Removability of sets for sub-polyharmonic functions

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Abstract. Our first aim in this paper is to generalize Bôcher's theorem for functions \( u \) whose Riesz measure \( \mu = \Delta^m u \) is nonnegative in the punctured unit ball \( B_0 \). In fact, if \( u \) satisfies a certain integral condition and \( \mu = \Delta^m u \geq 0 \) in \( B_0 \), then it is shown that \( u \) can be written as the sum of a generalized potential of \( \mu \) and a polyharmonic function on \( B \). This is nothing but the Laurent series expansion for \( u \).

The next aim is to give a polyharmonic version of the recent results by Riihentaus [11] concerning removability of sets for subharmonic functions.

1. Introduction and statement of results

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with a point \( x = (x_1, x_2, \ldots, x_n) \). For a multi-index \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), we set

\[
|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,
\]

\[
x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n},
\]

and

\[
D^\lambda = \left( \frac{\partial}{\partial x_1} \right)^{\lambda_1} \left( \frac{\partial}{\partial x_2} \right)^{\lambda_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\lambda_n}.
\]

We denote by \( B(x, r) \) the open ball centered at \( x \) with radius \( r > 0 \), whose boundary is written as \( S(x, r) = \partial B(x, r) \). We also denote by \( B \) the unit ball \( B(0, 1) \) and by \( B_0 \) the punctured unit ball \( B \setminus \{0\} \).

A real-valued function \( u \) on an open set \( G \subset \mathbb{R}^n \) is called polyharmonic of order \( m \) on \( G \) if \( u \in C^{2m}(G) \) and \( \Delta^m u = 0 \) on \( G \), where \( m \) is a positive integer, \( \Delta \) denotes the Laplacian and \( \Delta^m u = \Delta^{m-1}(\Delta u) \) (cf. [2], [10]). We denote by \( H^m(G) \) the space of polyharmonic functions of order \( m \) on \( G \). In particular, \( u \) is harmonic on \( G \) if \( u \in H^1(G) \).

The fundamental solution of \( \Delta^m \) is written as \( R_{2m} \), that is,

\[
R_{2m}(x) = \begin{cases} 
\frac{x_m |x|^{2m-n}}{n!} & \text{if } 2m - n \text{ is not an even nonnegative integer}, \\
\frac{x_m |x|^{2m-n} \log(1/|x|)}{n!} & \text{if } 2m - n \text{ is an even nonnegative integer},
\end{cases}
\]

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where the constant $a_m$ is chosen so that $\Delta^m R_{2m}$ is the Dirac measure $\delta$ at the origin. We denote by $R_{2m, L}$ the remainder term of Taylor expansion of $R_{2m}$:

$$R_{2m, L}(\zeta, x) = R_{2m}(\zeta - x) - \sum_{|\lambda| \leq L} \frac{\zeta^\lambda}{\lambda!} (D^\lambda R_{2m})(-x)$$

for a nonnegative integer $L$.

We say that a locally integrable function $u$ on an open set $G \subset \mathbb{R}^n$ is sub-polyharmonic of order $m$ in $G$ if $\Delta^m u \geq 0$ in $G$ in the weak sense, that is,

$$\int_G u(x) \Delta^m \varphi(x) dx \geq 0 \quad \text{for all nonnegative } \varphi \in C_0^\infty(G).$$

Our first aim in this note is to establish Bôcher's theorem for sub-polyharmonic functions $u \in L^1_{loc}(2B_0)$, where $2B_0 = B(0, 2) \setminus \{0\}$; for polyharmonic functions, we refer the reader to the previous paper [3] as a generalization of Armitage [1].

**Theorem 1.** Suppose that $u \in L^1_{loc}(2B_0)$ and $\mu = \Delta^m u$ is a nonnegative measure on $2B_0$. If $u$ satisfies

$$\int_{2B_0} |u(x)| |x|^s dx < \infty$$

for some number $s \geq \max\{-2m, -n\}$, then

$$u(x) = \int_{B_0} R_{2m, L}(\zeta, x) d\mu(\zeta) + h(x) + \sum_{|\lambda| \leq L} C(\lambda) D^\lambda R_{2m}(x)$$

for a.e. $x \in B_0$, where $L$ is the integer such that $s + 2m - 1 < L \leq s + 2m$, $h \in H^m(B)$ and $C(\lambda)$ denote constants.

The above expression is called the Laurent series expansion for $u$.

To prove Theorem 1, we first show that the generalized potential $\int_{B_0} R_{2m, L}(\zeta, x) d\mu(\zeta)$ satisfies condition (1) for $s' > s$, and then apply Bôcher's theorem for polyharmonic functions on $B_0$ given in [3].

Next we discuss removability of sets for sub-polyharmonic functions in $\mathbb{R}^n$.

We say that a continuous function $h$ on $[0, \infty)$ is a measure function if $h(0) = 0$, $h$ is nondecreasing and

$$h(2r) \leq Mh(r) \quad \text{for all } r > 0,$$

where $M$ is a positive constant. For $\varepsilon > 0$ and $E \subset \mathbb{R}^n$, write

$$E_\varepsilon = \{x \in \mathbb{R}^n : d(x, E) < \varepsilon\},$$

where $d(x, E)$ denotes the distance of $x$ from $E$, that is, $d(x, E) = \inf\{|x - y| : y \in E\}$. Then the upper Minkowski $h$-content of $E$ is defined by
\[ \mathcal{M}_h(E) = \limsup_{\varepsilon \to 0+} \frac{|E_\varepsilon|}{h(\varepsilon)}, \]

where $|F|$ denotes the $n$-dimensional Lebesgue measure of a set $F$. If $h(r) = r^{n-\alpha}$, $0 \leq \alpha < n$, then we write $\mathcal{M}_\alpha$ for $\mathcal{M}_h$.

We introduce the result by Riihentaus [11] (see also Gardiner [4]).

**Theorem A (Riihentaus).** Let $\alpha \in [0, n-2]$ and let $E$ be a closed set in $\Omega$ such that $\mathcal{M}_\alpha(E) = 0$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies
\[ f(x) \leq d(x, E)^{2+\alpha} \text{ for all } x \in \Omega \setminus E, \]
then $f$ has a subharmonic extension to $\Omega$.

Now we state the following theorem.

**Theorem 2.** Let $h$ be a measure function. Suppose $E$ is a closed set in $\Omega$ such that $\mathcal{M}_h(E) = 0$. If $u \in L^1_{\text{loc}}(\Omega \setminus E)$ is sub-polyharmonic of order $m$ in $\Omega \setminus E$ and satisfies
\[ |u(x)| \leq d(x, E)^{2mh(d(x, E))^{-1}} \text{ for all } x \in \Omega \setminus E, \]
then $u$ has a sub-polyharmonic extension to $\Omega$ of order $m$.

Let $h$ and $k$ be two measure functions on $[0, \infty)$ such that
\[ \lim_{r \to 0} \frac{k(r)}{h(r)} = 0. \]

In Theorem 2, if $\mathcal{M}_k(E) < \infty$ and
\[ |u(x)| \leq d(x, E)^{2mh(d(x, E))^{-1}} \text{ for all } x \in \Omega \setminus E, \]
then $u$ is shown to have a sub-polyharmonic extension to $\Omega$ (see also Riihentaus [11, Theorem 2]).

**2. Lemmas**

Throughout this paper, let $M$ denote various constants, not necessarily the same on any two occurrences.

We need several lemmas to prove Theorem 1.

**Lemma 1.** If $u$ and $\mu$ are as in Theorem 1, then
\[ \int_{A(r)} d\mu(\zeta) \leq Mr^{-2m} \int_{C(r)} |u(\zeta)| d\zeta \]
whenever $0 < r < \frac{1}{2}$, where $A(r) = \{ r \leq |x| < 2r \}$ and $C(r) = \{ r/2 < |x| < 4r \}$. 
Proof. Consider a function $\psi \in C^\infty_0(C(1))$ such that $\psi \geq 0$ and

$$\psi(x) = \begin{cases} 1 & \text{if } 1 \leq |x| \leq 2, \\ 0 & \text{if } |x| \leq 1/2 \text{ or } |x| \geq 4. \end{cases}$$

If we set $\psi_r(x) = \psi(x/r)$ for $0 < r < 1/2$, then

$$\int_{A(r)} d\mu(\zeta) \leq \int_{C(r)} \psi_r d\mu(\zeta) = \int_{C(r)} (A^m \psi_r) u d\zeta \leq \int_{C(r)} |A^m \psi_r| |u| d\zeta \leq Mr^{-2m} \int_{C(r)} |u| d\zeta.$$

This proves Lemma 1.

**Lemma 2.** If $u$ and $\mu$ are as above, then

$$\int_{B_0} |\zeta|^\ell d\mu(\zeta) < \infty$$

whenever $\ell \geq s + 2m$.

Proof. Let $A_j = A(2^{-j})$ and $C_j = C(2^{-j})$; then we have by Lemma 1

$$\int_{B_0} |\zeta|^\ell d\mu(\zeta) = \sum_{j=1}^\infty \int_{A_j} |\zeta|^\ell d\mu(\zeta) \leq \sum_{j=1}^\infty 2^{(-j+1)} \int_{A_j} d\mu(\zeta) \leq M \sum_{j=1}^\infty 2^{(-j+1)+2m} \int_{C_j} |u(\zeta)| d\zeta \leq M \sum_{j=1}^\infty \int_{C_j} |u(\zeta)| |\zeta|^\ell d\zeta \leq M \int_{2B_0} |u(\zeta)| |\zeta|^\ell d\zeta < \infty.$$

We put $I(x) = \int_{B_0} |R_{2m}(\zeta, x)| d\mu(\zeta)$, where $L$ is the integer such that $s + 2m - 1 < L \leq s + 2m$; note here that $L \geq 0$ and $L \geq 2m - n$ because $s \geq \max\{-2m, -n\}$. For $x \in B_0$, consider the sets
If $2m \geq n$, then we see from [6, Lemma 4.2] and [9, Lemmas 6, 8, 9] that

$$I(x) \leq M \int_{E_1} |\zeta|^{L+1} |x|^{2m-n-L-1} d\mu(\zeta)$$

$$+ M \int_{E_1} \left( |\zeta|^{2m-n} + |\zeta - x|^{2m-n} \log \frac{|\zeta|}{|\zeta - x|} \right) d\mu(\zeta)$$

$$+ M \int_{E_1} |\zeta| L |x|^{2m-n-L} \log \frac{4|\zeta|}{|x|} d\mu(\zeta)$$

$$= M \{ I_1(x) + I_2(x) + I_3(x) \};$$

if $2m < n$, then $I_2(x)$ is replaced by

$$I_2(x) = \int_{E_2} |\zeta - x|^{2m-n} d\mu(\zeta).$$

We prove the following lemma.

**Lemma 3.** If $\mu$ is a nonnegative measure on $B_0$ satisfying (6) and $s' > s \geq \max\{-2m, -n\}$, then

$$\int_B I(x)|x'|^s dx < \infty.$$  

**Proof.** We have only to treat $s'$ satisfying $s' > s$ and

$$s' - 1 < L - 2m < s'.$$

First, since $(2m-n-L-1+s') + n = s' - (L-2m+1) < 0$, we have

$$\int_B I_1(x)|x'|^s dx \leq M \left( \int_{E_1} |\zeta|^{L+1} |x|^{2m-n-L-1} d\mu(\zeta) \right) |x'|^s dx$$

$$\leq M \int_{B_0} |\zeta|^{L+1} \left( \int_{|x| \geq |\zeta|} |x|^{2m-n-L-1+s'} dx \right) d\mu(\zeta)$$

$$= M \int_{B_0} |\zeta|^{2m+s'} d\mu(\zeta) < \infty$$

with the aid of (6).
Next, noting that $|\zeta|/2 < |x| < 2|\zeta|$ when $\zeta \in E_2$, we have
\[
\int_B I_2(x)|x|' \, dx = \int_B \left\{ \int_{E_2} \left( |\zeta|^{2m-n} + |\zeta - x|^{2m-n} \log \frac{|\zeta|}{|\zeta - x|} \right) \mu(\zeta) \right\} |x|' \, dx
\]
\[
\leq \int_{B_0} |\zeta|^{2m-n} \left( \int_{\{|x|/2 < |x| < 2|\zeta|\}} |x|' \, dx \right) \mu(\zeta)
\]
\[
+ \int_{B_0} \left( \int_{\{|\zeta - x| \leq |\zeta|/2\}} |\zeta - x|^{2m-n} \log \frac{|\zeta|}{|\zeta - x|} |x|' \, dx \right) \mu(\zeta)
\]
\[
\leq M \int_{B_0} |\zeta|^{2m+n'} \mu(\zeta)
\]
\[
+ M \int_{B_0} |\zeta|^{n'} \left( \int_{|x|/2 < |x| < 2|\zeta|} |\zeta - x|^{2m-n} \log \frac{|\zeta|}{|\zeta - x|} \, dx \right) \mu(\zeta)
\]
\[
\leq M \int_{B_0} |\zeta|^{2m+n'} \mu(\zeta) < \infty.
\]
Finally, since $(2m - n - L + s') + n = s' - (L - 2m) > 0$, we establish
\[
\int_B I_3(x)|x|' \, dx = \int_B \left\{ \int_{E_2} |\zeta|^L |x|^{2m-n-L} \log \frac{4|\zeta|}{|x|} \mu(\zeta) \right\} |x|' \, dx
\]
\[
\leq \int_{B_0} |\zeta|^L \left( \int_{\{|x|/2 < 2|\zeta|\}} |x|^{2m-n-L+s'} \log \frac{4|\zeta|}{|x|} \, dx \right) \mu(\zeta)
\]
\[
\leq M \int_{B_0} |\zeta|^{2m+n'} \mu(\zeta) < \infty.
\]
Thus we have obtained
\[
\int_B I(x)|x|' \, dx < \infty,
\]
as required.

**Lemma 4.** If $u$ and $\mu$ are as above, then
\[
v(x) \equiv u(x) - \int_{B_0} R_{2m,L}(\zeta,x) \mu(\zeta) \in H^m(B_0)
\]
with $L$ as before.

**Proof.** It is sufficient to show that $D^m v = 0$ in $B_0$ in the weak sense.
Let $\varphi \in C_{0}^{\infty}(B_{0})$. In view of Lemma 3, we can apply Fubini’s theorem to obtain

$$
\langle u - v, A^{m}\varphi \rangle = \left( \int_{B_{0}} \left( R_{2m}(-x) - \sum_{|\lambda| \leq L} \frac{\zeta^{\lambda}}{\lambda!} (D^{\lambda}R_{2m})(-x) \right) d\mu(\zeta), A^{m}\varphi \right)
$$

$$
= \int_{B_{0}} \left( \int_{B_{0}} \left( R_{2m}(-x) - \sum_{|\lambda| \leq L} \frac{\zeta^{\lambda}}{\lambda!} (D^{\lambda}R_{2m})(-x) \right) A^{m}\varphi(x) dx \right) d\mu(\zeta)
$$

$$
= \int_{B_{0}} \varphi(\zeta) d\mu(\zeta)
$$

$$
= \langle u, A^{m}\varphi \rangle,
$$

since $\varphi$ vanishes in a neighborhood of the origin. This proves

$$
\langle v, A^{m}\varphi \rangle = 0,
$$

as required.

3. Proof of Theorem 1

From Lemmas 3 and 4, we see that $v \in H^{m}(B_{0})$ and

$$
\int_{B_{0}} |v(x)| |x|^{s'} dx < \infty
$$

for all $s' > s$. In view of [3], we can find $h \in H^{m}(B)$ and constants $C(\lambda)$ for which

$$
v(x) = h(x) + \sum_{|\lambda| \leq L} C(\lambda) D^{\lambda} R_{2m}(x)
$$

holds a.e. on $B_{0}$, where $L$ is the integer such that $s + 2m - 1 < L \leq s + 2m$. This implies that $u$ is of the form

$$
u(x) = \int_{B_{0}} R_{2m,L}(\zeta,x) d\mu(\zeta) + h(x) + \sum_{|\lambda| \leq L} C(\lambda) D^{\lambda} R_{2m}(x)
$$

for a.e. $x \in B_{0}$, as required.

In case $m = 1$, our theorem gives the following simple result.
COROLLARY. If \( u \) is a subharmonic function on \( 2B_0 \) satisfying

\[
\int_{2B_0} u^+(x)|x|^{-2}dx < \infty, \tag{7}
\]
then \( u \) can be extended to a subharmonic function on \( B \), where \( u^+(x) = \max\{u(x), 0\} \).

**Proof.** Since \( u^+ \) is subharmonic on \( 2B_0 \) and satisfies (1) with \( s = -2 \), we can take \( L = 0 \) in Theorem 1, and show that \( u^+ \) is of the form

\[
u^+(x) = \int_{B_0} R_2(\zeta - x)d\mu(\zeta) + h(x) + CR_2(x)
\]
for \( x \in B_0 \), where \( \mu = A u^+ \geq 0 \), \( \mu(B_0) < \infty \), \( h \) is harmonic in \( B \) and \( C \) is a constant. In view of (7),

\[
\liminf_{r \to 0} r^{-1} \int_{S(0, r)} u^+(x)dS(x) = 0.
\]
Moreover, by [8, Theorem 4.3.1] we see easily that

\[
\lim_{r \to 0} r^{-1} \left( \int_{S(0, r)} R_2(\zeta - x)d\mu(\zeta) \right) dS(x) = 0,
\]
where \( \kappa(r) = 1 \) for \( n \geq 3 \) and \( \kappa(r) = \log(1/r) \) for \( n = 2 \), which shows that \( C = 0 \). Thus \( u^+ \) is extended to a subharmonic function on \( B \). Since \( u \leq u^+ \), \( u \) is bounded above near the origin, so that \( u \) is extended to a subharmonic function on \( B \) by [6, Theorem 5.18].

4. Removability of sets

To prove Theorem 2, we need the following lemma, which is a version of partition of unity (cf. [7]).

**Lemma 5.** Let \( \{B_i : i = 1, \ldots, N\}, \ B_i = B(x_i, r_i) \), be a finite collection of balls such that \( \{S^{-1}B_i\} \) is mutually disjoint. Then there is a family of nonnegative functions \( \varphi_i \in C^\infty_c(\mathbb{R}^n) \) with support \( \text{supp} \varphi_i \subset 2B_i \) such that \( \sum_{i=1}^N \varphi_i(x) = 1 \) for \( x \in \bigcup_{i=1}^N B_i \). Furthermore, for each multi-index \( \lambda \), there is a constant \( C_{\lambda} \) such that

\[
|D^j \varphi_i(x)| \leq C_{\lambda} r_i^{-|\lambda|} \quad \text{for all } x \in \mathbb{R}^n \text{ and } i = 1, \ldots, N. \tag{8}
\]

**Proof of Theorem 2.** By our assumption that \( \mathcal{M}_h(E) = 0 \), for \( \varepsilon > 0 \), there is \( r_0, \ 0 < r_0 < 1 \), such that

\[
|E_r| \leq \varepsilon h(r) \quad \text{whenever } 0 \leq r \leq r_0. \tag{9}
\]
We first show that
\[
\int_{E \setminus E} d(x, E)^{2m} h(d(x, E))^{-1} \, dx \leq M r^{2m} e. \tag{10}
\]
If we put \( K_j = \{ x \in \mathbb{R}^n \mid d(x, E) < r 2^{-j} \} \), then
\[
E \setminus E = \bigcup_{j=0}^{\infty} (K_j \setminus K_{j+1}).
\]
Hence we have by (9)
\[
\int_{E \setminus E} d(x, E)^{2m} h(d(x, E))^{-1} \, dx = \sum_{j=0}^{\infty} \int_{K_j \setminus K_{j+1}} d(x, E)^{2m} h(d(x, E))^{-1} \, dx
\]
\[
\leq \sum_{j=0}^{\infty} (r 2^{-j})^{2m} h(r 2^{-j+1})^{-1} |K_j|
\]
\[
\leq M r^{2m} e \sum_{j=0}^{\infty} 2^{-2mj}
\]
\[
= M r^{2m} e. \tag{11}
\]
From (4) and (11) it follows that
\[
\int_{E \setminus E} |u| \, dx \leq M r^{2m} e. \tag{12}
\]
If we set \( u = 0 \) on \( E \), then we see that \( u \in L^1_{\text{loc}}(\Omega) \).

Next we show that
\[
\int_{\Omega} u(x) A^m \phi(x) \, dx \geq 0 \tag{13}
\]
for nonnegative \( \phi \in C^\infty_0(\Omega) \). We may assume that \( 0 \leq \phi \leq 1 \) and \( |D^j \phi| \leq 1 \) for every multi-index \( |\lambda| \leq 2m \). We put \( K = \text{supp} \phi \) and take \( r_0 > 0 \) such that \( K_{r_0} \subset \Omega \).

Let \( 0 < 4r < r_0 \). By a covering lemma, we can find a finite collection of balls \( B_i = B(x_i, r) \) such that \( \{ 5^{-1} B_i \} \) is mutually disjoint and
\[
\bigcup_{i=1}^{N} B_i \supset K.
\]
By re-indexing if necessary, we can find \( N^* \) such that
Let \( \varphi_i \) be as in Lemma 5. Since \( u \) is sub-polyharmonic of order \( m \) in \( \Omega \setminus E \), we see that

\[
\int_{2B_i} u A^m (\varphi \varphi_i) dx \geq 0
\]

for \( i = N^* + 1, \ldots, N \), so that

\[
\int_{\Omega} u A^m \varphi \ dx = \int_{\Omega} u A^m \left\{ \varphi \left( \sum_{i=1}^{N} \varphi_i \right) \right\} \ dx
\]

\[
= \sum_{i=1}^{N} \int_{2B_i} u A^m (\varphi \varphi_i) dx
\]

\[
\geq \sum_{i=1}^{N^*} \int_{2B_i} u A^m (\varphi \varphi_i) dx
\]

\[
\geq -\frac{M}{r^{2m}} \sum_{i=1}^{N^*} \int_{2B_i} |u| dx
\]

with the aid of (8). Thus by (12) we have

\[
\int_{\Omega} u A^m \varphi \ dx \geq -M \epsilon,
\]

which gives (13). Consequently, \( u \) is sub-polyharmonic of order \( m \) in \( \Omega \).

For a measure function \( h \) and \( f \in L^1_{loc}(\Omega) \), define

\[
A_{f,h}(x) = \sup_{B} r^{-2m} h(r)^{-1} \inf_{v} \int_{B} |f(y) - v(y)| dy,
\]

where the supremum is taken over all balls \( B = B(x,r) \subset \Omega \) and the infimum is taken over all \( v \in L^1_{loc}(\Omega) \) such that \( A^m v \geq 0 \) on \( B \). Further consider the set \( S_f \) of all \( x \in \Omega \) such that

\[
\lim_{r \to 0} \sup_{r} r^{-n-2m} \int_{B(x,r)} |f(y) - v(y)| dy > 0
\]

for all functions \( v \in L^1_{loc}(\Omega) \) satisfying \( A^m v \geq 0 \) on a neighborhood of \( x \).

As in [7] we can prove the following theorem, which gives an extension of a theorem in Gardiner [4].
Theorem 3. If \( A_{u,h} \in L^\infty(\Omega) \) and \( H_h(S_u) = 0 \), then \( u \) has a sub-polyharmonic extension to \( \Omega \), where \( H_h \) denotes the Hausdorff measure with a measure function \( h \).

5. Remarks on Theorem 1

Suppose that \( u \in L^1_{\text{loc}}(2B_0) \) and \( \mu = A^m u \) is a nonnegative measure on \( 2B_0 \). Then, as in the book of Hayman-Kennedy [6], \( u \) can be represented as

\[
u(x) = \int_{R_0} R_{2m,L(\|\cdot\|)}(\zeta,x)\,d\mu(\zeta) + h(x) + \sum_{\lambda} C(\lambda) D^{\ell} R_{2m}(x)
\]

for a.e. \( x \in B_0 \), where \( L(r) \) is a nonincreasing positive function on \((0,1] \), \( h \in H^m(B) \) and \( C(\lambda) \) denote constants. To prove this, we use the estimate

\[
|R_{2m,L(\|\cdot\|),\ell}(\zeta,x)| \leq M^{\ell+1} \frac{1}{|\zeta|^2m-n-\ell-1} \]

whenever \( 2|\zeta| \leq |x| \) and \( 2m-n < \ell + 1 \), where \( M \) is a positive constant depending only on \( m \) and \( n \).

Thus our theorem gives a condition which assures that \( L \) is bounded and the above sum contains only finite terms.

Remark. We do not know whether \( u \) has a similar Laurent expansion or not, if we replace condition (1) by

\[
\int_{2B_0} u^+(x)|x|^\ell \,dx < \infty.
\]

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