Subgroups of \( \pi_*(L^2 T(1)) \) at the prime two

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Abstract. Let \( T(1) \) be the Ravenel spectrum whose \( BP_* \)-homology is \( BP_*[t_1] \)(= \( BP_*T(1) \)), and let \( L_2 \) denote the Bousfield localization functor with respect to \( v_2^{-1}BP_* \).

In this paper, we show that the \( E_2 \)-term of the Adams-Novikov spectral sequence for \( \pi_*(L^2 T(1)) \) has horizontal vanishing line and is the \( E_{\infty} \)-term. We also find subgroups of the homotopy groups \( \pi_*(L^2 T(1)) \).

1. Introduction

In this paper, everything is localized at the prime two. Let \( BP_* \) denote the Brown-Peterson ring spectrum at the prime two. Then the homotopy groups \( \pi_*BP_* \) turn to the polynomial algebra \( BP_* = \mathbb{Z}[v_1, v_2, \ldots] \) over the Hazewinkel generators \( v_i \) with \( |v_i| = 2^{i+1} - 2 \). The Ravenel spectrum \( T(1) \) is characterized by the Brown-Peterson homology as \( BP_*(T(1)) = BP_*[t_1] \subset BP_*T(1) \). We consider the spectrum \( G = v_2^{-1}BP_* \). Let \( L_2 \) denote the Bousfield localization functor on the stable homotopy category of spectra with respect to \( G \). One of the methods to determine the homotopy groups \( \pi_*(L^2 T(1)) \) is the Adams-Novikov spectral sequence \( E^2_{*} = H^*v_2^{-1}BP_*[t_1] \Rightarrow E^\infty_{*} = \text{Ext}_{E_*(G)}(G, -) \). We study the \( E_2 \)-term by the chromatic spectral sequence \( \sum_{i=0}^2 \pi_iM_0^i[t_1] \Rightarrow \pi_iM_1^i[t_1] \) and the mod 2 Bockstein spectral sequences \( \pi^*M_0^i[t_1] \Rightarrow \pi^*M_1^i[t_1] \) and \( \pi^*M_0^i[t_1] \Rightarrow \pi^*M_1^i[t_1] \). Here, \( M_0^i = 2^{-i}BP_* \), \( M_1^i = v_2^{-1}BP_*/(2) \), \( M_0^i = v_2^{-1}BP_*/(2^\infty) \), \( M_1^i = v_2^{-1}BP_*/(2, v_1^\infty) \), and \( M_0^i = v_2^{-1}BP_*/(2^\infty, v_1^\infty) \).

The modules \( \pi^*M_0^i[t_1] \) and \( \pi^*M_1^i[t_1] \) are given by Ravenel in [7]. In [5], Mahowald and the second author determined \( H^*M_0^0[t_1] \) as the tensor product of the polynomial algebra \( K(2)_*[v_3, h_{20}] \) and the exterior algebra \( A(h_{21}, h_{30}, h_{31}, p_2) \), where \( K(2)_* = \mathbb{Z}/2[v_2^{-1}] \). In [8], the second author determined \( H^*M_1^1[t_1] \) by the \( v_1 \)-Bockstein spectral sequence \( H^*M_0^0[t_1] \Rightarrow H^*M_1^1[t_1] \) to be the tensor product of \( A(p_2) \) and the direct sum of modules \( A_i \).
A_0 = \left( v_1^{-1} K/K \oplus \sum_{n>1} x_n K/(v_1^n)[x_{n+1}] \otimes A(g_{n+1}) \right) \otimes A(h_{20}),
A_1 = v_2 K/(v_1^2)[x_2] \otimes A(h_{30}, h_{31}) \quad \text{and}
A_2 = v_3 K(2)_o[v_3^2, h_{20}] \otimes A(h_{21}, h_{30}, h_{31}).

Here \( K = \mathbb{Z}/2[v_1, v_2^{-1}] \), \( a_n \) denotes the integer \( 2^n + \frac{1}{2} (2^n - 2^{\text{even}}) \) for \( v(n) = (1 - (-1)^n)/2 \), and the elements \( x_n, g_n, h_{ij} \) and \( h_{20} \) denote the cohomology classes represented by the cocycles of the cobar complex \( \Omega^2_{G\langle G \rangle} G_s[t_1]/(2, v_1^2) \) for a suitable \( j > 0 \), whose leading terms are \( v_3^2, v_3^{(2^n-2^{\text{even}})/3} t_2^{2^{m+1}}, t_2^2 \) and \( v_3^2 t_2 \), respectively. Consider the submodule
\begin{align*}
A_{21} = v_3 K_2^2[v_3^2] \otimes A(h_{21}, h_{30}, h_{31}) \subset A_2,
\end{align*}
and put \( A_2^0 = A_2/A_{21} \) as a module. We see that there is a submodule
\begin{align*}
\widetilde{A}_2 = v_2 v_3 K_2^2[v_3^2, h_{20}] \otimes A(h_{21}, h_{30}, h_{31})
\end{align*}
of \( H^* M_0^2[t_1] \), where \( K_2^2 = \mathbb{Z}/2[v_2^{-1}] \) and \( x \in \widetilde{A}_2 \) is considered to be \( x/2v_1 \in H^* M_0^2[t_1] \). Then we show that the map \( \varphi : H^* M_1^2[t_1] \to H^* M_0^2[t_1] \) given by \( \varphi(x) = x/2 \) is restricted to \( \varphi : A_2^0 \to \widetilde{A}_2 \) and then the sequence \( 0 \to (A_2^0)^s \to (A_2^0)^t \to 0 \) for each \( s > 3 \) is exact, where \( (M)^s \) denotes the submodule of \( M \) consisting of elements of cohomology dimension \( s \), and \( \delta \) is the connecting homomorphism associated to the short exact sequence \( 0 \to M_1^2[t_1] \to M_0^2[t_1] \to M_0^2[t_1] \to 0 \). This shows our first result.

**Theorem 1.1.** \( H^* M_0^2[t_1] \) is isomorphic to \( (\widetilde{A}_2 \otimes A(\rho_2))^s \) for \( s > 4 \).

Furthermore, we show that the mod 2 Bockstein spectral sequence splits (see Lemma 3.6). A summand of the spectral sequence is \( A_2^0 \Rightarrow \widetilde{A}_2 \). It seems very complicated to determine the other parts \( A_1 = (A_0 \oplus A_1 \oplus A_{21}) \otimes A(\rho_2) \Rightarrow \widetilde{A}_1 \) (cf. [6], [2], [9]).

Let \( W \) be the spectrum such that \( BP_*(L_1 W) = M_0^2 \). Indeed, \( W \) is the cofiber of the localization map \( V \to L_1 V \), where \( V \) is the cofiber of the localization map \( S^0 \to SQ \). Then \( H^* M_0^2[t_1] \) is isomorphic to the \( E_2 \)-term of the Adams-Novikov spectral sequence for \( \pi_*(L_2 W \wedge T(1)) \). We consider the submodule
\begin{align*}
\widetilde{A}_{21} = v_3 K_2^2[v_3^2] \otimes A(h_{30}, h_{31}) \subset H^* M_0^2[t_1],
\end{align*}
and see that \( \widetilde{A}_{21} \otimes A(\rho_2) \subset \widetilde{A}_1 \) (see Corollary 4.4). We write \( \widetilde{A}_1^0 = \widetilde{A}_1/\left( \widetilde{A}_{21} \otimes A(\rho_2) \right) \) as a module. We compute the differentials of the Adams-Novikov spectral sequence on \( \widetilde{A}_2 \) and \( \widetilde{A}_{21} \), and then show that the differentials on \( \widetilde{A}_1^0 \) are zero after a modification of \( \widetilde{A}_1^0 \) (see Corollary 4.8).
Theorem 1.2. The Adams-Novikov $E_\infty$-term for the homotopy groups
$
\pi_*(L_2T(1) \wedge W)
$ is isomorphic to the direct sum of $A_1^0$ and $A_2 \otimes A(\rho_2)$, where

\[
A_2 = v_2v_3K_0^*[v_3] \otimes A(h_{20}, h_{21}, h_{30}, h_{31}) \oplus v_2v_3h_{20}^2K_0^*[v_3] \otimes A(h_{30}, h_{31}).
\]

Note that we do not determine the structure of $\tilde{A}_1^0$ of the theorem, though
we know that the Adams-Novikov differentials are trivial on it.

By the definition of $W$, we have the composite $\eta : W \to \Sigma V \to S^2$, which
induces the composite of connecting homomorphisms $\eta_* : H^*M_0^1[t_1] \to H^{*+1}v_2^{-1}BP_*/(2^\infty)[t_1] \to H^{*+2}v_2^{-1}BP_*/[t_1]$ in the long exact sequences

\[
H^*M_0^1[t_1] \to H^*M_0^2[t_1] \delta \to H^{*+1}v_2^{-1}BP_*/(2^\infty)[t_1] \to H^{*+1}M_0^1[t_1] \quad \text{and}
\]

\[
H^*M_0^0[t_1] \to H^*v_2^{-1}BP_*/(2^\infty)[t_1] \delta \to H^{*+1}v_2^{-1}BP_*/[t_1] \to H^{*+1}M_0^0[t_1].
\]

Since we see that both of $H^*M_0^0[t_1]$ and $H^*M_0^1[t_1]$ are zero for $s > 0$ (Theorem
2.5), we see that the connecting homomorphisms are isomorphisms for $s > 0$, and so is $\eta_*$. In Proposition 4.7, we show that the $E_4$-term is the $E_\infty$-term.

Since $\eta_*$ is a map of spectral sequences, we have the results on $\pi_*(L_2T(1))$.

Corollary 1.3. The Adams-Novikov spectral sequence converging to the
homotopy groups $\pi_*(L_2T(1))$ collapses from the $E_4$-term.

Corollary 1.4. The homotopy groups $\pi_*(L_2T(1))$ contain the subgroups
isomorphic to $A_2 \otimes A(\rho_2)$, which is the image of $A_2 \otimes A(\rho_2)$ under the map

$\eta_* : \pi_*(L_2T(1) \wedge W) \to \pi_*(L_2T(1))$.

In the next section, we show that $H^*M_0^1[t_1]$ is zero for $s > 0$ by deter-
mining it. In sections 3 and 4, we give proofs of Theorems 1.1 and 1.2, respectively. The authors would like to thank Professor Xiangjun Wang who
pointed out mistakes in Lemmas 3.3 and 4.3 in a draft version of this paper.

2. $H^*M_0^1[t_1]$

Let $BP$ denote the Brown-Peterson spectrum at the prime two. Then

$BP_\ast = \mathbb{Z}_2[v_1, v_2, \ldots]$ and $BP_\ast(BP) = BP_\ast[t_1, t_2, \ldots]$, and ($BP_\ast, BP_\ast(BP)$) is a
Hopf algebroid. Hereafter, we write

$H^*M = \text{Ext}_{BP_\ast(BP)}^{+}(BP_\ast, M)$

for a $BP_\ast(BP)$-comodule $M$. Consider the $BP_\ast(BP)$-comodule $M_1^0 = v_1^{-1}BP_\ast/(2)$. Then in [7, Th. 6.1.1 and Cor. 6.5.6], it is shown that

$H^*M_1^0[t_1] = K(1)_*[v_2] \otimes A(h_{20}).$

Here $H^*M$ for a $BP_\ast(BP)$-comodule $M$ denotes $\text{Ext}_{BP_\ast(BP)}^{+}(BP_\ast, M)$, $K(1)_* = \ldots$

Subgroups of $\pi_*(L_2T(1))$ at the prime two
Let $d: v^{-1}_1 A \rightarrow v^{-1}_1 A \otimes A \Gamma$ denote $\eta_R - \eta_L$. Then we have

**Lemma 2.2.** Let $x_{1,i}$ be the elements defined above. Then we see that $d(x_{1,n}) \equiv 2^{n+1} X_n t_2 \mod (2^{n+2})$ for $n \geq 0$, where $X_0 = 1$ and $X_n = x_{1,0} x_{1,1} \ldots x_{1,n-1}$ for $n > 0$.

**Proof.** For $n = 0$, it follows from Lemma 2.1. For $n = 1$, we obtain the equation from the computations:
Here, the underlined terms with the same subscript cancel out.

Inductively, suppose that \( d(x_{1,n}) \equiv 2^{n+1} X_n t_2 \mod (2^{n+2}) \). Then

\[
d(x_{1,n}^2) \equiv (x_{1,n} + 2^{n+1} X_n t_2)^2 - x_{1,n}^2 \mod (2^{n+3})
\]

\[
\equiv 2^{n+2} x_{1,n} X_n t_2 \mod (2^{n+3}),
\]

and obtain the congruence for \( n + 1 \).

**Lemma 2.3.** \( H^0 M_0^1 [t_1] \) is the tensor product of \( \mathbb{Z}[v_1^1] \) and the direct sum of \( Q/\mathbb{Z} [v_1] \) and \( \mathbb{Z} / (2^{n+1}) \) generated by \( x_{1,n}^1 / 2^{n+1} \) for each \( n \geq 0 \) and odd \( s > 0 \).

**Proof.** Let \( B \) denote the module of the lemma. Then we have a sequence \( H^* M_0^1 [t_1] \to B \) fitting in the commutative diagram

\[
0 \to H^0 M_0^1 [t_1] \to H^0 M_0^1 [t_1] \oplus 2 \to H^0 M_0^1 [t_1] \oplus 2 \to H^1 M_0^1 [t_1]
\]

Here \( \phi(x) = x/2 \). If the bottom sequence is exact, then the inclusion \( i \) is an isomorphism by [4, Remark 3.1]. To see the exactness, it suffices to show that \( \text{Ker} \delta \subset \text{Im} 2 \), which is seen by \( \delta(x_{1,n}^1 / 2^{n+1}) = v_2^2/n + 2^{n+1} h_{20} \) for odd \( s > 0 \).

**Corollary 2.4.** The image of \( \phi : H^1 M_0^1 [t_1] \to H^1 M_0^1 [t_1] \) is zero.

**Proof.** Note that each integer \( s \geq 0 \) is expressed uniquely as \( 2^{n+1} t + 2^n - 1 \) for some \( t, n \geq 0 \). Therefore, each generator \( v_2^2 h_{20} \in H^1 M_0^1 [t_1] \) for \( s \geq 0 \) is the image of \( x_{1,n}^2 / 2^{n+1} \) under \( \delta \).

**Theorem 2.5.** \( H^* M_0^1 [t_1] = 0 \) for \( s > 0 \).

3. **Proof of Theorem 1.1**

We will study \( H^* M_0^1 [t_1] \) for \( s \geq 0 \) by using the exact sequence

\[
\cdots \to H^* M_0^2 [t_1] \to H^* M_0^1 [t_1] \to H^* M_0^1 [t_1] \to H^{*+1} M_0^1 [t_1] \to \cdots
\]
associated to the short exact sequence

\[ 0 \to M_1^0[t_1] \xrightarrow{\varphi} M_2^0[t_1] \xrightarrow{\gamma} M_2^1[t_1] \to 0, \]

where \( \varphi(x) = x/2 \). Here, \( H^*M = \text{Ext}^*_BP_*(BP_*, M) \) as before. Consider the submodules

\[ A_2 = v_3K(2)[\frac{v_3^2}{v_2}, h_{20}] \otimes A(h_{21}, h_{30}, h_{31}) \quad \text{and} \]

\[ A_{21} = v_3K^2[v_2^3] \otimes A(h_{21}, h_{30}, h_{31}) \]

of \( H^*M_1^0[t_1] \), where \( K(2)_s = \mathbb{Z}/2\{v_2^s\} \), \( K^2_s = \mathbb{Z}/2\{v_2^s\} \) and an element \( x \) of the modules is considered to be an element \( x/v_1 \) of \( H^*M_1^0[t_1] \). Put \( A_2^0 = A_2/A_{21} \) as a module. Then, it is shown in [8, Th. 6.13] that

\[ H^*M_1^0[t_1] = (A_2^0 \otimes A(\rho_2))^s \]

for \( s > 4 \) and \( H^4M_1^0[t_1] = (A_2^0 \otimes A(\rho_2))^4 \otimes v_3K^2[v_2^3]\{h_{21}, h_{30}, h_{31}\}, \)

where \( (M)^s \) denotes the submodule of \( M \) consisting of elements of cohomology dimension \( s \).

The exact sequence (3.1) defines the Bockstein spectral sequence \( H^*M_1^0[t_1] \Rightarrow H^*M_1^0[t_1] \). The differential \( d_1 \) is defined to be \( d_1 = \partial \rho : H^*M_1^0[t_1] \to H^{*+1}M_1^0[t_1] \) for the maps \( \delta \) and \( \varphi \) in (3.1). Then we have the following lemma.

**Lemma 3.3.** The differential \( d_1 \) of the Bockstein spectral sequence acts on \( A_2^0 \) as follows:

\[ d_1(v_2^{2u+1}x) = v_2^uxh_{20} \]

for an integer \( u \) and \( x \in A_2^0 \) with \( v_2 \not| x \).

**Proof.** Each cohomology class is represented as follows:

\[ h_{20} = [t_2], \quad h_{21} = [t_2^2], \quad h_{30} = [t_3] \quad \text{and} \quad h_{31} = [t_2^3]. \]

For the diagonal map \( A \), Quillen’s formula \( A(t_n) = \Psi_0(n) + \sum_{k=1}^n m_k(\Psi_k(n) - A((t_n-k))^{1+1} \otimes 1 + 1 \otimes t_2 \) and \( A(t_3) = \Psi_0(3) - v_1 t_2 \otimes t_2 \equiv t_3 \otimes 1 + 1 \otimes t_3 \mod(4, v_1) \), where \( \Psi_k(l) = \sum_{i=0}^l t_i^{k+} \otimes t_{l-i}^{k+} \) and \( t_0 = 1 \). Thus this together with Lemma 2.1 shows that

\[ (3.4) \quad d(v_2) \equiv 2t_2, \quad d(v_3) \equiv 2t_3, \quad d(t_2^2) \equiv 2t_2 \otimes t_2 \quad \text{and} \quad d(t_3^2) \equiv 2t_3 \otimes t_3 \mod(4, v_1) \quad \text{in} \quad \Omega^*v_2^{-1}BP_*. \]

By the definition of the differential of the cobar complex, the element \( d(v_2^{2u+1}x/4) \) of \( \Omega^*M_2^0[t_1] \) is computed

\[ d(v_2^{2u+1}x/4) = d(v_2^{2u+1})x/4 + v_2^{2u+1}d(x/4) \]

\[ = v_2^{2u+1}t_2x/2 + v_2^{2u+1}d(x)/4 \]

\[ = v_2^uxh_{20}/2 + v_2^{2u+1}y/2, \]

\[ d(v_2^{2u+1}x/4) \],
where $y$ is an element of $\Omega^* v_2^{-1} BP_*/(4, v_1^\infty)[t_1]$ such that $d(x) = 2y$. We see that $y \neq \pm v_2^{-1} x h_{20}$ mod($4, v_1$) by (3.4). Note here that $t_3 \otimes t_5$ represents the cohomology class $h_{31}(v_2^{-1} h_{20} + v_2^{-2} h_{21}) + v_2^{-3} v_1^2 h_{20} h_{21}$ (see [5, p. 243, (1)]).

The lemma indicates that $h_{21}$ is redefined as

$$h_{21} = [t_2^2 + v_2 t_2]$$

and gives rise to the differential pattern on $A_2^0$:

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in which $x \mapsto y$ denotes the $d_i(x/v_1) = y/v_1$ for $x/v_1, y/v_1 \in A_2^0$.

Observe the long exact sequence (3.1), and note that the module $\tilde{A}_2$ given in the introduction is Im $\varphi$. Then the above differential pattern shows that $\delta$ is a monomorphism on $\tilde{A}_2$, since $\tilde{A}_2$ is generated by the elements at the tails of the arrows.

**Lemma 3.5.** The module $\tilde{A}_2$ given in the introduction fits in the short exact sequence

$$0 \to (\tilde{A}_2)^{s-1} \xrightarrow{\delta} (A_2^0)^s \xrightarrow{\varphi} (\tilde{A}_2)^s \to 0$$

for $s > 3$.

**Proof of Theorem 1.1.** Since $H^s M_1^1[t_1] = (A_2^0 \otimes A(p_2))^s$ for $s > 4$, we have the commutative diagram

$$
\begin{array}{ccc}
(A_2 \otimes A(p_2))^{s-1} & \xrightarrow{\delta} & (A_2^0 \otimes A(p_2))^s \\
\downarrow & & \downarrow \\
A_2 \otimes A(p_2) & \xrightarrow{\varphi} & \tilde{A}_2 \otimes A(p_2) \\
\uparrow & & \uparrow \\
H^{s-1} M_1^0[t_1] & \xrightarrow{\delta} & H^s M_1^1[t_1] \\
\end{array}
$$
of exact sequences by Lemma 3.5. If we show that the images of the left \( \delta \)'s agree, then the map \( g \) is an isomorphism by [4, Remark 3.11]. We denote the maps \( \delta \) and \( \varphi \) in the top sequence by \( \delta' \) and \( \varphi' \). Then \( \text{Im} \delta' \subset \text{Im} \delta \). For any \( x \notin \text{Im} \delta' \), \( \varphi'(x) = x/2 \neq 0 \) and \( \delta'(x/2) \neq 0 \), which shows \( g(x/2) \neq 0 \) since \( \delta' = \delta g \). Therefore, \( \varphi(x) = g(\varphi'(x)) = g(x/2) \neq 0 \), and \( x \notin \text{Im} \delta \). \( \square \)

**Lemma 3.6.** The Bockstein spectral sequence \( H^* M_1^1|t_1| \Rightarrow H^* M_0^1|t_1| \) splits into two spectral sequences \( A_1 = \langle A_0 \oplus A_1 \oplus A_2 \rangle \oplus A(\rho_2) \Rightarrow A_1 \) and \( A_0^0 \oplus A(\rho_2) \Rightarrow A_2 \oplus A(\rho_2) \). Here, the module \( A_1 \) denotes a module fitting in the long exact sequence

\[
0 \to (A_1)^0 \xrightarrow{\varphi} (A_1)^0 \xrightarrow{\delta} (A_1)^1 \xrightarrow{\varphi} \cdots \\
\delta (A_1)^s \xrightarrow{\varphi} (A_1)^s \xrightarrow{\delta} (A_1)^s+1 \xrightarrow{\varphi} \cdots
\]

**Proof.** By Lemma 3.5, we have the subspectral sequence \( A_0^0 \oplus A(\rho_2) \Rightarrow A_2 \oplus A(\rho_2) \). Furthermore, Lemma 3.5 implies that all elements of \( A_0^0 \oplus A(\rho_2) \) do not survive to the \( E_2 \)-term of the Bockstein spectral sequence. It follows that the differential \( d_r \) acts on \( A_1 \). Now \( A_1 \) is generated by elements \( \bar{x}_r \) such that \( 2^r \bar{x}_r = \bar{x}_1 = \varphi(x) \) and \( \delta(\bar{x}_r)' \)'s are linearly independent. \( \square \)

**Remark.** \( A_1 \) is not determined here. Even the 0-dimensional part \( (A_1)^0 \) of it is very complicated (see. [6], [9]), though \( (A_1)^s = 0 \) for \( s > 4 \).

### 4. Proof of Theorem 1.2

Recall [8] the spectrum \( C \) such that \( BP_* (C) = BP_*/(2, v_1^\infty)[t_1] \). Then \( C \) fits in the cofiber sequence

\[
C \xrightarrow{\varphi} W \wedge T(1) \xrightarrow{2} W \wedge T(1) \to \Sigma C,
\]

which induces the short exact sequence

\[
0 \to M_1^1|t_1| \xrightarrow{\varphi} M_0^1|t_1| \xrightarrow{2} M_0^1|t_1| \to 0
\]

by applying \( BP_* (L_2 -) \). Let \( E_*^{p,q}(X) \) denote the \( E_r \)-term of the \( v_2 \) \( BP \) based Adams spectral sequence converging to \( \pi_*(L_2 X) \). Then the \( E_2 \)-term is \( \text{Ext}^{*,*}_{BP_* (v_2 \wedge \infty BP)}(v_2 \wedge BP_* (v_2 \wedge BP_* (X))) \), which is isomorphic to \( H^* v_2 \wedge BP_* (X) \) by the change of rings theorem of Hovey and Sadofsky [1, Th. 3.1]. Indeed, we use the modified one [3, Th. 3.3]. In our case, we consider the spectral sequences \( E_2^v(C) = H^* M_1^1|t_1| \Rightarrow \pi_*(L_2 C) \) and \( E_2^v(W \wedge T(1)) = H^* M_0^1|t_1| \Rightarrow \pi_*(L_2 W \wedge T(1)) \).

For the sake of simplicity, we compute differentials by setting \( v_2^2 = 1 \). In [8, Lemma 7.4], it is shown that for any \( v_3^{d+3} x/v_1 \in E_2^{d-1}(C) \cap A_2 \),
The other differentials on $E_2^*(C)$ are trivial except for the differentials

(4.2) \[ d_3(x_{n+1}^s h_3^2 v_1^{n_0}) = \begin{cases} 
  v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0} & \text{n is even} \\
  v_2 v_3^{2^s(x-1)+8(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0} & \text{n is odd}
\end{cases} \]

for $n \geq 2$ and odd $s > 0$, and a $v_3$-multiple of them ([8, Lemmas 7.6 and 7.8]). Here $h_3^2$ is defined as the class represented by the cocycle $\hat{\omega}$ in the congruence $d(v_3^4) \equiv 2v_2^2 F_2 \bmod(4)$, whose leading term is $v_2^2 v_3^2 t_2$.

**Lemma 4.3.** In the Adams-Novikov $E_2^*$-term for $\pi_*(L_2 W \wedge T(1))$,

\[ d_3(v_3^s x/2 v_1) = v_2 v_3 x h_3^2 h_3^2 v_1^{n_0} \quad \text{and} \quad d_3(v_2 v_3^2 y/2 v_1) = v_2 v_3 y h_3^2 h_3^2 v_1^{n_0} \]

for $x \in K_2^1[v_3^4] \otimes A(h_3^2, h_3^2)$ and $y \in K_2^1[v_3^4, h_3^2] \otimes A(h_3^2, h_3^2, h_3^2)$, and

\[ \begin{align*}
  d_3(x_{n+1}^s h_3^2 v_1^{n_0}) &= \begin{cases} 
  v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is even} \\
  v_2 v_3^{2^s(x-1)+8(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is odd}
\end{cases} \\
  d_3(x_{n+1}^s g_3 h_3^2 v_1^{n_0}) &= \begin{cases} 
  v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is even} \\
  v_2 v_3^{2^s(x-1)+8(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is odd}
\end{cases} \\
  d_3(v_3^s x_{n+1}^s h_3^2 v_1^{n_0}) &= \begin{cases} 
  v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is even} \\
  v_2 v_3^{2^s(x-1)+8(2^{n-2}-1)/3+1} h_3^2 h_3^2 v_1^{n_0} & \text{n is odd}
\end{cases}
\]

for positive integers $s$ and $n$ with $n > 1$. Here the equations are all up to sign.

**Proof.** Note that $v_3 x h_3^2 v_1^{n_0} = v_2 v_3 x h_3^2 h_3^2 v_1^{n_0}$ in $E_1^*(W \wedge T(1))$, since $\delta(v_3 x h_3^2 v_1^{n_0}) = v_3 x h_3^2 v_1^{n_0} + v_2 v_3 x h_3^2 h_3^2 v_1^{n_0}$ by Lemma 3.3. In the same manner as this, we have the relations $v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0} = v_2 v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0}$ and $v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0} = v_2 v_3^{2^s(x-1)+4(2^{n-2}-1)/3+1} h_3^2 h_3^2 h_3^2 h_3^2 v_1^{n_0}$, since $h_3^2 = h_3^2$. Then the differentials in (4.1) and (4.2) of the form $d_3(x) = y$ (resp. $d_3(x) = y$) yield differentials $d_3(x/2) = v_3 z/2$ and $d_3(x/2) = v_3 z/2$ (resp. $d_3(x/2) = v_3 z/2$ and $d_3(x/2) = v_3 z/2$) of $E_1^*(W \wedge T(1))$, where $z$ is an element such that $\delta(w) = y - z \in H^* M_1^1[t_1]$ for an element $w$ of $H^* M_1^1[t_1]$. \[\Box\]
Corollary 4.4. The module $\tilde{A}_{21}$ given in Introduction is a submodule of $H^*M_3^*[t_1]$. In other words, the map sending an element $x \in \tilde{A}_{21}$ to $x/2v_1 \in H^*M_3^*[t_1]$ is a monomorphism.

Proof. It suffices to show that $x/2v_1 \neq 0 \in H^*M_3^*[t_1]$ for $x \in \tilde{A}_{21}$. The first equation of Lemma 4.3 shows $d_5(x/2v_1) \neq 0$. □

Corollary 4.5. After a suitable modification of $\tilde{A}_{1}^{0}$, the $v_2^{-1}BP$ based Adams differentials $d_5$ originating in $\tilde{A}_{1}^{0}$ are all zero.

Proof. The only non-trivial differentials originating in $\tilde{A}_{1}^{0}$ are given in Lemma 4.3, and their targets are all in the image of $d_5$ originating in $(A_2 \oplus A_2) \otimes A(\rho_2)$. □

Remark. This modification of $\tilde{A}_{1}^{0}$ does not change the additive structure of $\tilde{A}_{1}^{0}$ nor the $E_2$-term $H^*M_3^*[t_1]$. In fact, each generator $x \in \tilde{A}_{1}^{0}$ is just replaced by $x + y$ for some $y \in (A_2 \oplus A_2) \otimes A(\rho_2)$.

Theorem 4.6. The $E_4$-term of the $v_2^{-1}BP$ based Adams spectral sequence contains $A_2 \otimes A(\rho_2)$, which is obtained from the subgroup $A_2 \otimes A(\rho_2)$ of the $E_2$-term. Here, $A_2$ is the module given in Theorem 1.2.

Proof. The $v_2^{-1}BP$ based Adams differential $d_5$ makes $(A_2, d_5)$ a differential module by Lemma 4.3, whose homology is

$$\tilde{A}_2 = v_2v_3K_2^2[v_3^2, h_{30}]/(h_{30}^2) \otimes A(h_{21}, h_{30}, h_{31}).$$

We decompose $\tilde{A}_2$ into the direct sum of the two modules

$$\tilde{A}_{21} = v_2v_3K_2^2[v_3^2, h_{20}, h_{30}, h_{31}] \otimes A(h_{21}, h_{30}, h_{31}) \oplus v_2v_3h_{20}^2K_2^2[v_3^2] \otimes A(h_{30}, h_{31});$$

and

$$\tilde{A}_{22} = v_2v_3h_{21}^2K_2^2[v_3^2] \otimes A(h_{30}, h_{31}).$$

The first differential in Lemma 4.3 gives the isomorphism $d_5 : \tilde{A}_{21} \cong \tilde{A}_{22}$, and we obtain the theorem by setting

$$\tilde{A}_2 = \tilde{A}_{21}'.$$

Proposition 4.7. The $v_2^{-1}BP$ based Adams spectral sequence converging to $\pi_*(L_2W \wedge T(1))$ collapses from the $E_4$-term. That is, $E_4 = E_{\infty}^s$.

Proof. Since $(\tilde{A}_1)' = 0$ for $s > 4$ and $(\tilde{A}_2 \otimes A(\rho_2))' = 0$ for $s > 5$, we see that $E_4 = 0$ for $s > 5$. Therefore, the differentials $d_s$ are all trivial for $r > 5$. Suppose that $d_5(x/2^l) = y/2$ for $x/2^l \in \tilde{A}_1$. Then $y/2 \in \tilde{A}_{21}'$, and so $\delta(y/2) \neq 0 \in E_2^s(C)$. Send the relation $d_5(x/2^l) = y/2$ by $\delta$, and we see that $d_5(\delta(x/2^l)) = \delta(y/2) \in E_2^s(C)$. Since $E_2^s(C) = 0$ by [8, Corollary 7.9], there is
an element \( z \neq 0 \in E_3^1 \) such that \( d_3(z) = \delta(y/2) \). Then, \( \varphi_*(z) \) must be hit by \( x/2^{i+1} \) under \( d_3 \). By Lemma 4.3, there is no such differential.

From the proof of this together with Corollary 4.5, we obtain the following:

**Corollary 4.8.** The differentials \( d_0 \) of the \( v_2^{-1}BP \) based Adams spectral sequence for \( \pi_*(L_2W \wedge T(1)) \) are trivial on \( A_1^{0} \subset E_2^* \).

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