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Asymptotic behaviors of least-energy solutions to Matukuma type equations with an inverse square potential

Dedicated to Professors Mitsuru Ikawa and Sadao Miyatake on the occasion of their sixtieth birthdays

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Abstract. Behavior of least-energy solutions to Matukuma type equations with an inverse square potential are discussed. The difference of the behavior of solutions are obtained. We also consider the behavior of scaled solutions and obtain a limiting function.

1. Introduction

This is a note on the behavior of least-energy solutions to

$$\Delta u + K(x)u^{1+\varepsilon} = 0 \qquad \text{in } \mathbf{R}^n \tag{1.1}$$

as $\varepsilon \downarrow 0$ under the condition

$$\text{(K)} \qquad \begin{cases} K(x) \in C^1(\mathbf{R}^n), K(x) > 0 & \text{in } \mathbf{R}^n, \\ x \cdot \nabla K(x) + 2K(x) \geq 0, \not\equiv 0 & \text{on } \mathbf{R}^n, \\ \lim\limits_{|x|\to\infty} |x|^2 K(x) = c_0 > 0. \end{cases}$$

In some cases, we assume further that $K(x)$ satisfies

$$\text{(K.1)} \qquad K(x) = |x|^{-2}(c_0 + c_1|x|^{-1} + k_1(x)) \qquad \text{on } |x| \geq R_*$$

with $R_* > 0$, $c_1 \in \mathbf{R}$, where k_1 satisfies $k_1(|x|) = O(|x|^{-\mu})$ and $x \cdot \nabla k_1(x) = O(|x|^{-\mu})$ for some $\mu > 1$.

A typical example of $K(x)$ which satisfies (K) with $c_0 = 1$ is that

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$$K(x) = \frac{1}{1+|x|^2},$$

which is exactly the same $K(x)$ appearing in the original Matukuma equation. Moreover, this $K(x)$ also satisfies (K.1) with $c_1 = 0$, since $K(x) = |x|^{-2}(1 - 1/\{(1+|x|^2)|x|^2\}$.

By the terminology "least-energy" solution, we mean a positive solution to (1.1) determined by the minimization problem

$$S_\varepsilon := \inf_{u \in \mathscr{D}, u \not\equiv 0} \frac{\int_{\mathbf{R}^n} |\nabla u|^2 dx}{(\int_{\mathbf{R}^n} K(x) u^{2+\varepsilon}\, dx)^{2/(2+\varepsilon)}}, \tag{1.2}$$

where $\mathscr{D}$ is the space which is the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|\nabla \cdot\|_2$. Then a function u_ε which attains (1.2) is a solution to

$$\Delta u_\varepsilon + \frac{S_\varepsilon}{(\int_{\mathbf{R}^n} K(x)|u_\varepsilon|^{2+\varepsilon})^{\varepsilon/(2+\varepsilon)}} K(x) u_\varepsilon^{1+\varepsilon} = 0. \tag{1.3}$$

Thus, by setting

$$\tilde{u}_\varepsilon := \frac{S_\varepsilon^{1/\varepsilon}}{(\int_{\mathbf{R}^n} K(x)|u_\varepsilon|^{2+\varepsilon} dx)^{1/(2+\varepsilon)}} u_\varepsilon,$$

$\tilde{u}_\varepsilon$ satisfies (1.1) and

$$\int_{\mathbf{R}^n} |\nabla \tilde{u}_\varepsilon|^2 dx = S_\varepsilon^{(2+\varepsilon)/\varepsilon}. \tag{1.4}$$

Related to (1.2), we introduce the value

$$S(\ell) := \inf_{u \in \mathscr{D}, u \not\equiv 0} \frac{\int_{\mathbf{R}^n} |\nabla u|^2 dx}{(\int_{\mathbf{R}^n} |x|^{-\ell} |u|^{2(n-\ell)/(n-2)} dx)^{(n-2)/(n-\ell)}} \tag{1.5}$$

with $0 < \ell \le 2$. As is known by Egnell [2] or Horiuchi [5, 6], $S(\ell)$ is the best constant of the embedding

$$\mathscr{D} \hookrightarrow L_\ell^p := \left\{ u \,|\, \int_{\mathbf{R}^n} |x|^{-\ell} |u|^p dx < \infty \right\}$$

with $p = 2(n-\ell)/(n-2)$. The extremal function is

$$U(x) = \left(1 + \frac{1}{(n-2)(n-\ell)} |x|^{2-\ell}\right)^{-(n-2)/(2-\ell)}. \tag{1.6}$$

However, the pointwise limit of $U(x)$ as $\ell \uparrow 2$ is

$$\lim_{\ell \uparrow 2} U(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

which is never a minimizer for $S(2)$. Note that $S(2) = (n-2)^2/4$ in case of $\ell = 2$ in view of the Hardy inequality

$$\frac{(n-2)^2}{4} \int_{\mathbf{R}^n} \frac{u^2}{|x|^2}\, dx \leq \int_{\mathbf{R}^n} |\nabla u|^2 dx.$$

We also note that there exists no minimizer for $S(2)$ (see, e.g., [2]).

In this paper, we investigate the behavior of S_ε and solutions u_ε as well as their scaled properties as $\varepsilon \downarrow 0$.

Theorem 1.1. *Under* (K), *the behavior of* S_ε *is as follows:*

$$\lim_{\varepsilon \downarrow 0} S_\varepsilon = \frac{(n-2)^2}{4c_0}.$$

Similar to Kabeya [7] for slowly decaying $K(x)$ ($K(x) \sim |x|^{-\ell}$ with $0 < \ell < 2$), the behavior of the norm of a least-energy solution is obtained. In view of the Pohozaev identity (see Lemma 2.2 of [7] or Proposition 1 of Naito [12]) yields

$$\int_{\mathbf{R}^n} \left\{ \left(\frac{n-2}{2} - \frac{n}{2+\varepsilon} \right) - \frac{x \cdot \nabla K(x)}{(2+\varepsilon)K(x)} \right\} K(x) |u_\varepsilon|^{2+\varepsilon} dx = 0. \tag{1.7}$$

Under (K), if $\int_{\mathbf{R}^n} K(x)u^2\, dx < \infty$ (especially $u \in \mathscr{D}$), then $u \equiv 0$ for $\varepsilon = 0$. However, depending on c_0, any least-energy solution blows up in this case. This is different from the case where $K(x)$ is slowly decaying as studied in [7]. One explanation is that the limiting problem for the slowly decaying case is still a nonlinear one, while this one is a linear one. The limiting problem (linear problem) in this case does not admit any scalings which erase c_0. Thus the dependence on c_0 arises.

The case for the faster decaying $K(x)$ will be discussed in Kabeya and Yanagida [8].

For radial solutions, the blowup or vanishing behaviors are obtained in Theorem 2.5 of Yanagida and Yotsutani [15]. We also see more precise behaviors of solutions than those obtained in [15].

Theorem 1.2. *Under* (K), *the norms* $\|\nabla u_\varepsilon\|_2$ *and* $\|u_\varepsilon\|_\infty$ *of any least-energy solution* u_ε *to* (1.1) *blow up if* $0 < c_0 < (n-2)^2/4$ *and vanish if* $c_0 > (n-2)^2/4$. *Moreover, in either case,* u_ε *satisfies*

$$\lim_{\varepsilon \downarrow 0} \|\nabla u_\varepsilon\|_2^\varepsilon = \frac{(n-2)^2}{4c_0}.$$

Note that the blowup and vanishing are determined by the limit of S_ε in Theorem 1.1.

In the "critical case" $c_0 = (n-2)^2/4$, we need a careful calculation and we impose further assumptions on $K(x)$.

THEOREM 1.3. *In the case where* $c_0 = (n-2)^2/4$, *suppose that* (K) *and* (K.1) *hold. Then the norms* $\|\nabla u_\varepsilon\|_2$ *and* $\|u_\varepsilon\|_\infty$ *of any least-energy solution blow up.*

Using Theorems 1.2 and 1.3, by a scaling, we see a limiting behavior of a least-energy solution. Unfortunately, the scaling is only valid for any domain that are the exterior of a ball centered at the origin.

THEOREM 1.4. *Suppose that* (K) *and* (K.1) *hold. For any least-energy solution* $u_\varepsilon(x)$ *and for any* $R > 0$, *let*

$$v_\varepsilon(y) := \frac{u_\varepsilon(x)}{\max_{|x| \geq R/\varepsilon} u_\varepsilon(x)} \qquad \text{with } x = \frac{y}{\varepsilon}. \tag{1.8}$$

Then there exists a subsequence $\{\varepsilon_j\}$ *such that the maximum point* Q_{ε_j} *of* v_{ε_j} *converges to* Q_* *and* $v_{\varepsilon_j}(y)$ *converges locally uniformly to* $V(y)$ *on* $\{y \in \mathbf{R}^n \mid R \leq |y| \leq R'\}$, *where* V *is a positive solution to*

$$\begin{cases} \Delta_y V + \dfrac{\tilde{c}}{|y|^2} V = 0, \\ V(Q_*) = 1, \ \lim_{|y|\to\infty} V(y) = 0, \end{cases} \tag{1.9}$$

with any $R' > R$ *and some* $0 < \tilde{c} \leq (n-2)^2/4$ *and* Δ_y *being the Laplacian with respect to* y.

REMARK 1.1. In Theorem 1.2, we see that $\|\nabla u_\varepsilon\|_2$ blows up or vanishes as $\varepsilon \downarrow 0$. By the scaling (1.8), despite the difference of the behavior of $\|\nabla u_\varepsilon\|_2$, we can extract a special limiting function, to which the scaled solution converges.

If we scale

$$\bar{v}_\varepsilon(y) := \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_\infty}, \qquad x = \frac{y}{\varepsilon},$$

then we only have the limiting function

$$\overline{V}(y) = \begin{cases} 1, & y = 0, \\ 0, & y \neq 0. \end{cases}$$

Thus we have used the scaling as in Theorem 1.4 to derive a useful information.

If $K(x)$ is radial and $K_r \le 0$, then u_ε must be radial by Gidas, Ni and Nirenberg [3]. In this case, we can take $|Q_*| = R$ without extracting a subsequence and the limiting solution is a radial one since the local uniform limit of a radial function is also radial. However, we do not know whether positive solutions of (1.9) are necessarily radially symmetric or not.

In Section 2, we give a proof of Theorem 1.1. Fundamental Lemmas for proofs of Theorems 1.2, 1.3 and 1.4 are given in Section 3. Proofs of Theorems 1.2, 1.3 and 1.4 are given in Section 4. In Section 5, we give a proof of Lemma 2.1 for the sake of the reader's convenience as an appendix.

2. Proof of Theorem 1.1

First we note that $S(\ell)$ is expressed in terms of the gamma functions and the exact value is obtained in Lemma 3.1 of Horiuchi [6] and Theorem 1.1 of Catrina and Wang [1]. The continuity of $S(\ell)$ at $\ell = 2$ is shown also in [1]. We summarize their results as below. We remark that they studied wider class of the weighted Sobolev type embeddings.

Lemma 2.1 (Horiuchi [6], Catrina and Wang [1]). *The explicit form of $S(\ell)$ is given by*

$$S(\ell) = (n-2)^2 \frac{\left(\dfrac{\omega_n}{2-\ell}\right)^{(2-\ell)/(n-\ell)} \Gamma\left(\dfrac{n-2\ell+2}{2-\ell}\right)\Gamma\left(\dfrac{n-2}{2-\ell}\right)}{\left\{\Gamma\left(\dfrac{n-\ell}{2-\ell}\right)\right\}^{2(n-2)/(n-\ell)}\left\{\Gamma\left(\dfrac{2(n-\ell)}{2-\ell}\right)\right\}^{(2-\ell)/(n-\ell)}}$$

for $\ell < 2$ and $S(\ell) \to (n-2)^2/4 = S(2)$ as $\ell \uparrow 2$, where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.

For the sake of self-containedness, we will give a proof in the Appendix.

The following is the estimate of the supremum norm, which will be useful for the uniform estimate. The estimate is essentially due to Lemma B.3 of Struwe [13] (see also Lemma 7 of Han [4]).

Lemma 2.2. *For any classical solution $u_\varepsilon \in \mathscr{D}$ to* (1.1)*, there exists a constant $\hat{C}_\varepsilon = \hat{C}(\|\nabla u_\varepsilon\|_2^\varepsilon) > 0$ such that $\|u_\varepsilon\|_\infty \le \hat{C}_\varepsilon \|\nabla u_\varepsilon\|_2$.*

Proof. We regard (1.1) as

$$\Delta u_\varepsilon + (K(x)u_\varepsilon^\varepsilon)u_\varepsilon = 0.$$

Since $K(x) \in L^\infty(\mathbf{R}^n)$ and since $\varepsilon > 0$ is small, we see that $K(x)u_\varepsilon^\varepsilon \in L^\alpha(B(Q_\varepsilon, 1))$ with $\alpha > n/2$. Then as in Lemma B.3 (pp. 244–245) of [13] or as in Lemma 7 of [4], we have

$$\max_{B(Q_\varepsilon,1/2)} u_\varepsilon \le C(\|K(x)u_\varepsilon^\varepsilon\|_\alpha, \alpha, n)\|\nabla u_\varepsilon\|_2, \tag{2.1}$$

where C depends on L^α-norm of $K(x)u_\varepsilon^\varepsilon$, α and n. For L^α-norm of $K(x)u_\varepsilon^\varepsilon$, by the Hölder and Sobolev inequalities, for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_{B(Q_\varepsilon,1)} (K(x)u_\varepsilon^\varepsilon)^\alpha dx \\
&\quad \le \|K\|_\infty^\alpha \int_{B(Q_\varepsilon,1)} u_\varepsilon^{\varepsilon\alpha}\, dx \\
&\quad \le \|K\|_\infty^\alpha |B(Q_\varepsilon,1)|^{\{2n-(n-2)\varepsilon\alpha\}/(2n)} \left(\int_{B(Q_\varepsilon,1)} u_\varepsilon^{2n/(n-2)}\, dx\right)^{(n-2)\varepsilon\alpha/(2n)} \\
&\quad \le \bar{C}\|K\|_\infty^\alpha |B(Q_\varepsilon,1)|\, \|\nabla u_\varepsilon\|_2^{\varepsilon\alpha}
\end{aligned}$$

where $\bar{C}$ is a constant independent of ε. Since $u_\varepsilon \in \mathscr{D}$ by assumption, we have the desired estimate (The dependence of $\hat{C}_\varepsilon$ on ε comes from the ε-dependence of $\|\nabla u_\varepsilon\|_2$ and note that α and n are independent of ε). □

Using Lemma 2.2, we prove Theorem 1.1.

Proof of Theorem 1.1. Before proving the equality, we easily see that S_ε is uniformly bounded. Then as in the proof of Lemma 2.4 in Kabeya [7], we prove

$$\liminf_{\varepsilon\downarrow 0} S_\varepsilon \ge \frac{1}{c_0} S(2) \qquad \text{and} \qquad \limsup_{\varepsilon\downarrow 0} S_\varepsilon \le \frac{1}{c_0} S(2).$$

First we prove $\liminf_{\varepsilon\downarrow 0} S_\varepsilon \ge (c_0)^{-1} S(2)$. Let $u_\varepsilon(x)$ be a function which attains S_ε with $\int_{\mathbf{R}^n} K(x)u_\varepsilon^{2+\varepsilon}\, dx = 1$ and let $v_\varepsilon(y) = u_\varepsilon(x)$ with $x = y/\varepsilon$. Then, u_ε is a solution to

$$\Delta u_\varepsilon + S_\varepsilon K(x) u_\varepsilon^{1+\varepsilon} = 0$$

and we have

$$\int_{\mathbf{R}^n} |\nabla_x u_\varepsilon|^2 dx = \frac{1}{\varepsilon^{n-2}} \int_{\mathbf{R}^n} |\nabla_y v_\varepsilon|^2 dy = S_\varepsilon$$

and

$$\int_{\mathbf{R}^n} K(x)u_\varepsilon^{2+\varepsilon}\, dx = \frac{1}{\varepsilon^{n-2}} \int_{\mathbf{R}^n} \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) v_\varepsilon^{2+\varepsilon}\, dy = 1,$$

where ∇_x, ∇_y denote the gradient with respect to x and y, respectively. Hence we get

$$S_\varepsilon = \frac{\int_{\mathbf{R}^n} |\nabla_x u_\varepsilon|^2 dx}{\left(\int_{\mathbf{R}^n} K(x) u_\varepsilon^{2+\varepsilon}\, dx\right)^{2/(2+\varepsilon)}} \tag{2.2}$$

$$= \frac{\varepsilon^{-(n-2)} \int_{\mathbf{R}^n} |\nabla_y v_\varepsilon|^2 dy}{\varepsilon^{-2(n-2)/(2+\varepsilon)} \left(\int_{\mathbf{R}^n} \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) v_\varepsilon^{2+\varepsilon}\, dy \right)^{2/(2+\varepsilon)}}$$

$$\geq \frac{\int_{\mathbf{R}^n} |\nabla_y v_\varepsilon|^2 dy}{\varepsilon^{(n-2)\varepsilon/(2+\varepsilon)} \left(\int_{\mathbf{R}^n} \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) v_\varepsilon^2\, dy \right)^{2/(2+\varepsilon)} \left(\max_{\mathbf{R}^n} v_\varepsilon \right)^{2\varepsilon/(2+\varepsilon)}}.$$

By Lemma 2.2 and since $\|\nabla u_\varepsilon\|_2 = S_\varepsilon^{1/2}$ is uniformly bounded, we see that the right-hand side of (2.1) for $K(x)$ replaced by $S_\varepsilon K(x)$ is uniformly bounded. Hence, in view of $\|u_\varepsilon\|_\infty = \|v_\varepsilon\|_\infty$, we have $\limsup_{\varepsilon\downarrow 0} (\max_{\mathbf{R}^n} v_\varepsilon)^{2\varepsilon/(2+\varepsilon)} \leq 1$. Thus, taking a limit infimum, we obtain

$$\liminf_{\varepsilon\downarrow 0} S_\varepsilon \geq \liminf_{\varepsilon\downarrow 0} \frac{\int_{\mathbf{R}^n} |\nabla_y v_\varepsilon|^2 dy}{c_0 \int_{\mathbf{R}^n} |y|^{-2} v_\varepsilon^2\, dy} \geq \frac{1}{c_0} S(2) = \frac{(n-2)^2}{4c_0}$$

in view of

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) = c_0 |y|^{-2},$$

where the convergence is locally uniformly in $\mathbf{R}^n \backslash \{0\}$ by (K), and the Hardy inequality.

To prove $\limsup_{\varepsilon\downarrow 0} S_\varepsilon \leq (1/c_0) S(2)$, we set $w(x) = |x|^{-(n-2)/2} \varphi_\varepsilon(|x|)$. Here, $\varphi_\varepsilon(\geq 0) \in C_0^\infty((0,\infty))$ satisfies $\operatorname{supp} \varphi_\varepsilon \subset [1, 2\varepsilon^{-1}]$, $\max_{[0,\infty)} \varphi = 1$, $\operatorname{supp} \varphi' \subset [1,2] \cup [\varepsilon^{-1}, 2\varepsilon^{-1}]$, $\varphi_\varepsilon(x) \equiv f(x)$ on $[1,2]$ such that $f \in C^\infty(0,\infty)$, fulfills $f(1) = 0$, $f(x) > 0$ in $(1,2]$ and $f(2) = 1$, and $\varphi_\varepsilon(x) \equiv g(\varepsilon x)$ on $[\varepsilon^{-1}, 2\varepsilon^{-1}]$, where $g(x) \in C^\infty(0,\infty)$ satisfies $g(1) = 1$, $g(x) > 0$ in $[1,2)$ and $g(2) = 0$.

Since $w' = -\{(n-2)/2\} r^{-n/2} \varphi_\varepsilon + r^{-(n-2)/2} \varphi_\varepsilon'$, we have

$$\int_{\mathbf{R}^n} |\nabla w|^2 dx$$

$$= \omega_n \int_0^\infty \left\{ \left(\frac{n-2}{2}\right)^2 r^{-n} \varphi_\varepsilon^2 - (n-2) r^{-(n-1)} \varphi_\varepsilon \varphi_\varepsilon' + r^{-(n-2)} (\varphi_\varepsilon')^2 \right\} r^{n-1}\, dr$$

$$= \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^\infty r^{-1} \varphi_\varepsilon^2\, dr - (n-2)\omega_n \int_0^\infty \varphi_\varepsilon \varphi_\varepsilon'\, dr + \omega_n \int_0^\infty r (\varphi_\varepsilon')^2 dr$$

$$= \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^\infty r^{-1} \varphi_\varepsilon^2\, dr - (n-2)\omega_n \int_0^\infty \varphi_\varepsilon \varphi_\varepsilon'\, dr + \omega_n \left(\int_1^2 + \int_{\varepsilon^{-1}}^{2\varepsilon^{-1}} \right) r (\varphi_\varepsilon')^2 dr.$$

The second and third terms yield

$$\int_0^\infty \varphi_\varepsilon \varphi_\varepsilon' \, dr = \left[\frac{1}{2}\varphi_\varepsilon^2\right]_0^\infty = 0$$

and

$$\left(\int_1^2 + \int_{\varepsilon^{-1}}^{2\varepsilon^{-1}}\right) r(\varphi_\varepsilon')^2 dr \le 6 + 4\varepsilon^2 \int_{\varepsilon^{-1}}^{2\varepsilon^{-1}} r \, dr = 12,$$

respectively. Thus we get

$$S_\varepsilon \le \frac{\left(\frac{n-2}{2}\right)^2 \omega_n \int_1^{2\varepsilon^{-1}} r^{-1}\varphi_\varepsilon^2 \, dr + 12\omega_n}{\left(\int_{\mathbf{R}^n} K(x) w^{2+\varepsilon} \, dx\right)^{2/(2+\varepsilon)}}. \tag{2.3}$$

As for the denominator, for any $\eta > 0$, take $R > 0$ so large that

$$|x|^2 K(x) \ge c_0 - \eta$$

holds for any $x \in \mathbf{R}^n \backslash B_R$. Then we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} K(x) w^{2+\varepsilon} \, dx \\
&\quad = \int_{B_{2\varepsilon^{-1}} \backslash B_1} K(x)|x|^{-(n-2)(2+\varepsilon)/2} \varphi_\varepsilon^{2+\varepsilon} \, dx \\
&\quad \ge \int_{B_{2\varepsilon^{-1}} \backslash B_R} |x|^2 K(x)|x|^{-n-(n-2)\varepsilon/2} \varphi_\varepsilon^{2+\varepsilon} \, dx + \int_{B_R \backslash B_2} K(x)|x|^{-(n-2)(2+\varepsilon)/2} dx \\
&\quad \ge \left(\frac{2}{\varepsilon}\right)^{-(n-2)\varepsilon/2} (c_0 - \eta)\omega_n \int_R^{2\varepsilon^{-1}} r^{-1}\varphi_\varepsilon^{2+\varepsilon} \, dr + \int_{B_R \backslash B_2} K(x)|x|^{-(n-2)(2+\varepsilon)/2} dx
\end{aligned}$$

for any $\varepsilon^{-1} > R$. The second term is uniformly bounded. Since $\varphi_\varepsilon^\varepsilon \to 1$ locally uniformly in $(1, \infty)$ by the definition of φ_ε, we see that

$$\lim_{\varepsilon \downarrow 0} \frac{\int_1^{2\varepsilon^{-1}} r^{-1}\varphi_\varepsilon^2 \, dr}{\left(\int_R^{2\varepsilon^{-1}} r^{-1}\varphi_\varepsilon^{2+\varepsilon} \, dr\right)^{2/(2+\varepsilon)}} = 1.$$

Thus taking a limit supremum of (2.3) as $\varepsilon \downarrow 0$, we have

$$\limsup_{\varepsilon \downarrow 0} S_\varepsilon \le \frac{1}{c_0 - \eta}\left(\frac{n-2}{2}\right)^2.$$

Since $S(2) = (n-2)^2/4$ and since $\eta > 0$ is arbitrary, we obtain $\lim_{\varepsilon\downarrow 0} S_\varepsilon = (1/c_0)S(2)$. □

3. Fundamental properties of solutions

To prove Theorems 1.3 and 1.4, we need to estimate the location of the maximum point of $u_\varepsilon(x)$ and $\|u\|_\infty$ from above. First we need a uniform a priori estimate for u_ε satisfying (1.4), almost identical to Lemma 2.2.

Lemma 3.1. *For any least-energy solution u_ε to* (1.1), *there exists a constant $\tilde{C} > 0$ independent of ε such that $\|u_\varepsilon\|_\infty^\varepsilon \le \tilde{C}$.*

Proof. The proof is almost identical to that of Lemma 2.2. In the proof of Lemma 2.2, we just note here that

$$\begin{aligned}\int_{B(Q_\varepsilon,1)} (K(x)u_\varepsilon^\varepsilon)^\alpha dx &\le C\|K\|_\infty^\alpha |B(Q_\varepsilon,1)|\,\|\nabla u_\varepsilon\|_2^{\varepsilon\alpha} \\ &\le 2C\|K\|_\infty^\alpha |B(Q_\varepsilon,1)| \left(\frac{(n-2)^2}{4c_0}\right)^\alpha,\end{aligned}$$

with $\alpha > n/2$ by Theorem 1.1, where C is a constant independent of ε. Thus, the L^α norm $\|K(x)u_\varepsilon^\varepsilon\|_{L^\alpha(B(Q_\varepsilon,1/2))}$ is bounded independent of ε. Hence, the constant in (2.1) is independent of ε. By Theorem 1.1 and (1.4), $\|\nabla u_\varepsilon\|_2^\varepsilon$ is uniformly bounded. Thus, we have the desired estimate. □

Next, we show a decay property of u_ε.

Lemma 3.2. *Under* (K), *there exists $R_1 > 0$ independent of ε such that*

$$u_\varepsilon(x) = \tilde{C}_\varepsilon |x|^{-(n-2)} + h_\varepsilon(x)$$

holds on $|x| \ge R_1$ with $\tilde{C}_\varepsilon > 0$ and a higher order term $h_\varepsilon(x) = O(|x|^{-(n-2)(1+\varepsilon)})$ at infinity.

Proof. As in Lemma 3.5 of Kabeya [7] (if $K(x)$ is radial, the decay order is obtained by Li and Ni [9, 11]), we deduce the decay order of u_ε. Using the Green function of $-\Delta$ in $\mathbf{R}^n$, we have

$$u_\varepsilon(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbf{R}^n} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy. \tag{3.1}$$

By the Hölder inequality, we have

$$\int_{\mathbf{R}^n} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy \le \int_{|x-y|\le 1} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy$$
$$+ \left\{\int_{|x-y|\ge 1}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy\right\}^{(n+2-(n-2)\varepsilon)/(2n)}$$
$$\times \left(\int_{|x-y|\ge 1} u_\varepsilon^{2n/(n-2)}\,dy\right)^{(n-2)(1+\varepsilon)/(2n)}.$$

Since $K(y) \sim |y|^{-2}$ at $|y| = \infty$, $\|u_\varepsilon\|_\infty$ is finite and since $u_\varepsilon \in \mathscr{D}$, the right-hand side is finite.

From now on, $R_0 > 0$ is supposed to be large so that

$$\frac{c_0}{2} \le |x|^2 K(x) \le 2c_0 \tag{3.2}$$

on $|x| \ge R_0$, and we take x so that $|x| \ge R_1 := \max\{2R_0, R_0 + 1\}$, and C denotes the generic constant independent of ε. x may be taken even larger if necessary.

For $|x| \ge R_1$, first we note that

$$\frac{\max_{|x-y|\le 1} K(y)}{K(x)} \le C$$

with $C > 0$ independent of x. Indeed, $|x-y| \le 1$ implies $|x| - 1 \le |y| \le |x| + 1$. Thus we have

$$K(y) \le \frac{2c_0}{|y|^2} \le \frac{2c_0}{(|x|-1)^2}$$

in view of (3.2) for $|x| \ge R_1$. Again from (3.2), there holds

$$K(x) \ge \frac{c_0}{2|x|^2}$$

and we get

$$\frac{\max_{|x-y|\le 1} K(y)}{K(x)} \le \frac{4|x|^2}{(|x|-1)^2}$$

for $|x| \ge R_1$. Note that the right-hand side is uniformly bounded for $|x| \ge R_1$. Thus we see that

$$\int_{|x-y|\le 1} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy \le C\|u_\varepsilon\|_\infty K(x)\int_{|x-y|\le 1}\frac{1}{|x-y|^{n-2}}\,dy \le C\|\nabla u_\varepsilon\|_2 |x|^{-2}$$

holds by (2.1) and Lemma 3.1 ($\|u_\varepsilon\|_\infty^\varepsilon$ is uniformly bounded). Next, we decompose

$$\int_{|x-y|\geq 1}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy = I_1 + I_2 + I_3 + I_4$$

with

$$I_1 = \int_{1\leq|x-y|\leq|x|/2}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy,$$

$$I_2 = \int_{|x|/2\leq|x-y|\leq 2|x|}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy,$$

$$I_3 = \int_{\substack{|x-y|\geq 2|x|\\ 2|x|\geq|y|\geq|x|}}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy,$$

$$I_4 = \int_{\substack{|x-y|\geq 2|x|\\ |y|\geq 2|x|}}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy.$$

On $1\leq|x-y|\leq|x|/2$, we see $|y|\geq|x|/2\geq R_0$ and get

$$\begin{aligned}I_1 &\leq C|x|^{-4n/\{n+2-(n-2)\varepsilon\}}\int_1^{|x|/2} r^{-2n(n-2)/(n+2-(n-2)\varepsilon)+n-1}\,dr\\ &\leq C(|x|^{-4n/(n+2-(n-2)\varepsilon)} + |x|^{-n(n-2)(1+\varepsilon)/(n+2-(n-2)\varepsilon)}).\end{aligned}$$

For I_2, we have

$$\begin{aligned}I_2 &\leq C|x|^{-2n(n-2)/(n+2-(n-2)\varepsilon)}\left(\int_{|y|\leq R_1} + \int_{R_1\leq|y|\leq 3|x|}\right)K(y)^{2n/(n+2-(n-2)\varepsilon)}dy\\ &\leq C(|x|^{-2n(n-2)/(n+2-(n-2)\varepsilon)} + |x|^{-n(n-2)(1+\varepsilon)/(n+2-(n-2)\varepsilon)}),\end{aligned}$$

since $|x-y|\leq 2|x|$ implies $|y|\leq 3|x|$. Similarly, for I_3, we get

$$\begin{aligned}I_3 &\leq C|x|^{-2n(n-2)/(n+2-(n-2)\varepsilon)}|x|^{-4n/(n+2-(n-2)\varepsilon)}\int_{|x|\leq|y|\leq 2|x|} dy\\ &\leq C|x|^{-n(n-2)(1+\varepsilon)/(n+2-(n-2)\varepsilon)}.\end{aligned}$$

Finally, for I_4, we note that $|x-y|\geq 2|x|$ with $|y|\geq 2|x|$ implies $|x-y|\geq|y|/2$ (indeed, $|x-y|\geq|y|-|x|\geq|y|-|y|/2$). Thus we have

$$I_4 \le C \int_{\substack{|x-y| \ge 2|x| \\ |y| \ge 2|x|}} |y|^{-2n^2/(n+2-(n-2)\varepsilon)} dy \le C|x|^{-n(n-2)(1+\varepsilon)/(n+2-(n-2)\varepsilon)}.$$

Hence we obtain

$$\begin{aligned}\left(\int_{|x-y|\ge 1}\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(n+2-(n-2)\varepsilon)} dy\right)^{(n+2-(n-2)\varepsilon)/(2n)} \\ = (I_1 + I_2 + I_3 + I_4)^{(n+2-(n-2)\varepsilon)/(2n)} \\ \le C(|x|^{-2} + |x|^{-(n-2)(1+\varepsilon)/2} + |x|^{-(n-2)}).\end{aligned}$$

Thus we have

$$\begin{aligned}u_\varepsilon(x) &\le C\|\nabla u_\varepsilon\|_2 |x|^{-2} + C\|\nabla u_\varepsilon\|_2 |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}} \\ &= C_\varepsilon |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}}\end{aligned} \tag{3.3}$$

for $|x| \ge R_1$, with $C_\varepsilon = C\|\nabla u_\varepsilon\|_2$, again by (2.1) and Lemma 3.1 for the estimate of the second term.

By (3.3), we can estimate the right-hand side of (3.1) directly. We again decompose

$$\int_{\mathbf{R}^n} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$J_1 = \int_{|x-y|\le 1} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy,$$

$$J_2 = \int_{1\le|x-y|\le|x|/2} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy,$$

$$J_3 = \int_{|x|/2\le|x-y|\le 2|x|} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy,$$

$$J_4 = \int_{\substack{|x-y|\ge 2|x| \\ 2|x|\ge|y|\ge|x|}} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy,$$

$$J_5 = \int_{\substack{|x-y|\ge 2|x| \\ |y|\ge 2|x|}} \frac{K(y)u_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}} dy.$$

Then, denoting the generic constant (may dependent on ε) by C_ε, we have, via the step similar to the previous one,

$$
\begin{aligned}
J_1 &\le C_\varepsilon |x|^{-2-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)} \int_{|x-y|\le 1} \frac{1}{|x-y|^{n-2}}\, dy \\
&\le C_\varepsilon |x|^{-2-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}, \\
J_2 &\le C_\varepsilon |x|^{-2-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)} \int_1^{|x|/2} r\, dr \\
&\le C_\varepsilon |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}, \\
J_3 &\le C_\varepsilon |x|^{-(n-2)} \left(\int_{|y|\le R_1} + \int_{R_1\le |y|\le 3|x|} \right) K(y) u_\varepsilon(y)^{1+\varepsilon} dy \\
&\le C_\varepsilon (|x|^{-(n-2)} + |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}), \\
J_4 &\le C_\varepsilon |x|^{-(n-2)} |x|^{-2-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)} \int_{2|x|\ge |y|\ge |x|} dy \\
&\le C_\varepsilon |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}, \\
J_5 &\le C_\varepsilon \int_{|x|}^{\infty} |y|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)-1} dy \\
&\le C_\varepsilon |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}.
\end{aligned}
$$

Thus we obtain

$$u_\varepsilon(x) \le C_\varepsilon |x|^{-\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)}$$

for $|x| \ge R_1$ if $\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon) \le n-2$. If $\min\{2,(n-2)\cdot(1+\varepsilon)/2\}(1+\varepsilon) > n-2$, then we are done. For fixed $\varepsilon > 0$, repeating this process ℓ times so that $\min\{2,(n-2)(1+\varepsilon)/2\}(1+\varepsilon)^\ell > n-2$, we have

$$u_\varepsilon(x) \le C_\varepsilon |x|^{-(n-2)} \tag{3.4}$$

for $|x| \ge R_1$. We should note here that the decay rate $|x|^{-(n-2)}$ is never improved in view of the estimate in J_3.

Then as in Theorem 2.4 of Li and Ni [10] (the estimate of the second order of the expansion), we obtain

$$u_\varepsilon(x) = \tilde{C}_\varepsilon |x|^{-(n-2)} + h_\varepsilon(x). \tag{3.5}$$

with

$$\tilde{C}_\varepsilon := \lim_{|x|\to\infty} |x|^{n-2} u_\varepsilon(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbf{R}^n} K(x) u_\varepsilon^{1+\varepsilon}\, dx, \tag{3.6}$$

and $h_\varepsilon(x) = O(|x|^{-(n-2)(1+\varepsilon)})$ being a higher order term. Since (3.4) holds for $|x| \ge R_1$ with R_1 independent of ε, (3.5) holds also for $|x| \ge R_1$. □

REMARK 3.1. The reason why we have obtained the exact decay rate is that the dominant term in the estimate of J_3 is $|x|^{-(n-2)}$ and that the iteration is no longer effective to gain the decay rate due to this term.

C_ε may go to infinity as $\varepsilon \downarrow 0$ because $\|\nabla u_\varepsilon\|_2$ may go to infinity and the iteration needs more times (unbounded) as $\varepsilon \downarrow 0$.

REMARK 3.2. Under (K) and (K.1), according to Theorem 2.16 of Li and Ni [10], u_ε is expanded as follows:

$$u_\varepsilon(x) = \frac{C_\varepsilon}{|x|^{n-2}} + \frac{1}{|x|^{n-2}} \sum_{m=1}^{2k_\varepsilon+1} \frac{C_{1,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}} + \frac{1}{|x|^{n-1}} \sum_{m=1}^{k_\varepsilon} \frac{C_{2,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}} + \frac{a_\varepsilon \cdot x}{|x|^n}\left(1 + \sum_{m=1}^{k_\varepsilon} \frac{C_{3,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}}\right) + R_\varepsilon\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{n-1}} \tag{3.7}$$

for $|x| \geq R_1$, where k_ε is an integer such that $(n-2)k_\varepsilon\varepsilon \leq 1 < (n-2)(k_\varepsilon+1)\varepsilon$, $C_{1,m,\varepsilon}$, $C_{2,m,\varepsilon}$, $C_{3,m,\varepsilon}$ are constants (not necessarily positive), $a_\varepsilon \in \mathbf{R}^n$ is a constant vector, and $R_\varepsilon(t)$ is a Lipschitz continuous function near $t = 0$ with $R_\varepsilon(0) = 0$. Note that R_1 can be taken larger than R_*. The assumption (K.1) is needed to have the exact expansion as above. Without (K.1), it is hard to obtain the expansion (3.7).

Carefully following the proof of Theorem 2.16 of [10], we can obtain the constants $C_{1,m,\varepsilon}$.

LEMMA 3.3. *The constant* $C_{1,m,\varepsilon}$ *in* (3.7) *satisfies*

$$C_{1,m,\varepsilon} = \frac{(-1)^m C_\varepsilon^{(1+\varepsilon)^m} c_0^m}{(n-2)^{2m}\varepsilon^m m! \prod_{\ell=1}^m (1+\ell\varepsilon)} \tag{3.8}$$

for $m = 1, 2, \ldots, k_\varepsilon$.

PROOF. A proof is done by following the proof of Theorem 2.16 of [10]. So we give a sketchy proof. The essential part is to express (1.1) as

$$\Delta u_\varepsilon + |x|^{-2}(c_0 + c_1|x|^{-1} + k_1(x)) \frac{C_\varepsilon^{1+\varepsilon}}{|x|^{(n-2)(1+\varepsilon)}} \left(\frac{|x|^{n-2}u_\varepsilon}{C_\varepsilon}\right)^{1+\varepsilon} = 0 \tag{3.9}$$

on $|x| \geq R_1$ and expand $C_\varepsilon^{-1}|x|^{n-2}u_\varepsilon$ step by step.

In what follows, f_i and $u_{i,\varepsilon}$ represent the remainder terms. First, we note that the equation (1.1) is expressed as

$$\Delta u_\varepsilon + C_\varepsilon^{1+\varepsilon} c_0 |x|^{-(n-2)(1+\varepsilon)} + f_1(u_\varepsilon) = 0$$

on $|x| \geq R_1$. Note that the original proof is done via the Kelvin transformation. However, we can prove this lemma directly since we know the decay order. Then, as in (2.22) of [10] (p. 203), we have

$$u_\varepsilon = C_\varepsilon |x|^{-(n-2)} - \frac{C_\varepsilon^{(1+\varepsilon)} c_0}{(n-2)^2(1+\varepsilon)\varepsilon} |x|^{-(n-2)(1+\varepsilon)} + u_{1,\varepsilon}.$$

The expression is seemingly a contradictory one since the coefficient of the second term is apparently larger than that of the first term. However, the second term is eventually almost cancelled out by $u_{1,\varepsilon}$ since u_ε is always positive.

Next, using the expansion, we see that u_ε satisfies

$$\Delta u_\varepsilon + C_\varepsilon^{1+\varepsilon} c_0 |x|^{-(n-2)(1+\varepsilon)} - \frac{C_\varepsilon^{(1+\varepsilon)^2} c_0^2}{(n-2)^2(1+\varepsilon)\varepsilon} |x|^{-(n-2)(1+2\varepsilon)} + f_2(u_\varepsilon) = 0. \tag{3.10}$$

Then we have via the method of the deduction of (2.22) in [10],

$$\begin{aligned} u_\varepsilon = C_\varepsilon |x|^{-(n-2)} &- \frac{C_\varepsilon^{(1+\varepsilon)} c_0}{(n-2)^2(1+\varepsilon)\varepsilon} |x|^{-(n-2)-(n-2)\varepsilon} \\ &+ \frac{C_\varepsilon^{(1+\varepsilon)^2} c_0^2}{2(n-2)^4(1+\varepsilon)(1+2\varepsilon)\varepsilon^2} |x|^{-(n-2)(1+2\varepsilon)} + u_{2,\varepsilon}. \end{aligned}$$

Indeed, three terms from the top satisfy (3.10) with $f_2 \equiv 0$ (calculate $\Delta(u_\varepsilon - u_{2,\varepsilon})$ as a radial function). Moreover, the "uniqueness" of the top three terms are verified as in the proof of Theorem 2.16 of [10] (p. 203). Thus $C_{1,2,\varepsilon}$ is obtained. To obtain $C_{1,3,\varepsilon}$, we again repeat the argument. Thus u_ε satisfies

$$\begin{aligned} \Delta u_\varepsilon + C_\varepsilon^{1+\varepsilon} c_0 |x|^{-(n-2)(1+\varepsilon)} &- \frac{C_\varepsilon^{(1+\varepsilon)^2} c_0^2}{(n-2)^2(1+\varepsilon)\varepsilon} |x|^{-(n-2)(1+2\varepsilon)} \\ &+ \frac{C_\varepsilon^{(1+\varepsilon)^3} c_0^3}{2(n-2)^4(1+\varepsilon)(1+2\varepsilon)\varepsilon^2} |x|^{-(n-2)(1+3\varepsilon)} + f_3(u_\varepsilon) = 0. \end{aligned}$$

Then we have

$$C_{1,3,\varepsilon} = -\frac{C_\varepsilon^{(1+\varepsilon)^3} c_0^3}{6(n-2)^6(1+\varepsilon)(1+2\varepsilon)(1+3\varepsilon)\varepsilon^3}.$$

Inductively, we obtain the conclusion. $\square$

As for the other terms in (3.7), we have the following estimate. Let us define

$$C(\varepsilon) := \max\left\{ C_\varepsilon, \max_{1 \le m \le k_\varepsilon} |C_{1,m,\varepsilon}| \right\}.$$

LEMMA 3.4. *The constants in* (3.7) *of* $C_{1,m,\varepsilon}$ $(k_\varepsilon + 1 \le m \le 2k_\varepsilon + 1)$, $C_{2,m,\varepsilon}$, $|a_\varepsilon| C_{3,m,\varepsilon}$ *and* $R_\varepsilon(x)$ *for fixed* x *are at most of order* $C(\varepsilon)$ *in* ε.

PROOF. We again consider the process of the proof of Lemma 3.3. Also confer to the deduction of (2.22) in [10]. It is easy to see that $(1+\varepsilon)^m \le (1+\varepsilon)^{k_\varepsilon} \le (1+\varepsilon)^{1/\{(n-2)\varepsilon\}} \le 2e^{1/(n-2)}$ for any sufficiently small $\varepsilon > 0$.

For coefficients $C_{1,m,\varepsilon}$ with $1 \le m \le k_\varepsilon$, ε^m appears in the denominator to cancel out the previous term as mentioned in the deduction of $C_{1,2,\varepsilon}$ in the proof of Lemma 3.3. The powers in x of the terms in the coefficient $C_{1,m,\varepsilon}$ $(1 \le m \le k_\varepsilon)$ converge to $-(n-2)$ as $\varepsilon \downarrow 0$. These terms induce the higher ε dependence.

$C_{1,m,\varepsilon}$ with $k_\varepsilon + 1 \le m \le 2k_\varepsilon + 1$ is determined in the same way as in the proof of Lemma 3.3. But a new ε power does not appear since $(n-2)(1+m\varepsilon)$ with $k_\varepsilon + 1 \le m \le 2k_\varepsilon + 1$ never converges to $n-2$ as $\varepsilon \downarrow 0$.

$C_{2,m,\varepsilon}$, a_ε, $C_{3,m,\varepsilon}$ and $R_\varepsilon(x)$ are determined by the terms in (3.9) which are products of $|x|^{-(n-2)(1+m\varepsilon)}$ term and $c_1|x|^{-1}$ or $k_1(x)$ inductively. Indeed, when the terms up to $C_{1,k_\varepsilon,\varepsilon}|x|^{-(n-2)(1+k_\varepsilon\varepsilon)}$ are obtained, (3.9) can be written as

$$\Delta u_\varepsilon + \frac{c_0 + c_1|x|^{-1} + k_1(x)}{|x|^2}\left(\sum_{k=1}^{k_\varepsilon} \frac{C_{1,m,\varepsilon}}{|x|^{(n-2)(1+m\varepsilon)}}\right) + f_4(u_\varepsilon) = 0$$

with f_4 being a remainder term. Thus we see that $C_{2,m,\varepsilon}$, a_ε, $C_{3,m,\varepsilon}$ and $R_\varepsilon(x)$ are determined from the product of $|x|^{-1}$ or $k_\varepsilon(x)$ with $C_{1,m,\varepsilon}|x|^{-(n-2)(1+m\varepsilon)}$.

Since the powers of other terms never converge to $|x|^{-(n-2)}$, the above process shows that the coefficients and R_ε cannot create the higher ε dependence as in $C(\varepsilon)$. To keep u_ε positive on $|x| \ge R_1$, they must be at most of the order $C(\varepsilon)$. □

Using (3.7) and Lemma 3.4, we show the boundedness of the maximum point of u_ε.

LEMMA 3.5. *Suppose that* (K) *and* (K.1) *hold. Then the maximum point of the least-energy solution* u_ε *is uniformly bounded.*

PROOF. By (K.1), using the expansion (3.7), we can express u_ε as

$$u_\varepsilon(x) = \frac{C_\varepsilon}{|x|^{n-2}} + \frac{f_{1,\varepsilon}(x)}{|x|^{n-2}} + \frac{f_{2,\varepsilon}(x)}{|x|^{n-1}} + \frac{(a_\varepsilon \cdot x) f_{3,\varepsilon}(x)}{|x|^n} + R_\varepsilon\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{n-1}}, \tag{3.11}$$

with

$$f_{1,\varepsilon}(x) = \sum_{m=1}^{2k_\varepsilon+1} \frac{C_{1,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}}, \quad f_{2,\varepsilon}(x) = \sum_{m=1}^{k_\varepsilon} \frac{C_{2,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}}, \quad f_{3,\varepsilon}(x) = 1 + \sum_{m=1}^{k_\varepsilon} \frac{C_{3,m,\varepsilon}}{|x|^{(n-2)m\varepsilon}}.$$

Note that $f_{1,\varepsilon}(x)$, $f_{2,\varepsilon}$, $\tilde{f}_{3,\varepsilon}(x) := (a_\varepsilon \cdot x) f_{3,\varepsilon}(x)$ and $R_\varepsilon(x/|x|^2)$ in terms of ε are at most of the order of $C(\varepsilon)$ by Lemma 3.4. Then we can express

$$(3.12)\qquad u_\varepsilon(x) = \frac{C(\varepsilon)}{|x|^{n-2}}\left\{\frac{C_\varepsilon}{C(\varepsilon)} + \frac{f_{1,\varepsilon}(x)}{C(\varepsilon)} + \frac{f_{2,\varepsilon}(x)}{C(\varepsilon)|x|} + \frac{\tilde{f}_{3,\varepsilon}(x)}{C(\varepsilon)|x|^2} + \frac{\tilde{R}_\varepsilon(x)}{C(\varepsilon)|x|}\right\},$$

with $\tilde{R}_\varepsilon(x) := R_\varepsilon(x/|x|^2)$.

Suppose that the maximum point Q_j of u_{ε_j} ($\varepsilon_j \downarrow 0$ as $j \to \infty$) tends to infinity. We may suppose that $|Q_j| > R_1$ for any j and fix x_0 so that $|x_0| \geq R_1$. Then we have

$$(3.13)\qquad 1 < \frac{u_{\varepsilon_j}(Q_j)}{u_{\varepsilon_j}(x_0)}$$

$$= \left(\frac{|x_0|}{|Q_j|}\right)^{(n-2)} \left\{ \frac{\dfrac{C_\varepsilon}{C(\varepsilon)} + \dfrac{f_{1,\varepsilon_j}(Q_j)}{C(\varepsilon_j)} + \dfrac{f_{2,\varepsilon_j}(Q_j)}{C(\varepsilon_j)|Q_j|} + \dfrac{\tilde{f}_{3,\varepsilon_j}(Q_j)}{C(\varepsilon_j)|Q_j|^2} + \dfrac{\tilde{R}_{\varepsilon_j}(Q_j)}{C(\varepsilon_j)|Q_j|}}{\dfrac{C_\varepsilon}{C(\varepsilon)} + \dfrac{f_{1,\varepsilon_j}(x_0)}{C(\varepsilon_j)} + \dfrac{f_{2,\varepsilon_j}(x_0)}{C(\varepsilon_j)|x_0|} + \dfrac{\tilde{f}_{3,\varepsilon_j}(x_0)}{C(\varepsilon_j)|x_0|^2} + \dfrac{\tilde{R}_{\varepsilon_j}(x_0)}{C(\varepsilon_j)|x_0|}} \right\}$$

$$=: \left(\frac{|x_0|}{|Q_j|}\right)^{n-2} L_j,$$

We consider the behavior of L_j. Here we note that constants in the denominator and those in the numerator are the same and that the remainder terms $\tilde{f}_{3,\varepsilon_j}(Q_j)/(C(\varepsilon)|Q_j|^2)$ and $\tilde{R}_{\varepsilon_j}(Q_j)/(C(\varepsilon)|Q_j|)$ are negligible compared with three terms in the numerator due to their decay properties as $|Q_j| \to \infty$.

In view of (3.4), since each term has a decay order and since the absolute value of each coefficient in (3.7) are bounded by $C(\varepsilon_j)$, the case where the numerator goes to infinity while the denominator stays bounded is impossible.

In the case where the denominator converges to 0, if we can find suitable point x_* ($|x_*| \geq R_1$) independent of ε so that $u_{\varepsilon_j}(x_*)$ stay uniformly away from zero, we can replace x_0 by x_*.

If this is not the case, then the denominator converges locally uniformly to 0. In this case, the decay property of the expanded functions in (3.7) shows that the numerator decays faster than the denominator.

Similarly, if the both of the denominator and the numerator go to infinity, the decay order shows that the slower growth of the numerator. Thus, the inequality (3.13) is violated if $|Q_j| \to \infty$. We complete the proof. □

4. Proofs of Theorems 1.2, 1.3 and 1.4

Now we are in a position to prove Theorems 1.2, 1.3 and 1.4.

Proof of Theorem 1.2. Since

$$\lim_{\varepsilon \downarrow 0} S_\varepsilon = \frac{(n-2)^2}{4c_0}$$

by Theorem 1.1, we immediately see that

$$\lim_{\varepsilon \downarrow 0} \|\nabla u_\varepsilon\|_2^2 = \lim_{\varepsilon \downarrow 0} S_\varepsilon^{(2+\varepsilon)/\varepsilon} = \lim_{\varepsilon \downarrow 0} \left\{ \frac{(n-2)^2}{4c_0} \right\}^{(2+\varepsilon)/\varepsilon}$$

$$\to \begin{cases} 0, & \text{if } c_0 > (n-2)^2/4, \\ \infty, & \text{if } 0 < c_0 < (n-2)^2/4. \end{cases}$$

Thus we have the desired limiting behavior. Moreover, by (1.4), we have

$$\|\nabla u_\varepsilon\|_2^\varepsilon = S_\varepsilon^{(2+\varepsilon)/2} \to \frac{(n-2)^2}{4c_0}$$

as $\varepsilon \downarrow 0$.

Now, we consider the behavior of $\|u_\varepsilon\|_\infty$. If $\lim_{\varepsilon \downarrow 0} \|\nabla u_\varepsilon\|_2 = 0$, then by (2.1) and the proof of Lemma 3.1, we see that $\|u_\varepsilon\|_\infty \to 0$ as $\varepsilon \downarrow 0$.

When $\lim_{\varepsilon \downarrow 0} \|\nabla u_\varepsilon\|_2 = \infty$, suppose that $\limsup_{\downarrow 0} \|u_\varepsilon\|_\infty < \infty$. Letting

$$\frac{u_\varepsilon(x)}{\|u_\varepsilon\|_\infty} = W_\varepsilon(y), \qquad x = \frac{y}{\varepsilon},$$

we see that $W_\varepsilon(y)$ is a solution to

$$\Delta_y W_\varepsilon + \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) \|u_\varepsilon\|_\infty^\varepsilon W_\varepsilon^{1+\varepsilon} = 0.$$

Since $\lim_{\varepsilon \downarrow 0} \varepsilon^{-2} K(y/\varepsilon) = c_0 |y|^{-2}$ locally uniformly in $\mathbf{R}^n \backslash \{0\}$, by choosing a subsequence if necessary (still denoted by ε), W_ε converges locally uniformly in $\mathbf{R}^n \backslash \{0\}$ to W, where W is a solution to

$$\Delta_y W + \frac{\bar{C}}{|y|^2} W = 0 \qquad \text{in } \mathbf{R}^n$$

with $\bar{C} := c_0 \lim_{\varepsilon \downarrow 0} \|u_\varepsilon\|_\infty^\varepsilon < (n-2)^2/4$. The limiting equation does not have any positive solution which is bounded near the origin unless $\bar{C} = 0$. Thus $W \equiv 0$ if $0 < \bar{C} < (n-2)^2/4$. However, concerning the constant $\tilde{C}_\varepsilon$ in (3.5), since we have

$$u_\varepsilon(x) = \|u_\varepsilon\|_\infty W_\varepsilon(y) = \tilde{C}_\varepsilon \varepsilon^{n-2} |y|^{-(n-2)} + h_\varepsilon\left(\frac{y}{\varepsilon}\right),$$

we obtain $\lim_{\varepsilon \downarrow 0} \varepsilon^{n-2} \tilde{C}_\varepsilon = 0$ in view of the local uniform convergence of W_ε to

$W \equiv 0$. Since the boundedness of u_ε implies the boundedness of ∇u_ε in view of the equation, we have

$$\lim_{\varepsilon\downarrow 0} \|\nabla u_\varepsilon\|_2^2 = \lim_{\varepsilon\downarrow 0} \int_{\mathbf{R}^n\setminus B_R} |\nabla u_\varepsilon|^2 dx = \infty,$$

for any $R > 0$. In view of (3.5), we see that

$$\int_{\mathbf{R}^n\setminus B_R} |\nabla u_\varepsilon|^2 dx \le C\tilde{C}_\varepsilon^2$$

as $\varepsilon \downarrow 0$ with $C > 0$. Thus we have

$$\int_{\mathbf{R}^n\setminus B_R} |\nabla u_\varepsilon|^2 dx = o(\varepsilon^{-2(n-2)})$$

as $\varepsilon \downarrow 0$.

However, we have seen

$$\lim_{\varepsilon\downarrow 0} \|\nabla u_\varepsilon\|_2^2 = \lim_{\varepsilon\downarrow 0} \left(\frac{(n-2)^2}{4c_0}\right)^{(2+\varepsilon)/\varepsilon} = \infty,$$

thus the growth order of $\|\nabla u_\varepsilon\|_2$ is faster than $\varepsilon^{-2(n-2)}$. Hence we get a contradiction for $0 < \bar{C} < (n-2)^2/4$.

If $\bar{C} = 0$, then there is a possibility of $W \equiv 1$. In this case, as in the last part of the proof of Lemma 3.2, we have

$$\frac{W_\varepsilon(2y)}{W_\varepsilon(y)} = 2^{-(n-2)} + o(1)$$

as $\varepsilon \downarrow 0$. This contradicts the uniform convergence of W_ε to 1. The case where $W \equiv 0$ is proved as in $0 < \bar{C} < (n-2)^2/4$.

Thus we have reached a contradiction for $0 \le \bar{C} < (n-2)^2/4$, that is, we have proved $\|u_\varepsilon\|_\infty \to \infty$ as $\varepsilon \downarrow 0$ if $0 < c_0 < (n-2)^2/4$. □

Proof of Theorem 1.3. In this case, since $\lim_{\varepsilon\downarrow 0}\|\nabla u_\varepsilon\|_2^\varepsilon = 1$, we need careful calculations. Suppose that $\limsup_{\varepsilon\downarrow 0}\|\nabla u_\varepsilon\|_2 < \infty$. Then by (2.1), we see that $\|u_\varepsilon\|_\infty$ is bounded. Then, choosing a subsequence if necessary, we see that u_{ε_j} converges to $\tilde{U} \in \mathscr{D}$ locally uniformly in $\mathbf{R}^n$, where $\tilde{U}$ is a nonnegative solution to

$$\Delta\tilde{U} + K(x)\tilde{U} = 0.$$

Note that this equation has only $\tilde{U} \equiv 0$ as a nonnegative solution by the Pohozaev identity (1.7). Thus, the convergence does not depend on subsequences. Moreover, by Lemma 3.5, the maximum point of u_ε is bounded. Thus, $\|u_\varepsilon\|_\infty \to 0$ as $\varepsilon \downarrow 0$.

Let $\hat{v}_\varepsilon(x) := u_\varepsilon(x)/\|u_\varepsilon\|_\infty$. Then $\hat{v}_\varepsilon$ satisfies

$$\Delta\hat{v}_\varepsilon + K(x)\|u_\varepsilon\|_\infty^\varepsilon \hat{v}_\varepsilon^{1+\varepsilon} = 0, \qquad \|\hat{v}_\varepsilon\|_\infty = 1.$$

Suppose that $\|u_\varepsilon\|_\infty^\varepsilon \to c > 0$ as $\varepsilon \downarrow 0$. Then, since the maximum point is bounded, $\hat{v}_\varepsilon$ converges to a nontrivial function (along a subsequence). However, again by the Pohozaev identity, the limiting equation has only $\hat{v} \equiv 0$ as a solution, which is a contradiction.

Suppose that $\|u_\varepsilon\|_\infty^\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then, $\hat{v}_\varepsilon$ converges to 1 locally uniformly in $\mathbf{R}^n$. However, as in the last part of the proof of Theorem 1.2 for $0 < c_0 < (n-2)^2/4$, there exists large $R > 0$ independent of ε such that

$$\frac{\hat{v}_\varepsilon(2x)}{\hat{v}_\varepsilon(x)} = 2^{-(n-2)} + o(1) \tag{4.1}$$

for $|x| \geq R$. Hence, the local uniform convergence of $\hat{v}_\varepsilon$ to 1 is impossible. Thus we see that $\lim_{\varepsilon\downarrow 0}\|u_\varepsilon\|_\infty = \infty$ and $\lim_{\varepsilon\downarrow 0}\|\nabla u_\varepsilon\|_2 \to \infty$.

By Theorem 1.1, $\|\nabla u_\varepsilon\|_2^{2\varepsilon/(2+\varepsilon)} = S_\varepsilon \to 1$ as $\varepsilon \downarrow 0$, i.e., $\|\nabla u_\varepsilon\|_2^\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Thus the proof is complete for any case. □

Using Lemmas 3.1, 3.2 and 3.5, we prove Theorem 1.4.

PROOF OF THEOREM 1.4. It is easy to see that $v_\varepsilon(y)$ satisfies

$$\Delta_y v_\varepsilon(y) + \frac{1}{\varepsilon^2}K\left(\frac{y}{\varepsilon}\right)\left(\max_{|x|\geq R/\varepsilon} u_\varepsilon(x)\right)^\varepsilon v_\varepsilon^{1+\varepsilon} = 0.$$

Since $\max_{|y|\geq R} v_\varepsilon(y) = 1$ and since $\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}K(y/\varepsilon) = c_0/|y|^2$ locally uniformly on $\{y \mid |y| \geq R\}$, by choosing a subsequence if necessary, $v_\varepsilon(y)$ converges to $V(y)$ locally uniformly on $\{y \mid |y| \geq R\}$, where $V(y)$ is a solution to

$$\Delta V + \frac{c_0 c_*}{|y|^2} V = 0 \tag{4.2}$$

with c_* being an accumulation point of $(\max_{|x|\geq R/\varepsilon} u_\varepsilon(x))^\varepsilon$ as $\varepsilon \downarrow 0$.

Suppose that c_* can be taken as $c_* = 0$. As in the proof of Lemma 3.5, the maximum point $\tilde{Q}_\varepsilon$ of $v_\varepsilon(y)$ $(\max_{|x|\geq R/\varepsilon} u_\varepsilon(x) = v_\varepsilon(\tilde{Q}_\varepsilon))$ is uniformly bounded in view of the decay (3.5). Thus there exists y_0 $(|y_0| \geq R)$ such that $V(y_0) = 1$. Since (4.2) yields $\Delta V = 0$ in this case, V might satisfy $V \equiv 1$. However, by (3.5), the local uniform convergence to 1 is absurd as in (4.1) (consider the ratio $v_\varepsilon(2y_1)/v_\varepsilon(y_1)$ with sufficiently large $|y_1|$). Thus V cannot be a positive constant in this case. Hence c_* must be positive.

As for the estimate of the upper bound of c_*, we use (2.1) and Lemma 3.1. By them, we have

$$\|u_\varepsilon\|_\infty \leq \tilde{C}\|\nabla u_\varepsilon\|_2$$

with $\tilde{C} > 0$ independent of ε. Combining Theorem 1.1 with (1.4), we see that $\|\nabla u_\varepsilon\|_2^\varepsilon \to (n-2)^2/(4c_0)$ as $\varepsilon \downarrow 0$. Thus we obtain $\limsup_{\varepsilon\downarrow 0}\|u_\varepsilon\|_\infty^\varepsilon \le (n-2)^2/(4c_0)$. Since

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} K\left(\frac{y}{\varepsilon}\right) = \frac{c_0}{|y|^2},$$

choosing a subsequence, we see that $v_\varepsilon(y)$ converges locally uniformly in $\{y \mid |y| \ge R\}$ to V, which is a solution to

$$\begin{cases} \Delta V + \dfrac{\tilde{c}}{|y|^2} V = 0, \\ V(Q_*) = 1, \lim\limits_{|y|\to\infty} V(y) = 0, \end{cases} \tag{4.3}$$

where $0 < \tilde{c} \le (n-2)^2/4$ and $|Q_*| \ge R$. Note that (4.3) has a solution (at least a radial one). Thus we obtain the desired conclusion. $\square$

5. Appendix

Here we give a proof of Lemma 2.1 for the sake of self-containedness.

Proof of Lemma 2.1. Since $S(\ell)$ is attained by $U(x) = (1+|x|^{2-\ell})^{-(n-2)/(2-\ell)}$, we have

$$\int_{\mathbf{R}^n} |\nabla U|^2 dx = (n-2)^2 \omega_n \int_0^\infty r^{n-1} r^{2-2\ell}(1+r^{2-\ell})^{-2(n-\ell)/(2-\ell)} dr$$

since $U_r = -(n-2)(1+|x|^{2-\ell})^{-(n-2)/(2-\ell)-1}|x|^{1-\ell}$. Letting $r = \rho^{2/(2-\ell)}$, we get

$$\begin{aligned} &\int_0^\infty r^{n+1-2\ell}(1+r^{2-\ell})^{-2(n-\ell)/(2-\ell)} dr \\ &\quad = \frac{2}{2-\ell}\int_0^\infty \rho^{(2n-3\ell+2)/(2-\ell)}(1+\rho^2)^{-2(n-\ell)/(2-\ell)} d\rho. \end{aligned}$$

Letting $\rho = \tan\theta$, we have

$$\begin{aligned} &\int_0^\infty r^{n+1-2\ell}(1+r^{2-\ell})^{-2(n-\ell)/(2-\ell)} dr \\ &\quad = \frac{2}{2-\ell}\int_0^{\pi/2} \sin^{2(n-2\ell+2)/(2-\ell)-1}\theta \cos^{2(n-2)/(2-\ell)-1}\theta \, d\theta. \end{aligned}$$

Since the beta function $B(p,q)$ is defined as

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta$$

and since $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, we obtain

$$\int_{\mathbf{R}^n} |\nabla U|^2 dx = \frac{(n-2)^2\omega_n}{2-\ell} B\left(\frac{n-2\ell+2}{2-\ell}, \frac{n-2}{2-\ell}\right)$$

$$= \frac{(n-2)^2\omega_n \Gamma\left(\frac{n-2\ell+2}{2-\ell}\right)\Gamma\left(\frac{n-2}{2-\ell}\right)}{(2-\ell)\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right)}.$$

Similarly, we have

$$\int_{\mathbf{R}^n} |x|^{-\ell} U^{2((n-\ell)/(n-2))}\, dx$$

$$= \omega_n \int_0^\infty r^{n-1-\ell}(1+r^{2-\ell})^{-2(n-\ell)/(2-\ell)} dr$$

$$= \frac{2\omega_n}{2-\ell}\int_0^\infty \rho^{(2(n-1)-\ell)/(2-\ell)}(1+\rho^2)^{-2(n-\ell)/(2-\ell)} d\rho$$

$$= \frac{2\omega_n}{2-\ell}\int_0^{\pi/2} \sin^{2(n-\ell)/(2-\ell)-1}\theta \cos^{2(n-\ell)/(2-\ell)-1}\theta\, d\theta$$

$$= \frac{\omega_n}{2-\ell} B\left(\frac{n-\ell}{2-\ell}, \frac{n-\ell}{2-\ell}\right)$$

$$= \frac{\frac{\omega_n}{2-\ell}\left(\Gamma\left(\frac{n-\ell}{2-\ell}\right)\right)^2}{\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right)},$$

where $r = \rho^{2/(2-\ell)}$ and $\rho = \tan\theta$ as before. Thus we obtain

$$S(\ell) = \frac{\int_{\mathbf{R}^n} |\nabla U|^2 dx}{(\int_{\mathbf{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx)^{(n-2)/(n-\ell)}}$$

$$= \frac{(n-2)^2\omega_n}{2-\ell} \frac{\Gamma\left(\frac{n-2\ell+2}{2-\ell}\right)\Gamma\left(\frac{n-2}{2-\ell}\right)}{\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right)}$$

$$\times \left\{ \left(\frac{\omega_n}{2-\ell}\right)^{(n-2)/(n-\ell)} \frac{\left(\Gamma\left(\frac{n-\ell}{2-\ell}\right)\right)^{2(n-2)/(n-\ell)}}{\left(\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right)\right)^{(n-2)/(n-\ell)}} \right\}^{-1}$$

$$= (n-2)^2\left(\frac{\omega_n}{2-\ell}\right)^{(2-\ell)/(n-\ell)}$$

$$\times \frac{\Gamma\left(\dfrac{n-2\ell+2}{2-\ell}\right)\Gamma\left(\dfrac{n-2}{2-\ell}\right)}{\left(\Gamma\left(\dfrac{n-\ell}{2-\ell}\right)\right)^{2(n-2)/(n-\ell)}\left(\Gamma\left(\dfrac{2(n-\ell)}{2-\ell}\right)\right)^{(2-\ell)/(n-\ell)}}.$$

Thus the first part is proved.

Since our aim is to let $\ell \uparrow 2$, we need the asymptotic expansion of the gamma function. The expansion formula (Stirling's formula, see e.g., Taylor [14], p. 267, (A.39)) yields

$$\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-1/2} + L(z),$$

near $z = \infty$ with $\lim_{|z|\to\infty} e^z z^{-(z-1/2)}L(z) = 0$. The formula yields

$$\Gamma\left(\frac{n-2\ell+2}{2-\ell}\right) = \sqrt{2\pi}e^{-(n-2\ell+2)/(2-\ell)}\left(\frac{n-2\ell+2}{2-\ell}\right)^{(n-2\ell+2)/(2-\ell)-1/2} + L_1(\ell),$$

$$\Gamma\left(\frac{n-2}{2-\ell}\right) = \sqrt{2\pi}e^{-(n-2)/(2-\ell)}\left(\frac{n-2}{2-\ell}\right)^{(n-2)/(2-\ell)-1/2} + L_2(\ell),$$

$$\Gamma\left(\frac{n-\ell}{2-\ell}\right) = \sqrt{2\pi}e^{-(n-\ell)/(2-\ell)}\left(\frac{n-\ell}{2-\ell}\right)^{(n-\ell)/(2-\ell)-1/2} + L_3(\ell),$$

$$\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right) = \sqrt{2\pi}e^{-2(n-\ell)/(2-\ell)}\left(\frac{2(n-\ell)}{2-\ell}\right)^{2(n-\ell)/(2-\ell)-1/2} + L_4(\ell),$$

where $L_i(\ell)$ $(i = 1, 2, 3, 4)$ are lower order terms as $\ell \uparrow 2$. Thus we have

$$\Gamma\left(\frac{n-2\ell+2}{2-\ell}\right)\Gamma\left(\frac{n-2}{2-\ell}\right) = 2\pi e^{-2(n-\ell)/(2-\ell)}(n-2\ell+2)^{(n-2\ell+2)/(2-\ell)-1/2}$$
$$\times (n-2)^{(n-2)/(2-\ell)-1/2}(2-\ell)^{1-2(n-\ell)/(2-\ell)} + L_5(\ell)$$

and

$$\left(\Gamma\left(\frac{n-\ell}{2-\ell}\right)\right)^{2(n-2)/(n-\ell)}\left(\Gamma\left(\frac{2(n-\ell)}{2-\ell}\right)\right)^{(2-\ell)/(n-\ell)}$$
$$= (2\pi)^{(2n-\ell-2)/(2(n-\ell))}2^{2-(2-\ell)/(2(n-\ell))}e^{-2-2(n-2)/(2-\ell)}$$
$$\times \left(\frac{n-\ell}{2-\ell}\right)^{2+2(n-2)/(2-\ell)-(2n-\ell-2)/(2(n-\ell))} + L_6(\ell)$$
$$= 8\pi e^{-2(n-\ell)/(2-\ell)}\left(\frac{n-\ell}{2-\ell}\right)^{2(n-\ell)/(2-\ell)-(2n-\ell-2)/(2(n-\ell))} + L_7(\ell),$$

where $L_i(\ell)$ $(i = 5, 6, 7)$ are lower order terms. Hence we see that

$$S(\ell) = (n-2)^2 \left(\frac{\omega_n}{2-\ell}\right)^{(2-\ell)/(n-\ell)}$$

$$\times \frac{e^{-2(n-\ell)/(2-\ell)}(n-2\ell+2)^{(n-2\ell+2)/(2-\ell)-1/2}(n-2)^{(n-2)/(2-\ell)-1/2}(2-\ell)^{1-2(n-\ell)/(2-\ell)}}{4e^{-2(n-\ell)/(2-\ell)}\left(\dfrac{n-\ell}{2-\ell}\right)^{2(n-\ell)/(2-\ell)-(2n-\ell-2)/(2(n-\ell))}}$$

$$+ L_8(\ell)$$

$$= \frac{n-2}{4}\left(\frac{\omega_n}{2-\ell}\right)^{(2-\ell)/(n-\ell)} \left\{\frac{(n-2\ell+2)^{n-2\ell+2}(n-2)^{n-2}}{(n-\ell)^{2(n-\ell)}}\right\}^{1/(2-\ell)}$$

$$\times (2-\ell)^{(2-\ell)/(2(n-\ell))}(n-\ell)^{(2n-\ell-2)/(2(n-\ell))} + L_9(\ell),$$

where $L_8(\ell)$ and $L_9(\ell)$ are lower order terms with $\lim_{\ell\uparrow 2} L_i(\ell) = 0$ $(i = 8, 9)$. Note that

$$\lim_{\ell\uparrow 2} \frac{1}{2-\ell}\left\{\log \frac{(n-2\ell+2)^{n-2\ell+2}(n-2)^{n-2}}{(n-\ell)^{2(n-\ell)}}\right\}$$

$$= \lim_{\ell\uparrow 2} \frac{(n-2\ell+2)\log(n-2\ell+2) + (n-2)\log(n-2) - 2(n-\ell)\log(n-\ell)}{2-\ell}$$

$$= -\lim_{\ell\uparrow 2}\{-2\log(n-2\ell+2) - 2 + 2\log(n-\ell) + 2\} = 0$$

by l'Hospital's rule. Thus we have

$$\lim_{\ell\uparrow 2}\left\{\frac{(n-2\ell+2)^{n-2\ell+2}(n-2)^{n-2}}{(n-\ell)^{2(n-\ell)}}\right\}^{1/(2-\ell)} = 1.$$

Moreover, by $\lim_{x\to+0} x^x = 1$, we see that

$$\lim_{\ell\uparrow 2} S(\ell) = \frac{(n-2)^2}{4} = S(2)$$

as desired. □

Remark 5.1. Since the area of the unit sphere is given by $\omega_n = 2\pi^{n/2}(\Gamma(n/2))^{-1}$, $S(\ell)$ coincides with $S_R(p, q, \alpha, \beta, n)$ in Lemma 3.1 of [6] with $p = 2$, $q = 2(n-\ell)/(n-2)$, $\alpha = 0$, $\beta = -(n-2)\ell/\{2(n-\ell)\}$ and $\gamma = n(2-\ell)/\{2(n-\ell)\}$.

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