Hiroshima Math. J.
**34** (2004), 21–56

# Mean curvature 1 surfaces in hyperbolic 3-space with low total curvature I

*Dedicated to Katsuhiro Shiohama on the occasion of his sixtieth birthday*

Wayne Rossman, Masaaki Umehara and Kotaro Yamada

(Received May 9, 2003)
(Revised October 28, 2003)

**Abstract.** A complete surface of constant mean curvature 1 (CMC-1) in hyperbolic 3-space with constant curvature $-1$ has two natural notions of "total curvature"—one is the *total absolute curvature* which is the integral over the surface of the absolute value of the Gaussian curvature, and the other is the *dual total absolute curvature* which is the total absolute curvature of the dual CMC-1 surface. In this paper, we completely classify CMC-1 surfaces with dual total absolute curvature at most $4\pi$. Moreover, we give new examples and partially classify CMC-1 surfaces with dual total absolute curvature at most $8\pi$.

With the developments of the last decade on constant mean curvature 1 (CMC-1) surfaces in hyperbolic 3-space $H^3$ (the complete simply-connected 3-manifold of constant sectional curvature $-1$), and with so many examples now known, it is a natural next step to classify all such surfaces with low total absolute curvature.

As CMC-1 surfaces in $H^3$ share quite similar properties with minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$, let us first comment that the total absolute curvature of a minimal surface in $\boldsymbol{R}^3$ is equal to the area (counted with multiplicity) of the Gauss image of the surface, and that complete minimal surfaces in $\boldsymbol{R}^3$ with total curvature at most $8\pi$ have been classified. (See Lopez [6] and also Table 2.) Furthermore, as the Gauss map of a complete conformally parametrized minimal surface is holomorphic, and has a well-defined limit at each end when the surface has finite total curvature, the area of the Gauss image must be an integer multiple of $4\pi$.

The question of classifying low total curvature CMC-1 surfaces in $H^3$ is analogous—however, unlike minimal surfaces in $\boldsymbol{R}^3$, CMC-1 surfaces in $H^3$ have two Gauss maps: the hyperbolic Gauss map $G$ and the secondary Gauss map $g$. So there are two ways to pose the question in $H^3$, with two very

2000 *Mathematics Subject Classification*. Primary 53A10; Secondary 53A35, 53A42.
*Key words and phrases.* hyperbolic space, constant mean curvature, total curvature.

different answers. One way is to consider the true total absolute curvature, which is the area of the image of $g$, but since $g$ might not be single-valued on the surface, the total curvature might not be an integer multiple of $4\pi$. This allows for many more possibilities and makes the problem more difficult than for minimal surfaces in $\boldsymbol{R}^3$. The authors take up this question in a separate paper [13].

The second way, which is the theme of this paper, is to study the area of the image of $G$, which we call the *dual* total absolute curvature, as it is the true total curvature of the dual CMC-1 surface (which we define in Section 1) in $H^3$. This way has the advantage that $G$ is single-valued on the surface, and so the dual total curvature is always an integer multiple of $4\pi$, like the case of minimal surfaces in $\boldsymbol{R}^3$. Furthermore, the dual total curvature satisfies not only the Cohn-Vossen inequality, but also the hyperbolic analogue of the Osserman inequality (which cannot be said about the true total curvature) [19, 23] (see also (2.1) in Section 2). So the dual total curvature shares more properties with the total curvature of minimal surfaces in $\boldsymbol{R}^3$, motivating our interest in it.

In this paper, we classify CMC-1 surfaces with dual total absolute curvature at most $4\pi$, and we go much of the way toward classifying CMC-1 surfaces with dual total absolute curvature at most $8\pi$ (as a first step to a full classification of the $8\pi$ case). In Section 1, we give a summary of the results, and in Section 2 we give preliminaries for the latter sections. The classification of CMC-1 surfaces with dual total absolute curvature less than or equal to $4\pi$ is given in Section 3. Surfaces with dual total absolute curvature $8\pi$ are discussed in Section 4—and there we find new examples, we classify certain cases, and we show nonexistence in certain other cases. In Section 5, from deformations of corresponding minimal surfaces in $\boldsymbol{R}^3$, we produce two classes of new CMC-1 surfaces with dual total absolute curvature $8\pi$. For the readers' convenience, we attach Appendix A to explain the computation of log-term coefficients of second order linear ordinary differential equations with regular singularities.

**Acknowledgement**

The authors thank the referee for his careful reading of this paper and for his comments.

## 1. Summary of the results

To state our results precisely, we begin with some notations. Let $f: M \to H^3$ be a complete conformal CMC-1 immersion of a Riemann surface $M$ into $H^3$. By Bryant's representation formula, there is a holomorphic null immersion $F: \tilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$ such that $f = FF^*$, where $\tilde{M}$ is the universal cover of $M$ and $F^* = {}^t\overline{F}$. ("null" means $\det(F^{-1}\,dF) = 0$.) Here, we consider $H^3 = \mathrm{SL}(2,\boldsymbol{C})/\mathrm{SU}(2) = \{aa^* \mid a \in \mathrm{SL}(2,\boldsymbol{C})\}$ [1, 15]. We call $F$ the *lift* of $f$, and $F$ satisfies

$$dF = F\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\frac{Q}{dg} \tag{1.1}$$

on $\tilde{M}$, where $g$ (the *secondary Gauss map*) is a meromorphic function defined on $\tilde{M}$ and $Q$ (the *Hopf differential*) is a holomorphic 2-differential on $M$. Then the induced metric $ds^2$ and complexification of the second fundamental form $h$ are

$$ds^2 = (1+|g|^2)^2\left|\frac{Q}{dg}\right|^2, \qquad h = -Q - \bar{Q} + ds^2.$$

By (1.1), the secondary Gauss map satisfies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}, \qquad \text{where } F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}.$$

The map $g$ is determined uniquely up to a Möbius transformation $g \mapsto a \star g$ by $a \in \mathrm{SU}(2)$, where, for general $a = (a_{ij}) \in \mathrm{SL}(2,\boldsymbol{C})$, we denote

$$a \star g := \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}}.$$

The *hyperbolic Gauss map* $G$ of $f$ is defined by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}},$$

which can be interpreted as stereographic projection of the endpoints in the sphere at infinity of $H^3$ of the oriented normal geodesics emanating from the surface. In particular, $G$ is a meromorphic function on $M$.

The inverse matrix $F^{-1}$ is also a holomorphic null immersion, and produces a new CMC-1 immersion $f^\# = F^{-1}(F^{-1})^* : \tilde{M} \to H^3$, called the *dual* of $f$ [19]. The induced metric $ds^{2\#}$ and the Hopf differential $Q^\#$ of $f^\#$ are

$$ds^{2\#} = (1+|G|^2)^2\left|\frac{Q}{dG}\right|^2, \qquad Q^\# = -Q. \tag{1.2}$$

So $ds^{2\#}$ and $Q^\#$ are well-defined on $M$ itself, even though $f^\#$ might be defined only on $\tilde{M}$. This duality between $f$ and $f^\#$ interchanges the roles of the hyperbolic Gauss map $G$ and secondary Gauss map $g$. In particular, one has

$$dF \cdot F^{-1} = -(F^{-1})^{-1}d(F^{-1}) = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix}\frac{Q}{dG}. \tag{1.3}$$

Hence $dFF^{-1}$ is single-valued on $M$, whereas $F^{-1}\,dF$ generally is not.

Since $ds^{2\#}$ is single-valued on $M$, we can define the *dual total absolute curvature*

$$\mathrm{TA}(f^{\#}) := \int_M (-K^{\#}) dA^{\#},$$

where $K^{\#}$ $(\leq 0)$ and $dA^{\#}$ are the Gaussian curvature and area element of $ds^{2\#}$, respectively. As

$$d\sigma^{2\#} := (-K^{\#}) ds^{2\#} = \frac{4\, dG d\bar{G}}{(1+|G|^2)^2} \tag{1.4}$$

is a pseudo-metric of constant curvature 1 with developing map $G$, $\mathrm{TA}(f^{\#})$ is the area of the image of $G$ on $\boldsymbol{CP}^1 = S^2$. The following assertion is important for us:

LEMMA 1.1 ([19, 22]). *The Riemannian metric $ds^{2\#}$ is complete (resp. nondegenerate) if and only if $ds^2$ is complete (resp. nondegenerate).*

So from now on, we suppose $f$ is complete and has $\mathrm{TA}(f^{\#}) < +\infty$. By Lemma 1.1, the conformal metric $ds^{2\#}$ is complete. As $\mathrm{TA}(f^{\#}) < +\infty$, $M$ is biholomorphic to a compact Riemann surface $\overline{M}_\gamma$ of some genus $\gamma$ with finitely many points excluded [8, Theorem 9.1]:

$$M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \qquad (p_1, \dots, p_n \in \overline{M}_\gamma). \tag{1.5}$$

The points $p_j$ are called the *ends* of the immersion $f$.

If $G$ has an essential singularity at any end $p_j$, then $\mathrm{TA}(f^{\#}) = +\infty$, since $\mathrm{TA}(f^{\#})$ is the area of $G(M)$ in $\boldsymbol{CP}^1 = S^2$. Since we have assumed $\mathrm{TA}(f^{\#}) < +\infty$, $G$ is meromorphic on all of $\overline{M}_\gamma$. In particular, $\mathrm{TA}(f^{\#}) = 4\pi \deg G \in 4\pi\boldsymbol{Z}$.

Since the dual immersion has finite total curvature, the Hopf differential $Q^{\#} = -Q$ can be extended to $\overline{M}_\gamma$ as a meromorphic 2-differential [1, Proposition 5]. Let

$$d_j = \mathrm{ord}_{p_j} Q = \text{order of } Q \text{ at the end } p_j$$

for each $j = 1, \dots, n$. We say that $f$ is a surface of *type* $\boldsymbol{\Gamma}(d_1, \dots, d_n)$ if $M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$ and $Q$ has order $d_j$ at each end $p_j$. We use $\boldsymbol{\Gamma}$ because it is the capitalized letter corresponding to $\gamma$, the genus of $\overline{M}_\gamma$. For instance, the class $\mathbf{I}(-4)$ (resp. $\mathbf{O}(-2,-3)$) means the class of surfaces of genus 1 (resp. genus 0) with 1 end (resp. 2 ends) so that $Q$ has a pole of order 4 at the single end (resp. a pole of order 2 at one end and order 3 at the other). Then our results are shown in Table 1. In the table,

Table 1. CMC-1 surfaces in $H^3$ with $\mathrm{TA}(f^{\#}) \le 8\pi$. (The corresponding results for minimal surfaces in $\boldsymbol{R}^3$ are shown in Table 2.)

<table>
<tr><th>Type</th><th>TA($f^{\#}$)</th><th>Reducibility</th><th>Status</th><th>c.f.</th></tr>
<tr><td>$\mathbf{O}(0)$</td><td>0</td><td>$\mathscr{H}^3$-reducible</td><td>classified$^0$</td><td>Horosphere</td></tr>
<tr><td>$\mathbf{O}(-4)$</td><td>$4\pi$</td><td>$\mathscr{H}^3$-reducible</td><td>classified</td><td>Duals of Enneper cousins [10, Example 5.4]</td></tr>
<tr><td>$\mathbf{O}(-2,-2)$</td><td>$4\pi$</td><td>reducible</td><td>classified</td><td>Catenoid cousins and warped catenoid cousins with embedded ends [1, Example 2], [15], [13]</td></tr>
<tr><td>$\mathbf{O}(-5)$</td><td>$8\pi$</td><td>$\mathscr{H}^3$-reducible</td><td>classified</td><td>Theorem 4.14</td></tr>
<tr><td>$\mathbf{O}(-6)$</td><td>$8\pi$</td><td>$\mathscr{H}^3$-reducible</td><td>classified</td><td>Theorem 4.14</td></tr>
<tr><td>$\mathbf{O}(-2,-2)$</td><td>$8\pi$</td><td>reducible</td><td>classified</td><td>Double covers of catenoid cousins and warped catenoid cousins with $m=2$ in [15, Theorem 6.2], [13]</td></tr>
<tr><td>$\mathbf{O}(-1,-4)$</td><td>$8\pi$</td><td>$\mathscr{H}^3$-reducible</td><td>classified$^0$</td><td>Theorem 4.13</td></tr>
<tr><td>$\mathbf{O}(-2,-3)$</td><td>$8\pi$</td><td>$\mathscr{H}^1$-reducible</td><td>classified</td><td>Theorems 4.11, 4.12</td></tr>
<tr><td>$\mathbf{O}(-2,-4)$</td><td>$8\pi$</td><td>$\mathscr{H}^1$-reducible<br>$\mathscr{H}^3$-reducible</td><td>classified<br>classified</td><td>Theorem 4.9<br>Theorem 4.10</td></tr>
<tr><td>$\mathbf{O}(-3,-3)$</td><td>$8\pi$</td><td>reducible</td><td>existence</td><td>Proposition 4.8</td></tr>
<tr><td>$\mathbf{O}(-1,-1,-2)$</td><td>$8\pi$</td><td>$\mathscr{H}^3$-reducible</td><td>classified$^0$</td><td>Theorem 4.7</td></tr>
<tr><td>$\mathbf{O}(-1,-2,-2)$</td><td>$8\pi$</td><td>$\mathscr{H}^1$-reducible<br>$\mathscr{H}^3$-reducible</td><td>classified<br>classified</td><td>Theorem 4.5<br>Theorem 4.6</td></tr>
<tr><td>$\mathbf{O}(-2,-2,-2)$</td><td>$8\pi$</td><td>irreducible<br>$\mathscr{H}^1$-reducible<br>$\mathscr{H}^3$-reducible</td><td>classified<br>existence$^+$<br>existence$^+$</td><td>[20, Theorem 2.6]<br>Example 4.3<br>Example 4.4</td></tr>
<tr><td>$\mathbf{I}(-3)$</td><td>$8\pi$</td><td></td><td>unknown</td><td></td></tr>
<tr><td>$\mathbf{I}(-4)$</td><td>$8\pi$</td><td></td><td>existence</td><td>Proposition 4.2</td></tr>
<tr><td>$\mathbf{I}(-1,-1)$</td><td>$8\pi$</td><td></td><td>unknown$^+$</td><td>Proposition 4.1</td></tr>
<tr><td>$\mathbf{I}(-2,-2)$</td><td>$8\pi$</td><td></td><td>existence</td><td>Genus 1 catenoid cousins [9]</td></tr>
</table>

- *classified* means the complete list of the surfaces in such a class is known (and this means not only that we know all the possibilities for the form of the data $(G,Q)$, but that we also know exactly for which $(G,Q)$ the period problems of the immersions are solved).

Table 2. The classification of complete minimal surfaces in $\boldsymbol{R}^3$ with TA $\le 8\pi$ ([6]), for comparison with Table 1.

| Type | TA | The surface | c.f. |
|---|---|---|---|
| $\mathbf{O}(0)$ | 0 | Plane | |
| $\mathbf{O}(-4)$ | $4\pi$ | Enneper's surface | |
| $\mathbf{O}(-5)$ | $8\pi$ | | [6, Theorem 6] |
| $\mathbf{O}(-6)$ | $8\pi$ | | [6, Theorem 6] |
| $\mathbf{O}(-2,-2)$ | $4\pi$<br>$8\pi$ | Catenoid<br>Double cover of the catenoid | |
| $\mathbf{O}(-1,-3)$ | $8\pi$ | | [6, Theorem 5] |
| $\mathbf{O}(-2,-3)$ | $8\pi$ | | [6, Theorem 4, 5] |
| $\mathbf{O}(-2,-4)$ | $8\pi$ | | [6, Theorem 5] |
| $\mathbf{O}(-3,-3)$ | $8\pi$ | | [6, Theorem 4] |
| $\mathbf{O}(-1,-2,-2)$ | $8\pi$ | | [6, Theorem 5] |
| $\mathbf{O}(-2,-2,-2)$ | $8\pi$ | | [6, Theorem 5] |
| $\mathbf{I}(-4)$ | $8\pi$ | Chen-Gackstatter surface | [6, Theorem 5], [2] |

- *classified*$^0$ means there exists a unique surface (up to isometries of $H^3$ and deformations that come from its reducibility).
- *existence* means that examples exist, but they are not yet classified.
- *existence*$^+$ means that all possibilities for the data $(G, Q)$ are determined in this paper, but the period problems are solved only for special cases.
- *unknown* means that neither existence nor non-existence is known yet.
- *unknown*$^+$ means that all possibilities for the data $(G, Q)$ are determined in this paper, but the period problems are still unsolved.

Any class and type of reducibility not listed in Table 1 cannot contain surfaces with $\mathrm{TA}(f^{\#}) \le 8\pi$. For example, any irreducible or $\mathscr{H}^3$-reducible surface of type $\mathbf{O}(-2,-3)$ must have dual total absolute curvature at least $12\pi$. (See Section 2 for the definitions of irreducibility, $\mathscr{H}^1$-reducibility, and $\mathscr{H}^3$-reducibility.)

## 2. Preliminaries

Before we begin proving the results, we prepare some fundamental properties and tools, which will play important roles in the latter sections.

**Analogue of the Osserman inequality.** The second and third authors showed [19]:

$$\frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) \geq -\chi(M) + n = 2(\gamma + n - 1). \tag{2.1}$$

Moreover, equality holds exactly when all the ends are properly embedded: This follows by noting that equality is equivalent to all ends being regular and embedded ([19]), and that any properly embedded end must be regular (proved recently by Collin, Hauswirth and Rosenberg [4]).

**Formulas for** $\mathrm{TA}(f^{\#})$**.** Let $\mu_j^{\#} \in \boldsymbol{Z}$ be the branching order of $G$ at the end $p_j$ for each $j = 1, \dots, n$. Since $G$ is a $(\mu_j^{\#} + 1)$-to-1 mapping in a neighborhood of $p_j$,

$$\mu_j^{\#} \leq \deg G - 1 = \frac{1}{4\pi}\,\mathrm{TA}(f^{\#}) - 1. \tag{2.2}$$

The umbilic points of $f$ are the zeroes of $Q = -Q^{\#}$, which are also the umbilic points of $f^{\#}$. Moreover, the order of $Q$ equals the branching order of $G$ at each point in $M$, since $ds^{2\#}$ in (1.2) is non-degenerate. Let $q_1, \dots, q_k$ be the umbilic points of $f$ and set

$$\xi_l := \mathrm{ord}_{q_l} Q = [\text{the branching order of } G \text{ at } q_l] \qquad (l = 1, \dots, k). \tag{2.3}$$

The pseudometric $d\sigma^{2\#}$ in (1.4) is said to have order $\beta$ at $p$ if it is asymptotic to $|z - z(p)|^{2\beta} dz d\bar{z}$, where $z$ is a complex coordinate around $p$. Then the branching order of $G$ is equal to the order of the metric $d\sigma^{2\#}$ in (1.4), the Gauss-Bonnet theorem implies that

$$\frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) = \chi(\overline{M}_\gamma) + \sum_{j=1}^{n} \mu_j^{\#} + \sum_{l=1}^{k} \xi_l, \tag{2.4}$$

where $\chi(\cdot)$ is the Euler characteristic. (This also follows from the Riemann-Hurwitz formula, since $\overline{M}_\gamma$ is a branched cover of $S^2$ via the map $G$.)

Since $Q$ is a meromorphic 2-differential, the total order of $Q$ satisfies

$$\sum_{l=1}^{k} \xi_l + \sum_{j=1}^{n} d_j = -2\chi(\overline{M}_\gamma). \tag{2.5}$$

By (2.4) and (2.5), we have

$$\frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) = -\chi(\overline{M}_\gamma) + \sum_{j=1}^{n} (\mu_j^{\#} - d_j) = 2\gamma - 2 + \sum_{j=1}^{n} (\mu_j^{\#} - d_j). \tag{2.6}$$

Completeness of the metric $ds^{2\#}$ at $p_j$ implies $\mu_j^{\#} - d_j \geq 1$. However, the case $\mu_j^{\#} - d_j = 1$ cannot occur ([19, Lemma 3]), so

$$\mu_j^{\#} - d_j \geq 2. \tag{2.7}$$

**Effects of transforming the lift $F$.** Here we consider the change $\hat{F} = aFb^{-1}$ of the lift $F$, where $a, b \in \mathrm{SL}(2, \boldsymbol{C})$. Then $\hat{F}$ is also a holomorphic null immersion, and the hyperbolic Gauss map $\hat{G}$, the secondary Gauss map $\hat{g}$ and the Hopf differential $\hat{Q}$ of $\hat{F}$ are given by (see [17])

$$\hat{G} = a \star G, \qquad \hat{g} = b \star g, \qquad \hat{Q} = Q. \tag{2.8}$$

In particular, the change $\hat{F} = aF$ moves the surface by a rigid motion of $H^3$, and does not change $g$ and $Q$. By choosing a suitable rigid motion $a \in \mathrm{SL}(2, \boldsymbol{C})$ of the surface in $H^3$, we shall frequently use the following change of the hyperbolic Gauss map to simplify its expression:

$$\hat{G} = a \star G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}, \qquad (a_{ij})_{i,j=1,2} \in \mathrm{SL}(2, \boldsymbol{C}). \tag{2.9}$$

**The Schwarzian derivative relation.** A direct computation implies that the secondary Gauss map $g$ depends on $G$ and $Q$ as follows ([15]):

$$S(g) - S(G) = 2Q, \tag{2.10}$$

where

$$S(g) = \left[\left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2\right] dz^2 \qquad \left({}' = \frac{d}{dz}\right)$$

is the *Schwarzian derivative* of $g$. Here, $z$ is a complex coordinate of $\overline{M}_\gamma$.

**SU(2)-monodromy conditions.** Here we recall from [10] the construction of CMC-1 surfaces with given hyperbolic Gauss map $G$ and Hopf differential $Q$, which will play a crucial role in this paper. Let $\overline{M}_\gamma$ be a compact Riemann surface and $M := \overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$. Let $G$ and $Q$ be a meromorphic function and meromorphic 2-differential on $\overline{M}_\gamma$. The pair $(G, Q)$ must satisfy the following two compatibility conditions:

(2.11) For all $q \in M$, $\mathrm{ord}_q\, Q$ is equal to the branching order of $G$, and

$$\text{for each end } p_j, \mu_j^{\#} - d_j \geq 2. \tag{2.12}$$

The first condition implies that the metric $ds^{2\#}$ is (and hence $ds^2$ is also, by Lemma 1.1) non-degenerate at $q \in M$. The second condition implies that the metric $ds^{2\#}$ is complete (and hence $ds^2$ is also, again by Lemma 1.1) at $p_j \in \overline{M}_\gamma$ $(j = 1, \dots, n)$.

For such a pair $(G, Q)$, a solution $g$ of equation (2.10) has singularities at the branch points of $G$ (umbilic points or ends) and the poles of $Q$ (ends). However, regardless of whether $q \in M$ is a regular or umbilic point, $ds^{2\#}$ and $Q^{\#}$ as in (1.2) give a (non-degenerate) Riemann metric and holomorphic 2-differential in a neighborhood $U_q \subset M$ of $q$. Then, by the fundamental theorem of surfaces, there exists a CMC-1 immersion $f^{\#}$ of $U_q$ into $H^3$ with induced metric $ds^{2\#}$ and Hopf differential $Q^{\#}$. So the hyperbolic Gauss map $g$ of $f^{\#}$, which is a solution of (2.10), is a well-defined meromorphic function on $U_q$. Since the solution of (2.10) is unique up to Möbius transformations $g \mapsto a \star g$ $(a \in \mathrm{SL}(2, \boldsymbol{C}))$, for any solution $g$ of (2.10) defined on the universal cover $\tilde{M}$ of $M$, there exists a representation

$$\rho_g : \pi_1(M) \to \mathrm{PSL}(2) \qquad \text{such that} \qquad g \circ \tau^{-1} = \rho_g(\tau) \star g$$

for each covering transformation $\tau \in \pi_1(M)$.

We now consider when the dual $f = (f^{\#})^{\#}$ (with data $(G, Q)$) of $f^{\#}$ is well-defined on $M$. Choosing $F$ so that $F^{-1}$ is a lift of $f^{\#}$ (and then also $(F^{-1})^{-1} = F$ is a lift of $(f^{\#})^{\#} = f$), and noting that the representation $\rho_g : \pi_1(M) \to \mathrm{PSL}(2, \boldsymbol{C})$ can be lifted into $\mathrm{SL}(2, \boldsymbol{C})$ [10], (2.8) implies

$$F^{-1} \circ \tau^{-1} = \rho_g(\tau) F^{-1} \tag{2.13}$$

for each $\tau \in \pi_1(M)$. Thus

$$f \circ \tau^{-1} = (F \circ \tau^{-1})(F \circ \tau^{-1})^* = F(\rho_g(\tau))^{-1}((\rho_g(\tau))^{-1})^* F^*, \tag{2.14}$$

and so $f$ is well-defined on $M$ if $\rho_g(\tau) \in \mathrm{SU}(2)$ for all $\tau \in \pi_1(M)$. This is the crux of the following Lemma 2.1. Before stating this lemma, we need a definition:

Definition 1. A CMC-1 immersion $f : M \to H^3$ is *reducible* if $\{\rho_g(\tau)\}_{\tau \in \pi_1(M)}$ are simultaneously diagonalizable (i.e. if there exists a $P \in \mathrm{PSL}(2, \boldsymbol{C})$ such that $P \rho_g(\tau) P^{-1}$ is diagonal for all $\tau \in \pi_1(M)$). If $f$ is not reducible, it is called *irreducible*. When $f$ is reducible, it is either $\mathscr{H}^3$-*reducible* or $\mathscr{H}^1$-*reducible* [10], and $f$ is called $\mathscr{H}^3$-reducible if $\{\rho_g(\tau)\}_{\tau \in \pi_1(M)}$ are all the identity, and is called $\mathscr{H}^1$-reducible otherwise.

Clearly $f$ is $\mathscr{H}^3$-reducible if and only if the lift $F$ itself is single-valued on $M$, by (2.13). The name $\mathscr{H}^1$-reducibility (resp. $\mathscr{H}^3$-reducibility) comes from the fact that the surface has exactly a 1 (resp. 3) dimensional deformation through surfaces preserving $G$ and $Q$ and the mean curvature, which is identified with the 1 (resp. 3) dimensional hyperbolic space $\mathscr{H}^1$ (resp. $\mathscr{H}^3$) [10]. On the other hand, if $f$ is irreducible, $f$ has no deformation preserving mean curvature and $(G, Q)$ (see [17, 10]).

LEMMA 2.1 ([17]). *Let $G$ and $Q$ be a meromorphic function and a meromorphic* 2*-differential on $\overline{M}_\gamma$ satisfying* (2.11) *and* (2.12). *Assume $g$ is a solution of* (2.10) *such that the image of $\rho_g$ lies in* $\mathrm{PSU}(2)$. *Then there exists a complete* CMC-1 *immersion $f : M \to H^3$ with hyperbolic Gauss map $G$, Hopf differential $Q$, and secondary Gauss map $g$.*

*If $f$ is irreducible, then $f$ is the unique surface with data $(G, Q)$. If $f$ is $\mathscr{H}^1$-reducible (resp. $\mathscr{H}^3$-reducible), then there exists exactly a* 1 *(resp.* 3*) parameter family of* CMC-1 *surfaces with data $(G, Q)$.*

In the case that $M$ is of genus $\gamma = 0$ with at most two ends, $f$ is reducible, as the fundamental group is commutative. More generally, for the case $\gamma = 0$ with $n$ ends, by Lemma 2.1 and the theory of linear ordinary differential equations (see Appendix A), we have:

PROPOSITION 2.2. *Let $\overline{M}_0 = \boldsymbol{C} \cup \{\infty\}$ and $M = \overline{M}_0 \backslash \{p_1, \dots, p_n\}$ with $p_1, \dots, p_{n-1} \in \boldsymbol{C}$. Let $G$ and $Q$ be a meromorphic function and a meromorphic* 2*-differential on $\boldsymbol{C} \cup \{\infty\}$ satisfying* (2.11) *and* (2.12). *Consider the linear ordinary differential equation*

$$\frac{d^2u}{dz^2} + r(z)u = 0, \tag{E.0}$$

*where $r(z)dz^2 := (S(G)/2) + Q$. Suppose $n \geq 2$, and also $d_j = \mathrm{ord}_{p_j} Q \geq -2$ and the indicial equation of* (E.0) *at $z = p_j$ has the two roots $\lambda_1^{(j)}, \lambda_2^{(j)}$ and log-term coefficient $c_j$, for $j = 1, 2, \dots, n-1$.*

(1) *Suppose that $\lambda_1^{(j)} - \lambda_2^{(j)} \in \boldsymbol{Z}^+$ and $c_j = 0$ for $j \leq n-1$. Then there is exactly a* 3*-parameter family of complete conformal* CMC-1 *immersions of $M$ into $H^3$ with hyperbolic Gauss map $G$ and Hopf differential $Q$. Moreover, such surfaces are $\mathscr{H}^3$-reducible.*
(2) *Suppose that $\lambda_1^{(j)} - \lambda_2^{(j)} \in \boldsymbol{Z}^+$ and $c_j = 0$ for $j \leq n-2$, and that $\lambda_1^{(n-1)} - \lambda_2^{(n-1)} \in \boldsymbol{R} \backslash \boldsymbol{Z}$. Then there exists exactly a* 1*-parameter family of complete conformal* CMC-1 *immersions of $M$ into $H^3$ with hyperbolic Gauss map $G$ and Hopf differential $Q$. Moreover, such surfaces are $\mathscr{H}^1$-reducible.*

*Here we denoted by $\boldsymbol{Z}^+$ the set of positive integers.*

The ordinary differential equation (E.0) has also been applied in [7] for constructing certain classes of $\mathscr{H}^3$-reducible CMC-1 surfaces.

PROOF. The general theory of Schwarzian derivatives shows ([21, Chapter 4]) that for a linearly independent pair $u_1, u_2$ of solutions of (E.0), the function $g := u_1/u_2$ satisfies (2.10). Conversely, any function $g$ satisfying $S(g) = 2r(z)dz^2$ is obtained in this way.

If $\lambda_1^{(j)} - \lambda_2^{(j)} = m \in \boldsymbol{Z}^+$ and $c_j = 0$, then there is a fundamental system of solutions of (E.0) in a neighborhood of $p_j$ of the form

$$u_1 = (z - p_j)^{\lambda_1^{(j)}} \varphi_1(z), \qquad u_2 = (z - p_j)^{\lambda_1^{(j)} - m} \varphi_2(z), \tag{2.15}$$

where $\varphi_1(z)$ and $\varphi_2(z)$ are holomorphic and nonzero at $z = p_j$. Then $g := u_1/u_2$ satisfies

$$g \circ \tau_j^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \star g, \tag{2.16}$$

where $\tau_j$ is the covering transformation which corresponds to a small loop around $z = p_j$, implying $\rho_g(\tau_j) = \text{identity}$. So for case (1), we have $\rho_g(\tau_j) = \text{identity}$ for all $j = 1, \dots, n-1$, and therefore also for $j = n$, which implies that $g$ is a meromorphic function on $\boldsymbol{C} \cup \{\infty\}$. By Lemma 2.1, there exists a conformal CMC-1 immersion $f_a$ on $M$ with the secondary Gauss map $a \star g$ for all $a \in \mathrm{SL}(2, \boldsymbol{C})$. If $a \in \mathrm{SU}(2)$, then $f_a$ coincides with $f_{\text{identity}}$ by (2.14), so we have that the 3-parameter family $(f_{[a]})_{[a] \in \mathrm{SL}(2,\boldsymbol{C})/\mathrm{SU}(2)}$ are complete conformal CMC-1 immersions with hyperbolic Gauss map $G$ and Hopf differential $Q$.

We remark here that if $\lambda_1^{(j)} - \lambda_2^{(j)} = m \in \boldsymbol{Z}^+$ and $c_j \neq 0$, then the monodromy matrix $\rho_g(\tau_j)$ defined by $g \circ \tau_j^{-1} = \rho_g(\tau_j) \star g$ is not diagonalizable and is not even in $\mathrm{SU}(2)$. So any CMC-1 immersion on $\tilde{M}$ (with $G$ and $Q$) cannot be well-defined on $M$ when some $c_j \neq 0$.

Next we consider case (2), that is $\lambda_1^{(n-1)} - \lambda_2^{(n-1)} \notin \boldsymbol{Z}$. There exists a fundamental system of solutions of (E.0) of the form

$$u_1 = (z - p_{n-1})^{\lambda_1^{(n-1)}} \varphi_1(z), \qquad u_2 = (z - p_{n-1})^{\lambda_2^{(n-1)}} \varphi_2(z), \tag{2.17}$$

where $\varphi_1(z)$ and $\varphi_2(z)$ are holomorphic and nonzero at $z = p_{n-1}$. When $\tau_{n-1}$ is the covering transformation induced from a small loop about $z = p_{n-1}$, $g := u_1/u_2$ satisfies

$$g \circ \tau_{n-1}^{-1} = \begin{bmatrix} e^{\pi i(\lambda_1^{(n-1)} - \lambda_2^{(n-1)})} & 0 \\ 0 & e^{\pi i(\lambda_2^{(n-1)} - \lambda_1^{(n-1)})} \end{bmatrix} \star g. \tag{2.18}$$

In particular, $\rho_g(\tau_{n-1}) \in \mathrm{SU}(2)$. On the other hand, in the proof of (1), we have seen that $\rho_g(\tau_j) = \text{identity}$ for $j \in (1, \dots, n-2)$. Hence $\rho_g(\tau_j) \in \mathrm{SU}(2)$ and are diagonal matrices for all $j \in (1, \dots, n)$, and we are in the $\mathscr{H}^1$-reducible case. Note that this remains true when $g$ is replaced by

$$sg(z) = a(s) \star g, \qquad \text{where } a(s) := \begin{bmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{bmatrix}, \text{ with } s \in \boldsymbol{R}^+,$$

where $\boldsymbol{R}^+$ is the set of positive reals. So we have a one-parameter family of complete conformal CMC-1 immersions with hyperbolic Gauss map $G$ and

Hopf differential $Q$ and secondary Gauss maps $sg$ for $s \in \boldsymbol{R}^+$. ($s_1 g$ and $s_2 g$ for $s_1 \neq s_2$ will not produce equivalent surfaces, as $a(s_1)(a(s_2))^{-1} \notin \mathrm{SU}(2)$.) Furthermore, Lemma 2.1 implies there is *only* a one-parameter family of CMC-1 immersions with data $(G, Q)$. $\square$

By (1.3), we have

$$(F^{-1})^{-1} d(F^{-1}) = \begin{pmatrix} g^{\#} & -g^{\#2} \\ 1 & -g^{\#} \end{pmatrix} \omega^{\#},$$

where

$$g^{\#} = G, \qquad \omega^{\#} = -\frac{Q}{dG}.$$

By Lemma 2.1 of [15] (replacing $F$ with $F^{-1}$), we have that $X = F_{21}(z), F_{22}(z)$ satisfies the equation

$$X'' - (\log(\hat{\omega}^{\#}))' X' + \hat{Q} X = 0, \tag{E.1$^{\#}$}$$

and $Y = F_{11}(z), F_{12}(z)$ satisfies the equation

$$Y'' - (\log(G^2 \hat{\omega}^{\#}))' Y' + \hat{Q} Y = 0, \tag{E.2$^{\#}$}$$

where $Q(z) = \hat{Q}(z) dz^2$ and $\omega^{\#} = \hat{\omega}^{\#}(z) dz$. (We call them (E.1)$^{\#}$ and (E.2)$^{\#}$ because they are the dual versions of equations (E.1) and (E.2) in [15].) These two equations have been shown in [23] as a modification of the corresponding equations in [15]. As we will see later, equations (E.1)$^{\#}$ and (E.2)$^{\#}$ are sometimes more convenient than equation (E.0) for solving monodromy problems. In fact, we will have use for the following lemma:

LEMMA 2.3. *Let $G$ and $Q$ be a meromorphic function and a holomorphic* 2-*differential on $D^* = \{z \in \boldsymbol{C}; 0 < |z| < 1\}$ such that the metric $ds^{2\#}$ defined by* (1.2) *is positive definite on $D^*$ and complete at* 0. *Assume* $\mathrm{ord}_{z=0}\, Q \geq -2$ *and $Q$ is not identically zero. Then the following three conditions are all equivalent.*

(1) *The difference of the solutions of the indicial equation of* (E.1)$^{\#}$ *at $z = 0$ is a positive integer and the log-term coefficient of* (E.1)$^{\#}$ *vanishes.*

(2) *The difference of the solutions of the indicial equation of* (E.2)$^{\#}$ *at $z = 0$ is a positive integer and the log-term coefficient of* (E.2)$^{\#}$ *vanishes.*

(3) *The difference of the solutions of the indicial equation of* (E.0) *at $z = 0$ is a positive integer and the log-term coefficient of* (E.0) *vanishes.*

PROOF. The hyperbolic Gauss map of the dual surface $f^{\#} = F^{-1}(F^{-1})^*$ is equal to the secondary Gauss map $g$ of $f = FF^*$. Thus conditions (1) and (2) are equivalent to the condition that $g$ is single valued at $z=0$, by Lemma 2.2 of [15]. On the other hand, as seen in the proof of Proposition 2.2, condition (3) is also equivalent to the condition that $g$ is single valued at $z=0$. $\square$

Here is a natural place to include the next lemma, which we shall use in the sequel [13], to this paper.

Lemma 2.4. *With the same assumptions as in Lemma* 2.3, *the following three conditions are all equivalent.*

(1) *The difference of the solutions of the indicial equation of* $(\mathrm{E.1})^{\#}$ *at* $z=0$ *is a real number.*
(2) *The difference of the solutions of the indicial equation of* $(\mathrm{E.2})^{\#}$ *at* $z=0$ *is a real number.*
(3) *The difference of the solutions of the indicial equation of* (E.0) *at* $z=0$ *is a real number.*

Proof. We write

$$G(z)=z^{\mu}\hat{G}(z), \qquad \omega^{\#}(z)=z^{\nu}\hat{\omega}_0^{\#}(z)dz,$$

where $\hat{G}$ and $\hat{\omega}_0^{\#}$ are nonzero and holomorphic at $z=0$, for some integers $\mu$ and $\nu$.

If $\mathrm{ord}_{z=0}\,Q=-2$, so $\mu+\nu=-1$ and $Q=(\theta z^{-2}+\cdots)dz^2$ for some $\theta\neq 0$, then the difference of the solutions of the indicial equations is $\sqrt{\mu^2-4\theta}$ in all three cases, hence the three statements are clearly equivalent.

If $\mathrm{ord}_{z=0}\,Q\geq -1$, then the indicial equation in the first case (resp. second case, third case) is

$$t(t-1)-\nu t=0, \qquad \left(\text{resp. } t(t-1)-(2\mu+\nu)t=0,\ t(t-1)+\frac{1-\mu^2}{4}=0\right).$$

Hence the difference of the roots is $|\nu+1|$ (resp. $|2\mu+\nu+1|, |\mu|$), and so all three statements hold. $\square$

## 3. The classification of surfaces with $\mathrm{TA}(f^{\#})\leq 4\pi$

We begin our consideration of classification with this simple case:

Theorem 3.1. *A complete* CMC-1 *immersion* $f$ *with* $\mathrm{TA}(f^{\#})\leq 4\pi$ *is congruent to one of the following:*

(1) *a horosphere,*
(2) *an Enneper cousin dual,* $(g,Q)=(\tan\sqrt{\theta}z,\theta\,dz^2)$ $(\theta\in\boldsymbol{C}\setminus\{0\})$,
(3) *a catenoid cousin,*

$$(g,Q)=\left(az^{\mu},\frac{1-\mu^2}{4z^2}\,dz^2\right) \qquad (a\in\boldsymbol{R}^{+},\ \mu\in\boldsymbol{R}^{+}\setminus\{1\}),$$

(4) *a warped catenoid cousin that has a degree* 1 *hyperbolic Gauss map,*

$$(g, Q) = \left(az^l + b, \frac{1-l^2}{4z^2}\, dz^2\right) \qquad (a, b \in \boldsymbol{C}\setminus\{0\},\ l \in \boldsymbol{Z}^+\setminus\{1\}).$$

PROOF. Since $\mathrm{TA}(f^{\#}) \in 4\pi\boldsymbol{Z}$, we need to consider only the cases $\mathrm{TA}(f^{\#}) = 0$ and $4\pi$. If $\mathrm{TA}(f^{\#}) = 0$, then the hyperbolic Gauss map is constant, so (1.4) implies $K^{\#} \equiv 0$. Thus $f^{\#}$ is a totally umbilic CMC-1 immersion, so both $f^{\#}$ and $f$ are horospheres. So we consider the remaining case $\mathrm{TA}(f^{\#}) = 4\pi$. Then $G$ is meromorphic of degree 1 on $\overline{M}_\gamma$, which implies $\gamma = 0$. Hence we may choose $\overline{M}_0 = \boldsymbol{C} \cup \{\infty\}$, and by (2.9), we may assume $G = z$. Since $G$ has no branch points, (2.3) implies there are no umbilic points, and (2.2) implies

$$\mu_j^{\#} = 0 \tag{3.1}$$

at each end $p_j$. By (2.6) and (3.1) and the fact that $\gamma = 0$, we have

$$2 = \frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) = -2 - \sum_{j=1}^{n} d_j. \tag{3.2}$$

By (2.7), we have $2 \geq -2 + 2n$, so $n = 1$ or $2$.

**The case** $n = 1$**.** In this case, (3.2) implies $d_1 = -4$. We may put the end at $p_1 = \infty$, and then $Q$ has a single pole of order 4 at $\infty$ and no zeroes. Thus $Q = \theta\, dz^2$ for some $\theta \in \boldsymbol{C}\setminus\{0\}$.

A CMC-1 surface in $H^3$ with secondary Gauss map $g = z$ and Hopf differential $Q = \theta\, dz^2$ is called an Enneper cousin [1]. So a surface with data $(G, Q) = (z, \theta\, dz^2)$ is the dual of an Enneper cousin [10, Example 5.4]. (Recall that dualizing switches the two Gauss maps, and changes the Hopf differential only by a sign.)

**The case** $n = 2$**.** In this case, (3.2) becomes $4 = -d_1 - d_2$. Then $d_j = -2$ $(j = 1, 2)$, by (2.7). Hence the immersion $f$ is a CMC-1 surface of genus 0 whose two ends must both be regular [15], and this type of surface is classified in [15]. In particular, $f$ is in the case $m = 1$ of Theorem 6.2 in [15]. So the surface is either a *catenoid cousin* [1, Example 2] or a *warped catenoid cousin* with embedded ends (the case $m = 1$ in Theorem 6.2 in [15]). $\square$

The warped catenoid cousins are described in detail in [13].

## 4. Surfaces with $\mathrm{TA}(f^{\#}) = 8\pi$

We now assume $f$ has $\mathrm{TA}(f^{\#}) = 8\pi$. Then, by (2.6) and (2.7),

$$6 = 2\gamma + \sum_{j=1}^{n} (\mu_j^{\#} - d_j) \geq 2(\gamma + n) \tag{4.1}$$

holds. Thus the possible cases are

$$(\gamma, n) = (0,1), (0,2), (0,3), (1,1), (1,2), \text{ and } (2,1).$$

Since $\mathrm{TA}(f^{\#}) = 8\pi$, $G$ is meromorphic on $\overline{M}_\gamma$ of degree 2. Hence (2.2) implies

$$\mu_j^{\#} \leq 1 \qquad (j = 1, 2, \ldots, n), \tag{4.2}$$

and at each umbilic point $q_l$,

$$\xi_l = 1 \qquad (l = 1, 2, \ldots, k). \tag{4.3}$$

**The case** $(\gamma, n) = (2, 1)$**.** Since equality holds in (2.1), the single end $p_1$ is embedded. By (4.1), $\mu_1^{\#} - d_1 = 2$. Thus the possible cases are

$$(\mu_1^{\#}, d_1) = (0, -2) \text{ or } (1, -1),$$

by (4.2). If $(\mu_1^{\#}, d_1) = (0, -2)$, the end $p_1$ is of type I in the sense of [11], so the flux about this end does not vanish [11, Proposition 2]. If $(\mu_1^{\#}, d_1) = (1, -1)$, then, since the end is embedded, Corollary 5 in [11] implies that the flux about the end again does not vanish. But non-vanishing flux at a single end contradicts the balancing formula [11, Theorem 1], so the case $(\gamma, n) = (2, 1)$ does not occur.

**The case** $(\gamma, n) = (1, 2)$**.** In this case, (4.1) implies $4 = (\mu_1^{\#} - d_1) + (\mu_2^{\#} - d_2)$. By (2.7), we have $\mu_j^{\#} - d_j = 2$ for $j = 1, 2$. Hence (4.2) implies

$$(\mu_j^{\#}, d_j) = (0, -2) \text{ or } (1, -1) \qquad (j = 1, 2).$$

Assume $d_1 = -2$ and $d_2 = -1$. Then, by the transformation (2.9) if necessary, we may assume the hyperbolic Gauss map has a zero or pole at each end. In this case, the end $p_1$ is regular of type I, and $p_2$ is regular of type II in the sense of [11], contradicting Theorem 7 in [11]. Hence this case is impossible, leaving the two remaining possibilities:

$$(\mu_1^{\#}, d_1) = (\mu_2^{\#}, d_2) = (0, -2), \tag{4.4}$$

$$(\mu_1^{\#}, d_1) = (\mu_2^{\#}, d_2) = (1, -1). \tag{4.5}$$

For the case (4.4), the first author and Sato [9] constructed a one-parameter family of "genus one catenoid cousins". Note that such surfaces cannot exist as minimal surfaces in $\boldsymbol{R}^3$, by Schoen's result [14].

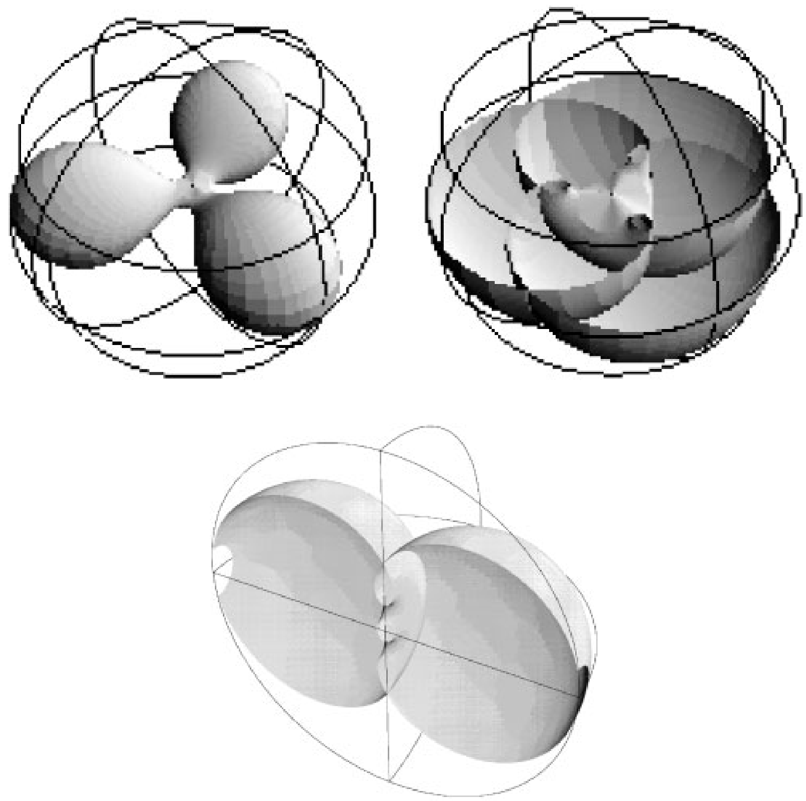

**Fig. 1.** Two CMC-1 trinoids in $H^3$, which are surfaces of type $\mathbf{O}(-2,-2,-2)$, and a genus 1 catenoid cousin, which is a surface of type $\mathbf{I}(-2,-2)$, shown in the Poincaré model of $H^3$. Only one of two congruent pieces of the right-most two surfaces is shown, and the other half of each surface is the reflection (in the plane containing the boundary curves seen here) of the piece shown.

**Surfaces of type $\mathbf{I}(-1,-1)$.** For the case (4.5), we can determine the candidates of $(G,Q)$ explicitly as follows (however, the period problem is unsolved and no example is known):

PROPOSITION 4.1. *Let $\overline{M}_1 = \boldsymbol{C}/\Gamma$, where $\Gamma$ is a lattice on $\boldsymbol{C}$, and assume there exists a* CMC-1 *immersion $f : \overline{M}_1\backslash\{p_1,p_2\} \to H^3$ with $\mathrm{TA}(f^{\#}) = 8\pi$ of type $\mathbf{I}(-1,-1)$. Then, after a suitable rigid motion of $H^3$, there exists a generating pair $\{v_1,v_2\} \subset \boldsymbol{C}$ of $\Gamma$ such that the hyperbolic Gauss map $G$ and Hopf differential $Q$ of $f$ are given by*

$$(4.6)\quad G = \wp(z), \qquad Q(z) = \theta\frac{\sigma(z-v_1/2)\sigma(z-v_2/2)}{\sigma(z)\sigma(z-(v_1+v_2)/2)}\,dz^2 \qquad (\theta\in\boldsymbol{C}\backslash\{0\}),$$

*where* $\wp(z)$ *is the Weierstrass* $\wp$*-function and* $\sigma$ *is the entire function defined by*

$$\sigma(z) := z \prod_{v \in \Gamma \setminus \{0\}} \left\{ \left(1 - \frac{z}{v}\right) \exp\left(\frac{z}{v} + \frac{z^2}{2v^2}\right) \right\}.$$

Proof. In this case, the hyperbolic Gauss map $G$ is of degree 2. Without loss of generality, we may assume that $z = 0$ is an end of the surface. Moreover, by (2.9) we may assume that $z = 0$ is a pole of $G$. As $z = 0$ is a branch point of $G$ (since $\mu_j^{\#} = 1$), $G$ has a pole of order 2 at $z = 0$. Up to a constant multiple and an additive constant, the function $\wp(z)$ is uniquely characterized as a degree 2 meromorphic function on $\boldsymbol{C}/\Gamma$ with a pole of order 2 at the origin [5]. Thus we have $G(z) = c\wp(z) + b$, and we can normalize $c = 1$ and $b = 0$, by (2.9).

Suppose $\{v_1, v_2\}$ generates $\Gamma$. Then the branch points of $\wp$ are 0, $v_1/2$, $v_2/2$ and $(v_1 + v_2)/2$ modulo $\Gamma$, which are the ends and umbilic points. We assume 0 and $(v_1 + v_2)/2$ are the ends. (If $v_1/2$ is an end, for example, we may change the generator $\Gamma$ to $\{\tilde{v}_1 = v_1 - v_2, \tilde{v}_2 = v_2\}$.) Thus the umbilic points are $v_1/2$ and $v_2/2$.

Next we find the Hopf differential $Q(z) = q(z)dz^2$, using the following fact:

Fact ([5]). *Let* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *be points in* $\boldsymbol{C}$ *such that* $a_j \neq b_k$ (mod $\Gamma$), $j, k \in \{1, \dots, n\}$, *and* $\sum_{j=1}^n a_j = \sum_{k=1}^n b_k$ (mod $\Gamma$). *Then*

$$f(z) := \theta \frac{\sigma(z - a_1) \dots \sigma(z - a_n)}{\sigma(z - b_1) \dots \sigma(z - b_n)} \qquad (\theta \in \boldsymbol{C} \setminus \{0\})$$

*is a meromorphic function on* $\boldsymbol{C}/\Gamma$ *such that* $\{a_1, \dots, a_n\}$ *(resp.* $\{b_1, \dots, b_n\}$*) are the set of zeroes (resp. poles), i.e. the divisor of* $f$ *is* $a_1 + \dots + a_n - b_1 - \dots - b_n$. *Conversely, any elliptic function on* $\boldsymbol{C}/\Gamma$ *with the same divisor is of this form.*

The meromorphic function $q(z)$ should have poles of order 1 at $z = 0$, $(v_1 + v_2)/2$ (ends) and zeroes of order 1 at $z = v_1/2, v_2/2$ (umbilic points). Thus $Q(z)$ can be written as in (4.6). □

**The case** $(\gamma, n) = (1, 1)$**.** By (4.1) and (4.2), we have two possible cases:

$$(\mu_1^{\#}, d_1) = (0, -4) \text{ or } (1, -3).$$

The second of these cases (the $\mathbf{I}(-3)$ case) is still unknown, but for the first case $\mathbf{I}(-4)$, the following proposition provides examples, proven (in Section 5) by deforming from a complete minimal surface in $\boldsymbol{R}^3$ of genus 1 with one end satisfying $d_1 = -4$.

Proposition 4.2. *By deforming the Chen-Gackstatter surface in* $\boldsymbol{R}^3$ [2], *one* *obtains a one-parameter family of* CMC-1 *surfaces of type* $\mathbf{I}(-4)$ *with dual total absolute curvature* $8\pi$.

**The case** $(\gamma, n) = (0, 3)$**.** Here, (4.1) and (2.7) imply $\mu_j^\# - d_j = 2$ for $j = 1, 2, 3$. Moreover, (2.5) implies $d_1 + d_2 + d_3 \le -4$. So (4.2) implies that the possibilities are:

Type $\mathbf{O}(-2, -2, -2) : (d_1, d_2, d_3) = (-2, -2, -2)$ and $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (0, 0, 0)$,

Type $\mathbf{O}(-1, -2, -2) : (d_1, d_2, d_3) = (-1, -2, -2)$ and $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (1, 0, 0)$,

Type $\mathbf{O}(-1, -1, -2) : (d_1, d_2, d_3) = (-1, -1, -2)$ and $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (1, 1, 0)$.

In each case, equality holds in (2.1), so all ends are embedded. Since the genus of the surface is 0, we can set $\overline{M}_0 = \boldsymbol{C} \cup \{\infty\}$.

**Surfaces of type** $\mathbf{O}(-2, -2, -2)$**.** Such surfaces have three embedded ends with $d_j = -2$ $(j = 1, 2, 3)$, and the irreducible ones are classified in [20, Theorem 2.6]. So here we consider the reducible case.

We may set $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty$. By (2.5) and (4.3), there are two distinct umbilic points $q_1$ and $q_2$ of order 1. Then the Hopf differential $Q$ must have simple zeroes at $q_1$ and $q_2$ and poles of order 2 at $0, 1$ and $\infty$. Since all three $\mu_j^\# = 0$, $q_1$ and $q_2$ are the only branch points of $G$. Also, $G(q_1)$, $G(q_2)$, and $G(\infty)$ are all distinct, because $q_1$ and $q_2$ are double points of $G$ and $\deg G = 2$. Then, by (2.9), we can set $G(q_1) = 0$, $G(q_2) = \infty$, and $G(\infty) = 1$. Thus $G$ and $Q$ are written as

$$(4.7) \qquad G = \left(\frac{z - q_1}{z - q_2}\right)^2, \qquad Q = \theta \frac{(z - q_1)(z - q_2)}{z^2 (z - 1)^2}\, dz^2 \qquad (\theta \in \boldsymbol{C} \setminus \{0\}).$$

EXAMPLE 4.3 ($\mathscr{H}^1$-reducible examples of type $\mathbf{O}(-2, -2, -2)$). For $s \in \boldsymbol{R}$ such that

$$(4.8) \qquad -4 \frac{1 + 4s + s^2}{1 + 10s + s^2} \in \boldsymbol{R} \setminus \boldsymbol{Z},$$

let

$$(4.9) \qquad q_1 = \frac{1 + 10s + s^2}{4s(1 - s)}, \qquad q_2 = \frac{1 + 10s + s^2}{4(s - 1)}, \qquad \text{and} \qquad \theta = -\frac{3}{4 q_1 q_2}.$$

Consider (E.0) for $r(z) dz^2 = (S(G)/2) + Q$, with $G$ and $Q$ determined by (4.7) and (4.9). Then the roots of the indicial equation of (E.0) at $z = 0$ are $-1/2$ and $3/2$, so their difference is $2 \in \boldsymbol{Z}$, and one can check by (A.15) that the log-term coefficient vanishes. Moreover, the difference of the roots of the indicial equation at $z=1$ equals the value in (4.8). Hence, by (2) of Proposition 2.2, there exists an $\mathscr{H}^1$-reducible CMC-1 immersion $f : \boldsymbol{C}\setminus\{0,1\} \to H^3$ with $G$ and $Q$ as in (4.7) and (4.9). Since each surface is $\mathscr{H}^1$-reducible (this follows from the fact that the difference of the roots of the indicial equation is an integer at $z=0$ and not an integer at $z=1$), there exists a one-parameter family of CMC-1 surfaces for each $s$, with this $G$ and $Q$. Thus, we have found a 2-parameter family of $\mathscr{H}^1$-reducible CMC-1 surfaces of type $\mathbf{O}(-2,-2,-2)$.

Example 4.4 ($\mathscr{H}^3$-reducible examples of type $\mathbf{O}(-2,-2,-2)$). For $m \geq 2$, $m \in \boldsymbol{Z}$, let

$$q_1 = \frac{1}{2}\left(1+\frac{1}{\sqrt{m}}\right), \qquad q_2 = \frac{1}{2}\left(1-\frac{1}{\sqrt{m}}\right), \qquad \text{and} \qquad \theta = -m(m+1).$$

Then a meromorphic function $g$ on $\boldsymbol{C}\cup\{\infty\}$ such that

$$dg = z^{m-1}(z-1)^{m-1}(z-q_1)(z-q_2)dz$$

satisfies equation (2.10) for $G$ and $Q$ as in (4.7). Since $g$ is meromorphic, $\rho_g(\tau)$ is the identity for all $\tau \in \pi_1(\boldsymbol{C}\setminus\{0,1\})$, so Lemma 2.1 implies there exists an $\mathscr{H}^3$-reducible CMC-1 immersion $f : \boldsymbol{C}\setminus\{0,1\} \to H^3$ whose hyperbolic Gauss map, Hopf differential, and secondary Gauss map are $G, Q$, and $g$, respectively.

**Surfaces of type $\mathbf{O}(-1,-2,-2)$.** In this case, we will see that there is a 2-parameter family of $\mathscr{H}^1$-reducible surfaces, and countably many $\mathscr{H}^3$-reducible families. By (2.5), there exists one umbilic point of order 1. Without loss of generality, we can set the ends to be $(p_1,p_2,p_3)=(0,1,p)$ $(p \in \boldsymbol{C}\setminus\{0,1\})$ and the umbilic point to be $q_1=\infty$. Then the Hopf differential $Q$ has a pole of order 2 (resp. order 1) at $z=1,p$ (resp. $z=0$) and has no zeroes on $\boldsymbol{C}$, so it has the form

$$Q = \frac{\theta\, dz^2}{z(z-1)^2(z-p)^2} \qquad (\theta \in \boldsymbol{C}\setminus\{0\}).$$

By (2.3) and the fact $\mu_1^{\#}=1$, $G$ has branch points of order 1 at $z=0$ and $\infty$. Then, by (2.9), we may assume $G=z^2$, because $\deg G = 2$. Consider the ordinary differential equation (E.0) with $r(z)dz^2 = (S(G)/2)+Q$. At the singularity $z=0$, $r(z)$ expands as

$$r(z) = -\frac{3}{4}\frac{1}{z^2}+\frac{\theta}{p^2}\frac{1}{z}+\frac{2\theta(p+1)}{p^3}+O(z).$$

Thus the difference of the roots of the indicial equation of (E.0) at $z=0$ is 2.

Then, by (A.15), the log-term coefficient of (E.0) at $z = 0$ vanishes if and only if $\theta = -2p(p+1)$. Hence, if such a surface exists, $G$ and $Q$ are

$$G = z^2, \qquad Q = \frac{-2p(p+1)}{z(z-1)^2(z-p)^2}\, dz^2 \qquad (p \in \boldsymbol{C}\setminus\{0,1\}). \tag{4.10}$$

For $G$ and $Q$ as in (4.10), $r(z)$ expands at the singularity $z = 1$ as

$$r(z) = \frac{-2p(p+1)}{(1-p)^2} \frac{1}{(z-1)^2} + O((z-1)^{-1}).$$

Then the roots of the indicial equation of (E.0) at $z = 1$ are

$$\lambda_1 = 2 + \frac{2}{p-1}, \qquad \lambda_2 = -1 - \frac{2}{p-1}.$$

So $\lambda_1 - \lambda_2 \in \boldsymbol{Z}$ exactly when $4/(p-1) \in \boldsymbol{Z}$. Then, by Proposition 2.2, we have

THEOREM 4.5. *Let $p \in \boldsymbol{R}$ such that $p \neq 1$ and $4/(p-1) \notin \boldsymbol{Z}$. Then there exists a conformal $\mathscr{H}^1$-reducible* CMC-1 *immersion $f : M = \boldsymbol{C} \cup \{\infty\}\setminus\{0,1,p\} \to H^3$ with $\mathrm{TA}(f^{\#}) = 8\pi$ and hyperbolic Gauss map and Hopf differential as in* (4.10). *Moreover, all $\mathscr{H}^1$-reducible surfaces with $\mathrm{TA}(f^{\#}) = 8\pi$ of type* $\mathbf{O}(-1,-2,-2)$ *are given in this manner.*

The above discussion yields that all CMC-1 surfaces of type $\mathbf{O}(-1,-2,-2)$ are reducible. So it only remains to classify the $\mathscr{H}^3$-reducible case:

THEOREM 4.6. *Let $r \geq 3$ be an integer and $p = (r+2)/(r-2)$. Then there exists a conformal $\mathscr{H}^3$-reducible* CMC-1 *immersion $f : M = \boldsymbol{C} \cup \{\infty\}\setminus \{0,1,p\} \to H^3$ with $\mathrm{TA}(f^{\#}) = 8\pi$ whose hyperbolic Gauss map and Hopf differential are as in* (4.10). *Moreover, all $\mathscr{H}^3$-reducible surfaces with* $\mathrm{TA}(f^{\#}) = 8\pi$ *of type* $\mathbf{O}(-1,-2,-2)$ *are given in this manner.*

PROOF. For given $r \geq 3$, there is a meromorphic function $g$ on $\boldsymbol{C} \cup \{\infty\}$ so that

$$dg = \frac{z(z-p)^{r-2}}{(z-1)^{r+2}}\, dz, \tag{4.11}$$

since the right-hand side of (4.11) has no residue. One can check that $S(g) - S(G) = 2Q$ when $p = (r+2)/(r-2)$. Hence, by Lemma 2.1, there exists an $\mathscr{H}^3$-reducible CMC-1 immersion $f : \boldsymbol{C} \cup \{\infty\}\setminus\{0,1,p\} \to H^3$ with $G$ and $Q$ as in (4.10) and secondary Gauss map $g$ satisfying (4.11).

Conversely, let $f : \boldsymbol{C} \cup \{\infty\}\setminus\{0,1,p\} \to H^3$ be an $\mathscr{H}^3$-reducible CMC-1 immersion of type $\mathbf{O}(-1,-2,-2)$ with $\mathrm{TA}(f^{\#}) = 8\pi$. Then $G$ and $Q$ are as in (4.10). Let $m_2$ (resp. $m_3$) be the difference of the roots of the indicial equation of (E.0) at $z=1$ (resp. $z=p$) for such $G$ and $Q$. Then we have $m_2 = |3+(4/(p-1))|$ and $m_3 = |1+(4/(p-1))|$. Since $f$ is $\mathscr{H}^3$-reducible, $m_2$ and $m_3$ are positive integers (so also $4/(p-1)\in \boldsymbol{Z}$). We may assume $m_2 \geq m_3$. (If not, we can exchange the two ends $p$ and 1, by changing $p$ and $z$ to $1/p$ and $z/p$. Using (2.9), we see that (4.10) is unchanged.)

Suppose that $m_2 = m_3 = 1$, then $g$ is not branched at both 1 and $p$. Noting that the branching orders of $g$ and $G$ are equal at any finite point of the surface (this follows from equation (2.10)), we see that $g$ has branch points of order 1 at 0 and $\infty$ and no other branch points. So $g$ has degree 2 and $g = a \star z^2$ for some $a \in \mathrm{SL}(2,\boldsymbol{C})$ and so $Q = (1/2)(S(g)-S(G)) = 0$, which is impossible.

Thus $m_2 \geq 2$, and it follows that $4/(p-1)$ is a positive integer. By setting $r = 2+(4/(p-1)) \geq 3$, we have

$$m_2 = 3+\frac{4}{p-1} = r+1, \qquad m_3 = 1+\frac{4}{p-1} = r-1, \qquad \text{and} \qquad p = \frac{r+2}{r-2}.$$

Thus $G$ and $Q$ are as in (4.10) with $p = (r+2)/(r-2)$. $\square$

**Surfaces of type $\mathbf{O}(-1,-1,-2)$.** In this case, by (2.5), the surface has no umbilic points. We set the ends $(p_1,p_2,p_3) = (0,1,\infty)$. The Hopf differential is then

$$Q = \frac{\theta\, dz^2}{z(z-1)}, \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.12}$$

The hyperbolic Gauss map $G$ is a meromorphic function on $\boldsymbol{C}\cup\{\infty\}$ of degree 2 with branch points of order 1 at $z=0$ and $z=1$. Hence we may set

$$G = \left(\frac{z-1}{z}\right)^2. \tag{4.13}$$

Theorem 4.7. *Any complete* CMC-1 *immersion that is of type* $\mathbf{O}(-1,-1,-2)$ *with* $\mathrm{TA}(f^{\#}) = 8\pi$ *is congruent to an* $\mathscr{H}^3$*-reducible* CMC-1 *immersion* $f: M = \boldsymbol{C}\setminus\{0,1\} \to H^3$ *with hyperbolic Gauss map and Hopf differential*

$$G = \left(\frac{z-1}{z}\right)^2, \qquad Q = \frac{-2\, dz^2}{z(z-1)}.$$

Proof. Consider equation (E.0) for $G$ and $Q$ in (4.13) and (4.12) respectively. Then the roots of the indicial equations of (E.0) are $-1/2$ and $3/2$ at both $z=0$ and $z=1$. By (A.15), the log-term coefficients at $z=0$ and at $z=1$ both vanish if and only if $\theta = -2$. By Proposition 2.2, the corresponding 3-

parameter family of CMC-1 immersions consists of immersions that are all well-defined on $M = \boldsymbol{C}\setminus\{0,1\}$ and are $\mathscr{H}^3$-reducible. $\square$

**The case** $(\gamma, n) = (0, 2)$**.** In this case, (4.1) and (2.7) imply that

$$(\mu_1^\# - d_1, \mu_2^\# - d_2) = (2, 4) \qquad \text{or} \qquad (\mu_1^\# - d_1, \mu_2^\# - d_2) = (3, 3).$$

Then, by (4.2), all possibilities are:

Type $\mathbf{O}(-2,-4) : (d_1, d_2) = (-2,-4)$ and $(\mu_1^\#, \mu_2^\#) = (0, 0)$,

Type $\mathbf{O}(-2,-3) : (d_1, d_2) = (-2,-3)$ and $(\mu_1^\#, \mu_2^\#) = (0, 1)$ or $(1, 0)$,

Type $\mathbf{O}(-1,-4) : (d_1, d_2) = (-1,-4)$ and $(\mu_1^\#, \mu_2^\#) = (1, 0)$,

Type $\mathbf{O}(-1,-3) : (d_1, d_2) = (-1,-3)$ and $(\mu_1^\#, \mu_2^\#) = (1, 1)$,

Type $\mathbf{O}(-3,-3) : (d_1, d_2) = (-1,-3)$ and $(\mu_1^\#, \mu_2^\#) = (0, 0)$,

Type $\mathbf{O}(-2,-2) : (d_1, d_2) = (-2,-2)$ and $(\mu_1^\#, \mu_2^\#) = (1, 1)$.

Since the surface has genus 0, we can set $\overline{M}_0 = \boldsymbol{C} \cup \{\infty\}$ and $M = \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, p_2\}$. Since $\pi_1(M)$ is commutative, all surfaces of these types are reducible.

**Surfaces of type** $\mathbf{O}(-3,-3)$**.** There exists a minimal surface in $\boldsymbol{R}^3$ of class $\mathbf{O}(-3,-3)$ with total absolute curvature $8\pi$ [6]. The following is proven in Section 5:

PROPOSITION 4.8. *By deforming the minimal surface of type* $\mathbf{O}(-3,-3)$ *in* $\boldsymbol{R}^3$*, one obtains a one-parameter family of* CMC-1 *surfaces of type* $\mathbf{O}(-3,-3)$ *with dual total absolute curvature* $8\pi$.

**Surfaces of type** $\mathbf{O}(-2,-4)$**.** In this case, by (2.5) and (4.3), such a surface has two distinct umbilic points of order 1. We may set the ends to be $(p_1, p_2) = (0, \infty)$ and the umbilic points to be $(q_1, q_2) = (1, q)$, $q \in \boldsymbol{C}\setminus\{0, 1\}$, on $\boldsymbol{C} \cup \{\infty\}$. Then we may assume

$$G = \left(\frac{z-q}{z-1}\right)^2, \qquad Q = \frac{\theta(z-1)(z-q)}{z^2}\, dz^2 \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.14}$$

For such $G$ and $Q$, the roots of the indicial equation of (E.0) at $z = 0$ are

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{1 - 4\theta q}), \qquad \lambda_2 = \frac{1}{2}(1 - \sqrt{1 - 4\theta q}).$$

Then, by (2) of Proposition 2.2, we have

Theorem 4.9. *Let $\theta \in \boldsymbol{C}\setminus\{0\}$ and $q \in \boldsymbol{C}\setminus\{0,1\}$ be complex numbers such that*

$$\sqrt{1-4\theta q} \in \boldsymbol{R}\setminus\boldsymbol{Z}.$$

*Then there exists a conformal $\mathscr{H}^1$-reducible* CMC-1 *immersion $f : \boldsymbol{C}\setminus\{0\} \to H^3$ of type $\mathbf{O}(-2,-4)$ with $\mathrm{TA}(f^\#) = 8\pi$ whose hyperbolic Gauss map and Hopf differential are as in* (4.14). *Moreover, all $\mathscr{H}^1$-reducible surfaces with $\mathrm{TA}(f^\#) = 8\pi$ of type $\mathbf{O}(-2,-4)$ are given in this manner.*

It only remains to consider the $\mathscr{H}^3$-reducible case:

Theorem 4.10. *Let $s \in \boldsymbol{R}$ such that $\sqrt{1-4s} \geq 2$ is an integer. Then there exists at least* 1 *and at most $\sqrt{1-4s}$ conformal $\mathscr{H}^3$-reducible* CMC-1 *immersions $f : \boldsymbol{C}\setminus\{0\} \to H^3$ of type $\mathbf{O}(-2,-4)$ with $\mathrm{TA}(f^\#) = 8\pi$ whose hyperbolic Gauss map and Hopf differential are as in* (4.14). *Moreover, all $\mathscr{H}^3$-reducible surfaces with $\mathrm{TA}(f^\#) = 8\pi$ of type $\mathbf{O}(-2,-4)$ are given in this manner.*

Proof. For $G$ and $Q$ in (4.14), equation $(\mathrm{E.1})^\#$ becomes

$$z^2X'' + z\left\{2 + \frac{4z}{1-z}\right\}X' + \{\theta(z-1)(z-q)\}X = 0. \tag{4.15}$$

By Lemma 2.3 and Proposition 2.2, it is enough to show that there exists data $(G, Q)$ such that the difference of the roots of the indicial equation of (4.15) at $z = 0$ is an integer and the log-term vanishes.

The coefficients of (4.15) expand as

$$z\left\{2 + \frac{4z}{1-z}\right\} = z\left\{2 + 4\sum_{j=1}^{\infty} z^j\right\} \qquad \text{and}$$

$$\theta(z-1)(z-q) = \theta q - \theta(1+q)z + \theta z^2$$

for $z$ sufficiently close to 0. Assume the roots $\lambda_1, \lambda_2$ of the indicial equation of (4.15) satisfy $\lambda_1 - \lambda_2 = m \in \boldsymbol{Z}^+$. Then

$$s := \theta q = \frac{1-m^2}{4} \qquad \text{and} \qquad \lambda_2 = -\frac{m+1}{2} \qquad (m \geq 2). \tag{4.16}$$

Let

$$\mu_j = \begin{cases} \dfrac{1}{j(m-j)} & (j = 1, 2, \dots, m-1) \\ -\dfrac{1}{m} & (j = m) \end{cases}.$$

Then by Proposition A.3 in Appendix A, the log-term coefficient $c$ of (4.15) is given by $c = a_m$, where

$$a_0 = 1, \qquad a_1 = \mu_1\left(\frac{1}{4}(m+1)(m-9) - \theta\right),$$

$$\begin{aligned} a_j = \mu_j\Biggl[&\left(\sum_{k=0}^{j-2}(4k-2m-2)a_k\right) + \theta a_{j-2} \\ &+ \left(\frac{1}{4}(m+1)(m-9) - 4 + 4j - \theta\right)a_{j-1}\Biggr] \qquad (j = 2, \dots, m). \end{aligned} \tag{4.17}$$

Hence $a_j$ is a polynomial in $\theta$ of order $j$. We now define $t_0 = 1$, and we define $t_j$ and $u_j$ for $j = 1, \dots, m$ by the relations

$$a_j = t_j\theta^j + u_j\theta^{j-1} + \cdots \qquad (j = 1, 2, \dots, m).$$

It follows that $t_j = -\mu_j t_{j-1}$, and hence $t_m \neq 0$. Then, defining $\Lambda_j := u_j/t_j$, we also have

$$\Lambda_j = \Lambda_{j-1} - \frac{(m+1)(m-9)}{4} - 4j + 4 + (j-1)(m-j+1)$$

for $j = 2, \dots, m$. Since $\Lambda_1 = -(m+1)(m-9)/4$, we have

$$\Lambda_m = \sum_{j=2}^{m}\left[-\frac{(m+1)(m-9)}{4} - 4j + 4 + (j-1)(m-j+1)\right] = \frac{m}{12}(49 - m^2).$$

If the only roots of the polynomial

$$c = t_m\theta^m + u_m\theta^{m-1} + \cdots = t_m(\theta^m + \Lambda_m\theta^{m-1} + \cdots) = 0$$

with respect to $\theta$ are 0 and $(1-m^2)/4 < 0$, then it follows that $\Lambda_m$ would be nonnegative. However, $\Lambda_m < 0$ for all $m \geq 8$, hence this polynomial must have some root $\theta \in \boldsymbol{C}\setminus\{0, (1-m^2)/4\}$, and then $q = (1-m^2)/(4\theta) \in \boldsymbol{C}\setminus\{0, 1\}$. For this $\theta$ and $q$, we have $c = 0$, and thus we have at least one surface for each $m \geq 8$. Since $c$ is a polynomial of degree $m$ in $\theta$, there are at most $m$ roots, and hence at most $m$ surfaces.

For $m \leq 7$, one can check by explicitly computing the polynomial for $c$ that there is always at least one root $\theta \in \boldsymbol{C}\setminus\{0, (1-m^2)/4\}$. $\square$

**Surfaces of type $\mathbf{O}(-2,-3)$ with $\mu_1^\# = 0$.** Here, by (2.5), there exists only one umbilic point of order 1. We set the ends to be $(p_1, p_2) = (1, \infty)$ and the umbilic point to be $q_1 = 0$. We may assume

$$G = z^2, \qquad Q = \frac{\theta z\, dz^2}{(z-1)^2} \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.18}$$

Then the roots of the indicial equation of (E.0) at $z = 1$ are

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{1-4\theta}), \qquad \lambda_2 = \frac{1}{2}(1 - \sqrt{1-4\theta}).$$

Hence, by Proposition 2.2, we have

Theorem 4.11. *Let $\theta \in \boldsymbol{R}$ such that $\sqrt{1-4\theta} \in \boldsymbol{R}\setminus\boldsymbol{Z}$. Then there exists a conformal $\mathscr{H}^1$-reducible* CMC-1 *immersion $f : \boldsymbol{C}\setminus\{1,\infty\} \to H^3$ of type* $\mathbf{O}(-2,-3)$ *with* $\mathrm{TA}(f^{\#}) = 8\pi$ *whose hyperbolic Gauss map and Hopf differential are as in* (4.18). *Moreover, all $\mathscr{H}^1$-reducible surfaces of type* $\mathbf{O}(-2,-3)$ *with* $(\mu_1^{\#}, \mu_2^{\#}) = (0,1)$ *and* $\mathrm{TA}(f^{\#}) = 8\pi$ *are given in this manner.*

Now we will show that there are no $\mathscr{H}^3$-reducible surfaces of this type, by showing that the log-term coefficient at $z = 1$ of $(\mathrm{E}.1)^{\#}$ cannot be zero. With the data as in (4.18), equation $(\mathrm{E}.1)^{\#}$ becomes

$$(z-1)^2 X'' + 2(z-1)X' + \theta(1 + (z-1))X = 0,$$

and so $p_0 = 2$, $q_0 = q_1 = \theta$, $p_j = 0$ for $j \geq 1$, and $q_j = 0$ for $j \geq 2$, where the $p_j$ and $q_j$ are as defined in (A.3). Therefore, by Proposition A.3, we have $c = -\theta^m/(m!(m-1)!) \neq 0$.

**Surfaces of type $\mathbf{O}(-2,-3)$ with $\mu_1^{\#} = 1$.** In this case, we set the ends to be $(p_1, p_2) = (0, \infty)$ and the only umbilic point to be $q_1 = 1$. Then we may assume

$$G = \left(\frac{z-1}{z}\right)^2, \qquad Q = \frac{\theta(z-1)dz^2}{z^2} \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.19}$$

Thus the roots of the indicial equation of (E.0) at $z = 0$ are

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{4+4\theta}), \qquad \lambda_2 = \frac{1}{2}(1 - \sqrt{4+4\theta}).$$

So, by Proposition 2.2, we have

Theorem 4.12. *Let $\theta \in \boldsymbol{R}$ such that $\sqrt{4+4\theta} \in \boldsymbol{R}\setminus\boldsymbol{Z}$. Then there exists a conformal $\mathscr{H}^1$-reducible* CMC-1 *immersion $f : \boldsymbol{C}\setminus\{0\} \to H^3$ of type* $\mathbf{O}(-2,-3)$ *with* $\mathrm{TA}(f^{\#}) = 8\pi$ *whose hyperbolic Gauss map and Hopf differential are as in* (4.19). *Moreover, all $\mathscr{H}^1$-reducible surfaces of type* $\mathbf{O}(-2,-3)$ *with* $(\mu_1^{\#}, \mu_2^{\#}) = (1,0)$ *and* $\mathrm{TA}(f^{\#}) = 8\pi$ *are given in this manner.*

Now we will show that there are no $\mathscr{H}^3$-reducible surfaces of this type as well, again by showing that a log-term coefficient cannot be zero. With $G$ and $Q$ as in (4.19), equation $(\mathrm{E}.1)^{\#}$ becomes

$$z^2X'' - zX' + \theta(z-1)X = 0,$$

and so $p_0 = -1$, $-q_0 = q_1 = \theta$, $p_j = 0$ for $j \geq 1$, and $q_j = 0$ for $j \geq 2$, where the $p_j$ and $q_j$ are as defined in (A.3). Hence again, by Proposition A.3, we have $c \neq 0$.

**Surfaces of type O$(-1,-4)$.** We set the ends to be $(p_1, p_2) = (0, 1)$ and the single umbilic point to be $q_1 = \infty$, then we may assume

$$G = z^2, \qquad Q = \frac{\theta\, dz^2}{z(z-1)^4} \qquad (\theta \in \boldsymbol{C}\backslash\{0\}). \tag{4.20}$$

The roots of the indicial equation of (E.0) for such $G$ and $Q$ at $z=0$ are $3/2$ and $-1/2$. Then, by Lemma A.15, the log-term coefficient at $z=0$ vanishes if and only if $\theta = -4$. Thus

THEOREM 4.13. *Any complete* CMC-1 *immersion of type* $\mathbf{O}(-1,-4)$ *with* $\mathrm{TA}(f^{\#}) = 8\pi$ *is congruent to an* $\mathscr{H}^3$*-reducible* CMC-1 *immersion* $f : M = \boldsymbol{C} \cup \{\infty\}\backslash\{0,1\} \to H^3$ *with hyperbolic Gauss map and Hopf differential*

$$G = z^2, \qquad Q = \frac{-4\, dz^2}{z(z-1)^4}.$$

**Surfaces of type O$(-1,-3)$.** In this case, there are no umbilic points, by (2.5). Then, if we set the ends to be $(p_1, p_2) = (0, \infty)$, we may assume

$$G = z^2, \qquad Q = \frac{\theta}{z}\, dz^2 \qquad (\theta \in \boldsymbol{C}\backslash\{0\}).$$

The roots of the indicial equation of (E.0) at $z=0$ are $3/2$ and $-1/2$, and the log-term coefficient vanishes if and only if $\theta = 0$, by (A.15). So this case is impossible.

**Surfaces of type O$(-2,-2)$.** This is the case of genus zero CMC-1 surfaces with 2 regular ends. Such surfaces are completely classified in [15, Theorem 6.2] see also [13, Theorem 2.1]. As seen in [13], a complete CMC-1 immersion $f : \boldsymbol{C}\backslash\{0\} \to H^3$ is an $m$-fold cover of a catenoid cousin, or a warped catenoid cousin with

$$g = \frac{m^2 - l^2}{4l} z^l + b, \qquad \omega = z^{-l-1}\, dz \qquad (l, m \in \boldsymbol{Z}^+,\ l \neq m,\ b \geq 0). \tag{4.21}$$

(Here, we use "$m$" instead of the "$\delta$" in [13], following the notation of [15].) In both of these two cases, the degree of the hyperbolic Gauss map $G$ is $m$. Hence a complete CMC-1 surface with $\mathrm{TA}(f^{\#}) = 8\pi$ is a double cover of a catenoid cousin, or a warped catenoid cousin with $m = 2$ in (4.21).

**The case** $(\gamma, n) = (0, 1)$**.** In this case, we can set $M = \boldsymbol{C}$. Since $M$ is simply connected, we have no period problem. By (4.1) and (4.2), $d_1 = -5$ or $-6$.

For the case of $\mathbf{O}(-5)$, there is one umbilic point, which we may suppose is at $q_1 = 0$. By (4.1), we have $\mu_1^{\#} = 1$, so we may assume

$$G = z^2, \qquad Q = \theta z\, dz^2 \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.22}$$

For the case of $\mathbf{O}(-6)$, there are two umbilic points of order 1. Without loss of generality, we can set them to be $(q_1, q_2) = (0, 1)$. So, since $\mu_1^{\#} = 0$, we may assume

$$G = \left(\frac{z-1}{z}\right)^2, \qquad Q = \theta z(z-1) dz^2 \qquad (\theta \in \boldsymbol{C}\setminus\{0\}). \tag{4.23}$$

Theorem 4.14. *A* CMC-1 *surface of genus zero with one end such that* $\mathrm{TA}(f^{\#}) = 8\pi$ *is congruent to an immersion* $f : \boldsymbol{C} \to H^3$ *with hyperbolic Gauss map and Hopf differential as in* (4.22) *or* (4.23). *Moreover, such a surface is* $\mathscr{H}^3$*-reducible.*

## 5. Deformation of minimal surfaces to CMC-1 surfaces

In this section, we prove Propositions 4.2 and 4.8. For this, we will need a method from [10] that produces a 1-parameter family of CMC-1 surfaces in $H^3$ from a corresponding minimal surface in $\boldsymbol{R}^3$, so we describe that method first.

We start with a complete minimal surface $f_0 : M \to \boldsymbol{R}^3$ of finite total curvature. We require the immersion to be symmetric in the following sense, a condition that generically eliminates virtually all minimal surfaces, but eliminates none of the better known surfaces, which all have symmetries:

> **Symmetry condition:** There is a disk $D \subset M$ so that $f_0(D)$ is bounded by non-straight planar geodesics.

If $f_0$ is symmetric with respect to a disk $D$, then $f_0(D)$ generates the full surface by reflections across planes containing the boundary planar geodesics of $\partial f_0(D)$, by the Schwarz reflection principle [8]. Since the surface has finite total curvature, it is shown in [10] that the boundary $\partial f_0(D)$ is contained entirely in only either one plane $P_1$, or in two intersecting planes $P_1, P_2$, or

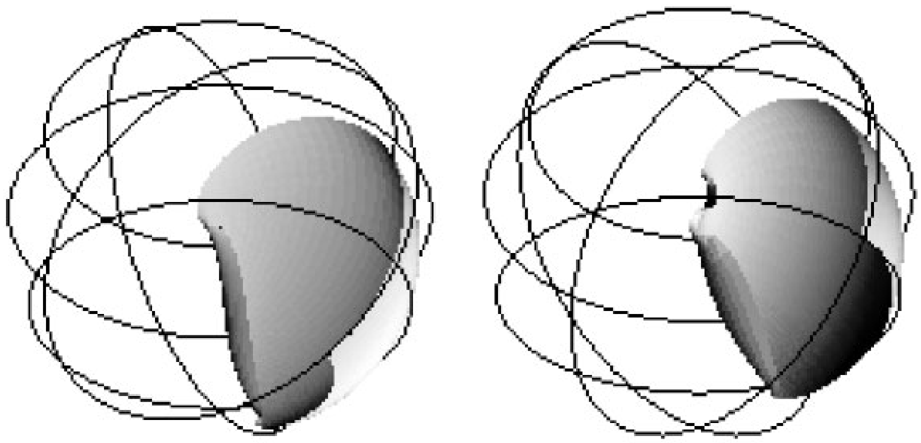

**Fig. 2.** Genus 0 and genus 1 Enneper cousin duals. Each surface has a single end that triply wraps around its limiting point at the south pole of the sphere at infinity. These surfaces are of type $\mathbf{O}(-4)$ and $\mathbf{I}(-4)$, and have $\mathrm{TA}(f^{\#}) = 4\pi$ and $\mathrm{TA}(f^{\#}) = 8\pi$. In both cases only one of four congruent pieces (bounded by planar geodesics) of the surface is shown.

in three planes $P_1, P_2$, and $P_3$ in general position. Let the boundary planar geodesics of $f_0(D)$ contained in $P_j$ be called $S_{j,1}, S_{j,2}, \dots, S_{j,\delta_j}$ ($j = 1, \dots, s$, for $s = 1, 2$ or $3$).

We now define non-degeneracy of the period problems. Let $\delta$ be the number of $S_{j,l}$ minus the number of planes ($\delta = \delta_1 + \delta_2 + \delta_3 - 3$ if $s = 3$, $\delta = \delta_1 + \delta_2 - 2$ if $s = 2$, and $\delta = \delta_1 - 1$ if $s = 1$).

**Nondegeneracy condition:** There exists a continuous $\delta$-parameter family of minimal disks $f_{0,v}(D)$ (where $v$ lies in a small neighborhood of the origin $\vec{0} \in \boldsymbol{R}^\delta$) such that

(1) $f_{0,\vec{0}}(D) = f_0(D)$.

(2) $\partial f_{0,v}(D) = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{\delta_j} S_{j,l}(v))$ holds, and each $S_{j,l}(v)$ is a planar geodesic lying in a plane $P_{j,l}(v)$ parallel to $P_j$.

(3) Letting $\mathrm{Per}_{j,l}(v)$ ($j = 1, \dots, s, l = 2, \dots, \delta_j$) be the oriented distance between the plane $P_{j,l}(v)$ and $P_{j,1}(v)$, the map from $v$ in $\boldsymbol{R}^\delta$ to $(\mathrm{Per}_{j,l}(v))$ in $\boldsymbol{R}^\delta$ is an open map onto a small neighborhood of $\vec{0} \in \boldsymbol{R}^\delta$.

Reflecting the minimal disk $f_{0,v}(D)$ along its boundary, $f_{0,v}$ is extended as a minimal immersion $f_{0,v} : \tilde{M} \to \boldsymbol{R}^3$, where $\tilde{M}$ is the universal cover of $M$. We denote by $G_v$ and $Q_v$ the Gauss map and the Hopf differential of the minimal surface $f_{0,\mu}$, respectively. Then by solving the equation (1.3) with $G = G_v$ and $Q = cQ_v$ ($c \in \boldsymbol{R}$), we have a CMC-1 imersion $f_{c,v} = FF^* : \tilde{M} \to H^3$ with the hyperbolic Gauss map $G_v$ and the Hopf differential $cQ_v$. If $f_{0,v}$ satisfies the above conditions, one can take a one parameter family $v = v(c)$ ($|c| < \varepsilon$ for sufficiently small positive number $\varepsilon$) such that $f_{c,v(c)}$ are CMC-1 immersion of $M$ into $H^3$. Namely,

THEOREM 5.1 ([10]). *If the minimal immersion $f_0$ is symmetric and non-degenerate, then there exists a one-parameter family $f_c$ of* CMC-1 *immersions of $M$ into $H^3$ such that the hyperbolic Gauss map (resp. the Hopf differential) of each $f_c$ coincides with the Gauss map (resp. a constant multiple of the Hopf differential) of $f_{0,v}$ for some $v \in \boldsymbol{R}^\delta$.*

We now consider two applications of this theorem:

**Existence of surfaces of type $\mathbf{I}(-4)$ with $\mathrm{TA}(f^\#) = 8\pi$.** We construct a deformation of the Chen-Gackstatter minimal surface defined on the elliptic curves

$$\overline{M}_1(v_1) = \{(z,w) \in (\boldsymbol{C} \cup \{\infty\})^2 \mid w^2 = z(z-1)(z+v_1)\} \qquad (v_1 \in \boldsymbol{R}^+),$$

with the point $p_1$ corresponding to $z = \infty$ removed ($p_1$ will be the single end of the surfaces). Let

$$g = \frac{v_2 w}{z}, \qquad \omega = \frac{z\,dz}{w} \qquad (v_2 \in \boldsymbol{R}^+).$$

We choose the fundamental pieces of the surfaces to be the images under the Weierstrass representation

$$\mathrm{Re}\int_{z_0=0}^{z} ((1-g^2, i(1+g^2), 2g)\omega \tag{5.1}$$

of the half sheets

$$\{(z, w_1 w_2 w_3) \in \overline{M}_1(v_1) \mid z \in \boldsymbol{C}, \mathrm{Im}(z) \ge 0, w_1^2 = z,$$
$$w_2^2 = z-1, w_3^2 = z+v_1, \arg(w_j) \in [0,\pi), j = 1,2,3\}.$$

The fundamental pieces are bounded by four planar geodesics, two of which lie in planes parallel to the $x_1x_3$-plane and two of which lie in planes parallel to the $x_2x_3$-plane. Thus $\delta = 2$. Note that the period problem is solved, and the Chen-Gackstatter surface is produced, if $v_1 = 1$ and $v_2 = \sqrt{B}$, where

$$B := \left(\int_0^1 \frac{x\,dx}{\sqrt{x(1-x^2)}}\right) \Big/ \left(\int_0^1 \frac{(1-x^2)dx}{\sqrt{x(1-x^2)}}\right) = \frac{2\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)} \simeq 0.68542.$$

The oriented distance functions (between the two pairs of parallel planes containing boundary curves of the fundamental pieces) are given by

$$\mathrm{Per}_1(v_1, v_2) = \int_0^1 (1 - v_2^2 x^{-1}(1-x)(x+v_1)) \frac{\sqrt{x}\, dx}{\sqrt{(1-x)(x+v_1)}},$$

$$\mathrm{Per}_2(v_1, v_2) = \int_0^1 \left(1 - v_1 v_2^2 x^{-1}(1-x)\left(x + \frac{1}{v_1}\right)\right) \frac{\sqrt{v_1}\sqrt{x}\, dx}{\sqrt{(1-x)(x+(1/v_1))}}.$$

To see that the period problem is nondegenerate, it is sufficient to check that the Jacobian matrix $(\partial(\mathrm{Per}_1, \mathrm{Per}_2)/\partial(v_1, v_2))$ has nonzero determinant at $(v_1, v_2) = (1, \sqrt{B})$. It is easy to check that $|\partial\, \mathrm{Per}_1/\partial v_2| = |\partial\, \mathrm{Per}_2/\partial v_2| \neq 0$ at $(v_1, v_2) = (1, \sqrt{B})$. We have

$$\left.\frac{\partial\, \mathrm{Per}_1}{\partial v_1}\right|_{(v_1, v_2)=(1, \sqrt{B})} = \int_0^1 \frac{x + B(1-x^2)}{2(x-1)(1+x)^2\sqrt{x}} \sqrt{1-x^2}\, dx,$$

$$\left.\frac{\partial\, \mathrm{Per}_2}{\partial v_1}\right|_{(v_1, v_2)=(1, \sqrt{B})} = \int_0^1 \frac{-x(x+2) + B(2+3x)(1-x^2)}{2(x-1)(1+x)^2\sqrt{x}} \sqrt{1-x^2}\, dx.$$

These integrals can be expressed in terms of elliptic integrals and the Gamma function, and we have

$$\left.\frac{\partial\, \mathrm{Per}_1}{\partial v_1}\right|_{(v_1, v_2)=(1, \sqrt{B})} \simeq -0.844, \qquad \left.\frac{\partial\, \mathrm{Per}_2}{\partial v_1}\right|_{(v_1, v_2)=(1, \sqrt{B})} \simeq -0.354.$$

Hence

$$\left|\frac{\partial\, \mathrm{Per}_1}{\partial v_1}\right| \neq \left|\frac{\partial\, \mathrm{Per}_2}{\partial v_1}\right|$$

holds at $(v_1, v_2) = (1, \sqrt{B})$. Thus the determinant of the Jacobian is nonzero, and the period problem is nondegenerate. Hence Theorem 5.1 implies existence of associated CMC-1 surfaces in $H^3$ of type $\mathbf{O}(-4)$. Furthermore, as Theorem 5.1 also implies that the hyperbolic Gauss maps will be $v_2 w/z$, these surfaces have dual total absolute curvature $8\pi$.

**Existence of surfaces of type $\mathbf{O}(-3, -3)$ with $\mathrm{TA}(f^\#) = 8\pi$.** Let $M = \boldsymbol{C} \cup \{\infty\} \setminus \{0, \infty\}$ and

$$g = \frac{2z^2 + 2az - a^2 - 1}{2(z+1)} + v, \qquad \text{and} \qquad \omega = \frac{(z+1)^2}{z^3}\, dz, \tag{5.2}$$

where $a, v \in \boldsymbol{R}$.

When $v = 0$, the Weierstrass representation (5.1) determines a minimal immersion $f_0 : M \to \boldsymbol{R}^3$ with finite total curvature of type $\mathbf{O}(-3, -3)$ ([6, Theorem 4]). For the metric to be nondegenerate at $z = -1$, we must assume $a \neq -1 \pm \sqrt{2}$.

Since the Hopf differential $Q = \omega\, dg$ satisfies $\overline{Q(\bar{z})} = Q(z)$, these minimal surfaces each have two planar geodesics that are the images of the positive and negative real axes of $\boldsymbol{C}$ under the Weierstrass representation (5.1), and their fundamental pieces are the images of the upper half plane of $\boldsymbol{C}$ under (5.1). The two planar geodesics comprise the boundaries of each of the fundamental pieces, and both lie in planes parallel to the $x_1x_3$-plane, since $g$ is real-valued on the real axis. So $\delta = 1$, and the oriented distance between the two planes containing the two geodesics is

$$\mathrm{Per}(v) := \mathrm{Re}\left(2\pi i \operatorname*{Res}_{z=0} i(1+g^2)\omega\right) = -2\pi v(2+2a+v),$$

so $d\,\mathrm{Per}(v)/dv$ is non-vanishing at $v = 0$ when $a \neq -1$. Thus Theorem 5.1 implies existence of a 1-parameter family of CMC-1 surfaces of type $\mathbf{O}(-3,-3)$ in $H^3$ for each $a \neq -1, -1 \pm \sqrt{2}$ with dual total absolute curvature $8\pi$ (as $g$ has degree 2).

**Appendix A.**

Here we review some elementary facts in the theory of linear ordinary differential equations. Define a differential operator

$$L[u] := z^2u'' + zp(z)u' + q(z)u \qquad \left({}' = \frac{d}{dz}\right). \tag{A.1}$$

In this note, we shall consider the solution of the ordinary differential equation with a regular singularity at the origin:

$$L[u] = 0, \tag{A.2}$$

where

$$p(z) = \sum_{j=0}^{\infty} p_j z^j, \qquad q(z) = \sum_{j=0}^{\infty} q_j z^j. \tag{A.3}$$

It is well-known (and we will see it in this appendix) that (A.2) has two linearly independent solutions $\{X_1, X_2\}$ of the form

$$X_1 = z^{\lambda_1} \sum_{j=0}^{\infty} \eta_{1,j} z^j, \qquad X_2 = \left(z^{\lambda_2} \sum_{j=0}^{\infty} \eta_{2,j} z^j\right) + cX_1 \log z,$$

where $\eta_{1,0} \neq 0$ and $\eta_{2,0} \neq 0$, and where $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1 = \frac{1}{2}\{(1-p_0)+m\}, \qquad \lambda_2 = \frac{1}{2}\{(1-p_0)-m\}, \tag{A.4}$$
$$m = \sqrt{(1-p_0)^2 - 4q_0}.$$

The coefficient $c$ is called *the log-term coefficient* of differential equation (A.2), which may be nonzero only when $\lambda_1 - \lambda_2 \in \boldsymbol{Z}$.

We shall give a method for computing the coefficient $c$. First, we shall describe two linearly independent solutions $X_1, X_2$ as a formal power series. If we find a solution of (A.2) as a formal power series, a well-known existence theorem from the theory of ordinary differential equations says that it will converge in a sufficiently small neighborhood of the origin [3]. So the formal treatment is sufficient for the computation of $c$.

For a complex variable $\lambda$, define rational functions $\zeta_j(\lambda)$ for non-negative integers $j$ as

$$\text{(A.5)} \quad \zeta_0(\lambda) = 1, \qquad \text{and} \qquad \zeta_j(\lambda) = -\frac{1}{\varphi(\lambda+j)} \sum_{k=0}^{j-1} r_{j,k}(\lambda)\zeta_k(\lambda) \quad (j = 1, 2, \dots),$$

where

$$\varphi(t) = t(t-1) + tp_0 + q_0, \qquad r_{j,k}(\lambda) = (\lambda + k)p_{j-k} + q_{j-k},$$

and we set

$$\text{(A.6)} \qquad X(\lambda) := z^{\lambda} \sum_{n=0}^{\infty} \zeta_n(\lambda) z^n.$$

Applying the operator $L$ to $X(\lambda)$, we have

$$\text{(A.7)} \quad L[X(\lambda)] = z^{\lambda}\left\{\varphi(\lambda) + \sum_{j=1}^{\infty}\left(\varphi(\lambda+j)\zeta_j(\lambda) + \sum_{k=0}^{j-1} r_{j,k}(\lambda)\zeta_k(\lambda)\right)z^j\right\} = z^{\lambda}\varphi(\lambda).$$

The quadratic equation

$$\text{(A.8)} \qquad \varphi(t) = t(t-1) + tp_0 + q_0 = 0$$

is called the *indicial equation* of the equation (A.2), and we denote the solutions of (A.8) by $\lambda_1$ and $\lambda_2$.

First, we consider the case $\lambda_1 - \lambda_2 \notin \boldsymbol{Z}$. In this case, $\varphi(\lambda_l + j) \neq 0$ $(l = 1, 2)$ for any positive integer $j$, and then $\zeta_j(\lambda_l)$ $(l = 1, 2)$ in (A.5) are all well-defined. Moreover, by (A.7), $X_1 := X(\lambda_1)$ and $X(\lambda_2)$ are linearly independent solutions of (A.2).

Next, assume $m := \lambda_1 - \lambda_2$ is a non-negative integer. Since $\varphi(\lambda_1 + j) \neq 0$ for any positive integer $j$, $X_1 := X(\lambda_1)$ is a well-defined power series and a solution of (A.2).

**The case $m = 0$.** Assume $\lambda_1 = \lambda_2$. Since $\varphi(\lambda_1 + j) \neq 0$ for any positive integer $j$, $\lambda = \lambda_1$ is not a pole of $\zeta_j(\lambda)$ for each $j$. Hence

$$\zeta_j(\lambda_1) \qquad \text{and} \qquad \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_1} \zeta_j(\lambda) \qquad (j = 0, 1, 2, \ldots)$$

are well-defined. Let

$$X_2 := \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_1} X(\lambda) = z^{\lambda_1} \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_1} \zeta_n(\lambda) \right) z^n + X_1 \cdot \log z. \tag{A.9}$$

Proposition A.1. *If $m = \lambda_1 - \lambda_2 = 0$, $X_2$ in* (A.9) *is a solution of* (A.2). *Moreover, the log-term coefficient of* (A.2) *never vanishes.*

Proof. It is enough to show that $X_2$ is a solution of (A.2). In fact, by (A.7),

$$L[X_2] = \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_1} L[X(\lambda)] = z^{\lambda_1} \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_1} \varphi(\lambda) + z^{\lambda_1} \varphi(\lambda_1) \log z = 0,$$

because $\varphi(\lambda) = (\lambda - \lambda_1)^2$. □

**The case** $m > 0$**.** Assume $m = \lambda_1 - \lambda_2$ is a positive integer. Since $\varphi(t) = (t - \lambda_2 - m)(t - \lambda_2)$, $\varphi(\lambda_2 + j)$ does not vanish for each positive integer $j$, except for $j = m$. Then $\zeta_j(\lambda)$ has no pole at $\lambda = \lambda_2$ for $j = 1, 2, \ldots, m-1$, and may have a pole of order one at $\lambda = \lambda_2$ for $j \geq m$. Hence

$$\lim_{\lambda \to \lambda_2} \{(\lambda - \lambda_2)\zeta_j(\lambda)\} \qquad \text{and} \qquad \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_2} [(\lambda - \lambda_2)\zeta_j(\lambda)]$$

are well-defined. Moreover,

$$\lim_{\lambda \to \lambda_2} \{(\lambda - \lambda_2)\zeta_j(\lambda)\} = 0 \qquad (j = 1, 2, \ldots, m-1) \tag{A.10}$$

holds. Let

$$\xi_j := \lim_{\lambda \to \lambda_2} \{(\lambda - \lambda_2)\zeta_{m+j}(\lambda)\} \qquad (j = 0, 1, 2 \ldots)$$

and set $c := \xi_0 = \lim_{\lambda \to \lambda_2} \{(\lambda - \lambda_2)\zeta_m(\lambda)\}$. Then by (A.5) and (A.10), we have

$$\xi_0 = c \qquad \text{and} \qquad \xi_j = \frac{-1}{\varphi(\lambda_2 + m + j)} \sum_{k=0}^{j-1} r_{j,k}(\lambda_2 + m)\xi_k \qquad (j = 1, 2, \ldots).$$

Comparing this with (A.5), we have $\xi_j = c\zeta_j(\lambda_1)$ $(j = 1, 2, \ldots)$, because $\lambda_1 = \lambda_2 + m$.

Let

$$X_2 := \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_2} [(\lambda - \lambda_2)X(\lambda)]. \tag{A.11}$$

Then by (A.10), we have

$$X_2 = z^{\lambda_2}\left(\sum_{j=0}^{\infty} \xi_j z^{j+m}\right) \log z + z^{\lambda_2} \sum_{j=0}^{\infty} \frac{\partial}{\partial\lambda}\bigg|_{\lambda=\lambda_2} \{(\lambda-\lambda_2)\zeta_j(\lambda)\} z^j$$

$$= c \log z X_1 + z^{\lambda_2} \sum_{j=0}^{\infty} \frac{\partial}{\partial\lambda}\bigg|_{\lambda=\lambda_2} \{(\lambda-\lambda_2)\zeta_j(\lambda)\} z^j.$$

PROPOSITION A.2. *If $m = \lambda_1 - \lambda_2$, is a positive integer, $X_2$ in* (A.11) *is a solution of* (A.2). *Moreover, the log-term coefficient $c$ of* (A.2) *is given by*

$$c := \xi_0 = \lim_{\lambda\to\lambda_2} \{(\lambda-\lambda_2)\zeta_m(\lambda)\}. \tag{A.12}$$

PROOF. By (A.7),

$$L[X_2] = \lim_{\lambda\to\lambda_2} \frac{\partial}{\partial\lambda}(z^{\lambda}(\lambda-\lambda_2)\varphi(\lambda)) = 0,$$

because $\varphi(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)$. □

We have established the following recursive formula for $c$, which follows immediately from equation (A.12):

PROPOSITION A.3. *If the difference $m$ of the roots of the indicial equation of* (A.2) *is a positive integer, then the log-term coefficient $c$ is*

$$c = -\frac{1}{m}\sum_{k=0}^{m-1}((\lambda_2+k)p_{m-k} + q_{m-k})a_k, \tag{A.13}$$

*where $a_0 = 1$ and*

$$a_j = \frac{1}{j(m-j)}\sum_{k=0}^{j-1}((\lambda_2+k)p_{j-k} + q_{j-k})a_k \qquad (j = 1, 2, \dots, m-1).$$

PROOF. Since $\varphi(t) = (t-\lambda_2)(t-\lambda_2-m)$, $\varphi(\lambda_2+j) \neq 0$ for $j = 1, \dots, m-1$ and then $a_j = \zeta_j(\lambda_2)$ $(j = 1, \dots, m-1)$ is well-defined. Hence, by (A.12),

$$c = \lim_{\lambda\to\lambda_2}\{(\lambda-\lambda_2)\zeta_m(\lambda)\}$$

$$= \lim_{\lambda\to\lambda_2} \frac{-(\lambda-\lambda_2)}{(\lambda+m-\lambda_2)(\lambda-\lambda_2)} \sum_{k=0}^{m-1} r_{m,k}(\lambda)\zeta_k(\lambda)$$

$$= -\frac{1}{m}\sum_{k=0}^{m-1}((\lambda_2+k)p_{m-k} + q_{m-k})a_k.$$

This completes the proof. □

Thus, in the case that $p(z) = 0$ and $m = 1, 2$, or 3, the solutions of $z^2 u''(z) + q(z)u(z) = 0$ have no log-term if and only if

$$q_1 = 0 \qquad (m = 1), \tag{A.14}$$

$$q_2 + (q_1)^2 = 0 \qquad (m = 2), \tag{A.15}$$

$$q_3 + q_1 q_2 + \frac{1}{4}(q_1)^3 = 0 \qquad (m = 3), \tag{A.16}$$

where $q(z) = \sum_{j=0}^{\infty} q_j z^j$, as in (A.3).

## References

[1] R. Bryant, Surfaces of mean curvature one in hyperbolic space, Astérisque **154–155** (1987), 321–347.

[2] C. C. Chen, F. Gackstatter, Elliptische und hyperelliptische Funktionen und vollständige Minimalflächen vom Enneperschen Typ, Math. Ann. **259** (1982), 359–369.

[3] E. A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

[4] P. Collin, L. Hauswirth and H. Rosenberg, The geometry of finite topology Bryant surfaces, Ann. of Math. (2) **153** (2001), 623–659.

[5] A. Hurwicz and R. Courant, Funktionen theorie, 4. Auflage, Springer, 1964.

[6] F. J. Lopez, The classification of complete minimal surfaces with total curvature greater than $-12\pi$, Trans. Amer. Math. Soc. **334** (1992), 49–74.

[7] C. McCune and M. Umehara, An analogue of the UP-iteration for constant mean curvature one surfaces in Hyperbolic 3-space, to appear in Geometry and its Applications.

[8] R. Osserman, A Survey of Minimal Surfaces, 2nd ed., Dover, 1986.

[9] W. Rossman, K. Sato, Constant mean curvature surfaces with two ends in hyperbolic space, Experimental Math. **7(2)** (1998), 101–119.

[10] W. Rossman, M. Umehara and K. Yamada, Irreducible constant mean curvature 1 surfaces in hyperbolic space with positive genus, Tôhoku Math. J. **49** (1997), 449–484.

[11] ———, A new flux for mean curvature 1 surfaces in hyperbolic 3-space, and applications, Proc. Amer. Math. Soc. **127** (1999), 2147–2154.

[12] ———, Mean curvature 1 surfaces with low total curvature in hyperbolic 3-space (an announcement), Advanced Studies in Pure Math., Minimal Surfaces, Geometric Analysis and Symplectic Geometry **34** (2002), 245–253.

[13] ———, Mean curvature 1 surfaces in hyperbolic 3-space with low total curvature II, Tôhoku Math. J. **55** (2003), 375–395.

[14] R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces, J. Diff. Geom. **18** (1982), 791–809.

[15] M. Umehara and K. Yamada, Complete surfaces of constant mean curvature-1 in the hyperbolic 3-space, Ann. of Math. **137** (1993), 611–638.

[16] ———, A parametrization of Weierstrass formulae and perturbation of some complete minimal surfaces of $\boldsymbol{R}^3$ into the hyperbolic 3-space, J. reine u. angew. Math. **432** (1992), 93–116.

[17] ———, Surfaces of constant mean curvature-$c$ in $H^3(-c^2)$ with prescribed hyperbolic Gauss map, Math. Ann. **304** (1996), 203–224.

[18] ———, Another construction of a CMC-1 surface in $H^3$, Kyungpook Math. J. **35** (1996), 831–849.
[19] ———, A duality on CMC-1 surface in the hyperbolic 3-space and a hyperbolic analogue of the Osserman Inequality, Tsukuba J. Math. **21** (1997), 229–237.
[20] ———, Metrics of constant curvature 1 with three conical singularities on the 2-sphere, Illinois J. Math. **44** (2000), 72–94.
[21] M. Yoshida, Fuchsian Differential Equations, Max-Plank-Institut für Mathematik, Friedr. Vieweg & Sohn, Bonn, 1987.
[22] Z. Yu, Value distribution of hyperbolic Gauss maps, Proc. Amer. Math. Soc. **125** (1997), 2997–3001.
[23] ———, The inverse surface and the Osserman Inequality, Tsukuba J. Math. **22** (1998), 575–588.

Wayne Rossman
*Department of Mathematics*
*Faculty of Science*
*Kobe University*
*Rokko*
*Kobe* 657-8501
*Japan*
*E-mail address:* *wayne@math.kobe-u.ac.jp*

Masaaki Umehara
*Department of Mathematics*
*Graduate School of Science*
*Osaka University*
*Toyonaka*
*Osaka* 560-0043
*Japan*
*E-mail address:* *umehara@math.wani.osaka-u.ac.jp*

Kotaro Yamada
*Faculty of Mathematics*
*Kyushu University* 36
*Higashi-ku*
*Fukuoka* 812-8581
*Japan*
*E-mail address:* *kotaro@math.kyushu-u.ac.jp*