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Multiple positive solutions for a semipositone fourth-order boundary value problem

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ABSTRACT. We consider the nonlinear fourth order boundary value problem

$$u^{(4)}(x) = \lambda f(x, u(x), u'(x))$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to (-\infty, \infty)$ is continuous with $f(x,u,p) \geq -M$ for some positive constant M. We show the existence and multiplicity of positive solutions by using a fixed point theorem in cones.

1. Introduction

The deformations of an elastic beam are described by a fourth-order two-point boundary value problem [6]. The boundary conditions are given according to the controls at the ends of the beam. For example, the nonlinear fourth order problem

$$u^{(4)}(x) = \lambda f(x, u(x), u'(x))$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0 \tag{1.1}$$

describes the deformations of an elastic beam whose one end fixed and the other end free.

The existence of solutions of (1.1) has been studied by Gupta [6]. But to the best of our knowledge, there are no any results concerning the existence of positive solutions of (1.1). In this paper, we will study the existence and multiplicity of positive solutions of (1.1).

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We will make the following assumptions:

(A1) $f:[0,1]\times[0,\infty)\times[0,\infty)\to(-\infty,\infty)$ is continuous and there exists $M>0$ such that

$$f(x,u,p)\geq -M,\qquad \text{for } (x,u,p)\in[0,1]\times[0,\infty)\times[0,\infty); \tag{1.2}$$

(A2) There exists a subinterval $[\alpha,\beta]\subset(0,1)$ with $\alpha<\beta$ such that

$$\lim_{p\to\infty}\frac{f(x,u,p)}{p}=\infty \tag{1.3}$$

holds uniformly for $(x,u)\in[\alpha,\beta]\times[0,\infty)$;

(A3)

$$f(x,u,0)>0,\qquad (x,u)\in[0,1]\times[0,\infty). \tag{1.4}$$

Remark 1. It is easy to see that (A3) implies that there exist two constants $a,b\in(0,\infty)$ such that

$$f(x,u,p)\geq b,\qquad (x,u,p)\in[0,1]\times[0,a]\times[0,a].$$

Very recently, Anuradha, Hai and Shivaji [1] studied the existence of positive solutions for second order boundary value problem

$$\begin{gathered}(p(t)u'(t))'+\lambda f(t,u(t))=0,\qquad r<t<R\\ au(r)-bp(r)u'(r)=0,\qquad cu(R)+dp(R)u'(R)=0\end{gathered} \tag{1.5}$$

under some superlinear semipositone conditions when $\lambda>0$ is small enough. Motivated by their work, we study the existence and multiplicity of positive solutions for fourth order problems (1.1). The main results of this paper are the following

Theorem 1. *Assume* (A1) *and* (A2) *hold. Then the problem* (1.1) *has at least one positive solution if* $\lambda>0$ *is small enough.*

Theorem 2. *Assume* (A1), (A2) *and* (A3) *hold. Then the problem* (1.1) *has at least two positive solutions if* $\lambda>0$ *is small enough.*

The proofs of above theorems are based upon the following Guo-Krasnoselskii fixed point theorem

Theorem 3. [5] *Let* E *be a Banach space, and let* $K\subset E$ *be a cone. Assume* Ω_1,Ω_2 *are open and bounded subsets of* E *with* $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, *and let*

$$A:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$$

be a completely continuous operator such that

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

For the results concerning the existence and multiplicity of positive solutions of fourth-order ordinary differential equations with other different conditions and nonnegative nonlinearities, one may refer, with further references therein, to Del Pino and Manásevich [2], Dunninger [3], Graef and Yong [4], Ma and Wang [7] and Zhang and Kong [10].

2. The preliminary lemmas

To prove Theorem 1 and Theorem 2, we need several preliminary results.

LEMMA 1. *For* $y \in C[0,1]$, *the problem*

$$u^{(4)}(x) = y(x), \qquad x \in (0,1)$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0 \tag{2.1}$$

is equivalent to the integral equation

$$u(x) = \int_0^x \left(\int_0^\tau \left(\int_r^1 \left[\int_s^1 y(t)dt \right] ds \right) dr \right) d\tau. \tag{2.2}$$

Moreover, if $y \ge 0$ *on* $[0,1]$, *then*
(i) $u(x) \ge 0$, $x \in [0,1]$;
(ii) $u'(x) \ge 0$, $x \in [0,1]$;
(iii) $u''(x) \ge 0$, $x \in [0,1]$;
(iv) $u'''(x) \le 0$, $x \in [0,1]$.

PROOF. It is easy to check that (2.1) is equivalent to (2.2).

If $y \ge 0$ on $[0,1]$, then (2.2) implies $u \ge 0$ on $[0,1]$. Moreover

$$u'(x) = \int_0^x \left(\int_r^1 \left[\int_s^1 y(t)dt \right] ds \right) dr \ge 0, \qquad x \in [0,1]$$

$$u''(x) = \int_x^1 \left[\int_s^1 y(t)dt \right] ds \ge 0, \qquad x \in [0,1]$$

and

$$u'''(x) = -\int_x^1 y(t)dt \le 0, \qquad x \in [0,1].$$

In the following, we will use the Banach space $C[0,1]$ and its sup norm $\|\cdot\|_0$.

LEMMA 2. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then the unique solution* u *of the problem* (2.1) *satisfies*

$$u'(x) \geq \|u'\|_0 q(x),$$

where

$$q(x) := x, \qquad x \in [0,1]. \tag{2.3}$$

PROOF. By Lemma 1, we know that $u'''(x) \leq 0$ on $[0,1]$. So, the graph of u' is concave down. This together with the facts that $u'(0) = (u')'(1) = 0$ and $u'' \geq 0$ imply

$$u'(x) \geq \|u'\|_0 q(x).$$

LEMMA 3. *The boundary value problem*

$$\begin{aligned} u^{(4)}(x) &= 1, \qquad x \in (0,1) \\ u(0) = u'(0) &= u''(1) = u'''(1) = 0 \end{aligned} \tag{2.4}$$

has a solution

$$w(x) = \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{24}. \tag{2.5}$$

Moreover,

$$w(x) \leq \frac{1}{8} q(x), \qquad x \in [0,1] \tag{2.6}$$

and

$$w'(x) \leq \frac{1}{2} q(x), \qquad x \in [0,1]. \tag{2.7}$$

PROOF. (2.5) is an immediate consequence of Lemma 1. Since the graph of w is concave upward and $\|w\|_0 = \frac{1}{8}$, we know that (2.6) holds. By (2.5), we have that

$$w'(x) = \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6} \tag{2.8}$$

$$w''(x) = \frac{1}{2} - x + \frac{x^2}{2} \tag{2.9}$$

and

$$w'''(x) = -1 + x \leq 0, \qquad x \in [0,1]. \tag{2.10}$$

(2.10) implies that the graph of w' is concave downward. Therefore $w'(x) \le w''(0)q(x) = \frac{1}{2}q(x)$ for $x \in [0,1]$.

3. Proof of the theorems

Set

$$C_0^1[0,1] = \{u \mid u \in C^1[0,1], u(0) = u'(0) = 0\}.$$

We furnish the set $C_0^1[0,1]$ with the norm

$$\|u\| = \sup\{|u'(x)| : x \in [0,1]\} = \|u'\|_0,$$

by which $C_0^1[0,1]$ is a Banach space.

Proof of Theorem 1. Let

$$z = \lambda M w \tag{3.1}$$

where w is defined by (2.5). Then (1.1) has a positive solution u if $u + z := \tilde{u}$ is a solution of

$$\begin{aligned} u^{(4)} &= \lambda g(x, u - z, u' - z'), \qquad x \in (0,1) \\ u(0) &= u'(0) = u''(1) = u'''(1) = 0 \end{aligned} \tag{3.2}$$

and $\tilde{u}'(x) > z'(x)$ for $x \in (0,1)$, where $g : [0,1] \times R \times R \to [0,\infty)$ is defined by

$$g(x,u,p) = \begin{cases} f(x,u,p) + M, & (x,u,p) \in [0,1] \times [0,\infty) \times [0,\infty) \\ f(x,u,0) + M, & (x,u,p) \in [0,1] \times [0,\infty) \times (-\infty,0) \\ f(x,0,p) + M, & (x,u,p) \in [0,1] \times (-\infty,0) \times (0,\infty) \\ f(x,0,0) + M, & (x,u,p) \in [0,1] \times (-\infty,0) \times (-\infty,0) \end{cases} \tag{3.3}$$

Let us denote

$$\begin{aligned} K = \{u \mid u \in C_0^1[0,1], u(x) \ge 0 \text{ on } [0,1], \\ u'(x) \ge 0 \text{ on } [0,1], u'(x) \ge \|u\| q(x)\}, \end{aligned} \tag{3.4}$$

where $q(x)$ is defined by (2.3). It is obvious that K is a cone in $C_0^1[0,1]$.

For $v \in K$, denote by Av the unique solution of

$$\begin{aligned} u^{(4)} &= \lambda g(x, v - z, v' - z') \\ u(0) &= u'(0) = u''(1) = u'''(1) = 0, \end{aligned} \tag{3.5}$$

then

$$Av(x) = \lambda \int_0^x \left(\int_0^\tau \left(\int_r^1 \left[\int_s^1 g(t, v - z, v' - z')dt \right] ds \right) dr \right) d\tau. \tag{3.6}$$

By Lemma 2, we know that $A(K) \subset K$.

Let

$$\lambda \in (0, \Lambda) \tag{3.7}$$

be fixed, where

$$\Lambda = \min\left\{ \frac{12}{M_1}, \frac{1}{M} \right\} \tag{3.8}$$

$$M_1 = \max\{g(x, u, p) \mid 0 \le x \le 1, 0 \le u \le 2, 0 \le p \le 2\}. \tag{3.9}$$

Choose $\Omega_1 = \{u \in C_0^1[0, 1] \mid \|u\| < 2\}$. Then for $u \in K \cap \partial\Omega_1$, we have from (3.8), (3.9), (3.3) and the facts $z \ge 0$ and $z' \ge 0$ that

$$\begin{aligned} (Au)'(x) &= \lambda \int_0^x \left(\int_r^1 \left[\int_s^1 g(t, u - z, u' - z')dt \right] ds \right) dr \\ &\le \lambda M_1 \int_0^1 \left(\int_r^1 \left[\int_s^1 1 \, dt \right] ds \right) dr \\ &= \lambda M_1 \frac{1}{6} \\ &< 2. \end{aligned} \tag{3.10}$$

Therefore,

$$\|Au\| \le \|u\|, \qquad u \in K \cap \partial\Omega_1.$$

We note that

$$\min_{\alpha \le s \le \beta} q(s) = \alpha. \tag{3.11}$$

Choose a real number $N > 0$, such that

$$N\lambda\alpha \frac{1}{2} \left[\frac{1}{2}(\beta - \alpha) - \frac{1}{2}(\beta^2 - \alpha^2) + \frac{1}{6}(\beta^3 - \alpha^3) \right] \ge 1 \tag{3.12}$$

(We note that $\frac{1}{2}(\beta - \alpha) - \frac{1}{2}(\beta^2 - \alpha^2) + \frac{1}{6}(\beta^3 - \alpha^3)$ is always positive for all $0 < \alpha < \beta < 1$). Choose $\bar{R} > 2$, such that $h \ge \frac{1}{2}\bar{R}\alpha$ implies

$$\frac{g(x, u, h)}{h} \ge N, \qquad \text{for } (x, u) \in [\alpha, \beta] \times (-\infty, \infty), \tag{3.13}$$

and

$$1 - \frac{\lambda M}{2\bar{R}} \geq \frac{1}{2}. \tag{3.14}$$

Let

$$\Omega_2 = \{u \in C_0^1[0,1] \mid \|u\| < \bar{R}\}. \tag{3.15}$$

Then for $u \in K \cap \partial\Omega_2$, we have that

$$z'(s) = \lambda M w'(s) \leq \lambda M \frac{1}{2} q(s) \leq \frac{1}{2} \lambda M \frac{u'(s)}{\|u\|} = \frac{\lambda M}{2\bar{R}} u'(s). \tag{3.16}$$

Thus

$$u'(s) - z'(s) \geq \left(1 - \frac{\lambda M}{2\bar{R}}\right) u'(s). \tag{3.17}$$

Combining (3.17) with (3.14) and (3.11), we conclude that

$$u'(s) - z'(s) \geq \frac{1}{2} u'(s) \geq \frac{1}{2} \|u\| q(s) \geq \frac{1}{2} \bar{R}\alpha, \qquad s \in [\alpha, \beta]. \tag{3.18}$$

This together with (3.13) implies

$$g(x, u - z, u' - z') \geq N(u' - z') \geq N \frac{\bar{R}\alpha}{2}, \qquad s \in [\alpha, \beta]. \tag{3.19}$$

Thus from (3.12) we get

$$\begin{aligned}
(Au)'(1) &= \lambda \int_0^1 \left(\int_r^1 \left[\int_s^1 g(t, u - z, u' - z') dt \right] ds \right) dr \\
&\geq \lambda N \frac{\bar{R}\alpha}{2} \int_\alpha^\beta \left(\int_r^1 \left[\int_s^1 1 \, dt \right] ds \right) dr \\
&\geq \lambda N \frac{\bar{R}\alpha}{2} \left[\frac{1}{2}(\beta - \alpha) - \frac{1}{2}(\beta^2 - \alpha^2) + \frac{1}{6}(\beta^3 - \alpha^3) \right] \\
&\geq \bar{R} = \|u\|
\end{aligned} \tag{3.20}$$

for $u \in K \cap \partial\Omega_2$. Therefore, it follows from the first part of Theorem 3 that A has a fixed point $\tilde{u}$ in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that

$$2 \leq \|\tilde{u}\| \leq \bar{R}. \tag{3.21}$$

Moreover, by combining (3.21) with (3.7) and (3.8) and using Lemma 2 and Lemma 3, we know that

$$\tilde{u}'(x) \geq \|\tilde{u}\| q(x) \geq 2q(x) > 2\lambda M q(x) \geq 2\lambda M w'(x) \equiv 2z'(x), \quad x \in (0,1). \tag{3.22}$$

So

$$u'(x) = \tilde{u}'(x) - z'(x) \geq \frac{1}{2}\tilde{u}'(x), \qquad x \in (0,1) \tag{3.23}$$

and moreover, since

$$\begin{aligned} u(x) &= \tilde{u}(x) - z(x) \\ &= \int_0^x (\tilde{u}'(s) - z'(s))ds \\ &\geq \frac{1}{2}\int_0^x \tilde{u}'(s)ds \\ &\geq \int_0^x z'(s)ds > 0, \qquad x \in (0,1) \end{aligned} \tag{3.24}$$

we get a positive solution $u(x) = \tilde{u}(x) - z(x)$ of (1.1).

Proof of Theorem 2. From (3.23), we have that (1.1) has a positive solution u_1 satisfying

$$\|u_1\| \geq \frac{1}{2}\|\tilde{u}\| \geq 1. \tag{3.25}$$

To find the second positive solution of (1.1), we set

$$f^*(x,u,p) = \begin{cases} f(x,u,p), & \text{for } (x,u,p) \in [0,1] \times [0,a] \times [0,a] \\ f(x,a,p), & \text{for } (x,u,p) \in [0,1] \times (a,\infty) \times [0,a] \\ f(x,u,a), & \text{for } (x,u,p) \in [0,1] \times [0,a] \times (a,\infty) \\ f(x,a,a), & \text{for } (x,u,p) \in [0,1] \times (a,\infty) \times (a,\infty). \end{cases} \tag{3.26}$$

Then $f^*(x,u,p) \geq b$ for $(x,u,p) \in [0,1] \times [0,a] \times [0,a]$, where a, b are given in Remark 1.

Now, we consider the auxiliary equation

$$\begin{aligned} u^{(4)} &= \lambda f^*(x,u,u'), \qquad x \in (0,1) \\ u(0) &= u'(0) = u''(1) = u'''(1) = 0. \end{aligned} \tag{3.27}$$

It is easy to check that (3.27) is equivalent to the fixed point problem

$$u = Fu \tag{3.28}$$

where

$$Fu(x) := \lambda \int_0^x \left(\int_0^\tau \left(\int_r^1 \left[\int_s^1 f^*(t, u(t), u'(t)) dt \right] ds \right) dr \right) d\tau. \tag{3.29}$$

It is easy to check that $F : K \to K$ is completely continuous and $F(K) \subset K$.

Set

$$H = \min\{0.9, a\}, \tag{3.30}$$

and

$$\Lambda_1 = \min\left\{ \frac{6H}{M_2}, \Lambda \right\} \tag{3.31}$$

and fix

$$\lambda \in (0, \Lambda_1), \tag{3.32}$$

where

$$M_2 = \max\{f^*(x, u, p) \mid 0 \le x \le 1, 0 \le u \le H, 0 \le p \le H\}. \tag{3.33}$$

Choose $\Omega_3 = \{u \in C_0^1[0,1] \mid \|u\| < H\}$. Then for $u \in K \cap \partial\Omega_3$, we have that

$$\begin{aligned} (Fu)'(x) &= \lambda \int_0^x \left(\int_r^1 \left[\int_s^1 f^*(t, u, u') dt \right] ds \right) dr \\ &\le \lambda M_2 \int_0^1 \left(\int_r^1 \left[\int_s^1 1 \, dt \right] ds \right) dr \\ &\le \lambda M_2 \frac{1}{6} \\ &\le H. \end{aligned} \tag{3.34}$$

Therefore

$$\|Fu\| \le \|u\|, \qquad u \in K \cap \partial\Omega_3. \tag{3.35}$$

From (A3) and Remark 1, we know that

$$\lim_{p \to 0^+} \frac{f^*(x, u, p)}{p} = +\infty \tag{3.36}$$

uniformly for $(x, u) \in [0,1] \times [0, a]$. This means that there exists a constant r_0 $(r_0 < H)$, such that

$$f^*(x, u, p) \ge \eta p, \qquad \text{for } (x, u, p) \in [0,1] \times [0, r_0] \times [0, r_0]$$

where

$$\frac{1}{8}\eta\lambda \geq 1. \tag{3.37}$$

Then for $u \in K$ and $\|u\| = r_0$, we have from (3.11) and (3.37) that

$$\begin{aligned}
(Fu)'(1) &= \lambda \int_0^1 \left(\int_r^1 \left[\int_s^1 f^*(t, u, u')dt \right] ds \right) dr \\
&\geq \lambda \int_0^1 \left(\int_r^1 \left[\int_s^1 \eta u'(t)dt \right] ds \right) dr \\
&\geq \lambda\eta \int_0^1 \left(\int_r^1 \left[\int_s^1 q(t)dt \right] ds \right) dr\|u\| \\
&\geq \lambda\eta \int_0^1 \frac{r^2}{2} q(r)dr\|u\| \\
&= \lambda\eta \frac{1}{8}\|u\| \\
&\geq \|u\|.
\end{aligned} \tag{3.38}$$

Thus, we may let $\Omega_4 = \{u \in C_0^1[0, 1] \mid \|u\| < r_0\}$ so that

$$\|Fu\| \geq \|u\|, \qquad u \in K \cap \partial\Omega_4. \tag{3.39}$$

By the second part of Theorem 3, it follows that (3.27) has a positive solution u_2 satisfying

$$r_0 \leq \|u_2\| \leq H. \tag{3.40}$$

Combining this with (3.26) and (3.30), we find that u_2 is also a solution of (1.1).

From (3.30), (3.25), (3.31) and (3.40), we know that (1.1) has two distinct positive solutions u_1 and u_2 for $\lambda \in (0, \Lambda_1)$.

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