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Fractal dimensions for the fibres of certain type of carpets

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Abstract. We define the family $\mathscr{F}$ of certain type of carpets, and calculate the fractal dimensions of fibres F_x for all $F \in \mathscr{F}$ and for almost all x.

1. Introduction and preliminaries

Let $\mathscr{F}$ be the family of all carpets obtained by partitioning the unit square into four subsquares, discarding one of them and repeating this on each of the remaining squares, with no constraints on the positions of the discarded subsquares. For $F \in \mathscr{F}$ and $x \in [0,1]$, set vertical fibres $F_x = \{y \in [0,1] \mid (x,y) \in F\}$.

H. Furstenberg conjectured that almost all vertical fibres of all $F \in \mathscr{F}$ have positive Hausdorff dimension [5].

In [2], I. Benjamini and Y. Peres showed for all $F \in \mathscr{F}$, the lower bound of Hausdorff dimension of F_x is $\frac{1}{2}$ for almost all $x \in [0,1]$ with respect to Lebesgue measure.

In this note, we consider the family $\mathscr{F}(a,b)$ of certain type of carpets which are constructed as follows:

Partition the unit square into four rectangles, with the ratio of side length, $a:b$, where $a,b \in \mathbf{R}^+$ (the set of positive real numbers), $a+b=1$, $a \le b$ and discard one of them. Again partition each of the three remaining rectangles into four subrectangles, with the same ratio of side length, $a:b$, and discard one of them. Apply the same operation to each of the remaining rectangles, with no constraints on the positions of the discarded subrectangles, and repeat this operations to obtain a limit set, a type of carpet F in $\mathscr{F}(a,b)$.

For above $F \in \mathscr{F}(a,b)$, we find a lower bound and an upper bound of fractal dimensions for the vertical fibres F_x, which gives us the result of I. Benjamini and Y. Peres [2] as corollary.

We now review some definitions and the known-results.

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DEFINITION 1.1. Suppose that F is a bounded subset of $\mathbf{R}^n$ and s is a nonnegative real number. The Hausdorff measure of F is defined as

$$H^s(F) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i, |U_i| < \delta \right\}$$

where $|U|$ denotes the diameter of a set U.

The Hausdorff dimension of the set F is defined as

$$\dim_H F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

DEFINITION 1.2. The upper box dimension of bounded $F \subset \mathbf{R}^n$ are defined as

$$\overline{\dim}_B F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ denotes the number of δ-mesh cubes that intersect F.

Here we note that

$$\dim_H F \le \overline{\dim}_B F.$$

2. Main results

We are now in a position to prove our main results.

THEOREM 2.1. *For all* $F \in \mathscr{F}(a,b)$, *with* $a,b \in \mathbf{R}^+$, $a+b=1$ *and* $a \le b$

$$\dim_H F_x \ge \frac{2ab\{b(\log b - \log a) + \log a\}}{2ab^2(\log b - \log a) + \log a}$$

for almost all $x \in [0,1]$ *with respect to Lebesgue measure.*

PROOF. For all $x \in [0,1)$, we can define $\hat{x}$ as follows; Bisect the interval $[0,1)$ with ratio $a:b$ and we write

$$x_1 = \begin{cases} 0, & \text{if } x \text{ lies in } [0,a), \\ 1, & \text{otherwise} \end{cases}$$

and repeat to bisect $[0,a)$ and $[a,1)$ with the same ratio $a:b$, and we write

$$x_2 = \begin{cases} 0, & \text{if } x \text{ lies in the resulting interval } [0,a^2) \text{ or } [a, a+ab), \\ 1, & \text{otherwise.} \end{cases}$$

Repeating this method, we have a sequence $\hat{x} = (x_1, x_2, \ldots)$ for $x \in [0,1)$. Let Φ be the family of functions

$$\varphi : \bigcup_{n=0}^{\infty}(\{0,1\}^n \times \{0,1\}^n) \to \{0,1\}^2$$

where $\{0,1\}^0 \times \{0,1\}^0 = \{\emptyset\}$.

Then for each $\varphi \in \Phi$, we construct $F^*(\varphi) \subset \{0,1\}^\infty \times \{0,1\}^\infty$ by

$$F^*(\varphi) = \{(\hat{x}, \hat{y}) = ((x_1, x_2, \ldots), (y_1, y_2, \ldots)) \mid \forall n \geq 1, x_n, y_n \in \{0,1\}$$
$$\text{and } \varphi((x_1, x_2, \ldots, x_{n-1}), (y_1, y_2, \ldots, y_{n-1})) \neq (x_n, y_n)\}$$

and define a map f on $F^*(\varphi)$ by

$$f(\hat{x}, \hat{y}) = \bigcap_{n=0}^{\infty}(C_n(\hat{x}) \times C_n(\hat{y}))$$

where $C_n(\hat{x})$ and $C_n(\hat{y})$ are the n-th step intervals containing $\hat{x}$ and $\hat{y}$, respectively. If we write $F(\varphi) = f(F^*(\varphi))$, then we easily see that $\mathscr{F}(a,b) = \{F(\varphi) \mid \varphi \in \Phi\}$.

Now consider $\mathscr{F}_o(a,b) = \{F(\varphi) : \varphi \in \Phi$ and $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n) \neq (*, 1), \forall n\}$, that is, $\mathscr{F}_o(a,b)$ is the subfamily of $\mathscr{F}(a,b)$ whose elements are constructed by discarding one of the two-top subrectangles for all steps. Note that for any $F \in \mathscr{F}(a,b)$, there exists $\tilde{F} \in \mathscr{F}_o(a,b)$ such that $\dim_H \tilde{F}_x \leq \dim_H F_x$ for all $x \in [0,1]$ (c.f. [3], [5]). Hence we will compute a lower bound of $\dim_H \tilde{F}_x$ instead of a the lower bound of $\dim_H F_x$.

Let $F = F(\varphi) \in \mathscr{F}_o$ and $x, z \in [0,1)$, and write $\hat{x} = (x_1, x_2, \ldots,)$, $\hat{z} = (z_1, z_2, \ldots,)$ corresponding to x and z, respectively. We define

$$\tilde{\Pi}_x : \{0,1\}^\infty \to \{0,1\}^\infty \qquad \text{by } \tilde{\Pi}_x(\hat{z}) = \hat{y},$$

where $\hat{y} = (y_1, y_2, \ldots, y_n, \ldots)$ are defined inductively; σ_1 is defined by $\varphi(\emptyset) = (\sigma_1, 1)$ and then

$$y_1 = \begin{cases} 0, & \sigma_1 = x_1, \\ z_1, & \sigma_1 \neq x_1 \end{cases}$$

If $y_1, y_2, \ldots, y_n$ are defined, then $\sigma_n = \sigma_n(x_1, x_2, \ldots, x_{n-1}, z_1, z_2, \ldots, z_{n-1})$ is defined by

$$\varphi(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) = (\sigma_n, 1)$$

and

$$y_n = \begin{cases} 0, & \sigma_n = x_n, \\ z_n, & \sigma_n \neq x_n. \end{cases}$$

Therefore for given $x, z \in [0,1)$, we can define

$$\Pi_x(z) = y, \qquad \text{where } y = \bigcap_{n=1}^{\infty} C_n(\hat{y})$$

and

$$F_x = \bigcup_{z \in [0,1)} \Pi_x(z)$$

where $C_n(\hat{y})$ is the n-th step interval containing $\hat{y}$ with length

$$|C_n(\hat{y})| = a^{(n-\sum_{k=1}^n y_k)} \cdot b^{\sum_{k=1}^n y_k}.$$

And if m denotes Lebesgue measure on $[0,1]$, then $m\Pi_x^{-1}$ is a measure on the fibre F_x.

Therefore choose $(x,z) \in [0,1)^2$ randomly according to Lebesgue measure $m \times m$ and then we have

$$m\Pi_x^{-1}(C_n(\hat{y})) = b^{\sum_{k=1}^n y_k} \cdot a^{(\sum_{k=1}^n (\sigma_k \oplus x_k) - \sum_{k=1}^n y_k)}$$

where $\oplus$ denotes the sum mod 2.

Since σ_k is a function of $(x_1, \dots, x_{k-1}, z_1, \dots, z_{k-1})$ and y_k is a function of $(x_1, \dots, x_k, z_1, \dots, z_k, \sigma_1, \dots, \sigma_k)$, $\{\sigma_k \oplus x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are independent, thus for a. a. $(x,z) \in [0,1]^2$,

$$\frac{1}{n}\sum_{k=1}^n (\sigma_k \oplus x_k) \to 2ab \qquad \text{and}$$

$$\frac{1}{n}\sum_{k=1}^n y_k \to 2ab^2, \qquad \text{as } n \to \infty.$$

Therefore for a. a. $(x,z) \in [0,1]^2$,

$$\frac{\log m\Pi_x^{-1}(C_n(\hat{y}))}{\log|C_n(\hat{y})|} = \frac{(\sum_{k=1}^n y_k)\log b + (\sum_{k=1}^n (\sigma_k \oplus x_k) - \sum_{k=1}^n y_k)\log a}{(n - (\sum_{k=1}^n y_k))\log a + (\sum_{k=1}^n y_k)\log b}$$

$$\to \frac{2ab\{b(\log b - \log a) + \log a\}}{2ab^2(\log b - \log a) + \log a} \equiv \theta, \qquad \text{as } n \to \infty.$$

If we write

$$\Lambda(x) = \left\{ y \in F_x \mid \lim_{n\to\infty} \frac{\log m\Pi_x^{-1}(C_n(\hat{y}))}{\log|C_n(\hat{y})|} = \theta \right\},$$

then we obtain that for a. a. x with respect to m, $m\Pi_x^{-1}(\Lambda(x)) = 1$ by Fubini's theorem. Hence

$$\dim_H F_x \geq \theta$$

for a. a. x with respect to m (see [3]). □

Corollary 2.2 [1]. *Let $\mathscr{F}$ be as in our introduction. Then for all $F \in \mathscr{F}$, $\dim_H F_x \geq \frac{1}{2}$, for a. a. x with respect to m.*

Proof. Put $a = b = 1$ and $M = 2$ in Theorem 2.1. □

On the other hand, an upper bound of Hausdorff dimension for fibres F_x of $F \in \mathbf{F}(a,b)$ goes as follows.

Theorem 2.3. *For all $F \in \mathscr{F}(a,b)$ with $a, b \in \mathbf{Q}^+$, $a = \frac{c}{M}$, $b = \frac{d}{M}$*

$$\overline{\dim}_B F_x \leq \frac{\log(M^2 - d^2)}{\log M} - 1$$

for a. a. $x \in [0,1]$ with respect to Lebesgue measure m.

Proof. We easily see that F is covered at most $(M^2 - d^2)^n$ squares of side length M^{-n}, for every $n \geq 1$.

Put

$$\beta = \frac{\log(M^2 - d^2)}{\log M}.$$

Suppose that there exists $F \in \mathscr{F}$ and $\varepsilon > 0$ such that

$$m\{x \mid \overline{\dim}_B(F_x) \geq \beta - 1 + 2\varepsilon\} \equiv \delta > 0.$$

Then by Egorov's theorem there exists $n_o \in \mathbf{N}$ such that

$$m\left\{x \mid \sup_{k \geq n} \frac{\log N_{1/M^k}(F_x)}{-n \log M} \geq \beta - 1 + \varepsilon\right\} \geq \frac{1}{M}\delta, \qquad \forall n \geq n_o.$$

Therefore we have at least $\left(\frac{1}{M}\right)\delta M^n$ intervals of length M^{-n} on the x-axis above which F intersects more than $M^{n(\beta-1+\varepsilon)}$ squares for infinitely many $n \geq n_o$.

Then it leads to contradiction since

$$\frac{1}{M}\delta M^{n(\beta+\varepsilon)} > (M^2 - d^2)^n, \qquad \text{for some } n > \frac{\log M - \log \delta}{\varepsilon \log M}.$$ □

Remark. We don't know yet whether Theorem 2.3 is true or not for the case of irrational number.

Corollary 2.4 [1]. *Let $\mathscr{F}$ be as in the introduction. Then for all $F \in \mathscr{F}$, $\overline{\dim}_B F_x \leq \frac{\log 3}{\log 2} - 1$, for a. a. x with respect to m.*

Proof. Put $a = b = 1$ and $M = 2$, in Theorem 2.3. □

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