Behavior of the life span for solutions to the system of reaction-diffusion equations

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Abstract. We consider the weakly coupled system of reaction-diffusion equations

\[ \begin{align*}
    u_t &= D_u u + a(x) v^p, \\
    v_t &= D_v v + b(x) u^q, \\
    u(x, 0) &= \lambda^p \phi(x), \\
    v(x, 0) &= \lambda^q \psi(x),
\end{align*} \]

where \( 0 \leq a(x), b(x) \in C(\mathbb{R}^N) \), \( \varphi(x), \psi(x) \geq 0 \) are bounded continuous functions in \( \mathbb{R}^N \), \( p, q > 1, \mu, \nu > 0 \), and \( \lambda > 0 \) are parameters. The existence of solutions, blow-up conditions, and global solutions of the above equations with \( a(x) \equiv |x|^\sigma_1, b(x) \equiv |x|^\sigma_2 \) \((0 \leq \sigma_1 < N(p-1), 0 \leq \sigma_2 < N(q-1))\) are studied by Mochizuki and Huang. In this paper, we consider an estimate of maximal existence time of blow-up solutions as \( \lambda \) goes to 0 or \( \infty \), when \( a(x), b(x) \) are more general functions.

1. Introduction and statement of results

We consider bounded, nonnegative solutions to the Cauchy problem for a weakly coupled system

\[ \begin{align*}
    u_t &= D_u u + a(x) v^p \quad (x \in \mathbb{R}^N, t > 0), \\
    v_t &= D_v v + b(x) u^q \quad (x \in \mathbb{R}^N, t > 0), \\
    u(x, 0) &= \lambda^p \phi(x) \quad (x \in \mathbb{R}^N), \\
    v(x, 0) &= \lambda^q \psi(x) \quad (x \in \mathbb{R}^N),
\end{align*} \]  

where \( 0 \leq a(x), b(x) \in C(\mathbb{R}^N), 0 \leq \varphi(x), \psi(x) \in BC(\mathbb{R}^N) \); here \( BC(\mathbb{R}^N) \) is the set of bounded continuous functions on \( \mathbb{R}^N \), \( p, q > 1, \mu, \nu > 0 \), and \( \lambda > 0 \) are parameters. Since the nonlinearities, \( a(x)v^p,b(x)u^q \), are locally continuous in \( x \) and locally Lipschitz in \( u,v \), it follows from standard results that any solution \( u(x,t), v(x,t) \geq 0 \) of the equation (1) is in fact classical; that is, \( u,v \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap C(\mathbb{R}^N \times [0,T]) \) for some \( T > 0 \). Thus, the comparison theorem holds from Theorem 1 in [1]; i.e. if

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1 2000 Mathematics Subject Classification. 35K57, 58J35.
2 Key Words and Phrases. reaction-diffusion equations, heat and other parabolic equation methods.
\[
\begin{align*}
\tilde{f}_0 \leq u(x,0) \leq \bar{f}_0, \quad \tilde{g}_0 \leq v(x,0) \leq \bar{g}_0,
\end{align*}
\]

it follows that for \( x \in \mathbb{R}^N, 0 \leq t \leq T, \)
\[
\begin{align*}
\tilde{f}(t) \leq u(x,t) \leq \bar{f}(t), \quad \tilde{g}(t) \leq v(x,t) \leq \bar{g}(t),
\end{align*}
\]

where \( (\tilde{f}(t), \tilde{g}(t)) \) and \( (\bar{f}(t), \bar{g}(t)) \) are subsolution and supersolution of (1) with initial value \((\tilde{f}_0, \tilde{g}_0)\) and \((\bar{f}_0, \bar{g}_0)\).

We let \( T^*_\lambda > 0 \) be the maximal existence time. From the general theory of evolution equation \([9]\), it follows that there exists a unique bounded solution \( u(x,t) \) to the equation
\[
\begin{align*}
\frac{du}{dt} &= Du + a(x)u^\sigma \
&\quad \quad \quad (x \in \mathbb{R}^N, t > 0), \\
u(x,0) &= \Lambda \varphi(x) \quad (x \in \mathbb{R}^N),
\end{align*}
\]
which satisfies
\[
\begin{align*}
\sup_{t \in [0,T]} ||u(t)||_\infty < \infty \quad \text{for} \quad 0 < \exists T \leq \infty,
\end{align*}
\]

where \( a(x) \) is a continuous function which satisfies that \( a(x)/|x|^\alpha \) (\( \sigma > -2 \)) is bounded when \( |x| \) is sufficiently large, and \( 0 \leq \varphi(x) \leq \delta \exp(-\gamma|x|^2) \) holds. So
we define \( T^*_\lambda \) as follows:
\[
T^*_\lambda := \sup \left\{ T > 0; \sup_{t \in [0,T]} \{ ||u(t)||_\infty + ||v(t)||_\infty \} < \infty \right\}.
\]

If \( T^*_\lambda = \infty \), the solutions are global. The global existence and nonexistence are studied by Escobedo-Herrero \([2]\) and Mochizuki \([7]\) in the case \( a(x) \equiv b(x) \equiv 1 \), and are extended in \([8]\) to the case \( a(x) = |x|^{\sigma_1}, \ b(x) = |x|^{\sigma_2}, \) where \( 0 \leq \sigma_1 < N(p-1), \ 0 \leq \sigma_2 < N(q-1) \).

In this paper, we shall consider a precise estimate of \( T^*_\lambda \) as \( \lambda \) goes to 0 or \( \infty \). This problem is studied in Huang-Mochizuki-Mukai \([5]\) and Mochizuki \([7]\) in the special case \( a(x) \equiv b(x) \equiv 1 \). On the other hand, Pinsky \([11]\) studied the life span of the single equation (2) where \( a(x) \) is some kind of function. We shall extend the results of \([5]\) and \([7]\) and prove by the same methods as \([11]\).

We put
\[
\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}.
\]

**Theorem 1.** Assume that \( a, b \) satisfy
\[
a(x) \sim |x|^{\sigma_1}, \quad b(x) \sim |x|^{\sigma_2} \quad \text{as} \quad |x| \to \infty,
\]
where \( \sigma_1, \sigma_2 > -2 \) if \( N \geq 2, \ \sigma_1, \sigma_2 > -1 \) if \( N = 1, \) and that initial data \( \varphi, \psi \) satisfy
for some $\delta, \gamma > 0$.

(i) Suppose that $\alpha + \delta_1 > N$ (or $\beta + \delta_2 > N$), where

$$\delta_1 = \frac{\sigma_2 p + \sigma_1}{pq - 1}, \quad \delta_2 = \frac{\sigma_1 q + \sigma_2}{pq - 1}.$$ 

Then there exist $\lambda_1 > 0$ and $C > 0$ such that

$$T_\lambda^* \leq C\lambda^{-2\mu/(\alpha + \delta_1 - N)} \quad (or \quad C\lambda^{-2\nu/(\beta + \delta_2 - N)}) \quad \text{for} \quad \lambda < \lambda_1.$$ 

(ii) Suppose that

$$p < p^* = 1 + \frac{2 + \sigma_1}{N}, \quad q < q^* = 1 + \frac{2 + \sigma_2}{N}.$$ 

Let $\mu, \nu$ be chosen to satisfy

$$\frac{\mu}{\nu} = \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}.$$ 

Then we have

$$T_\lambda^* \sim \lambda^{-2\mu/(\alpha + \delta_1 - N)} \sim \lambda^{-2\nu/(\beta + \delta_2 - N)} \quad \text{as} \quad \lambda \to 0.$$ 

**Theorem 2.** Assume that $0 \leq a, b, \varphi, \psi \in BC(\mathbb{R}^N)$ and that there is a smooth bounded domain $D \subset \mathbb{R}^N$ such that

$$\inf_{x \in D} a(x), \inf_{x \in D} b(x), \inf_{x \in D} \varphi(x), \inf_{x \in D} \psi(x) > 0.$$ 

(i) Suppose that $\nu > \mu, q\mu > \nu$. Then there exist $\lambda_1 > 0$ and $C > 0$ such that

$$T_\lambda^* \leq C\lambda^{-2\mu/\alpha} \quad (or \quad C\lambda^{-2\nu/\beta}) \quad \text{for} \quad \lambda > \lambda_1.$$ 

(ii) Let $\mu, \nu$ be chosen to satisfy $\mu/\nu = \alpha/\beta$. Then we have

$$T_\lambda^* \sim \lambda^{-2\mu/\alpha} \sim \lambda^{-2\nu/\beta} \quad \text{as} \quad \lambda \to \infty.$$ 

**Remark 1.** Theorems 1 and 2 are the extension of results of [11]. If we put $u = v, \varphi = \psi, a = b, p = q, \sigma_1 = \sigma_2, \mu = \nu = 1$ in these theorems, the same results as Theorem 1 (i) and Theorem 3 (i) in [11] are obtained respectively.

We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively. In the sequel, we will use the notation

$$P(x, t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right).$$
We conclude this section by noting the following well-known integral representation which holds for bounded solutions $u(x,t), v(x,t)$ to (1):

\[
\begin{align*}
    u(x,t) &= \lambda^\mu \int_{\mathbb{R}^n} P(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}^n} P(x-y,t-s)a(y)v(y,s)dyds, \\
v(x,t) &= \lambda^\nu \int_{\mathbb{R}^n} P(x-y,t)\psi(y)dy + \int_0^t \int_{\mathbb{R}^n} P(x-y,t-s)b(y)u(y,s)dyds.
\end{align*}
\]

(3)

\section{Proof of Theorem 1}

We begin with the proof of the upper bounds.

**Lemma 2.1.** Let $u(x,t), v(x,t)$ satisfy (1). Then for any $t_0 \in (0, T^*_1)$, there exists a $c > 0$ such that

\[
u(x,t) \geq \lambda^\nu c t^{-N/2} \exp\left(-\frac{|x|^2}{2t}\right),
\]

\[
u(x,t) \geq \lambda^\nu c t^{-N/2} \exp\left(-\frac{|x|^2}{2t}\right), \quad \text{for } t \in [t_0, T^*_1), x \in \mathbb{R}^N.
\]

**Proof.** We prove only the first inequality. Since $\varphi(x) \neq 0$, there exists $D_1 \subset \mathbb{R}^N$ such that

\[
\begin{align*}
c_1 &= \inf_{x \in D_1} \varphi(x) > 0.
\end{align*}
\]

From the inequality $|x-y|^2 \leq 2|x|^2 + 2|y|^2$ and (3), it follows that

\[
\begin{align*}
u(x,t) &\geq \lambda^\mu \int_{\mathbb{R}^n} P(x-y,t)\varphi(y)dy \\
&\geq \lambda^\mu (4\pi)^{-N/2} c_1 \int_{D_1} \exp\left(-\frac{|x|^2}{2t} - \frac{|y|^2}{2t}\right)dy
\end{align*}
\]

\[
\begin{align*}
&\geq \lambda^\mu (4\pi)^{-N/2} c_1 t^{-N/2} \exp\left(-\frac{|x|^2}{2t_0}\right) \int_{D_1} \exp\left(-\frac{|y|^2}{2t_0}\right)dy,
\end{align*}
\]

for $t \geq t_0$.

Let $D_n = \{x \in \mathbb{R}^N; n < |x| < 2n\}$ if $N \geq 2$, and $D_n = \{x \in \mathbb{R}^N; n < x < 2n\}$ if $N = 1$. Let $\theta_n > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $D_n$, and let $\omega_n(x)$ denote the corresponding positive eigenfunction, normalized by $\int_{D_n} \omega_n(x)dx = 1$. Note that since $D_n$ contains an $N$-dimensional
cube of length $kn$ for an appropriate constant $k \in (0, 1)$, it follows that there exists a constant $c > 0$ such that

$$\theta_n \leq cn^{-2}. \quad (4)$$

By assumption, there exist $n_0$ and $c_1 > 0$ such that

$$a(x) \geq c_1|x|^{\alpha_1}, \quad b(x) \geq c_1|x|^{\alpha_2}, \quad \text{for } |x| \geq n_0. \quad (5)$$

From now on, we will always assume that $n \geq n_0$. Define

$$F_n(t) = \int_{D_n} u(x, t)a_n(x)dx,$$

$$G_n(t) = \int_{D_n} v(x, t)a_n(x)dx, \quad \text{for } 0 \leq t < T^*_n.$$  

Then it follows that $F_n(t) \leq \|u(t)\|_\infty$, $G_n(t) \leq \|v(t)\|_\infty$ for all $n > 0$. Thus, $T^*_n$ is no more than the blow up time of $(F_n(t), G_n(t))$. Let $\partial_t/\partial n$ be the outward normal derivative to $D_n$ at $x \in \partial D_n$. From Green’s formula and the fact that $\omega_n(x) = 0$ and $\partial \omega_n/\partial n \leq 0$ on $\partial D_n$, we obtain

$$\int_{D_n} (Au(x, t)a_n(x) - u(x, t)\partial \omega_n(x))dx = \int_{\partial D_n} \left(\frac{\partial u}{\partial n} - u \frac{\partial \omega_n}{\partial n}\right)ds \geq 0.$$  

From Hölder’s inequality, the inequality

$$\int_{D_n} v(x, t)a_n(x)dx \leq \left(\int_{D_n} v(x, t)^p a_n(x)dx\right)^{1/p}$$

holds. Using (4), (5), we obtain from (1)

$$F_n'(t) = \int_{D_n} u_t(x, t)a_n(x)dx$$

$$= \int_{D_n} (Au(x, t) + a(x)v(x, t)^p)a_n(x)dx$$

$$\geq \int_{D_n} u(x, t)\partial a_n(x)dx + c_1 \int_{D_n} |x|^{\alpha_1}v(x, t)^p a_n(x)dx$$

$$\geq -\theta_n \int_{D_n} u(x, t)a_n(x)dx + c_0 n^{\alpha_1} \int_{D_n} v(x, t)^p a_n(x)dx$$

$$\geq -cn^{-2}F_n(t) + c_0 n^{\alpha_1} G_n(t)^p.$$  

Thus, we obtain the following inequalities:

$$F_n'(t) \geq -cn^{-2}F_n(t) + c_0 n^{\alpha_1} G_n(t)^p \quad (t > 0),$$

$$G_n'(t) \geq -cn^{-2}G_n(t) + c_0 n^{\alpha_2} F_n(t)^q \quad (t > 0). \quad (6)$$
By Lemma 2.1, there exists a \( C > 0 \) such that \( u(x, n^2) \geq C \lambda^n n^{-N}, \) \( v(x, n^2) \geq C \lambda^n n^{-N} \) for \( n < |x| < 2n, \) thus
\[
F_n(n^2) \geq C \lambda^n n^{-N}, \quad G_n(n^2) \geq C \lambda^n n^{-N}.
\]
Let \( f_n, g_n \in C^0([0, T^*_n)) \cap C^1((0, T^*_n)) \) be the solution to the system of ordinary differential equations
\[
\begin{aligned}
f'_n(t) &= -cn^{-2}fn(t) + c_0n^{n_1}gn(t) \quad (t > 0), \\
g'_n(t) &= -cn^{-2}gn(t) + c_0n^{n_2}fn(t) \quad (t > 0), \\
f_n(n^2) &= C \lambda^n n^{-N}, \\
g_n(n^2) &= C \lambda^n n^{-N}.
\end{aligned}
\] (7)
Then \( (F_n(t), G_n(t)) \) is a supersolution of (7). By the scaling
\[
\begin{aligned}
f(t) &= e^{-\gamma/2}c^{1/2}_0 n^{n_1+\delta_1}fn(c^{-1/2}n^2(t+c)), \\
g(t) &= e^{-\beta/2}c^{1/2}_0 n^{n_2+\delta_2}gn(c^{-1/2}n^2(t+c)),
\end{aligned}
\] (8)
we obtain the simpler system of equations
\[
\begin{aligned}
f'(t) &= -f(t) + g(t)^p \quad (t > 0), \\
g'(t) &= -g(t) + f(t)^q \quad (t > 0),
\end{aligned}
\] (9)
with the initial data
\[
f(0) = C_p \lambda^n n^{n_1+\delta_1-N}, \quad g(0) = C_q \lambda^n n^{n_2+\delta_2-N},
\]
where \( C_p = Ce^{-\gamma/2}c^{1/2}_0, \) \( C_q = Ce^{-\beta/2}c^{1/2}_0. \)

**Lemma 2.2.** Let \( (f(t), g(t)) \) be the solution to (9) with the initial data
\[
f(0) > 1, \quad g(0) = 0.
\]
If \( f(0) \) is sufficiently large, then \( (f(t), g(t)) \) blows up in finite time. Moreover, the life span \( T_0 \) of \( (f(t), g(t)) \) is estimated from above by
\[
T_0 \leq t_0 + \int_{f(t_0)g(t_0)}^{\infty} \left\{ C(p, q) \xi^{(p+1)(q+1)/(p+q+2)} - 2 \xi \right\}^{-1} d\xi,
\] (10)
where
\[
C(p, q) = \left( \frac{p + q + 2}{p + 1} \right)^{(p+1)/(p+q+2)} \left( \frac{p + q + 2}{q + 1} \right)^{(q+1)/(p+q+2)}
\]
and \( 0 < t_0 < T_0 \) is chosen to satisfy \( \{f(t_0)g(t_0)\}^{(p-1)/(p+q+2)} > 2. \)

**Proof.** See e.g., K. Mochizuki [7].

**Proof of Theorem 1 (i).** As is shown in the above lemma, there exist \( A_1 > 0 \) and \( B_1 > 0 \) such that if
then \((f(t), g(t))\) blows up in finite time. We see that (11) will be satisfied if \(n = n(\lambda)\) is chosen so that

\[
\lambda^\nu = \gamma n^{\frac{1}{2}} + N,
\]

where \(\gamma > 0\) is a constant which satisfies \(\gamma > C_p^{-1}A_1\). If \(\lambda\) is sufficiently small, \(n > n_0\), so we can apply this argument. From (8) and Lemma 2.2, there exists a \(\lambda_0 > 0\) such that

\[
T_\lambda^* \leq c^{-1}n^2(T_0 + c) = C\lambda^{-2\mu/(\alpha + \beta - N)}
\]

for \(\lambda < \lambda_0\).

Note that there is only one equilibrium of system (9) in \(\mathbb{R}^2_+\), say \(P = (1, 1)\). As is easily seen, \(P\) is a saddle point. One of the separatrix starts from 0 and runs to \(\infty\). Another one intersects \(f\)-axis and \(g\)-axis at \(A_1\) and \(B_1\), respectively. Moreover, every solution \((f(t), g(t))\) of (9) with the initial value \((f(0), g(0))\) lying above this separatrix runs into

\[
Q = \{(f, g) \in \mathbb{R}^2_+; f^{1/p} < g < f^q\},
\]

and then blows up in finite time. As for these arguments, see e.g., Galaktionov-Kurdyumov-Samarskii [3, 4] or Qi-Levine [12].

We now turn to the proof of the lower bound. For the proof, we will need the following two lemmas from advanced calculus which appear as Lemmas 5 and 6 in [10].

**Lemma 2.3.** For each \(\sigma > 0\), there exists a constant \(c > 0\) such that

\[
\int_{\mathbb{R}^N} P(x - y, t)(1 + |y|)\sigma dy \leq c(1 + t^{\sigma/2} + |x|^{\sigma}), \quad \text{for} \ x \in \mathbb{R}^N, t > 0.
\]

**Proof.** Using the inequality \(|a + b|^{\sigma} \leq 2^{\sigma}(|a|^{\sigma} + |b|^{\sigma})\) for \(\sigma > 0\), we obtain

\[
\int_{\mathbb{R}^N} P(x - y, t)(1 + |y|)^\sigma dy = \int_{\mathbb{R}^N} P(z, t)(1 + |x + z|)^\sigma dz
\]

\[
\leq 2^{\sigma} \int_{\mathbb{R}^N} P(z, t)(1 + |z|)^\sigma dz
\]

\[
\leq 2^{\sigma} + 2^{2\sigma} \int_{\mathbb{R}^N} P(z, t)(|z|^{\sigma} + |z|^{\sigma}) dz
\]

\[
= 2^{\sigma} + 2^{2\sigma} |x|^{\sigma} + 2^{2\sigma} c_\sigma t^{\sigma/2},
\]
where
\[ c(x) = (4\pi)^{-N/2} \int_{\mathbb{R}^N} |\xi|^\sigma \exp\left(-\frac{|\xi|^2}{4}\right) d\xi. \]

**Lemma 2.4.** For \( \sigma \leq 0 \) and \( t > 0 \), the function
\[ H(x) = \int_{\mathbb{R}^N} P(x - y, t)(1 + |y|)^\sigma dy \]
attains its maximum at \( x = 0 \).

**Proof.** \( H(x) \) depends only on \( |x| \), thus it is enough to show that \( (x, \nabla H(x)) \leq 0 \) for all \( x \in \mathbb{R}^N \). We have
\[
\nabla H(x) = \int_{\mathbb{R}^N} \nabla_x P(x - y, t)(1 + |y|)^\sigma dy
\]
\[
= -\int_{\mathbb{R}^N} \nabla_y P(x - y, t)(1 + |y|)^\sigma dy
\]
\[
= \int_{\mathbb{R}^N} P(x - y, t)\nabla(1 + |y|)^\sigma dy.
\]

Thus,
\[
(x, \nabla H(x)) = \sigma(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) \int_{\mathbb{R}^N} \exp\left(\frac{(x, y)}{2t}\right)(x, y)
\]
\[
\times \exp\left(-\frac{|y|^2}{4t}\right)(1 + |y|)^{\sigma-1}|y|^{-1} dy.
\]
(12)

Since \( (x, \nabla H(x)) \) depends only on \( |x| \), it is enough to show that \( \int_{|x|=r} (x, \nabla H(x)) dx \leq 0 \) for all \( r > 0 \). Considering symmetry of functions, we see
\[
\int_{|x|=r} \exp\left(\frac{(x, y)}{2t}\right)(x, y) dx
\]
\[
= \left\{ \int_{|x|=r, (x, y) \geq 0} + \int_{|x|=r, (x, y) \leq 0} \right\} \exp\left(\frac{(x, y)}{2t}\right)(x, y) dx
\]
\[
= \int_{|x|=r, (x, y) \geq 0} \left\{ \exp\left(\frac{(x, y)}{2t}\right) - \exp\left(-\frac{(x, y)}{2t}\right) \right\} (x, y) dx
\]
\[
\geq 0,
\]
(13)
for all \( y \in \mathbb{R}^N \). From (12) and (13), we obtain \( \int_{|x|=r} (x, \nabla H(x)) dx \leq 0. \)
To prove that a given number $T > 0$ provides a lower bound for $T_\alpha^+$, we will make the following argument. Define

$$u_0(x, t) = \lambda^\mu \int_{\mathbb{R}^N} P(x - y, t) \phi(y) dy,$$

$$v_0(x, t) = \lambda^\nu \int_{\mathbb{R}^N} P(x - y, t) \psi(y) dy,$$

where $\phi, \psi$ satisfy

$$0 \leq \phi(x), \quad \psi(x) \leq \delta P(x, k)$$

for some $\delta, k > 0$, and

$$u_{n+1}(x, t) = u_0(x, t) + \int_0^t \int_{\mathbb{R}^N} P(x - y, t - s) a(y) v_n(y, s)^p dy ds,$$

$$v_{n+1}(x, t) = v_0(x, t) + \int_0^t \int_{\mathbb{R}^N} P(x - y, t - s) b(y) u_n(y, s)^q dy ds,$$

for $n \geq 0$. By induction, $u_{n+1}(x, t) \geq u_n(x, t), \quad v_{n+1}(x, t) \geq v_n(x, t)$. If there exists a $T > 0$ such that

$$\sup_{n \geq 0} u_n(x, t), \sup_{n \geq 0} v_n(x, t) < \infty, \quad \text{for } x \in \mathbb{R}^N, t \in [0, T),$$

then

$$\bar{u}(x, t) \equiv \lim_{n \to \infty} u_n(x, t), \quad \bar{v}(x, t) \equiv \lim_{n \to \infty} v_n(x, t)$$

converge uniformly in $x \in \mathbb{R}^N, t \in [0, T)$, and it follows from the monotone convergence theorem and (15) that $\bar{u}, \bar{v}$ satisfy (3) for $x \in \mathbb{R}^N, t \in (0, T)$; hence $T_\alpha^+ \geq T$. Thus, to obtain an estimate of the form $T_\alpha^+ \geq T$, it is enough to show the following lemma:

**Lemma 2.5.** If (14) holds,

$$u_n(x, t) \leq 2\lambda^\mu \delta P(x, t + k), \quad v_n(x, t) \leq 2\lambda^\nu \delta P(x, t + k)$$

holds for all $n \geq 0$ in $x \in \mathbb{R}^N, t \in [0, T(\lambda))$, where

$$T(\lambda) = C \min\{\lambda^{2(-p\nu + \rho)}/N(p^- p), \lambda^{2(-q\nu + q)}/N(q^- q)\} - k.$$
Proof. From (14) and the relation
\[ \int_{\mathbb{R}^n} P(x - y, t)P(y, k)dy \]
\[ = (4\pi t)^{-N/2}(4\pi k)^{-N/2} \exp \left( -\frac{|x|^2}{4(t + k)} \right) \]
\[ \times \int_{\mathbb{R}^n} \exp \left( -\frac{t + k}{4tk} \right) \left| y - \frac{kx}{t + k} \right|^2 dy \]
\[ = (4\pi(t + k))^{-N/2} \exp \left( -\frac{|x|^2}{4(t + k)} \right) \int_{\mathbb{R}^n} P(z, k)dz \]
\[ = P(x, t + k), \]
it follows that
\[ u_0(x, t) \leq \lambda^\mu \delta P(x, t + k) \leq 2\lambda^\mu \delta P(x, t + k), \]
\[ v_0(x, t) \leq \lambda^\mu \delta P(x, t + k) \leq 2\lambda^\mu \delta P(x, t + k), \]
for all \( t \geq 0 \). Hence (16) holds for \( n = 0 \) when \( 0 \leq t < \infty \).

Next, we shall assume that (16) holds for some \( n \geq 0 \). In the sequel \( C \) will denote a positive constant whose value will change from term to term. Since \( a(x) \leq C(1 + |x|)^{\eta_1} \) for some \( C > 0 \) by assumption, using (15), (16), and (17), we obtain
\[ u_{n+1}(x, t) \leq \lambda^\mu \delta P(x, t + k) \]
\[ + (2\lambda^\mu \delta)^{p} \int_0^t \int_{\mathbb{R}^n} a(y)P(x - y, t - s)P(y, s + k)^p dyds \]
\[ \leq \lambda^\mu \delta P(x, t + k) \]
\[ + (2\lambda^\mu \delta)^p C \int_0^t \int_{\mathbb{R}^n} (t - s)^{-N/2} (s + k)^{-Np/2} \]
\[ \times (1 + |y|)^{\eta_1} \exp \left( -\frac{|x - y|^2}{4(t - s)} - \frac{p|y|^2}{4(s + k)} \right) dyds. \]

Using the relation
\[ \exp \left( -\frac{|x - y|^2}{4(t - s)} - \frac{p|y|^2}{4(s + k)} \right) \]
\[ = \exp \left( -\frac{|y - R(s, t)x|^2}{4(t - s)R(s, t)} \right) \exp \left( -\frac{pR(s, t)|x|^2}{4(s + k)} \right), \]
where $R(s,t) = (s+k)/(s+k+p(t-s))$, (18) can be rewritten as

$$u_{\alpha+1}(x,t) \leq \lambda^\alpha P(x,t+k) + (2\lambda^\alpha)^\beta C \int_0^t \int_{\mathbb{R}^N} P(R(s,t)x-y, R(s,t)(t-s))$$

$$\times (1 + |y|)^{\sigma_1} (s+k)^{-N\rho/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right) \, dy \, ds. \quad (19)$$

At this stage in the proof, we must consider two cases separately. The first case is when $\sigma_1 > 0$, and the second case is when $\sigma_1 \leq 0$. We treat the case $\sigma_1 > 0$ first. Carrying out the integration over $R^N$ in (19), and using Lemma 2.3 with $t, x$ and $\sigma$ being replaced by $R(s,t)(t-s)$, $R(s,t)x$ and $\sigma_1$ respectively, the final term on the right hand side of (19) reduces to

$$(2\lambda^\alpha)^\beta C \int_0^t \int_{\mathbb{R}^N} [1 + R(s,t)^{\sigma_1/2}(t-s)^{\sigma_1/2} + R(s,t)^{\sigma_1}]$$

$$\times (s+k)^{-N\rho/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right) ds. \quad (20)$$

Multiplying outside the integral in (20) by the factor $\exp(-|x|^2/4(t+k))$, multiplying inside the integral by its reciprocal, and simplifying the argument in the exponential term, (20) may be rewritten as

$$(2\lambda^\alpha)^\beta C \exp\left(-\frac{|x|^2}{4(t+k)}\right) \int_0^t (s+k)^{-N\rho/2} R(s,t)^{N/2}$$

$$\times [1 + R(s,t)^{\sigma_1/2}(t-s)^{\sigma_1/2} + R(s,t)^{\sigma_1}]$$

$$\times \exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right) ds. \quad (21)$$

We now write

$$R(s,t)^{\sigma_1} |x|^{\sigma_1} \exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right)$$

$$= R(s,t)^{\sigma_1/2} z^{\sigma_1/2} \exp\left(-\frac{(p-1)z}{4(t+k)}\right), \quad (22)$$

where $z = R(s,t)|x|^2$. Differentiating this as a function of $z > 0$, we have
By the inequality $p > 1$, the function (22) of $z$ attains its maximum at $z = 2\sigma_1(t + k)/(p - 1)$. The maximum value then is

$$R(s, t)^{\sigma_1/2} \left( \frac{2\sigma_1(t + k)}{p - 1} \right)^{\sigma_1/2} e^{-\sigma_1/2}.$$

From this it follows that

$$R(s, t)^{\sigma_1} |x|^{\sigma_1} \exp \left( -\frac{(p - 1)R(s, t)|x|^2}{4(t + k)} \right)$$

$$\leq CR(s, t)^{\sigma_1/2} (t + k)^{\sigma_1/2},$$

(23)

for all $x \in \mathbb{R}^N$, $t > 0$ and $0 < s < t$. From (23) and the fact that $p > 1$, it follows that the quantity in (21) is smaller than

$$(2\lambda^2\delta)^p C \exp \left( -\frac{|x|^2}{4(t + k)} \right) \left\{ t \int_0^t (s + k)^{-Np/2} R(s, t)^{N/2} ds + \int_0^t (s + k)^{-Np/2} R(s, t)^{(N + \sigma_1)/2} (t - s)^{\sigma_1/2} ds \right\}.$$

(24)

We now carry out the integration in (24). Recalling that $p < p^* = 1 + (2 + \sigma_1)/N$, recalling that $R(s, t) = (s + k)/(s + k + p(t - s))$, and noting that $t + k \leq s + k + p(t - s) < p(t + k)$ for $s \in [0, t]$, we have

$$\int_0^t (s + k)^{-Np/2} R(s, t)^{N/2} ds$$

$$= (t + k)^{-N/2} \int_0^t (s + k)^{N(1-p)/2} \left( \frac{t + k}{s + k + p(t - s)} \right)^{N/2} ds$$

$$\leq (t + k)^{-N/2} \int_0^t (s + k)^{N(1-p)/2} ds$$

$$\leq \begin{cases} C(t + k)^{1-Np/2}, & \text{if } p < 1 + 2/N, \\ C(t + k)^{-N/2} \log(t/k + 1), & \text{if } p = 1 + 2/N, \\ C(t + k)^{-N/2}, & \text{if } p > 1 + 2/N, \end{cases}$$

(25)

and
\[
\int_0^t (s+k)^{-Np/2} R(s,t)^{(N+\sigma_1)/2} \left[ (t-s)^{\sigma_1/2} + (t+k)^{\sigma_1/2} \right] ds \\
\leq C(t+k)^{-N/2} \int_0^t (s+k)^{(N(1-p)+\sigma_1)/2} \frac{(t+k)}{s+k+p(t-s)}^{(N+\sigma_1)/2} ds \\
\leq C(t+k)^{-N/2} \int_0^t (s+k)^{(N(1-p)+\sigma_1)/2} ds \\
\leq C(t+k)^{-N/2+\frac{N(p^*-p)}{2}}.
\]

(26)

From (20), (21), (24), (25) and (26), we conclude now that the final term on the right hand side of (19) is smaller than

\[
(2\lambda^*\delta)^p C(t+k)^{-N/2+\frac{N(p^*-p)}{2}} \exp\left(-\frac{|x|^2}{4(t+k)}\right).
\]

Substituting this in (19), we obtain

\[
u_{n+1}(x,t) \\
\leq \lambda^p\delta P(x,t+k) + (2\lambda^*\delta)^p C(t+k)^{-N/2+\frac{N(p^*-p)}{2}} \exp\left(-\frac{|x|^2}{4(t+k)}\right) \\
= (\lambda^p\delta + (2\lambda^*\delta)^p C(t+k)^{\frac{N(p^*-p)}{2}}) P(x,t+k),
\]

(27)

for \(x \in \mathbb{R}^N, t \geq 0\).

We now turn to the case \(\sigma_1 \leq 0\). It follows from Lemma 2.4 that the inside integral,

\[
\int_{\mathbb{R}^N} P(R(s,t)x-y, R(s,t)(t-s))(1+|y|)^{\sigma_1} dy,
\]

appearing on the right hand side of (19), attains its maximum as a function of \(x\) when \(x = 0\). Thus, the final term on the right hand side of (19) is less than or equal to

\[
(2\lambda^*\delta)^p C \int_0^t \int_{\mathbb{R}^N} P(y, R(s,t)(t-s))(1+|y|)^{\sigma_1} \\
\times (s+k)^{-Np/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right) dy ds.
\]

(28)

By the facts that \(\int_{\mathbb{R}^N} P(y,t)(1+|y|)^{\sigma_1} dy \leq 1\) for \(t \in [0,1]\), and that
We now carry out the integration in (31). Recalling that 
\[ p > 1, \quad R(s, t) \leq 1, \quad pR(s, t)/(s+k) = p/(s+k + p(t-s)) \geq 1/(t+k) \]
for \( s \in [0, t] \), the quantity in (30) is less than or equal to

\[ (2\lambda^\nu)^p C \int_0^t (1 + R(s, t)(t-s))^\sigma_1/2 ds. \] (30)

Since \( p > 1 \), \( R(s, t) \leq 1 \) and \( pR(s, t)/(s+k) = p/(s+k + p(t-s)) \geq 1/(t+k) \)
for \( s \in [0, t] \), the quantity in (30) is less than or equal to

\[ (2\lambda^\nu)^p C \int_0^t R(s, t)^{(N+\sigma_1)/2}(1 + t-s)^\sigma_1/2 ds. \] (31)

We now carry out the integration in (31). Recalling that \( p < p^* = 1 + (2 + \sigma_1)/N \), that \( \sigma_1 \in (-2, 0) \) if \( N \geq 2 \) or that \( \sigma_1 \in (-1, 0) \) if \( N = 1 \),
and that \( R(s, t) = (s+k)/(s+k+p(t-s)) \), and noting that \( t+k \leq s+k + p(t-s) < p(t+k) \) for \( s \in [0, t] \),
we have

\[
\int_0^t R(s, t)^{(N+\sigma_1)/2}(1 + t-s)^\sigma_1/2 ds \\
\leq (t+k)^{-(N+\sigma_1)/2} \int_0^t (s+k)^{(N(1-p)+\sigma_1)/2}(1 + t-s)^\sigma_1/2 ds \\
\leq C(t+k)^{-(N+\sigma_1)/2} \left\{ (t+k)^{\sigma_1/2} \int_0^{t/2} (s+k)^{(N(1-p)+\sigma_1)/2} ds \\
+ (t+k)^{(N(1-p)+\sigma_1)/2} \int_{t/2}^t (t-s)^{\sigma_1/2} ds \right\} \\
\leq C(t+k)^{-N/2+N(p'-p)/2}. \] (32)
From (19), (28), (30), (31) and (32), we conclude that

\[ u_{n+1}(x, t) \leq (\lambda t^{\delta} + (2\lambda t^{\delta})^P C(t + k)^{N(p - r)/2}) P(x, t + k), \]

for \( x \in \mathbb{R}^N, t \geq 0. \)

In the same way as (18) through (32), we conclude that

\[ u_{n+1}(x, t) \leq (\lambda t^{\delta} + (2\lambda t^{\delta})^P C(t + k)^{N(p - r)/2}) P(x, t + k), \]

\[ v_{n+1}(x, t) \leq (\lambda t^{\delta} + (2\lambda t^{\delta})^Q C(t + k)^{N(q - q)/2}) P(x, t + k) \]

for \( x \in \mathbb{R}^N, t \geq 0. \) From (34), we find that (16) with \( n \) being replaced by \( n + 1 \) holds as long as

\[ (2\lambda t^{\delta})^P C(t + k)^{N(p - r)/2} \leq \lambda t^{\delta}, (2\lambda t^{\delta})^Q C(t + k)^{N(q - q)/2} \leq \lambda t^{\delta}. \]

Thus, (16) holds for all \( n \geq 0 \) when

\[ t \leq \min\{((2\lambda t^{\delta})^{-P} C\lambda t^{\delta})^{2(N(p - r))}, ((2\lambda t^{\delta})^{-Q} C\lambda t^{\delta})^{2(N(q - q))}\} - k \]

\[ = C \min\{\lambda^{2(-p\mu + \varphi)/N(p - r)}, \lambda^{2(-q\mu + \varphi)/N(q - q)}\} - k = T(\lambda). \]

**Proof of Theorem 1 (ii).** Recall here that we have assumed

\[ \frac{\mu}{v} = \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}. \]

Then since \( p\beta - \alpha = q\alpha - \beta = 2, \ p\delta_2 - \delta_1 = \sigma_1, \ q\delta_1 - \delta_2 = \sigma_2, \) it follows that

\[ \frac{-p\mu + \nu}{2 + \sigma_1 + N(1 - p)} = \frac{-\nu}{2 + \sigma_1 + N(1 - p)} \left( \frac{p - \mu}{v} \right) = \frac{-\nu}{2 + \sigma_1 + N(1 - p)} \left( \frac{p - \alpha + \delta_1 - N}{\beta + \delta_2 - N} \right) = \frac{-\nu}{\beta + \delta_2 - N}, \]

\[ \frac{-q\mu + \varphi}{2 + \sigma_2 + N(1 - q)} = \frac{-\mu}{2 + \sigma_2 + N(1 - q)} = \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}. \]

Thus, we obtain

\[ T_2^* \geq T(\lambda) \geq C\lambda^{-2\mu/(\alpha + \delta_1 - N)} = C\lambda^{-2\nu/(\beta + \delta_2 - N)}. \]

when \( \lambda > 0 \) is sufficiently small.

**3. Proof of Theorem 2**

We begin with the proof of the upper bounds. Let \( D \subset \mathbb{R}^N \) be a smooth bounded domain such that

\[ \inf_{x \in D} a(x), \inf_{x \in D} b(x), \inf_{x \in D} \varphi(x), \inf_{x \in D} \psi(x) \geq c > 0. \]
Let \( \theta > 0 \) denote the principal eigenvalue of \(-A\) with Dirichlet problem in \( D \), and let \( \omega(x) \) denote the corresponding positive eigenfunction, normalized by \( \int_D \omega(x)dx = 1 \). Define

\[
F(t) = \int_D u(x, t)\omega(x)dx,
\]

\[
G(t) = \int_D v(x, t)\omega(x)dx, \quad \text{for } 0 \leq t < T_\lambda^*.
\]

Using (35), we obtain from (1) that

\[
F'(t) = \int_D u_t(x, t)\omega(x)dx
\]

\[
= \int_D (Au(x, t) + a(x)v(x, t)^p)\omega(x)dx
\]

\[
\geq -\theta F(t) + cG(t)^p.
\]

Thus, we obtain the following inequalities:

\[
\begin{cases}
F'(t) \geq -\theta F(t) + cG(t)^p & (t > 0), \\
G'(t) \geq -\theta G(t) + cF(t)^q & (t > 0).
\end{cases}
\]  

(36)

From (35), \( F(0) \geq \lambda_\mu, \ G(0) \geq \lambda_\nu^\mu \).

Let \( f, g \in C^0((0, T_\lambda^*)) \cap C^1((0, T_\lambda^*)) \) be the solution of the system of ordinary differential equations

\[
\begin{align*}
\begin{cases}
\dot{f}(t) = -\theta f(t) + cg(t)^p & (t > 0), \\
\dot{g}(t) = -\theta g(t) + cf(t)^q & (t > 0), \\
f(0) = \lambda_\mu, \ g(0) = \lambda_\nu^\mu.
\end{cases}
\end{align*}
\]  

(37)

Then \((F(t), G(t))\) is a supersolution of (37).

**Lemma 3.1.** Define

\[
Q = \{(f, g) \in \mathbb{R}_+^2; (20c^{-1})^{1/p} < g < (20)^{-1}cf^q\},
\]

and let \((f(t), g(t))\) be the solution to (37). If \((f(0), g(0)) \in Q\), then \((f(t), g(t)) \in Q\) for all \( t \in [0, T_\lambda^*) \).

**Proof.** We shall first show that

\[
f(t) > f(0) > (20c^{-1})^{3/2} \quad \text{and} \quad g(t) > g(0) > (20c^{-1})^{\beta/2}
\]  

(38)

hold for all \( t \in (0, T_\lambda^*) \). Since \( f(t), g(t) \) are continuous at \( t = 0 \) and

\[
-\theta f(0) + cg(0)^p > \theta f(0) > 0, \quad -\theta g(0) + cf(0)^q > \theta g(0) > 0,
\]

(39)
there exists an $\varepsilon_1 > 0$ such that
\[ f'(t) = -\theta f(t) + cg(t)^p > 0, \]
\[ g'(t) = -\theta g(t) + cf(t)^q > 0, \quad \text{for } 0 < t < \varepsilon_1. \]

So (38) holds for $0 < t < \varepsilon_1$. Assume contrarily that there exists a $t_1 \in (0, T^*_{1})$ such that (38) holds for $0 < t < t_1$ and $f(t_1) = f(0)$. From (37), it follows that
\[ (e^{\theta t}f(t))' = e^{\theta t}f'(t) + \theta e^{\theta t}f(t) = ce^{\theta t}g(t)^p. \]

Integrating the both sides of this equality from $0$ to $t_1$, we obtain
\[ e^{\theta t_1}f(0) - f(0) = c \int_0^{t_1} e^{\theta s}g(s)^p ds \geq cg(0)^p \theta^{-1}(e^{\theta t_1} - 1). \]

Since $e^{\theta t_1} > 1$, it follows that $0f(0) \geq cg(0)^p$. This leads to a contradiction to (39), so we obtain $f(t) > f(0)$ for all $t \in (0, T^*_{1})$. In the same way, we also obtain $g(t) > g(0)$ for all $t \in (0, T^*_{1})$.

Next, we shall show that $(f(t), g(t)) \in Q$ for all $t \in (0, T^*_{1})$. Since $f(t), g(t)$ are continuous at $t = 0$, there exists an $\varepsilon_2 > 0$ such that $(f(t), g(t)) \in Q$ for $0 \leq t < \varepsilon_2$. Assume contrarily that there exists a $t_2 \in (0, T^*_{1})$ such that $(f(t), g(t)) \in Q$ for $0 \leq t < t_2$ and $2\theta f(t_2) = cg(t_2)^p$. Since it follows from (38) that
\[ (2\theta)^{-1}cf(t_2)^q - g(t_2) = \{(2\theta)^{-1}c\}^{q+1}g(t_2)^{pq-1} - 1\}g(t_2) > 0, \]
we obtain
\[ cpg(t_2)^{pq-1}g'(t_2) - 2\theta f'(t_2) \]
\[ = cpg(t_2)^{pq-1}(cf(t_2)^q - \theta g(t_2)) - 2\theta(cg(t_2)^p - \theta f(t_2)) \]
\[ > \theta(cpg(t_2)^{pq-1} - 2\theta f(t_2)) = c\theta(p-1)g(t_2)^p > 0. \]

Considering the continuity of $f'(t), g'(t)$, there exists an $\varepsilon > 0$ such that
\[ cpg(t)^{pq-1}g'(t) - 2\theta f'(t) > 0, \quad \text{for } t_2 - \varepsilon < t < t_2. \]

Integrating the left hand side of this inequality from $t$ satisfying $t_2 - \varepsilon < t < t_2$ to $t_2$, it follows that
\[ 0 < c \int_t^{t_2} pg(s)^{pq-1}g'(s)ds - 2\theta \int_t^{t_2} f'(s)ds \]
\[ = cg(t_2)^p - cg(t)^p - 2\theta f(t_2) + 2\theta f(t) \]
\[ = 2\theta f(t) - cg(t)^p. \]
This leads to a contradiction, so we obtain $2\theta f(t) < cg(t)^p$ for all $t \in [0, T^*)$. In the same way, we also obtain $2\theta g(t) < cf(t)^q$ for all $t \in [0, T^*)$.

Proof of Theorem 2 (i). Choosing $\lambda_0 > 0$ to satisfy $\lambda_0^{\mu-\nu} \geq 2\theta c^{-p}$, $\lambda_0^{\mu-\nu} \geq 2\theta c^{-q}$, we easily see from the inequalities $p
u > \mu$, $q\mu > \nu$ that $(f(0), g(0)) \in Q$ holds if $\lambda > \lambda_0$. Then we can apply Lemma 3.1 to obtain $(f(t), g(t)) \in Q$ for all $t \in [0, T^*)$. From now on, we will always assume that $\lambda > \lambda_0$. It follows from (37) that

$$f'(t) = -\theta f(t) + c_1 g(t)^p$$

$$> -\frac{1}{2} c_1 g(t)^p + c_1 g(t)^p = \frac{1}{2} c_1 g(t)^p$$

(40)

$$g'(t) > \frac{1}{2} c_2 f(t)^q$$

for $t \in (0, T^*)$.

Let us consider the system of ordinary differential equations

$$\begin{cases}
  x' = (1/2)c y^p, 
  y' = (1/2)c x^q 
  (t > 0), \\
  x(0) = c \lambda^\mu, 
  y(0) = c \lambda^\nu.
\end{cases}$$

(41)

Then $(f(t), g(t))$ is a supersolution of (41). From equation (41), it follows that $x'x' = y'y'$. Integrate the both sides from 0 to $t$. Then we have

$$\frac{x(t)^{q+1} - x(0)^{q+1}}{q+1} = \frac{y(t)^{p+1} - y(0)^{p+1}}{p+1}.$$  

(42)

If $(q+1)^{-1}x(0)^{q+1} \geq (p+1)^{-1}y(0)^{p+1}$, it follows from (42) that

$$x(t) \geq \left(\frac{q+1}{p+1}\right)^{1/(q+1)} y(t)^{(p+1)/(q+1)}.$$  

Substitute this in the second equation of (41). Then we have

$$y'(t) \geq \frac{1}{2} C_1(p, q) y(t)^{q(p+1)/(q+1)},$$

where $C_1(p, q) = c((q+1)/((p+1))^{q/(q+1)}$. Multiplying $y(t)^{-q(p+1)/(q+1)}$ and integrating the both sides from 0 to $t$, we obtain

$$-\frac{\beta}{2} (y(t)^{-2/\beta} - (c \lambda^\nu)^{-2/\beta}) \geq \frac{1}{2} C_1(p, q) t,$$

$$\beta y(t)^{-2/\beta} \leq \beta (c \lambda^\nu)^{-2/\beta} - C_1(p, q) t.$$  

(43)
Since the right hand side of the second equation of (43) equals 0 when
\[ t = \beta C_1(p, q)^{-1}(c\lambda r)^{-2/\beta}, \]
it follows that \( y(t) \) must blow up by the above \( t \). This gives the upper bound
\[ T^*_\lambda \leq C\lambda^{-2\varepsilon/\beta}, \quad \text{for } \exists C > 0. \]
In the case when \( (q + 1)^{-1}x(0)^{q+1} \leq (p + 1)^{-1}y(0)^{p+1} \), we obtain by the same method
\[ T^*_\lambda \leq C\lambda^{-2\varepsilon/\lambda}, \quad \text{for } \exists C > 0. \]

We now turn to the proof the lower bound. We will use an idea of the same type as that used to prove the lower bound in Theorem 1. Define
\[
\begin{align*}
  u_0(x, t) &= \lambda^p \int_{\mathbb{R}^n} P(x - y, t)\varphi(y)dy, \\
  v_0(x, t) &= \lambda^q \int_{\mathbb{R}^n} P(x - y, t)\psi(y)dy,
\end{align*}
\]
where \( \varphi, \psi \) satisfy
\[ 0 \leq \varphi(x), \quad \psi(x) \leq \delta \quad (44) \]
for some \( \delta > 0 \), and
\[
\begin{align*}
  u_{n+1}(x, t) &= u_0(x, t) + \int_0^t \int_{\mathbb{R}^n} P(x - y, t - s)a(y)v_n(y, s)^pdyds, \\
  v_{n+1}(x, t) &= v_0(x, t) + \int_0^t \int_{\mathbb{R}^n} P(x - y, t - s)b(y)u_n(y, s)^qdyds,
\end{align*}
\]
for \( n \geq 0 \). By the same argument as in Section 2, it is enough to show the following lemma:

**Lemma 3.2.** If (44) holds, the inequalities
\[ u_n(x, t) \leq 2\lambda^p\delta, \quad v_n(x, t) \leq 2\lambda^q\delta \quad (46) \]
hold for all \( n \geq 0 \) in \( x \in \mathbb{R}^N, \ t \in [0, T(\lambda)] \), where
\[ T(\lambda) = C \min\{\lambda^{-m+p}, \lambda^{-q+q+q}\}. \]

**Proof.** From (44), we easily see that
\[ u_0(x, t) \leq \lambda^p\delta \leq 2\lambda^p\delta, \quad v_0(x, t) \leq \lambda^q\delta \leq 2\lambda^q\delta, \quad (47) \]
for all \( t \geq 0 \). Hence (46) holds for \( n = 0 \) when \( 0 \leq t < \infty \).
Next, we shall assume that (46) holds for some \( n \geq 0 \). In the sequel \( C \) will denote a positive constant whose value will change from term to term. Using (45), (46), and (47), we obtain

\[
\begin{align*}
\frac{u_{n+1}(x, t)}{u_0(x, t^0)} & \leq \lambda^n \delta + (2\lambda^n \delta)^p C t, \\
\frac{v_{n+1}(x, t)}{v_0(x, t^0)} & \leq \lambda^n \delta + (2\lambda^n \delta)^q C t,
\end{align*}
\]

for \( x \in \mathbb{R}^N, \ t \geq 0 \). From (48), we find that (46) with \( n \) being replaced by \( n + 1 \) holds as long as

\[
(2\lambda^{n+1} \delta)^p C t \leq \lambda^{n+1} \delta; \quad (2\lambda^{n+1} \delta)^q C t \leq \lambda^{n+1} \delta.
\]

Thus, (46) holds for all \( n \geq 0 \) when

\[
t \leq \min \left\{ (2\lambda^n \delta)^{-p} C \lambda^n \delta, (2\lambda^n \delta)^{-q} C \lambda^n \delta \right\}
\]

\[
= C \min \{ \lambda^{-p\mu_+}, \lambda^{-q\gamma_+} \} = T(\lambda). \quad \square
\]

**Proof of Theorem 2 (ii).** Recall here that we have assumed

\[
\frac{\mu}{\nu} = \frac{\alpha}{\beta}.
\]

Then since \( p\beta - \alpha = q\alpha - \beta = 2 \), it follows that

\[
-pv + \mu = -v \cdot \left( p - \frac{\mu}{\nu} \right) = -v \cdot \left( p - \frac{\alpha}{\beta} \right) = -\frac{2v}{\beta},
\]

\[
-q\mu + v = -\frac{2\mu}{\alpha}.
\]

Thus, we obtain

\[
T^*_\lambda \geq T(\lambda) \geq C \lambda^{-2\mu/\beta} = C \lambda^{-2\mu/\beta}
\]

when \( \lambda > 0 \) is sufficiently large. \quad \square

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