Radial growth of $C^2$ functions satisfying Bloch type condition

Dedicated to Professor Makoto Sakai on the occasion of his sixtieth birthday

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Abstract. The aim of this paper is to give a simple proof of results by González-Koskela concerning the radial growth of $C^2$ functions satisfying Bloch type condition. Our results also give generalizations of their results.

1. Introduction

Denote by $B$ the Bloch space of all holomorphic functions $f$ on the unit disk $U$ which satisfy

$$
\|f\|_B = |f(0)| + \sup_{z \in U} (1 - |z|^2) |f'(z)| < \infty.
$$

The radial growth of Bloch functions was extensively discussed by Clunie-MacGregor [2], Korenblum [4], Makarov [5] and Pommerenke [7]. The law of the iterated logarithm of Makarov [5] states that if $f \in B$, then

$$
\limsup_{r \to 1} \frac{|f(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log \log \frac{1}{1-r}}} \leq C\|f\|_B
$$

(1)

for almost every $\zeta \in \partial U$, where $C$ is a universal constant. Pommerenke [7] proved that this inequality is true for $C = 1$ and this inequality is false for $C \leq 0.685$. Recently, González and Koskela studied the radial growth of $C^2$ functions on the unit ball $B^n$ of $\mathbb{R}^n$ which satisfy

$$
|\nabla u(x)|^2 + |u(x)A u(x)| \leq c \left( \frac{2}{1-|x|} \right)^{2\gamma}
$$

(2)

for all $x \in B^n$, where $c > 0$ and $\gamma \leq 1$. They showed the following result ([3, Theorem 1.2]).

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THEOREM A. Let $u$ be a $C^2$ function on $B^n$ satisfying (2). Then, for almost all $\zeta$, $|\zeta| = 1$,

\[
\limsup_{r \to 1} \frac{|u(r\zeta)|}{(\log \frac{1}{1-r})^{1-\gamma} \log \log \frac{1}{1-r}} \leq c_1
\]

if $\gamma < 1$; and

\[
\limsup_{r \to 1} \frac{|u(r\zeta)|}{\log \log \frac{1}{1-r}} \leq c_2
\]

if $\gamma = 1$. Here the constants $c_1$ and $c_2$ depend only on $n, c, \gamma$.

We denote by $B(x,r)$ and $S(x,r)$ the open ball and the sphere of center $x$ and radius $r$, respectively. We set $B^n = B(0,1)$ and $S^{n-1} = S(0,1)$. The Hausdorff measure with a measure function $h$ is written as $\mathcal{H}_h$. In case $h(r) = r^\alpha$, we write $\mathcal{H}_\alpha$ for $\mathcal{H}_h$.

Our first aim in the present note is to extend Theorem A by González-Koskela. For this purpose, let $\varphi$ be a positive, continuous and non-decreasing function on the interval $[0,1)$ satisfying

\[
\varphi(1 - r/2) \leq A\varphi(1 - r) \quad \text{for every } r \in (0,1)
\]

with a constant $A \geq 1$ and

\[
\int_0^1 (1 - t)\varphi(t)dt = \infty.
\]

Set

\[
\Phi(r) = \int_0^r (1 - t)\varphi(t)dt.
\]

THEOREM 1. Let $u$ be a $C^2$ function on $B^n$ with $u(0) = 0$ such that

\[
\mathcal{A}_u(x) = |Vu(x)|^2 + |u(x)Au(x)| \leq \varphi(|x|) \quad \text{for all } x \in B^n.
\]

Then for $\mathcal{H}_{n-1}$-a.e. $\zeta \in S^{n-1}$,

\[
\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)} \log \log \frac{1}{1-r}} \leq \sqrt{A}.
\]

REMARK 1. If we take $\varphi(r) = c(1 - r)^{-\gamma} \{\log(2/(1-r))\}^{-\gamma}$ for $c > 0$ and $\gamma \leq 1$, then Theorem 1 gives Theorem A.

On the other hand, we have the lower limit result as follows:

THEOREM 2. If $u$ is as in Theorem 1, then
lim inf \( \frac{|u(r^\zeta)|}{\sqrt{\Phi(r) \log \log \Phi(r)}} \) \leq 2

for \( \mathcal{H}_{n-1}\)-a.e. \( \zeta \in S^{n-1} \).

By Theorems 1 and 2, we have the following corollary.

**COROLLARY 1.** Let \( u \) be a \( C^2 \) function on \( B^n \) satisfying

\[
\mathcal{A}_u(x) \leq c(1 - |x|)^{-2} \left( \log(1) \frac{1}{1 - |x|} \right)^{-1} \cdots \left( \log(\ell-1) \frac{1}{1 - |x|} \right)^{-1} \left( \log(\ell) \frac{1}{1 - |x|} \right)^{-\gamma},
\]

where \( c > 0, \gamma \leq 1 \) and \( \log(k+1)(t) = \log_k \circ \log_1(t) \) with \( \log_1(t) = \log(e + t) \).

Then for \( \mathcal{H}_{n-1}\)-a.e. \( \zeta \in S^{n-1} \),

\[
\limsup_{r \to 1} \frac{|u(r^\zeta)|}{\sqrt{\left( \log(\ell) \frac{1}{1 - r} \right)^{1-\gamma} \log_2(1) \frac{1}{1 - r}}} \leq c_1
\]

and

\[
\liminf_{r \to 1} \frac{|u(r^\zeta)|}{\sqrt{\left( \log(\ell) \frac{1}{1 - r} \right)^{1-\gamma} \log_3(1) \frac{1}{1 - r}}} \leq c_2
\]

when \( \gamma < 1 \):

\[
\limsup_{r \to 1} \frac{|u(r^\zeta)|}{\sqrt{\log(\ell+1) \frac{1}{1 - r} \log_2(1) \frac{1}{1 - r}}} \leq c_3
\]

and

\[
\liminf_{r \to 1} \frac{|u(r^\zeta)|}{\sqrt{\log(\ell+1) \frac{1}{1 - r} \log(\ell+3) \frac{1}{1 - r}}} \leq c_4
\]

when \( \gamma = 1 \). Here \( c_1, c_2, c_3 \) and \( c_4 \) are constants depending only on \( c, \gamma \) and \( \ell \).

**2. Exponential integral**

In this section, we present an exponential estimate for \( C^2 \) functions satisfying (5). For this we prepare the following lemma, which is a generalization of [3, Theorem 2.2].

**LEMMA 1.** Let \( \varphi \) be a positive continuous function on \( \mathbb{R}^\ell \), and set

\[
\Phi(r) = \int_0^r (1 - t) \varphi(t) dt.
\]
Let $u$ be a $C^2$ function in $B^n$ with $u(0) = 0$ which satisfies condition (5). Then

$$\int_{S^{n-1}} |u(r\zeta)|^{2k} dS(\zeta) \leq \sigma_n 4^k k! |\Phi(r)|^k$$

(6)

for all $k \in \{0, 1, 2, \ldots\}$ and all $r \in (0, 1)$, where $\sigma_n$ denotes the surface measure of $S^{n-1}$.

**Proof.** First we show that

$$\frac{d}{dt} \int_{S^{n-1}} v(t\zeta) dS(\zeta) = t^{1-n} \int_{B(0,1)} \Delta v(w) dw$$

(7)

for each $v \in C^2(B^n)$. Using the divergence theorem, we have

$$\frac{d}{dt} \int_{S^{n-1}} v(t\zeta) dS(\zeta) = \int_{S^{n-1}} v(t\zeta) \cdot \nabla v(t\zeta) dS(\zeta)$$

$$= t^{1-n} \int_{S(0,1)} \frac{w}{t} \cdot \nabla v(w) dS(w)$$

$$= t^{1-n} \int_{B(0,1)} \Delta v(w) dw.$$

Thus (7) holds.

We prove this lemma by induction on $k$. Clearly, (6) holds for $k = 0$. Suppose that (6) holds for $k$. Using (7) and the assumption on induction, we obtain

$$\frac{d}{dt} \int_{S^{n-1}} |u(t\zeta)|^{2(k+1)} dS(\zeta)$$

$$= 2(k+1) t^{1-n} \int_{B(0,1)} |u(w)|^{2k} (u(w)\Delta u(w) + (2k+1)|Vu(w)|^2) dw$$

$$\leq 4(k+1)^2 t^{1-n} \int_{B(0,1)} |u(w)|^{2k} \mathcal{J}_u(w) dw$$

$$\leq 4(k+1)^2 t^{1-n} \int_0^r \rho^{n-1} \varphi(\rho) \left( \int_{S^{n-1}} |u(\rho\zeta)|^{2k} dS(z) \right) d\rho$$

$$\leq \sigma_n 4^{k+1} k! (k+1)^2 t^{1-n} \int_0^r \rho^{n-1} \varphi(\rho) |\Phi(\rho)|^k d\rho.$$

Integrating both sides from 0 to $r$ and applying Fubini’s theorem, we have
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\[
\int_{S^{n-1}} |u(r^*_\zeta)|^{2(k+1)}dS(\zeta) \leq \sigma_n 4^{k+1}k!(k+1)^2 \int_0^1 t^{1-n} \int_0^t r^{n-1} \varphi(\rho)|\Phi(\rho)|^k d\rho dt \\
= \sigma_n 4^{k+1}k!(k+1)^2 \int_0^r \left( \int_0^t t^{1-n} \, dt \right) r^{n-1} \varphi(\rho)|\Phi(\rho)|^k d\rho \\
\leq \sigma_n 4^{k+1}(k+1)! \int_0^r (k+1)(1-\rho) \varphi(\rho)|\Phi(\rho)|^k d\rho \\
= \sigma_n 4^{k+1}(k+1)! \int_0^r \frac{d}{d\rho} |\Phi(\rho)|^{k+1} d\rho \\
= \sigma_n 4^{k+1}(k+1)! [\Phi(\rho)]^{k+1}.
\]

Hence (6) also holds for $k+1$. The induction is completed.

**Lemma 2.** Let $u$ be a function in $B^n$ satisfying condition (6). Then for all $c$, $0 < c < 1/4$, and for all $r$, $0 < r < 1$,

\[
\int_{S^{n-1}} \exp \left( \frac{c|u(r^*_\zeta)|^2}{\Phi(r)} \right) dS(\zeta) \leq \frac{\sigma_n}{1 - 4c}.
\]

**Proof.** If $k$ is a non-negative integer, then, by (6), we have

\[
\frac{1}{k!} \int_{S^{n-1}} \left( \frac{c|u(r^*_\zeta)|^2}{\Phi(r)} \right)^k dS(\zeta) \leq (4c)^k \sigma_n
\]

for $c > 0$. Hence it follows that

\[
\int_{S^{n-1}} \exp \left( \frac{c|u(r^*_\zeta)|^2}{\Phi(r)} \right) dS(\zeta) = \int_{S^{n-1}} \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{c|u(r^*_\zeta)|^2}{\Phi(r)} \right)^k dS(\zeta) \\
\leq \sigma_n \sum_{k=0}^\infty (4c)^k.
\]

The series on the right converges if $0 < c < 1/4$ and thus our lemma is proved.

3. **Proof of Theorem 1**

Let $\varphi$ and $\Phi$ be as in the Introduction, and let $u$ be as in Theorem 1. To prove Theorem 1, we need the following two lemmas.

**Lemma 3.** For every $0 < r < 1$,

\[
\Phi(1-r/2) \leq \frac{A}{4} \Phi(1-r) + \Phi(1/2).
\]
Lemma 4. Let $u$ be a $C^2$ function in $B^n$ such that $|\nabla u(x)|^2 \leq \varphi(|x|)$. Then for every $x \in B^n \backslash B(0, 1/2)$,

$$|u(y) - u(z)| \leq A|\Phi(|x|)|^{1/2}$$

whenever $y, z \in B(x, (1 - |x|)/2)$.

Proof. We see that

$$|u(y) - u(z)| \leq (1 - |x|)[\varphi((1 + |x|)/2)]^{1/2} \leq A^{1/2}(1 - |x|)[\varphi(|x|)]^{1/2}$$

for all $y, z \in B(x, (1 - |x|)/2)$. On the other hand, we have for $1/2 < t < 1$,

$$\Phi(t) \geq \int_{2t^{-1}}^t (1 - s)\varphi(s)ds \geq \varphi(2t - 1)\int_{2t^{-1}}^t (1 - s)ds \geq A^{-1}(1 - t)^2\varphi(t).$$

Thus Lemma 4 follows.

Proof of Theorem 1. From Lemma 2, we see that

$$\int_{B^n} (1 - |x|)^{-1} \left( \log \frac{2}{1 - |x|} \right)^{1-\delta} \exp \left( \frac{c|u(x)|^2}{\Phi(|x|)} \right) dx < \infty$$

for all $c$, $0 < c < 1/4$, and all $\delta > 0$. Then there exists a set $E \subset S^{n-1}$ such that $\mathcal{H}_{n-1}(E) = 0$ and

$$\int_0^1 (1 - r)^{-1} \left( \log \frac{2}{1 - r} \right)^{1-\delta} \exp \left( \frac{c|u(r\zeta)|^2}{\Phi(r)} \right) dr < \infty,$$

for each $\zeta \in S^{n-1}\setminus E$, $0 < c < 1/4$ and $\delta > 0$, which implies that

$$\lim_{r \to 1} \int_r^{1+r/2} (1 - t)^{-1} \left( \log \frac{2}{1 - t} \right)^{1-\delta} \exp \left( \frac{c|u(t\zeta)|^2}{\Phi(t)} \right) dt = 0. \quad (9)$$

Fix $\zeta \in S^{n-1}\setminus E$. For $0 < r < 1$, define $I_r = [r, (1 + r)/2)$. From (9), we obtain

$$\lim_{r \to 1} \left( \log \frac{1}{1 - r} \right)^{1-\delta} \exp \left( \frac{c \inf_{t \in I_r} |u(t\zeta)|^2}{\Phi((1 + r)/2)} \right) = 0,$$

which implies that

$$\frac{c \inf_{t \in I_r} |u(t\zeta)|^2}{4^{-1}A\Phi(r) + \Phi(1/2)} \leq (1 + \delta) \log \log \frac{1}{1 - r} \quad (10)$$

for $r$ near 1, by Lemma 3. Hence it follows from (10) and Lemma 4 that

$$\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r) \log \log \frac{1}{1 - r}}} \leq \sqrt{\frac{A(1 + \delta)}{4c}}.$$
Here, letting \( c \to 1/4 \) and \( \delta \to 0 \), we obtain
\[
\limsup_{r \to 1^-} \frac{|u(r\zeta)|}{\Phi(r) \log \log \frac{r}{1-r}} \leq \sqrt{A},
\]
which proves Theorem 1.

4. Proof of Theorem 2

In this section we complete the proof of Theorem 2. By Lemma 2, we see that
\[
\int_{B^n \setminus B(0, r_0)} (1 - |x|) \varphi(|x|) \Phi(|x|)^{-1} (\log \Phi(|x|))^{-1-\delta} \exp \left( \frac{c|u(x)|^2}{\Phi(|x|)} \right) dx < \infty
\]
for all \( c, 0 < c < 1/4 \), and \( \delta > 0 \), where \( r_0 = \Phi^{-1}(e) \). Consequently,
\[
\lim_{r \to 1^-} \int_r^1 (1 - t) \varphi(t) \Phi(t)^{-1} (\log \Phi(t))^{-1-\delta} \exp \left( \frac{c|u(t\zeta)|^2}{\Phi(t)} \right) dt = 0
\]
for \( \mathcal{H}^{n-1} \)-a.e. \( \zeta \in S^{n-1}, 0 < c < 1/4 \) and \( \delta > 0 \). This implies that
\[
\lim_{r \to 1^-} \int_r^1 (1 - t) \varphi(t) \Phi(t)^{-1} (\log \Phi(t))^{-1-\delta} \exp \left( \frac{c\varrho_r(\zeta)^2}{\Phi(t)} \right) dt = 0,
\]
where \( \varrho_r(\zeta) = \inf_{r < \rho < 1} |u(\rho \zeta)| \). Since \( e^{\delta x} \geq \delta x \) for \( x > 0 \), we have
\[
\frac{d}{dt} \left( -(\log \Phi(t))^{-1-\delta} \exp \left( \frac{(1 + \delta)^{-1} c\varrho_r(\zeta)^2}{\Phi(t)} \right) \right)
\]
\[
= (1 + \delta)(1 - t) \varphi(t) \Phi(t)^{-1} (\log \Phi(t))^{-2-\delta} \exp \left( \frac{(1 + \delta)^{-1} c\varrho_r(\zeta)^2}{\Phi(t)} \right)
\]
\[
+ (\log \Phi(t))^{-1-\delta}(1 - t) \varphi(t) \Phi(t)^{-1}
\]
\[
\times \left( \frac{(1 + \delta)^{-1} c\varrho_r(\zeta)^2}{\Phi(t)} \right) \exp \left( \frac{(1 + \delta)^{-1} c\varrho_r(\zeta)^2}{\Phi(t)} \right)
\]
\[
\leq (1 + \delta + \delta^{-1})(1 - t) \varphi(t) \Phi(t)^{-1} (\log \Phi(t))^{-1-\delta} \exp \left( \frac{c\varrho_r(\zeta)^2}{\Phi(t)} \right)
\]
for \( r_0 < t < 1 \). From (11), we obtain
\[
\lim_{r \to 1^-} (\log \Phi(r))^{-1-\delta} \exp \left( \frac{(1 + \delta)^{-1} c\varrho_r(\zeta)^2}{\Phi(r)} \right) = 0,
\]
which implies that
\[
\frac{(1 + \delta)^{-1}c g_r(\zeta)^2}{\Phi(r)} \leq (1 + \delta) \log \log \Phi(r)
\]
for \( r \) near 1. By letting \( c \to 1/4 \) and \( \delta \to 0 \), we have
\[
\limsup_{r \to 1} \frac{g_r(\zeta)^2}{\Phi(r) \log \log \Phi(r)} \leq 4,
\]
which completes the proof of Theorem 2.

**Corollary 2.** Let \( \varphi \) and \( \Phi \) be as in the Introduction. Let \( u \) be a harmonic function on \( B^n \) satisfying
\[
|\nabla u(x)|^2 \leq \varphi(|x|) \quad \text{for all } x \in B^n.
\]
Then
\[
\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r) \log \log \Phi(r)}} \leq 2
\]
for \( \mathcal{H}_{n-1}\)-a.e. \( \zeta \in S^{n-1} \).

**Proof.** Consider the radial maximal function of \( u \) defined by
\[
\mathcal{R}_u(r, \zeta) = \max_{0 \leq t \leq r} |u(t \zeta)| \quad \text{for } 0 < r < 1 \text{ and } \zeta \in S^{n-1}.
\]
By the Hardy-Littlewood maximal theorem [1, Chapter 6] and Lemma 1,
\[
\int_{S^{n-1}} |\mathcal{R}_u(r, \zeta)|^{2k} dS(\zeta) \leq c_1 \int_{S^{n-1}} |u(r\zeta)|^{2k} dS(\zeta) \leq c_1 \sigma_n 4^k k! [\Phi(r)]^k
\]
for all \( k \) and \( 0 < r < 1 \), where \( c_1 \) is a constant depending only on \( n \). As in the proof of Theorem 1, we have
\[
\lim_{r \to 1} \int_{0}^{1} (1 - t) \varphi(t) \Phi(t)^{-1} (\log \Phi(t))^{-1-\delta} \exp \left( \frac{c_2 |\mathcal{R}_u(t, \zeta)|^2}{\Phi(t)} \right) dt = 0
\]
for \( \mathcal{H}_{n-1}\)-a.e. \( \zeta \in S^{n-1} \), all \( c_2, 0 < c_2 < 1/4 \), and \( \delta > 0 \). Hence we see as in the proof of Theorem 2 that
\[
\limsup_{r \to 1} \frac{\mathcal{R}_u(r, \zeta)}{\sqrt{\Phi(r) \log \log \Phi(r)}} \leq 2
\]
holds for \( \mathcal{H}_{n-1}\)-a.e. \( \zeta \in S^{n-1} \). Since \( \mathcal{R}_u(r, \zeta) \geq |u(r\zeta)| \), we have
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$$\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r) \log \log \Phi(r)}} \leq 2$$

for $\mathcal{H}_{n-1}$-a.e. $\zeta \in S^{n-1}$, which yields the required conclusion.

**Remark 2.** Let $u$ be a harmonic function on $\mathbb{B}^n$ satisfying

$$\|u\| = \sup_{x \in \mathbb{B}^n} (1 - |x|)|\nabla u(x)| < \infty.$$  

Then Corollary 2 says that

$$\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log \log \frac{1}{1-r}}} \leq 2\|u\|$$

for $\mathcal{H}_{n-1}$-a.e. $\zeta \in S^{n-1}$.

5. Hausdorff measures and radial growth

Let $\varphi$ and $\Phi$ be as in the Introduction. Take a positive non-decreasing function $\Psi$ on $[0,1)$ satisfying

$$\frac{\Phi(r) \log \log (1/(1-r))}{[\Psi'(r)]^2} \to 0 \quad \text{as } r \to 1.$$  

For $\lambda > 0$, consider the measure function

$$h_\lambda(t) = t^{n-1} \exp \left( 4^3 A^{-4} \lambda \frac{[\Psi(1-t)]^2}{\Phi(1-t)} \right).$$

We finally establish the following result.

**Theorem 3.** If $\lambda > 0$ and $u$ is as in Theorem 1, then

$$\limsup_{r \to 1} \frac{|u(r\zeta)|}{\Psi'(r)} \leq \lambda$$

for $\mathcal{H}_h$-a.e. $\zeta \in S^{n-1}$.

**Proof.** In view of Lemma 2, we see that

$$\int_{\mathbb{B}^n} (1 - |x|)^{-1} \left( \log \frac{2}{1 - |x|} \right)^{-2} \exp \left( c_1 |u(x)|^2 \frac{\Phi(|x|)}{\Phi(|x|)} \right) dx < \infty$$

for all $c_1$, $0 < c_1 < 1/4$. By the covering theorem, there exists a set $F \subset S^{n-1}$ such that $\mathcal{H}_h(F) = 0$ and
\[ \lim_{t \to 0} [h_t(5t)]^{-1} \int_{B(\zeta, t) \cap B^n} (1 - |x|)^{-1} \left( \log \frac{2}{1 - |x|} \right)^{-2} \exp \left( \frac{c_1 |u(x)|^2}{\Phi(|x|)} \right) \, dx = 0 \quad (12) \]

for \( \zeta \in S^{n-1} \setminus F \) and \( 0 < c_1 < 1/4 \) (cf. [6, Lemma 5.8.2]).

Fix \( \zeta \in S^{n-1} \setminus F \). For \( 0 < t < 1 \), write \( D_t = B((1 - t)\zeta + 4^{-1}t\zeta, 4^{-1}t) \). Since \( D_t \subset B(\zeta, t) \cap B^n \), we have

\[
[h_t(5t)]^{-1} \int_{B(\zeta, t) \cap B^n} (1 - |x|)^{-1} \left( \log \frac{2}{1 - |x|} \right)^{-2} \exp \left( \frac{c_1 |u(x)|^2}{\Phi(|x|)} \right) \, dx \geq [h_t(5t)]^{-1} \int_{D_t} (1 - |x|)^{-1} \left( \log \frac{2}{1 - |x|} \right)^{-2} \exp \left( \frac{c_1 |u(x)|^2}{\Phi(|x|)} \right) \, dx \geq [h_t(5t)]^{-1} |D_t|^{-1} \left( \log \frac{4}{t} \right)^{-2} \exp \left( \frac{c_1 \inf_{x \in D_t} |u(x)|^2}{\Phi(1 - t/2)} \right) \geq c_2 \exp \left( -4^3A^{-4} \lambda^2 \frac{[\Psi(1 - 5t)]^2}{\Phi(1 - 5t)} - 2 \log \log (1/t) + \frac{c_1 \inf_{x \in D_t} |u(x)|^2}{\Phi(1 - t/2)} \right),
\]

where \( c_2 \) is a positive constant. From (12), we obtain

\[
\frac{c_1 \inf_{x \in D_t} |u(x)|^2}{\Phi(1 - t/2)} \leq 4^3A^{-4} \lambda^2 \frac{[\Psi(1 - 5t)]^2}{\Phi(1 - 5t)} + 2 \log \log (1/t)
\]

for sufficiently small \( t > 0 \). By Lemma 3, we have

\[
\frac{c_1 \inf_{x \in D_t} |u(x)|^2}{4^{-1}A\Phi(1 - t) + \Phi(1/2)} \leq A^{-1} \lambda^2 \frac{[\Psi(1 - t)]^2}{\Phi(1 - t)} - c_3 + 2 \log \log (1/t)
\]

where \( c_3 = \Phi(1/2)((A/4)^3 + (A/4)^2 + (A/4)) \), which implies that

\[
\limsup_{t \to 0} \frac{\inf_{x \in D_t} |u(x)|^2}{\Psi(1 - t)^2} \leq \frac{1}{4c_1} \lambda^2.
\]

Hence it follows from Lemma 4 that

\[
\limsup_{t \to 0} \frac{|u((1 - t)\zeta)|}{\Psi(1 - t)} \leq \frac{\lambda}{\sqrt{4c_1}}.
\]

Letting \( c_1 \to 1/4 \), we obtain

\[
\limsup_{t \to 0} \frac{|u((1 - t)\zeta)|}{\Psi(1 - t)} \leq \lambda,
\]

which proves Theorem 3.
Remark 3. If we take $\varphi(r) = (1 - r)^{-2} \{\log(2/(1 - r))\}^{-\gamma}$, $\gamma < 1$, then the conclusion of Theorem 3 says that

$$\limsup_{r \to 1} \frac{|\mu(r\zeta)|}{\{\log(1/(1 - r))\}^{\frac{1}{2}}} \leq \lambda$$

for $\mathcal{H}_h$-a.e. $\zeta \in S^{n-1}$, where $2x > 1 - \gamma$ and

$$h(t) = t^{n-1} \exp[4^3 A^{-4} \lambda^2 \{\log(1/t)\}^{2x+y-1}]$$

Thus Theorem 3 can not cover [3, Theorem 1.3] by González-Koskela.

References


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