-x)^{-(\alpha_1+\alpha_2-r-1/2)}) \\ &\le C(1+t)^{-(1/2)(\alpha_1+\alpha_2+r-1/2)} \sum_{\lambda=0,1} \varphi_{\alpha_1+\alpha_2-r-1/2}(x,t;\lambda) \\ &\le C(1+t)^{-(1/2)(\alpha_1+\alpha_2+r-1/2)} \sum_{\lambda=0,1} \varphi_{\alpha_1+\alpha_2-r-1/2}(x,t;\lambda)\psi_{\beta_4}(x,t;\lambda). \end{aligned}$$

Similarly, we can show that $I_3(x,t)$ is estimated as

$$I_3(x,t) \le C(1+t)^{-(1/2)(\alpha_1+\alpha_2+r-1/2)} \sum_{\lambda=0,1} \varphi_{\alpha_1+\alpha_2-r-1/2}(x,t;\lambda)\psi_{\beta_4}(x,t;\lambda).$$

Collecting the above estimates we obtain (8.4).

If $\alpha_1, \alpha_2 \le 3/2$, then we can take $r = \alpha_1 + \alpha_2 - \frac{3}{2}$ and the estimate (8.5) follows. The proof is complete. □

LEMMA 8.6. *Suppose that* $\beta_1, \beta_2 \ge 0$, $\alpha_1 > 0$ *and* $\alpha_2, \lambda \in \mathbf{R}^1$. *Then we have*

$$\int_0^t e^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}\varphi_{\beta_1}(x,\tau;\lambda)\psi_{\beta_2}(x,\tau;\lambda)d\tau \le C(1+t)^{-\alpha_2}\varphi_{\beta_1}(x,t;\lambda)\psi_{\beta_2}(x,t;\lambda)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

PROOF. It is sufficient to show the estimate in the cases $\lambda = 0$ and $\lambda = 1$. If $\lambda = 0$, then the desired estimate follows from Lemma 6.3 because we have $\varphi_{\beta_1}(x,\tau;0) \le \varphi_{\beta_1}(x,t;0)$ and $\psi_{\beta_2}(x,\tau;0) \le \psi_{\beta_2}(x,t;0)$ for $0 \le \tau \le t$.

Now, we assume that $\lambda = 1$ and consider the following two cases according to (x,t). We denote by $I(x,t)$ the left hand side of the above inequality.

Case 1. $|x-(1+t)| \le \sqrt{1+t}$.

$$\begin{aligned} I(x,t) &\le \int_0^t e^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}d\tau \le C(1+t)^{-\alpha_2} \\ &\le C(1+t)^{-\alpha_2}\varphi_{\beta_1}(x,t;1)\psi_{\beta_2}(x,t;1). \end{aligned}$$

Case 2. $|x-(1+t)| \ge \sqrt{1+t}$. We split the integral over $\{\tau \in (0,t);\ 2|t-\tau| < |x-(1+t)|\}$ and $\{\tau \in (0,t); 2|t-\tau| > |x-(1+t)|\}$, and write the respective integrals as $I_1(x,t)$ and $I_2(x,t)$. If $2|t-\tau| \le |x-(1+t)|$, then we have $|x-(1+\tau)| \ge \frac{1}{2}|x-(1+t)|$. Therefore, $I_1(x,t)$ is estimated as

$$\begin{aligned} I_1(x,t) &\le C\int_0^t e^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}d\tau\varphi_{\beta_1}(x,t;1)\psi_{\beta_2}(x,t;1) \\ &\le C(1+t)^{-\alpha_2}\varphi_{\beta_1}(x,t;1)\psi_{\beta_2}(x,t;1). \end{aligned}$$

If $2|t-\tau| \ge |x-(1+t)|$, then we have $|x-(1+t)| \le 2(1+t)$. Therefore, $I_2(x,t)$ is estimated as

$$\begin{aligned} I_2(x,t) &\le C\int_{(1/2)|x-(1+t)|}^{\infty} e^{-\alpha_1\tau}\,d\tau \le Ce^{-\alpha_1|x-(1+t)|/2} \\ &\le C(1+(x-(1+t))^2)^{-(|\alpha_2|+\beta_1/2)}\psi_{\beta_2}(x,t;1) \\ &\le C(1+t)^{-\alpha_2}\varphi_{\beta_1}(x,t;1)\psi_{\beta_2}(x,t;1). \end{aligned}$$

Collecting the above estimates we obtain the desired one. The proof is complete. □

LEMMA 8.7. *Suppose that* $\alpha_1 \le 1$, $\alpha_2 < 3/2$, $\beta_1 \ge 1$, $\beta_2, \beta_3 \ge 0$, $\delta > 0$ *and* $\lambda \in \mathbf{R}^1$. *Then we have*

$$\int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2} \mathrm{e}^{-\delta|x-y-\lambda(t-\tau)|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;\lambda)\psi_{\beta_2}(y,\tau;\lambda)dyd\tau$$
$$\le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\{\varphi_{\beta_1}(x,t;\lambda)\psi_{\beta_2}(x,t;\lambda)+\varphi_{\beta_3}(x,t;\lambda)\}$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

Proof. Noting that $\alpha_2 < 3/2$, we take $p, q \in (1, \infty)$ such that $\alpha_2 - 1/2q < 1$ and $1/p + 1/q = 1$. We denote by $I(x,t)$ the left hand side of the above inequality and evaluate it by considering the following two cases.

Case 1. $|x - \lambda(1+t)| \le \sqrt{1+t}$. By Hölder's inequality, we see that

$$\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y-\lambda(t-\tau)|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;\lambda)\psi_{\beta_2}(y,\tau;\lambda)dy$$
$$\le |\mathrm{e}^{-\delta|\cdot|/\sqrt{t-\tau}}|_p |\varphi_{\beta_1}(\cdot\,,\tau;\lambda)|_q = C(t-\tau)^{1/(2p)}(1+\tau)^{1/(2q)},$$

which together with Lemma 4.1 implies that

$$I(x,t) \le C\int_0^t (t-\tau)^{-(\alpha_1-1/(2p))}(1+\tau)^{-(\alpha_2-1/(2q))}d\tau$$
$$\le C(1+t)^{-(\alpha_1+\alpha_2-3/2)} \le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\varphi_{\beta_3}(x,t;\lambda).$$

Case 2. $|x - \lambda(1+t)| \ge \sqrt{1+t}$. By Lemma 8.2, we see that

$$\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y-\lambda(t-\tau)|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;\lambda)\psi_{\beta_2}(y,\tau;\lambda)dy$$
$$= \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-\lambda(1+t)-y|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;0)\psi_{\beta_2}(y,\tau;0)dy$$
$$\le C\{(t-\tau)^{1/(2p)}(1+\tau)^{1/(2q)}\tilde{\varphi}_{\beta_3}(x-\lambda(1+t),t,\tau;0)$$
$$+ (t-\tau)^{1/2}\varphi_{\beta_1}(x-\lambda(1+t),\tau;0)\psi_{\beta_2}(x-\lambda(1+t),\tau;0)\},$$

which together with Lemma 8.4 implies that

$$I(x,t) \le C\int_0^t (t-\tau)^{-(\alpha_1-1/(2p))}(1+\tau)^{-(\alpha_2-1/(2q))}\tilde{\varphi}_{\beta_3}(x-\lambda(1+t),t,\tau;0)d\tau$$
$$+ C\int_0^t (t-\tau)^{-(\alpha_1-1/2)}(1+\tau)^{-\alpha_2}$$
$$\times \varphi_{\beta_1}(x-\lambda(1+t),\tau;0)\psi_{\beta_2}(x-\lambda(1+t),\tau;0)d\tau$$
$$\le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\{\varphi_{\beta_3}(x,t;\lambda)+\varphi_{\beta_1}(x,t;\lambda)\psi_{\beta_2}(x,t;\lambda)\}.$$

This completes the proof. □

LEMMA 8.8. *Suppose that* $\alpha_1 \le 1$, $\alpha_2 < 3/2$, $\beta_1 \ge 1$ $\beta_2, \beta_3, \beta_4 \ge 0$, $\delta > 0$, $\lambda, \lambda' \in \mathbf{R}^1$ *and* $\lambda \ne \lambda'$. *Then we have*

$$\int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2} \mathrm{e}^{-\delta|x-y-\lambda(t-\tau)|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;\lambda')\psi_{\beta_2}(y,\tau;\lambda')dyd\tau$$

$$\le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}$$

$$\times \sum_{l=\lambda,\lambda'} \{\varphi_{\beta_3}(x,t;l) + \varphi_{\beta_1}(x,t;l)\psi_{\beta_2}(x,t;l) + \varphi_1(x,t;l)\psi_{\beta_4}(x,t;l)\}$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

PROOF. By changing the variables x and y into $\tilde{x}$ and $\tilde{y}$ in accordance with (6.9), we can assume that $\lambda = 0$ and $\lambda' = 1$. Noting that $\alpha_2 < 3/2$, we take $p, q \in (1, \infty)$ such that $\alpha_2 - 1/2q < 1$ and $1/p + 1/q = 1$. We denote by $I(x,t)$ the left hand side of the above inequality and evaluate it by considering the following three cases.

Case 1. $|x| \le \sqrt{1+t}$. By the similar calculation to that of Case 1 in the proof of Lemma 8.7, we see that

$$|I(x,t)| \le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\varphi_{\beta_3}(x,t;0).$$

Case 2. $|x-(1+t)| \le \sqrt{1+t}$. Similarly, we have

$$|I(x,t)| \le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\varphi_{\beta_3}(x,t;1).$$

Case 3. $|x| \ge \sqrt{1+t}$ and $|x-(1+t)| \ge \sqrt{1+t}$. By Lemma 8.2, we see that

$$\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;1)\psi_{\beta_2}(y,\tau;1)dy$$

$$= \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-(1+\tau)-y|/\sqrt{t-\tau}} \varphi_{\beta_1}(y,\tau;0)\psi_{\beta_2}(y,\tau;0)dy$$

$$\le C\{(t-\tau)^{1/(2p)}(1+\tau)^{1/(2q)}\tilde{\varphi}_{\beta_3}(x,t,\tau;1) + (t-\tau)^{1/2}\varphi_{\beta_1}(x,\tau;1)\psi_{\beta_2}(x,\tau;1)\},$$

which together with Lemma 8.5 yields the desired estimate. The proof is complete. □

LEMMA 8.9. *Suppose that* $\alpha_1, \delta > 0$, $\beta_1, \beta_2 \ge 0$ *and* $\alpha_2, \lambda \in \mathbf{R}^1$. *Then we have*

$$\int_0^t \int_{\mathbf{R}^1} \mathrm{e}^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}\mathrm{e}^{-\delta|x-y|}\varphi_{\beta_1}(y,\tau;\lambda)\psi_{\beta_2}(y,\tau;\lambda)dyd\tau$$

$$\le C(1+t)^{-\alpha_2}\varphi_{\beta_1}(x,t;\lambda)\psi_{\beta_2}(x,t;\lambda)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

Proof. By Lemma 8.3, we see that

$$\begin{aligned}\int_{\mathbf{R}^1} & \mathrm{e}^{-\delta|x-y|}\varphi_{\beta_1}(y,\tau;\lambda)\psi_{\beta_2}(y,\tau;\lambda)dy \\ &= \int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-\lambda(1+\tau)-y|}\varphi_{\beta_1}(y,\tau;0)\psi_{\beta_2}(y,\tau;0)dy \\ &\le C\varphi_{\beta_1}(x,\tau;\lambda)\psi_{\beta_2}(x,\tau;\lambda).\end{aligned}$$

This and Lemma 8.6 give the desired estimate. The proof is complete. □

Now, let $\Phi_\beta(x,t)$ the function defined in (2.22). By Lemmas 8.3, 8.6–8.9, we can easily obtain the following four lemmas.

Lemma 8.10. *Suppose that* $\beta = (\beta_1, \beta_2)$, $\beta_1, \beta_2 \ge 0$ *and* $\delta > 0$. *Then we have*

$$\int_{\mathbf{R}^1} \mathrm{e}^{-\delta|x-y|}\Phi_\beta(y,t)dy \le C\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t \ge 0$, *where* C *is a positive constant.*

Lemma 8.11. *Suppose that* $\beta = (\beta_1, \beta_2)$, $\beta_1, \beta_2 \ge 0$, $\alpha_1 > 0$ *and* $\alpha_2 \in \mathbf{R}^1$. *Then we have*

$$\int_0^t \mathrm{e}^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}\Phi_\beta(x-\lambda_i(t-\tau),\tau)d\tau \le C(1+t)^{-\alpha_2}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$, $t > 0$ *and* $i = 1, \dots, \sigma$, *where* C *is a positive constant.*

Lemma 8.12. *Suppose that* $\beta = (\beta_1, \beta_2)$, $\beta_1 \ge 1$, $\beta_2 \ge 0$, $\alpha_1 \le 1$, $\alpha_2 < 3/2$ *and* $\delta > 0$. *Then we have*

$$\begin{aligned}\int_0^t\int_{\mathbf{R}^1} & (t-\tau)^{-\alpha_1}(1+\tau)^{-\alpha_2}\mathrm{e}^{-\delta|x-y-\lambda_i(t-\tau)|/\sqrt{t-\tau}}\Phi_\beta(y,\tau)dyd\tau \\ &\le C(1+t)^{-(\alpha_1+\alpha_2-3/2)}\Phi_\beta(x,t)\end{aligned}$$

for $x \in \mathbf{R}^1$, $t > 0$ *and* $i = 1, \dots, \sigma$, *where* C *is a positive constant.*

Lemma 8.13. *Suppose that* $\beta = (\beta_1, \beta_2)$, $\beta_1, \beta_2 \ge 0$, $\alpha_1, \delta > 0$ *and* $\alpha_2 \in \mathbf{R}^1$. *Then we have*

$$\int_0^t\int_{\mathbf{R}^1} \mathrm{e}^{-\alpha_1(t-\tau)}(1+\tau)^{-\alpha_2}\mathrm{e}^{-\delta|x-y|}\Phi_\beta(y,\tau)dyd\tau \le C(1+t)^{-\alpha_2}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

In the following we assume that system (1.1) admits an entropy function and that the symmetric system (2.1) satisfies the stability condition at $(\bar{u}, 0)$. Let $G_1(x,t)$ and $G_1^*(x,t)$ be the corresponding Green's functions defined in (5.3) and (5.7).

LEMMA 8.14. *Let l be a non-negative integer, $\beta_1 \geq 1$ and $u \in W^{l,\infty}_{\beta_1} \cap L^1$. Then we have*

$$\left| \int_{\mathbf{R}^1} \partial_x^l G_1(x-y,t) u(y) dy \right| \leq C(|u|_1 + \|u\|_{l,\infty,\beta_1})(1+t)^{-(1/2)(l+1)} \sum_{i=1}^{\sigma} \varphi_{\beta_1}(x,t;\lambda_i)$$

for $x \in \mathbf{R}^1$ and $t > 0$, where C is a positive constant.

PROOF. By Proposition 5.1 and (8.3), we see that

$$\begin{aligned}
&\left| \int_{\mathbf{R}^1} \partial_x^l G_1(x-y,t) u(y) dy \right| \\
&\quad \leq \sum_{k=0}^{l} \sum_{j=1}^{\sigma'} \mathrm{e}^{c_{j,1}t} |Q_{j,k}(t) \partial_x^{l-k} u(x + c_{j,0}t)| + \int_{\mathbf{R}^1} |R_0^{(l)}(x-y,t) u(y)| dy \\
&\quad \leq C\mathrm{e}^{-\delta_1 t}(1+t)^l \sum_{k=0}^{l} \sum_{i=1}^{\sigma} |\partial_x^{l-k} u(x - \lambda_i t)| \\
&\qquad + C(1+t)^{-(1/2)(l+1)} \sum_{i=1}^{\sigma} \int_{\mathbf{R}^1} \mathrm{e}^{-\delta_1 |x-y-\lambda_i t|/\sqrt{t}} |u(y)| dy \\
&\qquad + C\mathrm{e}^{-t} \int_{\mathbf{R}^1} \mathrm{e}^{-\delta_1 |x-y|} |u(y)| dy \\
&\quad \leq C\|u\|_{l,\infty,\beta_1} \mathrm{e}^{-\delta_1 t}(1+t)^l \sum_{i=1}^{\sigma} (1 + (x - \lambda_i t)^2)^{-\beta_1/2} \\
&\qquad + C(|u|_1 + |u|_{(\beta_1)}) \left\{ (1+t)^{-(1/2)(l+1)} \sum_{i=1}^{\sigma} \varphi_{\beta_1}(x,t;\lambda_i) + \mathrm{e}^{-t}(1+x^2)^{-\beta_1/2} \right\},
\end{aligned}$$

which gives the desired estimate. The proof is complete. □

LEMMA 8.15. *Let l be a non-negative integer, $\beta = (\beta_1, \beta_2)$, $\beta_1 \geq 1$, $\beta_2 \geq 0$ and $\alpha < 3/2$. Suppose that a function $H(x,t)$ satisfies the estimates*

$$\begin{cases} |\partial_y^k H(y,\tau)| \leq \Lambda(t)(1+\tau)^{-\alpha} \Phi_\beta(y,\tau), & k = 0, 1, \dots, l+1, \\ |\partial_y^l H(y,\tau)| + |\partial_y^{l+1} H(y,\tau)| \leq \Lambda(t)(1+\tau)^{-l/2-\alpha} \Phi_\beta(y,\tau) \end{cases}$$

for $y \in \mathbf{R}^1$ *and* $0 < \tau < t$ *with a non-negative valued function* $\Lambda(t)$. *Then we have*

$$\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1} G_1(x-y, t-\tau) H(y,\tau) dy d\tau\right| \le C\Lambda(t)(1+t)^{-(1/2)(l-1)-\alpha} \Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

Proof. By Proposition 5.1, we have

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1} G_1(x-y, t-\tau) H(y,\tau) dy d\tau\right| \\
&\quad \le \sum_{k=0}^{l} \sum_{j=1}^{\sigma'} \int_0^{t/2} \mathrm{e}^{c_{j,1}(t-\tau)} |Q_{j,k}(t-\tau) \partial_x^{l+1-k} H(x + c_{j,0}(t-\tau), \tau)| d\tau \\
&\qquad + \sum_{k=0}^{1} \sum_{j=1}^{\sigma'} \int_{t/2}^{t} \mathrm{e}^{c_{j,1}(t-\tau)} |Q_{j,k}(t-\tau) \partial_x^{l+1-k} H(x + c_{j,0}(t-\tau), \tau)| d\tau \\
&\qquad + \int_0^{t/2} \int_{\mathbf{R}^1} |R_0^{(l+1)}(x-y, t-\tau) H(y,\tau)| dy d\tau \\
&\qquad + \int_{t/2}^{t} \int_{\mathbf{R}^1} |R_0^{(1)}(x-y, t-\tau) \partial_y^l H(y,\tau)| dy d\tau \\
&\quad \le C\Lambda(t)(1+t)^{-l/2} \Bigg\{ \sum_{i=1}^{\sigma} \int_0^t \mathrm{e}^{-(\delta_1/2)(t-\tau)} (1+\tau)^{-\alpha} \Phi_\beta(x - \lambda_i(t-\tau), \tau) d\tau \\
&\qquad\qquad + \sum_{i=1}^{\sigma} \int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-1} (1+\tau)^{-\alpha} \\
&\qquad\qquad \times \mathrm{e}^{-\delta_1 |x-y-\lambda_i(t-\tau)|/\sqrt{t-\tau}} \Phi_\beta(y,\tau) dy d\tau \\
&\qquad\qquad + \int_0^t \int_{\mathbf{R}^1} \mathrm{e}^{-(1/2)(t-\tau)} (1+\tau)^{-\alpha} \mathrm{e}^{-\delta_1|x-y|} \Phi_\beta(y,\tau) dy d\tau \Bigg\}.
\end{aligned}$$

This and Lemmas 8.11–8.13 give the desired estimate. □

Similarly, we can show the following lemmas.

Lemma 8.16. *Suppose that* $\beta = (\beta_1, \beta_2)$, $\beta_1 \ge 1$, $\beta_2 \ge 0$, $\alpha < 3/2$ *and that a function* $H(x,t)$ *satisfies the estimate*

$$|H(y,\tau)| \le \Lambda(t)(1+\tau)^{-\alpha} \Phi_\beta(y,\tau)$$

for $y \in \mathbf{R}^1$ *and* $0 < \tau < t$ *with a non-negative valued function* $\Lambda(t)$. *Then we have*

$$\left|\int_0^t \int_{\mathbf{R}^1} G_1(x-y,t-\tau)H(y,\tau)dyd\tau\right| \le CA(t)(1+t)^{-(\alpha-1)}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

LEMMA 8.17. *Let* l *be a non-negative integer,* $\beta_1 \ge 1$ *and* $u \in W^{l,\infty}_{\beta_1} \cap L^1$. *Then we have*

$$\left|\int_{\mathbf{R}^1} \partial_x^l (G_1 - G_1^*)(x-y,t)u(y)dy\right|$$
$$\le C(|u|_1 + \|u\|_{l,\infty,\beta_1})t^{-(1/2)(l+1)}(1+t)^{-1/2}\sum_{i=1}^{\sigma}\varphi_{\beta_1}(x,t;\lambda_i)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

LEMMA 8.18. *Let* l *be a non-negative integer,* $\beta = (\beta_1, \beta_2)$, $\beta_1 \ge 1$, $\beta_2 \ge 0$ *and* $\alpha < 3/2$. *Suppose that a function* $H(x,t)$ *satisfies the estimates*

$$\begin{cases} |\partial_y^k H(y,\tau)| \le A(t)(1+\tau)^{-\alpha}\Phi_\beta(y,\tau), & k = 0,1,\dots,l+1, \\ |\partial_y^{l+1} H(y,\tau)| \le A(t)(1+\tau)^{-(1/2)(l+1)-\alpha}\Phi_\beta(y,\tau) \end{cases}$$

for $y \in \mathbf{R}^1$ *and* $0 < \tau < t$ *with a non-negative valued function* $A(t)$. *Then we have*

$$\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1}(G_1 - G_1^*)(x-y,t-\tau)H(y,\tau)dyd\tau\right| \le CA(t)(1+t)^{-l/2-\alpha}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

LEMMA 8.19. *Let* l *be a non-negative integer,* $\beta = (\beta_1, \beta_2)$, $\beta_1 \ge 1$, $\beta_2 \ge 0$ *and* $\alpha < 3/2$. *Suppose that a function* $H(x,t)$ *satisfies the estimates*

$$|\partial_y^k H(y,\tau)| \le A(t)(1+\tau)^{-k/2-\alpha}\Phi_\beta(y,\tau)$$

for $y \in \mathbf{R}^1$, $0 < \tau < t$ *and* $k = 0, l$, *with a non-negative valued function* $A(t)$. *Then we have*

$$\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y,t-\tau)H(y,\tau)dyd\tau\right| \le CA(t)t^{-l/2}(1+t)^{-(\alpha-1/2)}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$, *where* C *is a positive constant.*

LEMMA 8.20. *Let* l *be a non-negative integer,* $\beta_1 \ge 1$, $\beta_2 \ge 0$ *and* $1 \le p \le \infty$. *Then we have*

$$\left|\int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y,t)u(y)dy\right| \le C(|u|_p + |u|_{(\beta_1)})t^{-(1/2)(l+1)-1/(2p)}\sum_{i=1}^{\sigma}\varphi_{\beta_1}(x,t;\lambda_i)$$

for $x \in \mathbf{R}^1$ *and* $t > 0$ *if* $u \in W_{\beta_1}^{0,\infty} \cap L^p$, *and*

$$\left| \int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y,t) u(y) dy \right|$$
$$\leq C \sum_{i=1}^{\sigma} \{ (|u|_p + |u|_{(1)}) t^{-(1/2)(l+1)-1/(2p)} \varphi_1(x,t;\lambda_i) \psi_{\beta_2}(x,t;\lambda_i)$$
$$+ (|u_x|_1 + |u_x|_{(\beta_1)}) t^{-(1/2)(l+1)-(1/2)(\beta_1-1)} \varphi_{\beta_1}(x,t;\lambda_i) \}$$

for $x \in \mathbf{R}^1$ *and* $t > 0$ *if* $u \in W_1^{0,\infty} \cap L^p$ *and* $u_x \in W_{\beta_1}^{0,\infty} \cap L^1$, *where* C *is a positive constant.*

Proof. By Proposition 5.3, the integral can be decomposed as

$$\int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y,t) u(y) dy = \sum_{i=1}^{\sigma} I_i(x,t),$$

and we have the estimates

$$|I_i(x,t)| \leq C t^{-(1/2)(l+2)} \int_{\mathbf{R}^1} e^{-\delta|x-y-\lambda_i t|/\sqrt{1+t}} |u(y)| dy \tag{8.7}$$

and

$$|I_i(x,t)| \leq C t^{-(1/2)(l+1)} \int_{\mathbf{R}^1} e^{-\delta|x-y-\lambda_i t|/\sqrt{1+t}} |u_y(y)| dy \tag{8.8}$$

for $x \in \mathbf{R}^1$, $t > 0$ and $i = 1, \dots, \sigma$. (8.2) and (8.7) give the first estimate of the lemma. In order to show the second one, we evaluate $I_i(x,t)$ by considering the following two cases.

Case 1. $|x - \lambda_i(1+t)| \leq 1 + t$. By (8.2) and (8.7), we have

$$|I_i(x,t)| \leq C t^{-(1/2)(l+1)-1/(2p)} (|u|_p + |u|_{(1)}) \varphi_1(x,t;\lambda_i)$$
$$\leq C t^{-(1/2)(l+1)-1/(2p)} (|u|_p + |u|_{(1)}) \varphi_1(x,t;\lambda_i) \psi_{\beta_2}(x,t;\lambda_i).$$

Case 2. $|x - \lambda_i(1+t)| \geq 1 + t$. By (8.2) and (8.8), we have

$$|I_i(x,t)| \leq C t^{-(1/2)(l+1)} (|u_x|_1 e^{-\delta\sqrt{1+t}/4} e^{-\delta|x-\lambda_i(1+t)|/8\sqrt{1+t}}$$
$$+ |u_x|_{(\beta_1)} (1+t)^{-(1/2)(\beta_1-1)} \varphi_{\beta_1}(x,t;\lambda_i))$$
$$\leq C (|u_x|_1 + |u_x|_{(\beta_1)}) t^{-(1/2)(l+1)-(1/2)(\beta_1-1)} \varphi_{\beta_1}(x,t;\lambda_i).$$

These give the desired estimate. The proof is complete. □

We proceed to prove Theorem 2.6. Let (w, q) be a solution to (1.1) and (1.2) obtained in Theorem 2.2, and (v, q) the corresponding solution to (2.9). We introduce a weight function $\phi_n = \phi_n(x)$ by

$$\phi_n(x) = \mathrm{e}^{-(1/n)\sqrt{1+x^2}} \qquad \text{for } x \in \mathbf{R}^1,\ n = 1, 2, 3, \dots, \tag{8.9}$$

and put

$$v^n(x, t) = \phi_n(x) v(x, t), \qquad q^n(x, t) = \phi_n(x) q(x, t). \tag{8.10}$$

We multiply (2.9) by ϕ_n to obtain

$$\begin{cases} v_t^n + A v_x^n + L^T q_x^n = (\phi_n H_1)_x + \phi_n'(Av + L^T q - H_1), \\ -q_{xx}^n + Rq^n + v(Lv_x^n + Jq_x^n) = \phi_n H_2 + v\phi_n'(Lv + Jq) + \phi_n'' q - 2(\phi_n' q)_x. \end{cases}$$

Therefore, by (5.4) derivatives of (v^n, q^n) can be expressed as

$$\begin{cases} \partial_x^l v^n(t) = \partial_x^l G_1(t) * v^n(0) \\ \qquad + \displaystyle\int_0^t \partial_x^{l+1} G_1(t-\tau) * (\phi_n H_1(\tau) + L^T G_2 * (\phi_n H_2(\tau))) d\tau \\ \qquad + \displaystyle\int_0^t \partial_x^l G_1(t-\tau) * H_4^n(\tau) d\tau, \\ \partial_x^l q^n(t) = G_{2x} * \partial_x^l (vLv^n(t) + 2\phi_n'' q(t)) - \partial_x^l G_2 * H_5^n(t), \end{cases} \tag{8.11}$$

where

(8.12)

$$\begin{cases} H_4^n = \phi_n'(Av + L^T q - H_1) - 2L^T G_{2xx} * (\phi_n' q) + L^T G_{2x} * (v\phi_n'(Lv + Jq)), \\ H_5^n = \phi_n H_2 + v\phi_n'(Lv + Jq) + \phi_n'' q. \end{cases}$$

We put

(8.13)

$$\tilde{M}_{l,\beta}^n(t) = \sup_{0 \le \tau \le t,\, x \in \mathbf{R}^1} \{(1+\tau)^{-(1/2)(l+1)} \Phi_\beta(x, \tau)\}^{-1} (|\partial_x^l v^n(x, \tau)| + |\partial_x^l q^n(x, \tau)|).$$

Note that we do not know a priori that $\sup_{x \in \mathbf{R}^1} \Phi_\beta(x, \tau)^{-1}(|\partial_x^l v(x, \tau)| + |\partial_x^l q(x, \tau)|)$ is finite for $t > 0$. However, $\tilde{M}_{l,\beta}^n(t)$ is finite for $t \ge 0$ because of the weight function ϕ_n.

LEMMA 8.21. *Under the same assumptions in Theorem* 2.6, *we have*

$$\tilde{M}^n_{0,\beta}(t) \le C\{|v(0)|_1 + |v(0)|_{(\beta_1)} + (E_3 + n^{-1}(1+t))\tilde{M}^n_{0,\beta}(t)\}$$

for $t \ge 0$ *and* $n = 1, 2, \ldots,$ *and*

$$\tilde{M}^n_{l,\beta}(t) \le C\left\{|v(0)|_1 + \|v(0)\|_{l,\infty,\beta_1} + (E_3 + n^{-1}(1+t)^{(1/2)(l+1)})\tilde{M}^n_{l,\beta}(t) + (E_s + n^{-1}(1+E_s)(1+t)^{(1/2)(l+1)})\sum_{k=0}^{l-1}\tilde{M}^n_{k,\beta}(t)\right\}$$

for $t \ge 0$, $l = 1, \ldots, s-2$ *and* $n = 1, 2, \ldots,$ *where* C *is a positive constant independent of* t *and* n.

Proof. We evaluate the right hand sides of the equations in (8.11). By Lemma 8.14, we have

$$\left|\int_{\mathbf{R}^1} \partial_x^l G_1(x-y,t)v^n(y,0)dy\right| \le C(|v^n(0)|_1 + \|v^n(0)\|_{l,\infty,\beta_1})(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t) \le C(|v(0)|_1 + \|v(0)\|_{l,\infty,\beta_1})(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t).$$

By (2.10), Lemma 4.1, Sobolev's inequality and the results of Theorem 2.3 (see also (2.8)), we see that

$$|H_1(x,t)| \le C(|v(t)|_\infty + |q(t)|_\infty)(|v(x,t)| + |q(x,t)|) \le CE_3(1+t)^{-1/2}(|v(x,t)| + |q(x,t)|),$$

$$|H_{1x}(x,t)| \le C(|v_x(t)|_\infty + |q_x(t)|_\infty)(|v(x,t)| + |q(x,t)|) \le CE_3(1+t)^{-1}(|v(x,t)| + |q(x,t)|),$$

$$\begin{aligned}|\partial_x^l H_1(x,t)| &\le C(|v_x(t)|_\infty + |q_x(t)|_\infty)(|\partial_x^{l-1}v(x,t)| + |\partial_x^{l-1}q(x,t)|) \\ &\quad + C\sum_{k=0}^{l-2}(|\partial_x^{l-k}v(t)|_\infty + |\partial_x^{l-k}q(t)|_\infty)(|\partial_x^k v(x,t)| + |\partial_x^k q(x,t)|) \\ &\le CE_3(1+t)^{-1}(|\partial_x^{l-1}v(x,t)| + |\partial_x^{l-1}q(x,t)|) \\ &\quad + CE_s\sum_{k=0}^{l-2}(1+t)^{-(1/2)(l+1-k)}(|\partial_x^k v(x,t)| + |\partial_x^k q(x,t)|), \\ &\quad 2 \le l \le s-2,\end{aligned}$$

and

$$|\partial_x^{s-1}H_1(x,t)| \le C(|v_x(t)|_\infty + |q_x(t)|_\infty)(|\partial_x^{s-2}v(x,t)| + |\partial_x^{s-2}q(x,t)|)$$
$$+ C\sum_{k=0}^{s-3}(\|\partial_x^{s-1-k}v(t)\|_1 + \|\partial_x^{s-1-k}q(t)\|_1)(|\partial_x^k v(x,t)| + |\partial_x^k q(x,t)|)$$
$$\le CE_3(1+t)^{-1}(|\partial_x^{s-2}v(x,t)| + |\partial_x^{s-2}q(x,t)|)$$
$$+ CE_s\sum_{k=0}^{s-3}(1+t)^{-(1/2)(s-1-k)}(|\partial_x^k v(x,t)| + |\partial_x^k q(x,t)|)$$

for $x \in \mathbf{R}^1$ and $t \ge 0$. On the other hand, by the definition (8.9) of ϕ_n, we have

$$\phi_n(x)^{-1}|\partial_x^l\phi_n(x)| + \phi_n(x)|\partial_x^l(\phi_n(x)^{-1})| \le Cn^{-1}$$

and then

$$|\phi_n(x)\partial_x^l u(x)| \le |\partial_x^l(\phi_n(x)u(x))| + Cn^{-1}\sum_{k=0}^{l-1}|\partial_x^k(\phi_n(x)u(x))|$$

for $x \in \mathbf{R}^1$ and $n, l = 1, 2, \ldots,$ where $C = C(l) > 0$. Therefore, we see that

$$|\phi_n(x)H_1(x,t)| \le CE_3\tilde{M}_{0,\beta}^n(t)(1+t)^{-1}\Phi_\beta(x,t),$$
$$|\partial_x(\phi_n(x)H_1(x,t))| \le C(1+n^{-1}t^{1/2})E_3\tilde{M}_{0,\beta}^n(t)(1+t)^{-3/2}\Phi_\beta(x,t),$$
$$|\partial_x^l(\phi_n(x)H_1(x,t))| \le C\left\{E_3\tilde{M}_{l-1,\beta}^n(t) + E_s(1+n^{-1}t^{l/2})\sum_{k=0}^{l-2}\tilde{M}_{k,\beta}^n(t)\right\}$$
$$\times (1+t)^{-(1/2)(l+2)}\Phi_\beta(x,t), \qquad 2 \le l \le s-2,$$

and

$$|\partial_x^{s-1}(\phi_n(x)H_1(x,t))| \le C\left\{E_3\tilde{M}_{s-2,\beta}^n(t) + E_s(1+n^{-1}t^{(1/2)(s-1)})\sum_{k=0}^{s-3}\tilde{M}_{k,\beta}^n(t)\right\}$$
$$\times (1+t)^{-s/2}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$, $t \ge 0$ and $n = 1, 2, \ldots.$ Thus, we can apply Lemma 8.15 with $\alpha = 1$ and $H = \phi_n H_1$ to obtain

$$\left|\int_0^t\int_{\mathbf{R}^1}\partial_x G_1(x-y,t-\tau)\phi_n(y)H_1(y,\tau)dyd\tau\right|$$
$$\le CE_3(1+n^{-1}t^{1/2})\tilde{M}_{0,\beta}^n(t)(1+t)^{-1/2}\Phi_\beta(x,t)$$

and

$$\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l+1} G_1(x-y,t-\tau)\phi_n(y)H_1(y,\tau)dyd\tau\right|$$
$$\leq C\left\{E_3\tilde{M}^n_{l,\beta}(t)+E_s(1+n^{-1}t^{(1/2)(l+1)})\sum_{k=0}^{l-1}\tilde{M}^n_{k,\beta}(t)\right\}(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t)$$

for $x\in\mathbf{R}^1$, $t>0$, $l=1,\dots,s-2$ and $n=1,2,\dots$. By similar evaluation, Proposition 5.2 and Lemma 8.10, we see that $\int_0^t\int_{\mathbf{R}^1}\partial_x^{l+1}G_1(x-y,t-\tau)\cdot(G_2*(\phi_nH_2(\cdot,\tau)))(y)dyd\tau$ satisfies the same estimates as above and that

$$\left|\int_0^t \int_{\mathbf{R}^1} G_1(x-y,t-\tau)H_4^n(y,\tau)dyd\tau\right|\leq Cn^{-1}(1+t)^{1/2}\tilde{M}^n_{0,\beta}(t)\Phi_\beta(x,t)$$

and

$$\left|\int_0^t \int_{\mathbf{R}^1} \partial_x^{l} G_1(x-y,t-\tau)H_4^n(y,\tau)dyd\tau\right|$$
$$\leq Cn^{-1}\left\{\tilde{M}^n_{l,\beta}(t)+(1+E_s)\sum_{k=0}^{l-1}\tilde{M}^n_{k,\beta}(t)\right\}\Phi_\beta(x,t)$$

for $x\in\mathbf{R}^1$, $t>0$, $l=1,\dots,s-1$ and $n=1,2,\dots$. Adding the above estimates together, we obtain

$$|v^n(x,t)|\leq C\{|v(0)|_1+|v(0)|_{(\beta_1)}+(E_3+n^{-1}(1+t))\tilde{M}^n_{0,\beta}(t)\}(1+t)^{-1/2}\Phi_\beta(x,t)$$

and

$$|\partial_x^l v^n(x,t)|\leq C\left\{|v(0)|_1+\|v(0)\|_{l,\infty,\beta_1}+(E_3+n^{-1}(1+t)^{(1/2)(l+1)})\tilde{M}^n_{l,\beta}(t)\right.$$
$$\left.+(E_s+n^{-1}(1+E_s)(1+t)^{(1/2)(l+1)})\sum_{k=0}^{l-1}\tilde{M}^n_{k,\beta}(t)\right\}$$
$$\times(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t)$$

for $x\in\mathbf{R}^1$, $t\geq 0$, $l=1,\dots,s-2$ and $n=1,2,\dots$. Moreover, by evaluating the right hand side of the second equation in (8.11), we see that $q^n(x,t)$ satisfies the same estimates as above. This completes the proof of the lemma. □

By Lemma 8.21, for each $t\geq 0$ there exists a number $N(t)>0$ such that the estimates

$$\begin{cases} \tilde{M}^n_{0,\beta}(t) \leq C(|v(0)|_1 + |v(0)|_{(\beta_1)}), \\ \tilde{M}^n_{l,\beta}(t) \leq C\left(|v(0)|_1 + \|v(0)\|_{l,\infty,\beta_1} + E_s \sum_{k=0}^{l-1} \tilde{M}^n_{k,\beta}(t)\right) \end{cases}$$

hold for $l = 1, \dots, s-2$, $n \geq N(t)$ and $t \geq 0$, provided that E_3 is suitable small. These imply the estimate

$$\tilde{M}^n_{l,\beta}(t) \leq C(1 + E_s)^l(|v(0)|_1 + \|v(0)\|_{l,\infty,\beta_1})$$

for $l = 0, 1, \dots, s-2$, $n \geq N(t)$ and $t \geq 0$. Taking the limit as $n \to \infty$ in the above estimate and using the relation (2.8), we obtain the former estimate in Theorem 2.6. In order to show the latter one, it is sufficient to evaluate the right hand sides of the equations (7.6) and (7.7) by using the pointwise estimates obtained above. We omit the details and finish the proof of Theorem 2.6. □

We proceed to prove Theorem 2.7. Let ω be the function defined by (2.19) and put

$$\tilde{M}_{l,\beta,\gamma}(t) = \sup_{0 \leq \tau \leq t, x \in \mathbf{R}^1} \{(1+\tau)^{-(1/2)(l+1)-\gamma}\Phi_\beta(x,\tau)\}^{-1}|\partial_x^l \omega(x,\tau)|, \tag{8.14}$$

where $\gamma < 1/2$. By (2.18), Proposition 2.3 and Theorem 2.6, we have

$$\begin{aligned} |\partial_x^l \omega(x,t)| &\leq |\partial_x^l \theta(x,t)| + |\partial_x^l v(x,t)| \\ &\leq C(E_3 + (1+E_s)^l E_{l,\beta_1})(1+t)^{-(1/2)(l+1)}\Phi_\beta(x,t) \end{aligned} \tag{8.15}$$

for $x \in \mathbf{R}^1$, $t \geq 0$ and $l = 0, 1, \dots, s-2$. Therefore, $\tilde{M}_{l,\beta,\gamma}(t)$ is finite for $t \geq 0$ and $l = 0, 1, \dots, s-2$, and we do not have to use the weight function ϕ_n.

LEMMA 8.22. *Under the same assumptions in Theorem* 2.7, *we have*

$$\tilde{M}_{0,\beta,1/4}(t) \leq C\{(1+E_s)(E_3 + \tilde{E}^{(1)}_{0,\beta_1}) + E_3\tilde{M}_{0,\beta,1/4}(t)\}$$

for $t \geq 0$ *and*

$$\tilde{M}_{l,\beta,1/4}(t) \leq C\left\{(1+E_s)^{l+1}(E_3 + \tilde{E}^{(1)}_{l,\beta_1}) + E_3\tilde{M}_{l,\beta,1/4}(t) + E_s\sum_{k=0}^{l-1}\tilde{M}_{k,\beta,1/4}(t)\right\}$$

for $t \geq 0$ *and* $l = 1, \dots, s-3$, *where we used the notation in Theorem* 2.7.

PROOF. We evaluate each integral $I_i(x,t)$, $i = 1, \dots, 5$, in the right hand side of (7.8). By (7.10), $I_1(x,t)$ can be expressed as

$$I_1(x,t) = \int_{\mathbf{R}^1} \partial_x^{l+1} G_1^*(x-y,t)\tilde{\omega}(y)dy. \tag{8.16}$$

This and Lemma 8.20 yield the estimate

(8.17)

$$|I_1(x,t)| \le \begin{cases} C(|\tilde{\omega}|_2 + |\tilde{\omega}|_{(\beta_1)})t^{-(1/2)(l+1)-1/4}\Phi_\beta(x,t) & \text{when } 1 \le \beta_1 < 3/2, \\ C(|\tilde{\omega}|_2 + |\tilde{\omega}|_{(1)} + |\omega(\cdot,0)|_1 + |\omega(\cdot,0)|_{(\beta_1)})t^{-(1/2)(l+1)-1/4}\Phi_\beta(x,t) & \\ & \text{when } \beta_1 \ge 3/2 \end{cases}$$

for $x \in \mathbf{R}^1$ and $t > 0$. Here, by (7.11), (7.17) and (2.20), we have $|\tilde{\omega}|_{(\beta_1)} \le C(|W_0|_{(\beta_1)} + E_3)$. Moreover, it holds that $|\omega(\cdot,0)|_p \le C(|w_0 - \overline{w}|_p + E_3)$ and $|\omega(\cdot,0)|_{(\beta_1)} \le C(|w_0 - \overline{w}|_{(\beta_1)} + E_3)$. Therefore, we obtain

$$|I_1(x,t)| \le C(E_3 + \tilde{E}^{(1)}_{0,\beta_1})t^{-(1/2)(l+1)-1/4}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ and $t > 0$. By Lemma 8.17, $I_2(x,t)$ is estimated as

$$|I_2(x,t)| \le C(|w_0 - \overline{w}|_1 + \|w_0 - \overline{w}\|_{l,\infty,\beta_1})t^{-(1/2)(l+2)}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ and $t > 0$. By the results of Theorems 2.3 and 2.6, we see that

$$|Q(v(x,t),v(x,t))| \le CE_{0,\beta_1}(1+t)^{-1}\Phi_\beta(x,t)$$

and

$$|\partial_x^l Q(v(x,t),v(x,t))| \\ \le C(1+E_s)^l E_{l-1,\beta_1}(1+t)^{-(1/2)(l+2)}\Phi_\beta(x,t), \qquad 1 \le l \le s-2.$$

Therefore, we can apply Lemma 8.18 with $\alpha = 1$ and $H = Q(v,v)$ to obtain

$$|I_4(x,t)| \le C(1+E_s)^{l+1}E_{l,\beta_1}(1+t)^{-(1/2)(l+2)}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ and $t > 0$. Similarly, we can get

$$|I_5(x,t)| \le C(1+E_s)^{l+1}E_{l,\beta_1}(1+t)^{-(1/2)(l+2)+\varepsilon}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ and $t > 0$, where ε is an arbitrary positive constant. It remains to estimate $I_3(x,t)$, which is decomposed as (7.15). By Proposition 6.1 and 6.2, we have

$$|I_{3,1}(x,t)| \le CE_3(1+t)^{-(1/2)(l+1)-1/4}\Phi_\beta(x,t)$$

and

$$|I_{3,2}(x,t)| \le CE_3(1+t)^{-(1/2)(l+2)}\Phi_\beta(x,t)$$

for $x \in \mathbf{R}^1$ and $t > 0$. By the results of Theorem 2.3 and (8.14), $I_{3,3}(x,t)$ is estimated as

$$|I_{3,3}(x,t)| \le \begin{cases} CE_3\tilde{M}_{0,\beta,\gamma}(t)(1+t)^{-1/2-\gamma}\Phi_\beta(x,t) & \text{when } l=0, \\ C\left(E_3\tilde{M}_{l,\beta,\gamma}(t)+E_s\sum_{k=0}^{l-1}\tilde{M}_{k,\beta,\gamma}(t)\right)(1+t)^{-(1/2)(l+1)-\gamma}\Phi_\beta(x,t) & \\ & \text{when } 1\le l\le s-2 \end{cases}$$

for $x\in\mathbf{R}^1$ and $t>0$. Adding the above estimates with $\gamma=1/4$ together and using (8.15), we obtain the desired estimates. The proof is complete. □

By Lemma 8.22, we obtain the estimate

$$\tilde{M}_{l,\beta,1/4}(t)\le C(1+E_s)^{l+1}(E_3+\tilde{E}^{(1)}_{l,\beta_1}) \tag{8.18}$$

for $t\ge 0$ and $l=0,1,\dots,s-3$, provided that E_3 is suitable small. This is just the estimate stated in Theorem 2.7. This completes the proof of Theorem 2.7. □

Finally, we also assume the condition (2.21). As we mentioned in section 7, in this case the integral $I_{3,1}(x,t)$ is identically zero. Moreover, by (8.16) and Lemma 8.20, we obtain, in place of (8.17),

$$|I_1(x,t)|\le\begin{cases} C(|\tilde{\omega}|_1+|\tilde{\omega}|_{(\beta_1)})t^{-(1/2)(l+2)}\Phi_\beta(x,t) & \text{when } 1\le\beta_1<2, \\ C(|\tilde{\omega}|_1+|\tilde{\omega}|_{(1)}+|\omega(0)|_1+|\omega(0)|_{(\beta_1)})t^{-(1/2)(l+2)}\Phi_\beta(x,t) & \\ & \text{when } \beta_1\ge 2 \end{cases}$$

for $x\in\mathbf{R}^1$ and $t>0$. Therefore, the estimate

$$|I_1(x,t)|\le C(E_3+\tilde{E}^{(2)}_{0,\beta_1})t^{-(1/2)(l+2)}\Phi_\beta(x,t)$$

holds for $x\in\mathbf{R}^1$ and $t>0$. Hence, in place of (8.18) we obtain

$$\tilde{M}_{l,\beta,\gamma}(t)\le C(1+E_s)^{l+1}(E_3+\tilde{E}^{(2)}_{l,\beta_1})$$

for $t\ge 0$ and $l=0,1,\dots,s-3$, provided that E_3 is suitable small. This is just the estimate stated in Theorem 2.8. This completes the proof of Theorem 2.8. □

9. Remark of the order of time decay

The aim in this section is to show that the decay rate with respect to time in Theorems 2.4 and 2.7 is optimal, by considering a particular hyperbolic-elliptic coupled system of the form

$$\begin{cases} v_{1t}+2(v_1v_2)_x+q_{1x}=0, \\ v_{2t}+v_{2x}+(v_1^2)_x+q_{2x}=0, \\ -q_{1xx}+q_1+v_{1x}=0, \\ -q_{2xx}+q_2+v_{2x}=0 \end{cases} \tag{9.1}$$

with the initial conditions

$$v_1(x,0) = \frac{\delta}{(4\pi)^{1/2}} e^{-x^2/4}, \qquad v_2(x,0) = \frac{\delta}{(4\pi)^{1/2}} e^{-(x-1)^2/4}, \tag{9.2}$$

where δ is a parameter. System (9.1) is symmetric and satisfies the stability condition at the zero state. However, it does not satisfy the condition (2.21). The equations for the corresponding self-similar solutions $\theta = (\theta_1, \theta_2)$ are of the forms

$$\begin{cases} \theta_{1t} = \theta_{1xx}, \\ \theta_{2t} + \theta_{2x} = \theta_{2xx}. \end{cases} \tag{9.3}$$

Therefore, θ can be written explicitly as

$$\begin{cases} \theta_1(x,t) = \dfrac{\delta}{(4\pi(1+t))^{1/2}} e^{-x^2/(4(1+t))}, \\[2ex] \theta_2(x,t) = \dfrac{\delta}{(4\pi(1+t))^{1/2}} e^{-(x-(1+t))^2/(4(1+t))}. \end{cases} \tag{9.4}$$

Now, we put $\omega = (\omega_1, \omega_2) = (v_1 - \theta_1, v_2 - \theta_2)$. Then, it holds that $\omega(x,0) \equiv 0$. Therefore, by Theorems 2.3 and 2.4 there exists a positive constant δ_0 such that if $|\delta| \le \delta_0$, then we have the L^p decay estimates

$$\begin{cases} |\partial_x^l \omega(t)|_p \le C|\delta|(1+t)^{-(1/2)(l+1-1/p)-1/4}, \\ |\partial_x^l q(t)|_p \le C|\delta|(1+t)^{-(1/2)(l+2-1/p)} \end{cases} \tag{9.5}$$

for $t \ge 0$ and $l = 0, 1, 2$, where C is a positive constant. Moreover, we have the following proposition which asserts that $|\omega_2(t)|_\infty$ does not decay faster than $(1+t)^{-3/4}$ as $t \to \infty$.

Proposition 9.1. *Let $1 \le p \le \infty$ and $\varepsilon > 0$. There exists a positive constant δ_0 depending only on ε such that if $|\delta| \le \delta_0$, then we have the estimates*

$$\begin{cases} |\omega_1(t)|_p \le C_{13}|\delta|(1+t)^{-(1/2)(2-1/p)+\varepsilon}, \\ C_{14}^{-1}|\delta|^2(1+t)^{-3/4} \le |\omega_2(t)|_\infty \le C_{14}|\delta|(1+t)^{-3/4} \end{cases}$$

for $t \ge T_0$, where $T_0 = T_0(\delta) > 0$, $C_{13} = C_{13}(p, \varepsilon) > 0$ and C_{14} is an absolute constant.

In the following we shall show this proposition. By (9.1) and (9.3), ω satisfies the system

$$\begin{cases} \omega_{1t} = \omega_{1xx} - 2((\theta_1 + \omega_1)(\theta_2 + \omega_2))_x - q_{1xxx}, \\ \omega_{2t} + \omega_{2x} = \omega_{2xx} - ((\theta_1 + \omega_1)^2)_x - q_{2xxx}. \end{cases}$$

Therefore, we can express ω as

$$(9.6)\quad \begin{cases} \omega_1(t) = -2\displaystyle\int_0^t K_{1x}(t-\tau) * ((\theta_1+\omega_1)(\theta_2+\omega_2))(\tau)d\tau \\ \qquad\qquad - \displaystyle\int_0^t K_{1xxx}(t-\tau) * q_1(\tau)d\tau, \\ \omega_2(t) = -\displaystyle\int_0^t K_{2x}(t-\tau) * (\theta_1+\omega_1)^2(\tau)d\tau - \int_0^t K_{2xxx}(t-\tau) * q_2(\tau)d\tau, \end{cases}$$

where $K_1(x,t)$ and $K_2(x,t)$ are the Green's functions of the first and the second equations in (9.3), respectively, and written as

$$(9.7)\qquad K_1(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/(4t)}, \qquad K_2(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-t)^2/(4t)}.$$

We decompose ω_2 as

$$(9.8)\qquad \omega_2 = \omega_2^{(1)} + \omega_2^{(2)},$$

where

$$\omega_2^{(1)}(x,t) = -\int_0^t K_{2x}(t-\tau) * \theta_1(\tau)^2 d\tau.$$

LEMMA 9.1. *There exist positive constants c and T such that for any $\delta \in \mathbf{R}^1$, we have*

$$|\omega_2^{(1)}(t)|_\infty \geq c\delta^2(1+t)^{-3/4}$$

for $t \geq T$.

PROOF. By (9.4) and (9.7), we have

$$\omega_2^{(1)}(x,t) = -\frac{\delta^2}{16\pi^{3/2}} \int_0^t \int_{\mathbf{R}^1} (t-\tau)^{-3/2}(1+\tau)^{-1}(x-y-(t-\tau)) \\ \times e^{-(x-y-(t-\tau))^2/(4(t-\tau))} e^{-y^2/(2(1+\tau))}\, dyd\tau.$$

Using the identity

$$\frac{(x-y-(t-\tau))^2}{4(t-\tau)} + \frac{y^2}{2(1+\tau)} \\ = \frac{2(1+t)-(1+\tau)}{4(t-\tau)(1+\tau)} \left(y - \frac{1+\tau}{2(1+t)-(1+\tau)}(x-(t-\tau)) \right)^2 \\ + \frac{(x-(t-\tau))^2}{2(2(1+t)-(1+\tau))},$$

we see that

$$(9.9)\quad \omega_2^{(1)}(x,t) = -\frac{\delta^2}{16\pi^{3/2}}\int_0^t\int_{\mathbf{R}^1}(t-\tau)^{-3/2}(1+\tau)^{-1}$$
$$\times\left(\frac{2(t-\tau)}{2(1+t)-(1+\tau)}(x-(t-\tau))-z\right)$$
$$\times\, e^{-(2(1+t)-(1+\tau))/(4(t-\tau)(1+\tau))z^2}e^{-(x-(t-\tau))^2/(2(2(1+t)-(1+\tau)))}\,dzd\tau$$
$$= -\frac{\delta^2}{4\pi}\int_0^t(1+\tau)^{-1/2}(2(1+t)-(1+\tau))^{-3/2}(x-(t-\tau))$$
$$\times\, e^{-(x-(t-\tau))^2/(2(2(1+t)-(1+\tau)))}\,d\tau.$$

Particularly, we have

$$|\omega_2^{(1)}(1+t,t)| = \frac{\delta^2}{4\pi}\int_0^t(1+\tau)^{1/2}(2(1+t)-(1+\tau))^{-3/2}e^{-(1+\tau)^2/(2(2(1+t)-(1+\tau)))}\,d\tau$$
$$\geq \frac{\delta^2}{8\sqrt{2}\pi}(1+t)^{-3/2}\int_0^t(1+\tau)^{1/2}e^{-(1+\tau)^2/(2(1+t))}\,d\tau$$
$$= \frac{2^{1/4}}{8\pi}\delta^2(1+t)^{-3/4}\int_{(2(1+t))^{-1/2}}^{((1+t)/\sqrt{2})^{1/2}} s^{1/2}e^{-s^2}\,ds,$$

which implies the desired estimate. The proof is complete. □

Lemma 9.2. *For any $\delta\in\mathbf{R}^1$ and $1\leq p\leq\infty$, we have*

$$\left|\int_0^t K_{1x}(t-\tau)(\theta_1\omega_2^{(1)})(\tau)d\tau\right|_p \leq C|\delta|^3(1+t)^{-(1/2)(2-1/p)}(1+\log(1+t))$$

for $t>0$, where $C=C(p)>0$.

Proof. We denote by $I(x,t)$ the integral in the left hand side of the above estimate. By (9.9), we have

$$|\omega_2^{(1)}(y,\tau)| \leq C\delta^2(1+\tau)^{-1}\int_0^\tau(1+s)^{-1/2}e^{-(y-(\tau-s))^2/(8(1+\tau))}\,ds.$$

This together with (9.4) and (9.7) implies that

$$|I(x,t)| \leq C|\delta|^3\int_0^t\int_0^\tau\int_{\mathbf{R}^1}(t-\tau)^{-1}(1+\tau)^{-3/2}(1+s)^{-1/2}$$
$$\times\, e^{-(x-y)^2/(8(t-\tau))}e^{-y^2/(4(1+\tau))}e^{-(y-(\tau-s))^2/(8(1+\tau))}\,dydsd\tau.$$

Using the identity

$$\begin{aligned}
&\frac{(x-y)^2}{8(t-\tau)}+\frac{y^2}{16(1+\tau)}+\frac{(y-(\tau-s))^2}{16(1+\tau)}\\
&\quad=\frac{1+t}{8(t-\tau)(1+\tau)}\left\{y-\frac{(t-\tau)(1+\tau)}{1+t}\left(\frac{x}{t-\tau}+\frac{\tau-s}{2(1+\tau)}\right)\right\}^2\\
&\qquad+\frac{(x-(\tau-s)/2)^2}{8(1+t)}+\frac{(\tau-s)^2}{32(1+\tau)},
\end{aligned}$$

we obtain

$$\begin{aligned}
|I(t)|_p &\le C|\delta|^3(1+t)^{-(1/2)(1-1/p)}\\
&\quad\times\int_0^t\int_0^\tau (t-\tau)^{-1/2}(1+\tau)^{-1}(1+s)^{-1/2}\mathrm{e}^{-(\tau-s)^2/(32(1+\tau))}\,dsd\tau.
\end{aligned}$$

Since

$$\int_0^\tau (1+s)^{-1/2}\mathrm{e}^{-(\tau-s)^2/(32(1+\tau))}\,ds\le C$$

for $\tau>0$ with a positive constant C, we see that

$$\begin{aligned}
|I(t)|_p &\le C|\delta|^3(1+t)^{-(1/2)(1-1/p)}\int_0^t (t-\tau)^{-1/2}(1+\tau)^{-1}d\tau\\
&\le C|\delta|^3(1+t)^{-(1/2)(2-1/p)}(1+\log(1+t))
\end{aligned}$$

for $t>0$. This completes the proof. □

LEMMA 9.3. *Let* $1\le p\le\infty$ *and* $\varepsilon>0$. *There exists a positive constant* $\delta_0=\delta_0(\varepsilon)$ *such that if* $|\delta|\le\delta_0$, *then we have*

$$|\omega_1(t)|_p+|\omega_2^{(2)}(t)|_p\le C|\delta|(1+t)^{-(1/2)(2-1/p)+\varepsilon},$$

for $t\ge0$, *where* $C=C(p,\varepsilon)>0$.

PROOF. Put

$$\tilde{M}_p(t)=\sup_{0\le\tau\le t}(1+\tau)^{(1/2)(2-1/p)-\varepsilon}(|\omega_1(\tau)|_p+|\omega_2^{(2)}(\tau)|_p).$$

By (9.6), we have

$$\begin{aligned}\omega_1(t) &= -2\int_0^t K_{1x}(t-\tau)(\theta_1\theta_2)(\tau)d\tau - 2\int_0^t K_{1x}(t-\tau)(\theta_1\omega_2^{(1)})(\tau)d\tau \\ &\quad - 2\int_0^t K_{1x}(t-\tau)(\theta_1\omega_2^{(2)} + (\theta_2+\omega_2)\omega_1)(\tau)d\tau - \int_0^t K_{1xxx}(t-\tau)q_1(\tau)d\tau \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t).\end{aligned}$$

We evaluate each integral $I_i(t)$, $i = 1, \dots, 4$, as follows. By Proposition 6.2 and Lemma 9.2, we have

$$\begin{cases} |I_1(t)|_p \le C\delta^2(1+t)^{-(1/2)(2-1/p)}, \\ |I_2(t)|_p \le C|\delta|^3(1+t)^{-(1/2)(2-1/p)}(1+\log(1+t)) \end{cases}$$

for $t > 0$ and $1 \le p \le \infty$. $I_3(t)$ and $I_4(t)$ are estimated as

$$\begin{aligned}|I_3(t)|_p &\le \int_0^{t/2} |K_{1x}(t-\tau)|_p(|\theta(t)|_1 + |\omega(t)|_1)(|\omega_1(t)|_\infty + |\omega_2^{(2)}(\tau)|_\infty)d\tau \\ &\quad + \int_{t/2}^t |K_{1x}(t-\tau)|_1(|\theta(t)|_p + |\omega(t)|_p)(|\omega_1(t)|_\infty + |\omega_2^{(2)}(\tau)|_\infty)d\tau \\ &\le C|\delta|t^{-(1/2)(1-1/p)}\int_0^t (t-\tau)^{-1/2}(1+\tau)^{-1+\varepsilon}d\tau \tilde{M}_\infty(t) \\ &\le C|\delta|(1+t)^{-(1/2)(2-1/p)+\varepsilon}\tilde{M}_\infty(t)\end{aligned}$$

and

$$\begin{aligned}|I_4(t)|_p &\le \int_0^{t/2} |K_{1xxx}(t-\tau)|_p|q_1(\tau)|_1 d\tau + \int_{t/2}^t |K_{1x}(t-\tau)|_1|q_{1xx}(\tau)|_p d\tau \\ &\le C|\delta|t^{-(1/2)(3-1/p)}\int_0^t (t-\tau)^{-1/2}(1+\tau)^{-1/2}d\tau \le C|\delta|t^{-(1/2)(3-1/p)}\end{aligned}$$

for $t > 0$ and $1 \le p \le \infty$. Therefore, we obtain

$$|\omega_1(t)|_p \le C|\delta|(1+t)^{-(1/2)(2-1/p)+\varepsilon}(1+\tilde{M}_\infty(t)) \tag{9.10}$$

for $t \ge 0$ and $1 \le p \le \infty$. In view of the relation

$$\omega_2^{(2)}(t) = -\int_0^t K_{2x}(t-\tau) * ((2\theta_1+\omega_1)\omega_1)(\tau)d\tau - \int_0^t K_{2xxx}(t-\tau) * q_2(\tau)d\tau,$$

we see that $\omega_2^{(2)}(t)$ also satisfies the estimate in (9.10). Thus, we get the estimate

$$\tilde{M}_p(t) \le C|\delta|(1+\tilde{M}_\infty(t))$$

for $t \geq 0$ and $1 \leq p \leq \infty$, which yields that the estimate $\tilde{M}_p(t) \leq C|\delta|$ holds for $t \geq 0$ and $1 \leq p \leq \infty$, provided that δ is suitable small. This shows the desired estimate. The proof is complete. □

Lemmas 9.1 and 9.3, and the relation (9.8) prove Proposition 9.1.

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