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# Asymptotic expansion of the null distribution of the modified normal likelihood ratio criterion for testing $\Sigma = \Sigma_0$ under nonnormality

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**Abstract.** This paper is concerned with the null distribution of the modified normal likelihood ratio criterion for testing the null hypothesis that a covariance matrix is a given one, i.e., $\Sigma = \Sigma_0$, under nonnormality. We obtain an asymptotic expansion of the null distribution of the test statistic up to the order $n^{-1}$, where $n$ is the sample size, under nonnormality by using an Edgeworth expansion of the density function of a sample covariance matrix.

## 1. Introduction

Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ be $p \times 1$ random vevtors, where $n$ is the sample size. It is assumed that each vector $\boldsymbol{x}_j$ is *i.i.d.* with the mean $\mathrm{E}(\boldsymbol{x}) = \boldsymbol{\mu}$ and the covariance matrix $\mathrm{Cov}(\boldsymbol{x}) = \Sigma$. Consider testing the null hypothesis that the covariance matrix is a given one, i.e.,

$$H_0 : \Sigma = \Sigma_0. \tag{1.1}$$

Then a commonly used test statistic is

$$T = -2 \log L, \tag{1.2}$$

which is a modified likelihood ratio statistic for a multivariate normal population, where

$$L = \left(\frac{e}{n-1}\right)^{p(n-1)/2} |S\Sigma_0^{-1}|^{(n-1)/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}(S\Sigma_0^{-1})\right\},$$

$$S = \sum_{j=1}^{n} (\boldsymbol{x}_j - \bar{\boldsymbol{x}})(\boldsymbol{x}_j - \bar{\boldsymbol{x}})', \qquad \bar{\boldsymbol{x}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x}_j.$$

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Under normality, it is well known that the null distribution of $T$ converges to the chi-squared distribution with $q$ $(= p(p+1)/2)$ degrees of freedom, as $n$ tends to infinity. Under nonnormality, the limiting distribution of the null distribution of $T$ depends on the fourth cumulants of the true population distribution (see, e.g., Ito, 1969). When $\boldsymbol{x}_j$ is distributed as an elliptical distribution, the null distribution of $T$ is asymptotically distributed as a weighted sum of two independent chi-squared variables, i.e.,

$$\mathrm{P}(T \le x) \xrightarrow{\mathscr{D}} \frac{(p+2)\kappa + 2}{2}\chi_1^2 + (\kappa+1)\chi_{q-1}^2 \qquad (n \to \infty),$$

where $\kappa$ is a kurtosis parameter, and $\chi_1^2$ and $\chi_{q-1}^2$ are chi-squared variables with 1 and $q-1$ degrees of freedoms, respectively (see, e.g., Muirhead and Waternaux, 1980, and Tyler, 1983). Under more general distribution, Satorra and Bentler (1988) and Yanagihara, Tonda and Matsumoto (2002) introduced an explicit asymptotic distribution of the null distribution of $T$. The null distribution of $T$ converges to a weighted sum of $q$ chi-squared variables, i.e.,

$$\mathrm{P}(T \le x) \xrightarrow{\mathscr{D}} \sum_{j=1}^{q} \lambda_j \chi_{1,j}^2 \qquad (n \to \infty),$$

where each $\chi_{1,\alpha}^2$ is independently and identically distributed as a chi-squared distribution with 1 degrees of freedom and $\lambda_\alpha$ is an eigenvalue of $\Omega_s$. Here $\Omega_s$ is an asymptotic covariance matrix of $q \times 1$ vector $\mathrm{vecs}(U)$ whose elements are only distinct ones of

$$U = \sqrt{n}\Sigma^{-1/2}\left(\frac{1}{n-1}S - \Sigma\right)\Sigma^{-1/2}, \tag{1.3}$$

and is defined (see, Hample, *et al.*, 1986, pp. 272) by

$$\mathrm{vecs}(U) = \left(\frac{1}{\sqrt{2}}u_{11}, \frac{1}{\sqrt{2}}u_{22}, \dots, \frac{1}{\sqrt{2}}u_{pp}, u_{12}, \dots, u_{1p}, u_{23}, \dots, u_{p-1p}\right)', \tag{1.4}$$

where $u_{ab}$ is the $(a,b)$th element of $U$. Our main purpose is to obtain an asymptotic expansion of the null distribution of $T$ up to the order $n^{-1}$ under general condition. Note that the test statistic in (1.2) is invariant linear transformation $\boldsymbol{x}$ to $\Sigma^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu})$ when the hull hypothesis (1.1) is true. Therefore, without loss of generality, we may assume $\Sigma = I_p$ and $\boldsymbol{\mu} = \mathbf{0}$. In the following, we shall do that, and we regard $\Sigma$ and $\boldsymbol{\mu}$ as $I_p$ and $\mathbf{0}$, respectively.

The present paper is organized in the following way. In Section 2, we prepare the Edgeworth expansion of the density function of $U$ in (1.3). In Section 3, we derive an expansion of the null distribution of $T$ by expanding a characteristic function of $T$. In Section 4, we give a simple form of expansion of $\mathrm{E}(T)$ under the null hypothesis by using an alternative method.

## 2. Edgeworth expansion of sample covariance matrix

In this section, we obtain an Edgeworth expansion of the density function of the transformed sample covariance matrix $U$ in (1.3) by using the random vector $\boldsymbol{z}$ and the random matrix $V$

$$\boldsymbol{z} = \sqrt{n}\bar{\boldsymbol{x}}, \qquad V = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\boldsymbol{x}_j\boldsymbol{x}_j' - I_p), \tag{2.1}$$

which have asymptotic normality. Note that $U$ can be expanded as

$$U = V + \frac{1}{\sqrt{n}}(I_p - \boldsymbol{z}\boldsymbol{z}') + \frac{1}{n}V + \mathrm{O}_p(n^{-3/2}). \tag{2.2}$$

Under the elliptical distribution, Iwashita (1997) and Wakaki (1997) obtained an expansion of the joint density function of $\boldsymbol{z}$ and $V$ up to the order $n^{-1}$. Under the nonnormality, Fujikoshi (2002) obtained an expansion of the joint density function of $(\boldsymbol{z}, V)$ and $(\boldsymbol{z}, U)$ up to the order $n^{-1/2}$. Also, Kano (1995), who derived an asymptotic expansion of the distribution of Hotelling's $T^2$-statistic under general distribution, considered about the joint characteristic function of $\boldsymbol{z}$ and $V$ in the derivation process. However, our result is an expansion on $U$. Further, we expand up to the order $n^{-1}$ under nonnormality, not only for the elliptical distribution.

We consider alternative type of vector staking the columns of a matrix $U$ whose elements are only distinct ones of $U$ as

$$\mathrm{vech}(U) = (u_{11}, u_{21}, \dots, u_{p1}, u_{22}, \dots, u_{p2}, u_{33}, \dots, u_{pp-1})', \tag{2.3}$$

(see, Henderson and Searle, 1979 and Magnus and Neudecker, 1999, pp. 48). In what following, we consider the distribution of $\boldsymbol{u}_h = \mathrm{vech}(U)$ as the distribution of $U$. From a simple calculation, we can see that $\boldsymbol{u}_h$ is asymptotically distributed as the $q$-dimensional normal distribution with the mean $\mathbf{0}$ and the covariance matrix $\Omega_h = \mathrm{Cov}(\boldsymbol{u}_h)$.

Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p)' = \Sigma^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu})$ and $\mu_{a_1a_2\dots a_l}$ be the $l$th moment of $\boldsymbol{\varepsilon}$ defined by

$$\mu_{a_1a_2\dots a_l} = \mathrm{E}(\varepsilon_{a_1}\varepsilon_{a_2}\dots\varepsilon_{a_l}).$$

Similarly the corresponding $l$th cumulant of $\boldsymbol{\varepsilon}$ is denoted by $\kappa_{a_1a_2\dots a_l}$. Note that there are the following relations between moments and cumulants.

$$\mu_{abc} = \kappa_{abc}, \qquad \mu_{abcd} = \kappa_{abcd} + \sum_{[3]}\delta_{ab}\delta_{cd},$$

$$\mu_{abcde} = \kappa_{abcde} + \sum_{[10]}\delta_{ab}\mu_{cde},$$

$$\mu_{abcdef} = \kappa_{abcdef} + \sum_{[10]} \kappa_{abc}\kappa_{def} + \sum_{[15]} \delta_{ab}\kappa_{cdef} + \sum_{[15]} \delta_{ab}\delta_{cd}\delta_{ef},$$

$$\mu_{abcdefgh} = \kappa_{abcdefgh} + \sum_{[56]} \kappa_{abc}\kappa_{defgh} + \sum_{[35]} \kappa_{abcd}\kappa_{efgh} + \sum_{[28]} \delta_{ab}\kappa_{cdefgh}$$
$$+ \sum_{[280]} \delta_{ab}\kappa_{cde}\kappa_{fgh} + \sum_{[210]} \delta_{ab}\delta_{cd}\kappa_{efgh} + \sum_{[105]} \delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh},$$

where $\delta_{ab}$ is the Kronecker delta, i.e., $\delta_{aa} = 1$ and $\delta_{ab} = 0$ for $a \neq b$ and $\sum_{[j]}$ means the sum of all $j$ possible combinations of indices $a, b, c, d, e, f, g$ and $h$, i.e., $\sum_{[3]} \delta_{ab}\delta_{cd} = \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$.

Let $D_p$ and $D_p^+$ be the duplication matrix and its Moore-Penrose inverse (see, Henderson and Searle, 1979 and Magnus and Neudecker, 1999, pp. 48), respectively, which are defined by

$$\text{vec}(U) = D_p \,\text{vech}(U), \qquad \text{vech}(U) = D_p^+ \,\text{vec}(U) = (D_p' D_p)^{-1} D_p' \,\text{vec}(U).$$

By using $D_p^+$, the covariance matrix $\Omega_h$ can be expressed as

$$\Omega_h = D_p^+ \Psi_\kappa D_p^{+\prime} + I_q + \text{vech}(I_p)\,\text{vech}(I_p)',$$

where

$$\Psi_\kappa = \sum_{abcd}^{p} \kappa_{abcd}\{(\boldsymbol{e}_{p,a}\boldsymbol{e}_{p,b}') \otimes (\boldsymbol{e}_{p,c}\boldsymbol{e}_{p,d}')\}.$$

Here $\boldsymbol{e}_{p,a}$ is the $p$-dimensional vector whose $a$th elements is 1 and others 0, i.e., $I_p = (\boldsymbol{e}_{p,1}, \boldsymbol{e}_{p,2}, \ldots, \boldsymbol{e}_{p,p})$ and the notation $\sum_{a_1 a_2 \ldots}^{p}$ means $\sum_{a_1=1}^{p} \sum_{a_2=1}^{p} \cdots$.

We obtain an asymptotic expansion of the density function of $\boldsymbol{u}_h$ by inverting the characteristic function of $\boldsymbol{u}_h$. The derivation of the characteristic function is the same as the one in Yanagihara (2001), and Wakaki, Yanagihara and Fujikoshi (2002). Let $\boldsymbol{t}_1 = (t_1^{(1)}, t_2^{(1)}, \ldots, t_q^{(1)})'$, $\boldsymbol{t}_2 = (t_1^{(2)}, t_2^{(2)}, \ldots, t_p^{(2)})'$ and

$$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_q)' = \text{vech}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}' - I_p),$$

then the joint characteristic function of $V$ and $\boldsymbol{z}$, $\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2)$, is expressed as

$$\begin{aligned}
\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2) &= \text{E}[\exp\{i\boldsymbol{t}_1' \,\text{vech}(V) + i\boldsymbol{t}_2'\boldsymbol{z}\}] \\
&= \text{E}\left[\exp\left\{\frac{i}{\sqrt{n}}\left(\sum_{a=1}^{q} t_a^{(1)}\epsilon_a + \sum_{a'=1}^{p} t_{a'}^{(2)}\varepsilon_{a'}\right)\right\}\right]^n \\
&= \text{E}[\exp\{iA(\boldsymbol{t}_1, \boldsymbol{t}_2)\}]^n = \{h(\boldsymbol{t}_1, \boldsymbol{t}_2)\}^n = \exp\{H(\boldsymbol{t}_1, \boldsymbol{t}_2)\}.
\end{aligned}$$

Noting that $H(\boldsymbol{t}_1, \boldsymbol{t}_2) = n \log\{h(\boldsymbol{t}_1, \boldsymbol{t}_2)\}$. The characteristic function of $U$, $C_{u_h}(\boldsymbol{t}_1)$, is given as

$$C_{u_h}(\boldsymbol{t}_1) = \mathrm{E}\left[\exp\{i\boldsymbol{t}_1' \operatorname{vech}(V)\}\left\{1 + \frac{i}{\sqrt{n}}\boldsymbol{t}_1' \operatorname{vech}(I_p - \boldsymbol{z}\boldsymbol{z}') + \frac{1}{n}\left(i\boldsymbol{t}_1' \operatorname{vech}(V) + \frac{i^2}{2}\{\boldsymbol{t}_1' \operatorname{vech}(I_p - \boldsymbol{z}\boldsymbol{z}')\}^2\right)\right\}\right] + \mathrm{o}(n^{-1})$$
$$= C_0(\boldsymbol{t}_1) + \frac{1}{\sqrt{n}}C_1(\boldsymbol{t}_1) + \frac{1}{n}C_2(\boldsymbol{t}_1) + \mathrm{o}(n^{-1}). \tag{2.4}$$

In order to calculate each term in (2.4), we define the functions $l_1(a)$ and $l_2(a)$ $(l_1(a) \geq l_2(a))$ denoting a relation between an index of an element of matrix $X$ and the one of $\operatorname{vech}(X)$ as follows. Through the functions, we assume the $a$th element of $\operatorname{vech}(X)$ is equal to the $(l_1(a), l_2(a))$th element of $X$. Then the relation among $a, l_1(a)$ and $l_2(a)$ is written as

$$a = \left(p - \frac{l_1(a)}{2}\right)(l_1(a) - 1) + l_2(a), \qquad l_1(a) \geq l_2(a).$$

By using $\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2)$, the functions $l_1(a)$ and $l_2(a)$ and differential operators, the each term in (2.4) can be rewritten as

$$C_0(\boldsymbol{t}_1) = \Psi(\boldsymbol{t}_1, \boldsymbol{0}), \tag{2.5}$$

$$C_1(\boldsymbol{t}_1) = i\sum_{a=1}^{q} t_a^{(1)}\left(\delta_{l_1(a)l_2(a)} - i^{-2}\frac{\partial^2}{\partial t^{(2)}_{l_1(a)}\partial t^{(2)}_{l_2(a)}}\right)\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2)|_{t_2=0} + \mathrm{o}(n^{-1/2}), \tag{2.6}$$

$$C_2(\boldsymbol{t}_1) = i\sum_{a=1}^{q} t_a^{(a)} i^{-1}\frac{\partial}{\partial t_a^{(1)}}\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2)|_{t_2=0} + \frac{i^2}{2}\left\{\sum_{a=1}^{q} t_a^{(1)}\left(\delta_{l_1(a)l_2(a)} - i^{-2}\frac{\partial^2}{\partial t^{(2)}_{l_1(a)}\partial t^{(2)}_{l_2(a)}}\right)\right\}^2\Psi(\boldsymbol{t}_1, \boldsymbol{t}_2)|_{t_2=0} + \mathrm{o}(1). \tag{2.7}$$

We prepare the following notation for represent (2.5), (2.6) and (2.7) with using the joint moments of $\boldsymbol{\epsilon}$ and $\boldsymbol{\varepsilon}$. The $(i + 2j)$th joint moment of $\boldsymbol{\epsilon}$ and $\boldsymbol{\varepsilon}$ is expressed as

$$\eta^{[i+2j]}_{a_1a_2\ldots a_i;\, b_1b_2\ldots b_j} = \mathrm{E}(\varepsilon_{a_1}\varepsilon_{a_2}\ldots\varepsilon_{a_i}\epsilon_{b_1}\epsilon_{b_2}\ldots\epsilon_{b_j})$$
$$= \mathrm{E}\left[\left\{\prod_{\alpha=1}^{i}\varepsilon_{a_\alpha}\right\}\left\{\prod_{\alpha=1}^{j}(\varepsilon_{l_1(b_\alpha)}\varepsilon_{l_2(b_\alpha)} - \delta_{l_1(b_\alpha)l_2(b_\alpha)})\right\}\right]. \tag{2.8}$$

Specially, the marginal moments are represented as follows.

$$\eta^{[i]}_{a_1 a_2 \dots a_i;\,-} = \mathrm{E}(\varepsilon_{a_1}\varepsilon_{a_2}\dots\varepsilon_{a_i}), \qquad \eta^{[2j]}_{-;\,b_1 b_1 b_2 \dots b_j} = \mathrm{E}(\epsilon_{b_1}\epsilon_{b_2}\dots\epsilon_{b_j}).$$

The number in a bracket shows the maximum number of degrees of moment on $\boldsymbol{\varepsilon}$ in the joint moment. By using this joint moment, (2.5), (2.6) and (2.7) are rewritten as

$$C_0(\boldsymbol{t}_1) = \exp\left\{-\frac{1}{2}\sum_{ab}^{q} t_a^{(1)} t_b^{(1)} \eta^{[4]}_{-;\,ab}\right\}\left[1 + \frac{i^3}{6\sqrt{n}}\sum_{abc}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} \eta^{[6]}_{-;\,abc}\right.$$
$$+ \frac{1}{n}\left\{\frac{i^4}{24}\sum_{abcd}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} \eta^{[8]}_{-;\,abcd}\right.$$
$$\left.\left. + \frac{i^6}{72}\sum_{abcdef}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} t_e^{(1)} t_f^{(1)} \eta^{[6]}_{-;\,abc}\eta^{[6]}_{-;\,def}\right\}\right] + \mathrm{o}(n^{-1}), \tag{2.9}$$

$$C_1(\boldsymbol{t}_1) = -\exp\left\{-\frac{1}{2}\sum_{ab}^{q} t_a^{(1)} t_b^{(1)} \eta^{[4]}_{-;\,ab}\right\}\left[i^3 \sum_{abc}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} \eta^{[3]}_{l_1(a);\,b}\eta^{[3]}_{l_2(a);\,c}\right.$$
$$+ \frac{1}{\sqrt{n}}\left\{i^2 \sum_{ab}^{q} t_a^{(1)} t_b^{(1)} \eta^{[4]}_{l_1(a) l_2(a);\,b}\right.$$
$$+ \frac{i^4}{2}\sum_{abcd}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} (\eta^{[3]}_{l_1(a);\,b}\eta^{[5]}_{l_2(a);\,cd} + \eta^{[3]}_{l_2(a);\,b}\eta^{[5]}_{l_1(a);\,cd})$$
$$\left.\left. + \frac{i^6}{6}\sum_{abcdef}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} t_e^{(1)} t_f^{(1)} \eta^{[3]}_{l_1(a);\,b}\eta^{[3]}_{l_2(a);\,c}\eta^{[6]}_{-;\,def}\right\}\right] + \mathrm{o}(n^{-1/2}), \tag{2.10}$$

$$C_2(\boldsymbol{t}_1) = \frac{1}{2}\exp\left\{-\frac{1}{2}\sum_{ab}^{q} t_a^{(1)} t_b^{(1)} \eta^{[4]}_{-;\,ab}\right\}\left\{i^2 \sum_{ab}^{q} t_a^{(1)} t_b^{(1)} (2\eta^{[4]}_{-;\,ab} + \delta_{l_1(a) l_1(b)}\delta_{l_2(a) l_2(b)}\right.$$
$$+ \delta_{l_1(a) l_2(b)}\delta_{l_1(b) l_2(a)}) + i^4 \sum_{abcd}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} (\delta_{l_1(a) l_1(b)}\eta^{[3]}_{l_2(a);\,c}\eta^{[3]}_{l_2(b);\,d}$$
$$+ \delta_{l_1(a) l_2(b)}\eta^{[3]}_{l_2(a);\,c}\eta^{[3]}_{l_1(b);\,d} + \delta_{l_1(b) l_2(a)}\eta^{[3]}_{l_1(a);\,c}\eta^{[3]}_{l_2(b);\,d} + \delta_{l_2(a) l_2(b)}\eta^{[3]}_{l_1(a);\,c}\eta^{[3]}_{l_1(b);\,d})$$
$$\left. + i^6 \sum_{abcdef}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} t_e^{(1)} t_f^{(1)} \eta^{[3]}_{l_1(a);\,c}\eta^{[3]}_{l_2(a);\,d}\eta^{[3]}_{l_1(b);\,e}\eta^{[3]}_{l_2(b);\,f}\right\} + \mathrm{o}(1). \tag{2.11}$$

Substituting (2.9), (2.10) and (2.11) into (2.4) yields

$$C(\boldsymbol{t}_1) = \exp\left\{-\frac{1}{2}\sum_{ab}^{q} t_a^{(1)} t_b^{(1)} \eta^{[4]}_{-;\,ab}\right\}\left[1 + \frac{1}{6\sqrt{n}}\sum_{abc}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} m^{[6]}_{abc}\right.$$

$$+\frac{1}{n}\left\{\frac{i^2}{2}\sum_{ab}^{q} t_a^{(1)} t_b^{(1)} m_{ab}^{[4]} + \frac{i^4}{24}\sum_{abcd}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} m_{abcd}^{[8]} + \frac{i^6}{72}\sum_{abcdef}^{q} t_a^{(1)} t_b^{(1)} t_c^{(1)} t_d^{(1)} t_e^{(1)} t_f^{(1)} m_{abcdef}^{[12]}\right\}\Bigg] + \mathrm{o}(n^{-1}), \tag{2.12}$$

where

$$\begin{aligned}
m_{ab}^{[4]} &= \delta_{l_1(a)l_1(b)}\delta_{l_2(a)l_2(b)} + \delta_{l_1(a)l_2(b)}\delta_{l_1(b)l_2(a)}, \\
m_{abc}^{[6]} &= \eta_{-;abc}^{[6]} - 6\eta_{l_1(a);b}^{[3]}\eta_{l_2(a);c}^{[3]}, \\
m_{abcd}^{[8]} &= \eta_{-;abcd}^{[8]} - 12(\eta_{l_1(a);b}^{[3]}\eta_{l_2(a);cd}^{[5]} + \eta_{l_2(a);b}^{[3]}\eta_{l_1(a);cd}^{[5]}) \\
&\quad + 12(\delta_{l_1(a)l_1(b)}\eta_{l_2(a);c}^{[3]}\eta_{l_2(b);d}^{[3]} + \delta_{l_1(a)l_2(b)}\eta_{l_2(a);c}^{[3]}\eta_{l_1(b);d}^{[3]} \\
&\quad + \delta_{l_1(b)l_2(a)}\eta_{l_1(a);c}^{[3]}\eta_{l_2(b);d}^{[3]} + \delta_{l_2(a)l_2(b)}\eta_{l_1(a);c}^{[3]}\eta_{l_1(b);d}^{[3]}),
\end{aligned} \tag{2.13}$$

$$m_{abcdef}^{[12]} = \eta_{-;abc}^{[6]}\eta_{-;def}^{[6]} - 12\eta_{l_1(a);b}^{[3]}\eta_{l_2(a);c}^{[3]}\eta_{-;def}^{[6]} + 36\eta_{l_1(a);c}^{[3]}\eta_{l_2(a);d}^{[3]}\eta_{l_1(b);e}^{[3]}\eta_{l_2(b);f}^{[3]}$$

By inversing (2.12), we have Theorem 2.1.

Theorem 2.1. *Suppose that the following two conditions on* $\boldsymbol{x}_j$ *are satisfied;*

A1. $\mathrm{E}(\|\boldsymbol{x}_j\|^8) < \infty$.

A2. *the joint characteristic function* $\boldsymbol{x}_j\boldsymbol{x}_j'$ *and* $\boldsymbol{x}_j$ *is absolutely integrable, i.e., for some* $c \geq 1$,

$$\int_{\Re^q}\int_{\Re^p} |\mathrm{E}[\exp\{i\boldsymbol{t}_1' \operatorname{vech}(\boldsymbol{x}_j\boldsymbol{x}_j' - \Sigma) + i\boldsymbol{t}_2'\boldsymbol{x}_j\}]|^c d\boldsymbol{t}_1 d\boldsymbol{t}_2 < \infty.$$

*Let* $g_q(\boldsymbol{u}_h, \Omega_h)$ *be the density function of the q-dimensional multivariate normal distribution with mean* $\mathbf{0}$ *and covariance* $\Omega_h$ *as*

$$g_q(\boldsymbol{u}_h, \Omega_h) = \left(\frac{1}{\sqrt{2\pi}}\right)^q |\Omega_h|^{-1/2} \exp\left\{-\frac{1}{2}\boldsymbol{u}_h'\Omega_h^{-1/2}\boldsymbol{u}_h\right\}. \tag{2.14}$$

*Then, an asymptotic expansion of the density function of* $\boldsymbol{u}_h$ *is given by*

$$g_q(\boldsymbol{u}_h; \Omega_h)\Bigg[1 + \frac{1}{6\sqrt{n}}\sum_{abc}^{q} m_{abc}^{[6]} H_{abc}(\boldsymbol{u}_h; \Omega_h)$$

$$+\frac{1}{n}\left\{\frac{1}{2}\sum_{ab}^{q} m_{ab}^{[4]} H_{ab}(\boldsymbol{u}_h; \Omega_h) + \frac{1}{24}\sum_{abcd}^{q} m_{abcd}^{[8]} H_{abcd}(\boldsymbol{u}_h; \Omega_h)\right.$$

$$+\frac{1}{72}\sum_{abcdef}^{q} m^{[12]}_{abcdef} H_{abcdef}(\boldsymbol{u}_h;\Omega_h)\Bigg\}\Bigg] + \mathrm{o}(n^{-1})$$

$$= g_q(\boldsymbol{u}_h;\Omega_h)\left\{1+\frac{1}{\sqrt{n}}Q_1^{(h)}(\boldsymbol{u}_h)+\frac{1}{n}Q_2^{(h)}(\boldsymbol{u}_h)\right\}+\mathrm{o}(n^{-1}), \tag{2.15}$$

*where the coefficient m's are defined by* (2.13). *Here* $H_{a_1a_2\dots a_j}(\boldsymbol{u}_h;\Omega_h)$ *is the jth general Hermite polynomial defined by*

$$H_{a_1a_2\dots a_j}(\boldsymbol{u}_h;\Omega_h) = (-1)^j \frac{\partial^j}{\partial u_{a_1}^{(h)}\partial u_{a_2}^{(h)}\dots\partial u_{a_j}^{(h)}} g_q(\boldsymbol{u}_h;\Omega_h),$$

*where* $\boldsymbol{u}_h = (u_1^{(h)}, u_2^{(h)}, \dots, u_q^{(h)})'$. *By letting* $\zeta_{ab}^{(h)}$ *be the* $(a,b)$*th element of* $\Omega_h^{-1}$, *we have*

$$H_{ab}(\boldsymbol{u};\Omega_h) = \sum_{ij}^{q} u_i^{(h)}u_j^{(h)}\zeta_{ai}^{(h)}\zeta_{bi}^{(h)} - \zeta_{ab}^{(h)},$$

$$H_{abc}(\boldsymbol{u};\Omega_h) = \sum_{ijk}^{q} u_i^{(h)}u_j^{(h)}u_k^{(h)}\zeta_{ai}^{(h)}\zeta_{bj}^{(h)}\zeta_{ck}^{(h)} - \sum_{i=1}^{q} u_i^{(h)}\sum_{[3]}\zeta_{ab}^{(h)}\zeta_{ci}^{(h)},$$

$$\begin{aligned} H_{abcd}(\boldsymbol{u};\Omega_h) &= \sum_{ijkl}^{q} u_i^{(h)}u_j^{(h)}u_k^{(h)}u_l^{(h)}\zeta_{ai}^{(h)}\zeta_{bj}^{(h)}\zeta_{ck}^{(h)}\zeta_{dl}^{(h)} \\ &\quad - \sum_{ij}^{q} u_i^{(h)}u_j^{(h)}\sum_{[6]}\zeta_{ab}^{(h)}\zeta_{ci}^{(h)}\zeta_{dj}^{(h)} + \sum_{[3]}\zeta_{ab}^{(h)}\zeta_{cd}^{(h)}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} H_{abcdef}(\boldsymbol{u};\Omega_h) &= \sum_{ijklmn}^{q} u_i^{(h)}u_j^{(h)}u_k^{(h)}u_l^{(h)}u_m^{(h)}u_n^{(h)}\zeta_{ai}^{(h)}\zeta_{bj}^{(h)}\zeta_{ck}^{(h)}\zeta_{dl}^{(h)}\zeta_{em}^{(h)}\zeta_{fn}^{(h)} \\ &\quad - \sum_{ijkl}^{q} u_i^{(h)}u_j^{(h)}u_k^{(h)}u_l^{(h)}\sum_{[15]}\zeta_{ab}^{(h)}\zeta_{ci}^{(h)}\zeta_{dj}^{(h)}\zeta_{ek}^{(h)}\zeta_{fl}^{(h)} \\ &\quad + \sum_{ij}^{q} u_i^{(h)}u_j^{(h)}\sum_{[45]}\zeta_{ab}^{(h)}\zeta_{cd}^{(h)}\zeta_{ei}^{(h)}\zeta_{fj}^{(h)} - \sum_{[15]}\zeta_{ab}^{(h)}\zeta_{cd}^{(h)}\zeta_{ef}^{(h)}. \end{aligned}$$

The conditions of validity can be obtain by the same way as in Bhattacharya and Rao (1976) and Bhattacharya and Ghosh (1978).

By changing the moment of $\mathrm{vech}(\boldsymbol{\varepsilon\varepsilon}' - I_p)$ to one of $\mathrm{vecs}(\boldsymbol{\varepsilon\varepsilon}' - I_p)$ in Theorem 2.1, we can obtain an asymptotic expansion of the density function of $\boldsymbol{u}_s = \mathrm{vecs}(U)$. The covariance matrix of $\boldsymbol{u}_s, \Omega_s$, is shown as

$$\Omega_s = \frac{1}{2}D_{s,p}'\Psi_\kappa D_{s,p} + I_q,$$

where $D_{s,p}$ is the standardized duplication matrix (see, Yanagihara, Tonda and Matsumoto, 2002) defined by

$$\mathrm{vec}(U) = \sqrt{2} D_{s,p} \,\mathrm{vecs}(U), \qquad \mathrm{vecs}(U) = \frac{1}{\sqrt{2}} D'_{s,p} \,\mathrm{vec}(U).$$

Let us define the functions $i_1(a)$ and $i_2(a)$ by $i_1(a) \le i_2(a)$ and

$$a = i_1(a)\delta_{i_1(a)i_2(a)} + (1 - \delta_{i_1(a)i_2(a)})\left\{\frac{1}{2}(2p - i_1(a) - 1) + i_2(a)\right\}.$$

Then the $a$th element of vector $\mathrm{vecs}(X)$ is equal to the $(i_1(a), i_2(a))$th element of matrix $X$. Moreover, we are necessary to define new joint moment of $\boldsymbol{\epsilon}_s = \mathrm{vecs}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}' - I_p)$ and $\boldsymbol{\varepsilon}$. The $(i+2j)$th complex moment of $\boldsymbol{\epsilon}_s$ and $\boldsymbol{\varepsilon}$, $\eta^{[i+2j],(s)}_{a_1 a_2 \dots a_i; b_1 b_2 \dots b_j}$, is represented by changing $\boldsymbol{\epsilon}$ to $\boldsymbol{\epsilon}_s = (\epsilon^{(s)}_1, \epsilon^{(s)}_1, \dots, \epsilon^{(s)}_q)'$ in (2.8) as

$$\begin{aligned}\eta^{[i+2j],(s)}_{a_1 a_2 \dots a_i; b_1 b_2 \dots b_j} &= \mathrm{E}(\varepsilon_{a_1}\varepsilon_{a_2}\dots\varepsilon_{a_i}\epsilon^{(s)}_{b_1}\epsilon^{(s)}_{b_2}\dots\epsilon^{(s)}_{b_j}) \\ &= \mathrm{E}\left[\left\{\prod_{\alpha=1}^{i}\varepsilon_\alpha\right\}\left\{\prod_{\alpha=1}^{j}\frac{\varepsilon_{i_1(b_\alpha)}\varepsilon_{i_2(b_\alpha)} - \delta_{i_1(b_\alpha)i_2(b_\alpha)}}{\sqrt{1+\delta_{i_1(b_\alpha)i_2(b_\alpha)}}}\right\}\right]. \qquad (2.17)\end{aligned}$$

Specially, in the case of moment of only $\boldsymbol{\epsilon}_s$, it is represented as

$$\eta^{[2j],(s)}_{-;b_1 b_1 b_2 \dots b_j} = \mathrm{E}(\epsilon^{(s)}_{b_1}\epsilon^{(s)}_{b_2}\dots\epsilon^{(s)}_{b_j}).$$

Therefore, an asymptotic expansion of the density function on $\boldsymbol{u}_s$ is shown as in the following Corollary.

Corollary 2.2. *Under the same assumption as in Theorem* 2, *an asymptotic expansion of the density function of* $\boldsymbol{u}_s$ *is given by*

$$\begin{aligned} g_q(\boldsymbol{u}_s; \Omega_s)\Bigg[1 &+ \frac{1}{6\sqrt{n}}\sum_{abc}^{q} m^{[6],(s)}_{abc} H_{abc}(\boldsymbol{u}_s;\Omega_s) \\ &+ \frac{1}{n}\left\{\frac{1}{2}\sum_{ab}^{q} m^{[4],(s)}_{ab} H_{ab}(\boldsymbol{u}_s;\Omega_s) + \frac{1}{24}\sum_{abcd}^{q} m^{[8],(s)}_{abcd} H_{abcd}(\boldsymbol{u}_s;\Omega_s) \right.\\ &\left. + \frac{1}{72}\sum_{abcdef}^{q} m^{[12],(s)}_{abcdef} H_{abcdef}(\boldsymbol{u}_s;\Omega_s)\right\}\Bigg] + \mathrm{o}(n^{-1}) \\ = g_q(\boldsymbol{u}_s;\Omega_s)&\left\{1 + \frac{1}{\sqrt{n}}Q^{(s)}_1(\boldsymbol{u}_s) + \frac{1}{n}Q^{(s)}_2(\boldsymbol{u}_s)\right\} + \mathrm{o}(n^{-1}), \qquad (2.18)\end{aligned}$$

*where the coefficient* $m^{[i],(s)}_{\dots;\dots}$*s are defined by changing* $\eta^{[j]}_{\dots;\dots}$ *to* $\eta^{[j],(s)}_{\dots;\dots}$ *in* (2.13).

## 3. Asymptotic expansion of the null distribution of $T$

In this section, we obtain an asymptotic expansion of the null distribution of $T$ up to the order $n^{-1}$ by using the Edgeworth expansion of the density function of $\boldsymbol{u}_s$ under nonnormality. Since $T$ is the modified normal likelihood ratio test statistic, it is known that its null distribution is asymptotically distributed as the chi-squared distribution with $q$ degrees of freedom under normality. Under normality, many authors introduced asymptotic expansions of the null distribution of $T$, see, e.g., Sugiura (1969). Under an elliptical distribution, Hayakawa (1986, 1999) obtained an expansion of non-null distribution of $T$ up to the order $n^{-1/2}$. However, asymptotic expansion of $T$ under general distribution has not been obtained at this time, though there are some results. Nagao and Srivastava (1992) derived an expansion of non-null distribution of test statistics for sphericity test up to the order $n^{-1/2}$ and Tonda and Wakaki (2003) obtained an expansion of the null distribution of test statistic for the equality of covariance for $k$-groups in univariate case up to the order $n^{-1}$.

In order to obtain the null distribution of $T$, we derive an expansion of the characteristic function $C_T(t)$ of $T$ up to the order $n^{-1}$. Our method is similar to the one as in Yanagihara (2001) and Wakaki, Yanagihara and Fujikoshi (2002). Let $\boldsymbol{u}_s = (u_1^{(s)}, u_2^{(s)}, \dots, u_q^{(s)})'$ and a $p \times p$ matrix

$$K_a = \frac{1}{\sqrt{1 + \delta_{i_1(a)i_2(a)}}}(\boldsymbol{e}_{p,i_1(a)}\boldsymbol{e}'_{p,i_2(a)} + \boldsymbol{e}_{p,i_2(a)}\boldsymbol{e}'_{p,i_1(a)}),$$

then the matrix $U$ is rewritten as

$$U = \sum_{a=1}^{q} u_a^{(s)} K_a.$$

Therefore, we have a perturbation expansion of the test statistic as follows.

$$\begin{aligned} T &= \frac{1}{2}\,\mathrm{tr}(U^2) - \frac{1}{3\sqrt{n}}\,\mathrm{tr}(U^3) + \frac{1}{4n}\{\mathrm{tr}(U^4) - 2\,\mathrm{tr}(U^2)\} + \mathrm{O}_p(n^{-3/2}) \\ &= \boldsymbol{u}_s'\boldsymbol{u}_s - \frac{1}{3\sqrt{n}}\sum_{abc}^{q} u_a^{(s)}u_b^{(s)}u_c^{(s)}\,\mathrm{tr}(K_aK_bK_c) \\ &\quad + \frac{1}{4n}\left\{\sum_{abcd}^{q} u_a^{(s)}u_b^{(s)}u_c^{(s)}u_d^{(s)}\,\mathrm{tr}(K_aK_bK_cK_d) - 4\boldsymbol{u}_s'\boldsymbol{u}_s\right\} + \mathrm{O}_p(n^{-3/2}). \end{aligned} \quad (3.1)$$

Then, $C_T(t)$ can be expended as

$$C_T(t) = \mathrm{E}[\exp(itT)]$$
$$= \mathrm{E}\left[\exp\{it\boldsymbol{u}_s'\boldsymbol{u}_s\}\left\{1 + \frac{1}{\sqrt{n}}R_1(\boldsymbol{u}_s) + \frac{1}{n}R_2(\boldsymbol{u}_s)\right\}\right] + \mathrm{o}(n^{-1}), \qquad (3.2)$$

where

$$R_1(\boldsymbol{u}_s) = -\frac{it}{3}\sum_{abc}^{q} u_a^{(s)}u_b^{(s)}u_c^{(s)}k_{abc},$$
$$R_2(\boldsymbol{u}_s) = \frac{it}{4}\left\{\sum_{abcd}^{q} u_a^{(s)}u_b^{(s)}u_c^{(s)}u_d^{(s)}k_{abcd} - 4\boldsymbol{u}_s'\boldsymbol{u}_s\right\}$$
$$+ \frac{(it)^2}{18}\sum_{abcdef}^{q} u_a^{(s)}u_b^{(s)}u_c^{(s)}u_d^{(s)}u_e^{(s)}u_f^{(s)}k_{abc}k_{def},$$

and

$$k_{abc} = \mathrm{tr}(K_aK_bK_c), \qquad k_{abcd} = \mathrm{tr}(K_aK_bK_cK_d).$$

From (2.18), (3.2) and

$$\exp(it\boldsymbol{u}_s'\boldsymbol{u}_s)\exp\left(-\frac{1}{2}\boldsymbol{u}_s'\Omega_s^{-1}\boldsymbol{u}_s\right) = \exp\left\{-\frac{1}{2}\boldsymbol{u}_s'(\Omega_s^{-1} - 2itI_q)\boldsymbol{u}_s\right\},$$

$C_T(t)$ is rewritten as

$$C_T(t) = \int_{\Re^q}\exp(it\boldsymbol{u}_s'\boldsymbol{u}_s)\left\{1 + \frac{1}{\sqrt{n}}R_1(\boldsymbol{u}_s) + \frac{1}{n}R_2(\boldsymbol{u}_s)\right\}$$
$$\times g_q(\boldsymbol{u}_s;\Omega_s)\left\{1 + \frac{1}{\sqrt{n}}Q_1^{(s)}(\boldsymbol{u}_s) + \frac{1}{n}Q_2^{(s)}(\boldsymbol{u}_s)\right\}d\boldsymbol{u}_s + \mathrm{o}(n^{-1})$$
$$= \prod_{j=1}^{q}(1 - 2it\lambda_j)^{-1/2} \times \mathrm{E}_{\boldsymbol{y}}\left[\left\{1 + \frac{1}{\sqrt{n}}R_1(\Gamma^{1/2}\boldsymbol{y}) + \frac{1}{n}R_2(\Gamma^{1/2}\boldsymbol{y})\right\}\right.$$
$$\left.\times\left\{1 + \frac{1}{\sqrt{n}}Q_1^{(s)}(\Gamma^{1/2}\boldsymbol{y}) + \frac{1}{n}Q_2^{(s)}(\Gamma^{1/2}\boldsymbol{y})\right\}\right] + \mathrm{o}(n^{-1}),$$

where $\lambda_j$'s $(1 \le j \le q)$ are the eigenvalues of $\Omega_s$, $\boldsymbol{y} \sim \mathrm{N}_q(\boldsymbol{0}, I_q)$ and $\Gamma^{-1} = \Omega_s^{-1} - 2itI_q$. Since $\Gamma^{1/2}\boldsymbol{y} \sim \mathrm{N}_q(\boldsymbol{0}, \Gamma)$, $C_T(t)$ is represented as

$$C_T(t) = \prod_{j=1}^{q}\varphi_j^{1/2}\mathrm{E}_{\boldsymbol{y}}\left[\left\{1 + \frac{1}{\sqrt{n}}R_1(\boldsymbol{y}) + \frac{1}{n}R_2(\boldsymbol{y})\right\}\right.$$
$$\left.\times\left\{1 + \frac{1}{\sqrt{n}}Q_1^{(s)}(\boldsymbol{y}) + \frac{1}{n}Q_2^{(s)}(\boldsymbol{y})\right\}\right] + \mathrm{o}(n^{-1}),$$

where $\varphi_j^{-1} = 1 - 2it\lambda_j$ and $\boldsymbol{y} \sim \mathrm{N}_q(\mathbf{0}, \Gamma)$. Thus, the characteristic function $C_T(t)$ is derived from calculations of moments on the multivariate normal distribution.

We consider the calculations of the moments on $\boldsymbol{y} \sim \mathrm{N}_q(\mathbf{0}, \Gamma)$. From the simple calculation, we obtain

$$\begin{aligned} &\mathrm{E}_{\boldsymbol{y}}[y_a y_b] = \gamma_{ab}, && \mathrm{E}_{\boldsymbol{y}}[y_a y_b y_c] = 0, \\ &\mathrm{E}_{\boldsymbol{y}}[y_a y_b y_c y_d] = \sum_{[3]} \gamma_{ab}\gamma_{cd}, && \mathrm{E}_{\boldsymbol{y}}[y_a y_b y_c y_d y_e y_f] = \sum_{[15]} \gamma_{ab}\gamma_{cd}\gamma_{ef}. \end{aligned} \tag{3.3}$$

Let a $q \times q$ orthogonal matrix $P$ be defined by $P'\Omega_s P = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_q) = \Lambda$ and $P'P = PP' = I_q$. Then $\Gamma = PWP'$, where

$$W = \mathrm{diag}(\lambda_1\varphi_1, \lambda_2\varphi_2, \dots, \lambda_q\varphi_q).$$

Therefore, $\gamma_{ab}$ for the $(a,b)$th element of $\Gamma$ and $\zeta_{ab}^{(s)}$ for the $(a,b)$th element of $\Omega_s^{-1}$ have the following relations.

$$\begin{aligned} \gamma_{ab} &= \sum_{i=1}^q \rho_{ai}\rho_{bi}\lambda_i\varphi_i, && (\Gamma = PWP'), \\ \zeta_{ab}^{(s)} &= \sum_{i=1}^q \rho_{ai}\rho_{bi}\lambda_i^{-1}, && (\Omega_s^{-1} = P\Lambda^{-1}P'), \\ \sum_{i=1}^q \zeta_{ai}^{(s)}\rho_{bi} &= \rho_{ab}\lambda_b^{-1}, && (\Omega_s^{-1}P = P\Lambda^{-1}). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), the expectations of the Hermite polynomials yields

$$\begin{aligned} \mathrm{E}_{\boldsymbol{y}}[H_{ab}(\boldsymbol{y}; \Omega_s)] &= \sum_{i=1}^q \rho_{ai}\rho_{bi}\lambda_i^{-1}\varphi_i - \zeta_{ab}^{(s)}, \\ \mathrm{E}_{\boldsymbol{y}}[H_{abc}(\boldsymbol{y}; \Omega_s)] &= 0, \\ \mathrm{E}_{\boldsymbol{y}}[H_{abcd}(\boldsymbol{y}; \Omega_s)] &= \sum_{ij}^q \left( \sum_{[3]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj} \right) \lambda_i^{-1}\lambda_j^{-1}\varphi_i\varphi_j \\ &\quad - \sum_{i=1}^q \left( \sum_{[6]} \rho_{ai}\rho_{bi}\zeta_{cd}^{(s)} \right) \lambda_i^{-1}\varphi_i + \sum_{[3]} \zeta_{ab}^{(s)}\zeta_{cd}^{(s)}, \\ \mathrm{E}_{\boldsymbol{y}}[H_{abcdef}(\boldsymbol{y}; \Omega_s)] &= \sum_{ijk}^q \left( \sum_{[15]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk} \right) \lambda_i^{-1}\lambda_j^{-1}\lambda_k^{-1}\varphi_i\varphi_j\varphi_k \end{aligned}$$

$$- \sum_{ij}^{q} \left( \sum_{[45]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\zeta_{ef}^{(s)} \right) \lambda_i^{-1}\lambda_j^{-1}\varphi_i\varphi_j$$

$$+ \sum_{i=1}^{q} \left( \sum_{[45]} \rho_{ai}\rho_{bi}\zeta_{cd}^{(s)}\zeta_{ef}^{(s)} \right) \lambda_i^{-1}\varphi_i - \sum_{[15]} \zeta_{ab}^{(s)}\zeta_{cd}^{(s)}\zeta_{ef}^{(s)}.$$

Besides, we have

$$\mathrm{E}_{\boldsymbol{y}}[\boldsymbol{y}'\boldsymbol{y}] = \sum_{i=1}^{q} \lambda_i\varphi_i,$$

$$\mathrm{E}_{\boldsymbol{y}}\left[ \sum_{abcd}^{q} y_a y_b y_c y_d k_{abcd} \right] = \sum_{abcd}^{q} k_{abcd} \sum_{ij}^{q} \left( \sum_{[3]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj} \right) \lambda_i\lambda_j\varphi_i\varphi_j,$$

$$\mathrm{E}_{\boldsymbol{y}}\left[ \sum_{abcdef}^{q} y_a y_b y_c y_d y_e y_f k_{abc} k_{def} \right]$$

$$= \sum_{abcdef}^{q} k_{abc}k_{def} \sum_{ijk}^{q} \left( \sum_{[15]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk} \right) \lambda_i\lambda_j\lambda_k\varphi_i\varphi_j\varphi_k,$$

$$\mathrm{E}_{\boldsymbol{y}}\left[ \sum_{abcdef}^{q} y_a y_b y_c k_{abc} m_{def}^{[6],(s)} H_{def}(\boldsymbol{y}; \Omega_s) \right]$$

$$= \sum_{abcdef}^{q} k_{abc} m_{def}^{[6],(s)} \left[ \sum_{ijk}^{q} \left\{ \left( {\sum_{[6]}}' \rho_{ai}\rho_{di}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{fk} \right) \lambda_i\lambda_k^{-1} \right. \right.$$

$$\left. + \left( {\sum_{[9]}}' \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk} \right) \right\} \varphi_i\varphi_j\varphi_k$$

$$\left. - \sum_{ij}^{q} \left( \sum_{[3]} \rho_{ai}\rho_{bi}\rho_{cj} \right) \left( \sum_{[3]} \rho_{dj}\zeta_{ef}^{(s)} \right) \lambda_i\varphi_i\varphi_j \right],$$

where ${\sum}'_{[6]} \rho_{ai}\rho_{di}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{fk}$ and ${\sum}'_{[9]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk}$ mean the following summations.

$$\sum_{[15]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk} = {\sum_{[6]}}' \rho_{ai}\rho_{di}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{fk} + {\sum_{[9]}}' \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk},$$

$$\sum_{[6]}{}' \rho_{ai}\rho_{di}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{fk} = \rho_{ai}\rho_{di}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{fk} + \rho_{ai}\rho_{di}\rho_{bj}\rho_{fj}\rho_{ck}\rho_{ek} + \rho_{ai}\rho_{ei}\rho_{bj}\rho_{dj}\rho_{ck}\rho_{fk}$$
$$+ \rho_{ai}\rho_{ei}\rho_{bj}\rho_{fj}\rho_{ck}\rho_{dk} + \rho_{ai}\rho_{fi}\rho_{bj}\rho_{dj}\rho_{ck}\rho_{ek} + \rho_{ai}\rho_{fi}\rho_{bj}\rho_{ej}\rho_{ck}\rho_{dk}.$$

From these expectations, we can obtain the characteristic function $C_T(t)$ as

$$C_T(t) = \prod_{\alpha=1}^{q} \varphi_\alpha^{1/2}\left\{1 + \frac{1}{n}\left(b^{(0)} + \sum_{i=1}^{q} b_i^{(1)}\varphi_i + \sum_{ij}^{q} b_{ij}^{(2)}\varphi_i\varphi_j + \sum_{ijk}^{q} b_{ijk}^{(3)}\varphi_i\varphi_j\varphi_k\right)\right\}$$
$$+ \mathrm{o}(n^{-1}),$$

where

$$b^{(0)} = \frac{q}{2} - \frac{1}{2}\sum_{ab}^{q} m_{ab}^{[4],(s)}\zeta_{ab}^{(s)} + \frac{1}{24}\sum_{abcd}^{q} m_{abcd}^{[8],(s)}\sum_{[3]}\zeta_{ab}^{(s)}\zeta_{cd}^{(s)}$$
$$+ \frac{1}{72}\sum_{abcdef}^{q} m_{abcdef}^{[12],(s)}\sum_{[15]}\zeta_{ab}^{(s)}\zeta_{cd}^{(s)}\zeta_{ef}^{(s)},$$

$$b_i^{(1)} = -\frac{1}{2} + \frac{1}{2}\sum_{ab}^{q} m_{ab}^{[4],(s)}\rho_{ai}\rho_{bi}\lambda_i^{-1} - \frac{1}{24}\sum_{abcd}^{q} m_{abcd}^{[8],(s)}\left(\sum_{[6]}\zeta_{ab}^{(s)}\rho_{ci}\rho_{di}\right)\lambda_i^{-1}$$
$$- \frac{1}{8}\sum_{abcd}^{q} k_{abcd}\left(\sum_{[3]}\delta_{ab}\rho_{ci}\rho_{di}\right)\lambda_i + \frac{1}{72}\sum_{abcdef}^{q} m_{abcdef}^{[12],(s)}\left(\sum_{[45]}\zeta_{ab}^{(s)}\zeta_{cd}^{(s)}\rho_{ei}\rho_{fi}\right)\lambda_i^{-1}$$
$$+ \frac{1}{72}\sum_{abcdef}^{q} k_{abc}k_{def}\left(\sum_{[15]}\zeta_{ab}^{(s)}\zeta_{cd}^{(s)}\rho_{ei}\rho_{fi}\right)\lambda_i$$
$$- \frac{1}{36}\sum_{abcdef} k_{abc}m_{def}^{[6],(s)}\left(\sum_{[3]}\delta_{ab}\rho_{ci}\right)\left(\sum_{[3]}\rho_{di}\zeta_{ed}^{(s)}\right), \tag{3.5}$$

$$b_{ij}^{(2)} = \frac{1}{24}\sum_{abcd}(m_{abcd}^{[8],(s)}\lambda_i^{-1}\lambda_j^{-1} + 3k_{abcd}\lambda_i)\left(\sum_{[3]}\rho_{\rho_{ai}}\rho_{bi}\rho_{cj}\rho_{dj}\right)$$
$$- \frac{1}{72}\sum_{abcdef}^{q} m_{abcdef}^{[12],(s)}\left(\sum_{[45]}\zeta_{ab}^{(s)}\rho_{ci}\rho_{di}\rho_{ej}\rho_{fj}\right)\lambda_i^{-1}\lambda_j^{-1}$$
$$+ \frac{1}{36}\sum_{abcdef}^{q} k_{abc}(k_{def}\lambda_i + m_{def}^{[6],(s)}\lambda_i^{-1})\left(\sum_{[15]}\delta_{ab}\rho_{ci}\rho_{di}\rho_{ej}\rho_{fj}\right)$$

$$+\frac{1}{36}\sum_{abcdef}^{q} k_{abc} m_{def}^{[6],(s)} \left(\sum_{[3]} \rho_{ai}\rho_{bi}\rho_{cj}\right)\left(\sum_{[3]} \rho_{dj}\zeta_{ef}^{(s)}\right),$$

$$b_{ijk}^{(3)} = \frac{1}{72}\sum_{abcdef}^{q} \{m_{abcdef}^{[12],(s)} \lambda_i^{-1}\lambda_j^{-1}\lambda_k^{-1} + k_{abc}(k_{def}\lambda_i - 2m_{def}^{[6],(s)}\lambda_i^{-1})\}$$

$$\times \left(\sum_{[15]} \rho_{ai}\rho_{bi}\rho_{cj}\rho_{dj}\rho_{ek}\rho_{fk}\right).$$

Finally, by inverting $C_T(t)$, we have a following Theorem.

THEOREM 3.1. *If $\boldsymbol{x}_j$ satisfies the condition* A1 *and*

A3. *the joint characteristic function $\boldsymbol{x}_j\boldsymbol{x}_j'$ and $\boldsymbol{x}_j$ satisfies the Cramér condition, i.e.,*

$$\limsup_{\boldsymbol{t}_1'\boldsymbol{t}_1 + \boldsymbol{t}_2'\boldsymbol{t}_2 \to \infty} |\mathrm{E}[\exp\{i\boldsymbol{t}_1' \operatorname{vech}(\boldsymbol{x}_j\boldsymbol{x}_j' - \Sigma) + i\boldsymbol{t}_2'\boldsymbol{x}_j\}]| < 1.$$

*Then the null distribution of $T$ can be expanded as*

$$\mathrm{P}(T \le x) = F_q(x;\Lambda) + \frac{1}{n}\Bigg\{ b^{(0)}F_q(x;\Lambda) + \sum_{i=1}^{q} b_i^{(1)}F_{q+2}(x;\Lambda_i)$$

$$+ \sum_{ij}^{q} b_{ij}^{(2)}F_{q+4}(x;\Lambda_{ij}) + \sum_{ikj}^{q} b_{ijk}^{(3)}F_{q+6}(x;\Lambda_{ijk})\Bigg\} + \mathrm{o}(n^{-1}), \qquad (3.6)$$

*where coefficients $b^{(0)}, b_i^{(1)}, b_{ij}^{(2)}$ and $b_{ijk}^{(3)}$ are defined by* (3.5) *and the diagonal matrices $\Lambda_i, \Lambda_{ij}$ and $\Lambda_{ijk}$ are given by*

$$\Lambda_i = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_q, \lambda_i, \lambda_i),$$

$$\Lambda_{ij} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_q, \lambda_i, \lambda_i, \lambda_j, \lambda_j),$$

$$\Lambda_{ijk} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_q, \lambda_i, \lambda_i, \lambda_j, \lambda_j, \lambda_k, \lambda_k).$$

*By denoting $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_m)$, we have*

$$F_m(x;\Omega) = \sum_{l=0}^{\infty} e_l(\Omega) G_{m+2l}(x/\beta),$$

*where $G_f(\cdot)$ is the distribution function of the chi-squared distribution with $f$ degrees of freedom,*

$$e_r(\Omega) = \begin{cases} \prod_{l=1}^{m} (\beta/\omega_l)^{1/2}, & (r = 0) \\ (2r)^{-1} \sum_{l=0}^{r-1} H_{r-l}(\Omega) e_l(\Omega) & (r \geq 1), \end{cases}, \qquad H_r(\Omega) = \sum_{l=1}^{m} (1 - \beta/\omega_l)^r,$$

*and a coefficient $\beta$ is a suitably chosen constant for a rapid convergence.*

The conditions of validity can be obtain by the same way as in Bhattacharya and Rao (1976) and Wakaki, Yanagihara and Fujikoshi (2002).

Let

$$T^{(\Lambda)} = \sum_{\alpha=1}^{q} \lambda_\alpha \chi^2_{1,\alpha}, \qquad T_i^{(\Lambda)} = \sum_{\alpha \neq i}^{q} \lambda_\alpha \chi^2_{1,\alpha} + \lambda_i \chi^2_{3,i},$$

$$T_{ij}^{(\Lambda)} = \sum_{\alpha \neq i,j}^{q} \lambda_\alpha \chi^2_{1,\alpha} + \lambda_i \chi^2_{3,i} + \lambda_j \chi^2_{3,j},$$

$$T_{ijk}^{(\Lambda)} = \sum_{\alpha \neq i,j,k}^{q} \lambda_\alpha \chi^2_{1,\alpha} + \lambda_i \chi^2_{3,i} + \lambda_j \chi^2_{3,j} + \lambda_k \chi^2_{3,k},$$

where each $\chi^2_{f,\alpha}$'s are independent chi-squared random variables with $f$ degrees of freedom. From Johnson and Kotz (1970), pp. 149, the following equations hold.

$$\mathrm{P}(T^{(w)} \leq x) = F_q(x; \Lambda), \qquad \mathrm{P}(T_i^{(w)} \leq x) = F_{q+2}(x; \Lambda_i),$$

$$\mathrm{P}(T_{ij}^{(w)} \leq x) = F_{q+4}(x; \Lambda_{ij}), \qquad \mathrm{P}(T_{ijk}^{(w)} \leq x) = F_{q+6}(x; \Lambda_{ijk}).$$

Explicit forms of $F(\cdot)$ can be given by

$$F_q(x; \Lambda) = \sum_{l=0}^{\infty} e_l(\Lambda) G_{q+2l}(x/\beta),$$

$$e_r(\Lambda) = \begin{cases} \prod_{l=1}^{q} (\beta/\lambda_l)^{1/2}, & (r = 0) \\ (2r)^{-1} \sum_{l=0}^{r-1} H_{r-l}(\Lambda) e_l(\Lambda), & (r \geq 1) \end{cases},$$

$$H_r(\Lambda) = \sum_{l=1}^{q} (1 - \beta/\lambda_l)^r,$$

$$F_{q+2}(x; \Lambda_i) = \sum_{l=0}^{\infty} e_l(\Lambda_i) G_{q+2+2l}(x/\beta),$$

$$e_r(\Lambda_i) = \begin{cases} (\beta/\lambda_i) \prod_{l=1}^{q} (\beta/\lambda_l)^{1/2}, & (r = 0) \\ (2r)^{-1} \sum_{l=0}^{r-1} H_{r-l}(\Lambda_i) e_l(\Lambda_i), & (r \geq 1) \end{cases},$$

$$H_r(\Lambda_i) = \sum_{l=1}^{q}(1-\beta/\lambda_l)^r + 2(1-\beta/\lambda_i)^r,$$

$$F_{q+4}(x;\Lambda_{ij}) = \sum_{l=0}^{\infty} e_l(\Lambda_{ij})G_{q+4+2l}(x/\beta),$$

$$e_r(\Lambda_{ij}) = \begin{cases} (\beta/\lambda_i)(\beta/\lambda_j)\prod_{l=1}^{q}(\beta/\lambda_l)^{1/2}, & (r=0) \\ (2r)^{-1}\sum_{l=0}^{r-1} H_{r-l}(\Lambda_{ij})e_l(\Lambda_{ij}), & (r \geq 1) \end{cases},$$

$$H_r(\Lambda_{ij}) = \sum_{l=1}^{q}(1-\beta/\lambda_l)^r + 2(1-\beta/\lambda_i)^r + 2(1-\beta/\lambda_j)^r,$$

$$F_{q+6}(x;\Lambda_{ijk}) = \sum_{l=0}^{\infty} e_l(\Lambda_{ijk})G_{q+6+2l}(x/\beta),$$

$$e_r(\Lambda_{ijk}) = \begin{cases} (\beta/\lambda_i)(\beta/\lambda_j)(\beta/\lambda_k)\prod_{l=1}^{q}(\beta/\lambda_l)^{1/2}, & (r=0) \\ (2r)^{-1}\sum_{l=0}^{r-1} H_{r-l}(\Lambda_{ijk})e_l(\Lambda_{ijk}), & (r \geq 1) \end{cases},$$

$$H_r(\Lambda_{ijk}) = \sum_{l=1}^{q}(1-\beta/\lambda_l)^r + 2(1-\beta/\lambda_i)^r + 2(1-\beta/\lambda_j)^r + 2(1-\beta/\lambda_k)^r.$$

## 4. Expansion of the expectation of $T$

The result in Theorem 3.1 has very complicated form. However, the expansion of $\mathrm{E}(T)$ under the null hypothesis can be expressed simply as follows.

Substituting (2.2) to (3.1) yields

$$\begin{aligned} T = {} & \frac{1}{2}\operatorname{tr}(V^2) - \frac{1}{\sqrt{n}}\left[\frac{1}{3}\operatorname{tr}(V^3) + \operatorname{tr}\{V(zz' - I_p)\}\right] \\ & + \frac{1}{n}\left[\frac{1}{2}\operatorname{tr}(V^2) + \frac{1}{4}\operatorname{tr}(V^4) + \frac{1}{2}\operatorname{tr}\{(zz' - I_p)^2\} + \operatorname{tr}\{V^2(zz' - I_p)\}\right] \\ & + \mathrm{O}_p(n^{-3/2}). \end{aligned} \tag{4.1}$$

Note that

$$\mathrm{E}[\operatorname{tr}(V^2)] = \kappa_4^{(1)} + 2q,$$

$$\mathrm{E}[\operatorname{tr}(V^3)] = \frac{1}{\sqrt{n}}\{\kappa_6^{(1)} + 2(2\kappa_{3,3}^{(1)} + 3\kappa_{3,3}^{(2)}) + 3(p+3)\kappa_4^{(1)} + p(p^2+3p+4)\},$$

$$\mathrm{E}[\mathrm{tr}(V^4)] = \kappa_{4,4}^{(1)} + 2\kappa_{4,4}^{(2)} + 4(p+2)\kappa_4^{(1)} + p(2p^2+5p+5),$$

$$\mathrm{E}[\mathrm{tr}\{(\boldsymbol{z}\boldsymbol{z}' - I_p)^2\}] = 2q + \mathrm{O}(n^{-1}),$$

$$\mathrm{E}[\mathrm{tr}\{V(\boldsymbol{z}\boldsymbol{z}' - I_p)\}] = \frac{1}{\sqrt{n}}(\kappa_4^{(1)} + 2q),$$

$$\mathrm{E}[\mathrm{tr}\{V^2(\boldsymbol{z}\boldsymbol{z}' - I_p)\}] = \kappa_{3,3}^{(1)} + \kappa_{3,3}^{(2)} + \mathrm{O}(n^{-1}),$$

where

$$\begin{aligned} \kappa_4^{(1)} &= \sum_{ab}^{p} \kappa_{aabb}, & \kappa_{3,3}^{(1)} &= \sum_{abc}^{p} \kappa_{abc}^2 & \kappa_{3,3}^{(2)} &= \sum_{abc}^{p} \kappa_{aab}\kappa_{bcc} \\ \kappa_6^{(1)} &= \sum_{abc}^{p} \kappa_{aabbcc}, & \kappa_{4,4}^{(1)} &= \sum_{abcd}^{p} \kappa_{abcd}^2, & \kappa_{4,4}^{(2)} &= \sum_{abcd}^{p} \kappa_{aabc}\kappa_{bcdd}. \end{aligned} \tag{4.2}$$

By using these results, we obtain the following Theorem.

Theorem 4.1. *If $\boldsymbol{x}_j$ has the sixth moment, the asymptotic expectation of $T$ can be given by*

$$\begin{aligned} \mathrm{E}(T) = q + \frac{\kappa_4^{(1)}}{2} + \frac{1}{12n}\{3(\kappa_{4,4}^{(1)} + 2\kappa_{4,4}^{(2)}) - 4(\kappa_6^{(1)} + \kappa_{3,3}^{(1)} + 3\kappa_{3,3}^{(2)}) \\ - 18\kappa_4^{(1)} + p(2p^2+3p-1)\} + \mathrm{o}(n^{-1}), \end{aligned} \tag{4.3}$$

*where each $\kappa$'s are given by* (4.2).

From Theorem 4.1, we can use the Bartlett correction of $T$ when estimators of higher order cumulants are obtained.

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