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Isolas: compact solution components separated away from a given equilibrium curve

Santiago CANO-CASANOVA, Julián LÓPEZ-GÓMEZ and
Marcela MOLINA-MEYER

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ABSTRACT. In this paper some continuation techniques based on the implicit function theorem are combined with the topological degree to show the existence of solution isolas, with respect to a given state, for a class of weighted boundary value problems of superlinear indefinite elliptic type. No result of this nature seems to be available in the literature. Further, pseudo-spectral methods coupled with path following solvers are used to compute these isolas in some simple one-dimensional prototype models.

1. Introduction

Unexpected events often provoke drastic changes of mind in human beings and mathematics do not escape from this general principle. As a matter of fact, within the context of Nonlinear Analysis, one of most surprising features is the emergence of solution *isolas* in semilinear elliptic boundary value problems; perhaps because at present no general scheme to generate them has been described in the specialized literature. As finding out an isola with respect to a given state requires the use of two parameters—one is needed for obtaining a bounded component of the solution set as bifurcating from the given state, the other for isolating the component itself from the given state—, the immersion of a given particular problem into a two-dimensional variety of problems is needed for generating solution isolas; perhaps a revolutionary idea, far from being well understood yet.

This paper addresses this problem in its full generality. Section 2 makes precise the concept of isola in the context of abstract Nonlinear Analysis and uses the topological degree to give a multiplicity result. Then, in Section 3, the abstract theory of Section 2 is applied to show the existence of a one-parameter family of isolas in a general class of weighted boundary value problems of super-linear indefinite elliptic type. Finally, in Section 4, we will discuss the

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results of some numerical computations that we have carried for a simple one-dimensional prototype model. Although the abstract results of this paper apply in much more general contexts than the one considered in Section 3, for the sake of clarity in the exposition, we have refrained ourselves of giving the most general results.

2. The concept of isola and a multiplicity result

Suppose U is a real Banach space, denote by $\mathscr{L}(U)$ the space of linear continuous operators in U, and consider a continuous map

$$\mathfrak{F} : \mathbf{R} \times U \to U$$

of the form

$$\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u + \mathfrak{N}(\lambda, u),$$

where

- $\mathfrak{L} : \mathbf{R} \to \mathscr{L}(U)$ is a continuous map such that $\mathfrak{L}(\lambda) - I$ is compact for each $\lambda \in \mathbf{R}$, where I denotes the identity operator of U.
- $\mathfrak{N} : \mathbf{R} \times U \to U$ is a compact operator such that

$$\lim_{u \to 0} \sup_{\lambda \in K} \frac{\mathfrak{N}(\lambda, u)}{\|u\|} = 0 \tag{2.1}$$

  for every compact interval $K \subset \mathbf{R}$.

Our main goal in this section is analyzing some fine properties of the *compact isolas* of the set of non-trivial solutions of

$$\mathfrak{F}(\lambda, u) = 0. \tag{2.2}$$

Note that $(\lambda, u) = (\lambda, 0)$ solves Equation (2.2) for each $\lambda \in \mathbf{R}$. This is why any solution of the form $(\lambda, 0)$ will be referred to as a *trivial solution*, while solutions of the form (λ, u) with $u \neq 0$ are said to be non-trivial solutions of (2.2). Most precisely, though it may contain some trivial solution, the set of non-trivial solutions of Equation (2.2) is defined by

$$\mathfrak{S} := \{(\lambda, u) \in \mathbf{R} \times (U \setminus \{0\}) : \mathfrak{F}(\lambda, u) = 0\} \cup (\Sigma \times \{0\}), \tag{2.3}$$

where $\Sigma \subset \mathbf{R}$ stands for the *real spectrum of the family* $\mathfrak{L}(\lambda)$, i.e., $\sigma \in \Sigma$ if and only if $\sigma \in \mathbf{R}$ and $\mathfrak{L}(\sigma)$ has a non-trivial kernel. Since $\mathfrak{L}(\lambda)$ is Fredholm of index zero, by the open mapping theorem, $\mathfrak{L}(\lambda)$ is an isomorphism if $\lambda \in \mathbf{R} \setminus \Sigma$. Within the general setting of this paper, it is well known that Σ is a closed subset of $\mathbf{R}$ and that all bifurcation values of λ to non-trivial solutions of (2.2) from the trivial solution $(\lambda, 0)$ must lie in Σ. Actually, by the continuity of $\mathfrak{F}$, $\mathfrak{S}$ is a closed subset of $\mathbf{R} \times U$ (cf. [10, Section 6.1]). The set $\mathfrak{S}$ consists of all non-trivial solutions of (2.2) plus all possible bifurcation points from the trivial solution curve $(\lambda, 0)$.

Throughout this work, as usual within the context of global bifurcation theory, by a *component* of $\mathfrak{S}$ it is meant a maximal (for the inclusion) closed and connected subset of $\mathfrak{S}$. A bounded component $\mathfrak{C}$ of $\mathfrak{S}$, necessarily compact, is said to be an *isola*—with respect to the trivial solution $(\lambda, 0)$—if

$$\mathfrak{C} \cap (\Sigma \times \{0\}) = \emptyset, \tag{2.4}$$

i.e., if it is bounded away from the trivial solution. All isolas considered in this paper will be understood as isolas with respect to the trivial solution curve $(\lambda, 0)$. The following concept plays a crucial role in the abstract theory developed in this section.

Definition 2.1. *Suppose $\mathfrak{C} \subset \mathfrak{S}$ is an isola. A bounded open set $\Omega \subset \mathbf{R} \times U$ is said to be an open isolating neighborhood of $\mathfrak{C}$ if*

$$\mathfrak{C} \subset \Omega \qquad \textit{and} \qquad \partial\Omega \cap \mathfrak{S} = \emptyset. \tag{2.5}$$

The following result establishes the existence of open isolating neighborhoods. Throughout this paper, we will denote by $B_R(\lambda, u)$ the ball of radius $R > 0$ centered at $(\lambda, u) \in \mathbf{R} \times U$, and by $B_R(u)$ the ball of radius R centered at $u \in U$. If $u = 0$, we simply set $B_R := B_R(0)$.

Proposition 2.2. *Suppose $\mathfrak{C} \subset \mathfrak{S}$ is an isola. Then, for each $\varepsilon > 0$, $\mathfrak{C}$ possesses an open isolating neighborhood $\Omega \subset \mathbf{R} \times U$ such that*

$$\Omega \subset \mathfrak{C} + B_\varepsilon(0, 0).$$

Proof. Thanks to (2.4), for any sufficiently small $\delta > 0$, the open neighborhood

$$\mathscr{U} := \mathfrak{C} + B_\delta(0, 0)$$

satisfies

$$\bar{\mathscr{U}} \cap (\Sigma \times \{0\}) = \emptyset.$$

Fix one of these δ's. If $\partial\mathscr{U} \cap \mathfrak{S} = \emptyset$, then $\mathscr{U}$ provides us with an open isolating neighborhood of $\mathfrak{C}$, but, in general, this does not occur. So, suppose

$$\partial\mathscr{U} \cap \mathfrak{S} \neq \emptyset$$

and set

$$M := \bar{\mathscr{U}} \cap \mathfrak{S}, \qquad A := \mathfrak{C}, \qquad B := \partial\mathscr{U} \cap \mathfrak{S}.$$

Then, M is a compact metric space and A, B are two disjoint compact non-empty subsets of M. By the maximality of $\mathfrak{C}$, no subcontinuum of M connects A with B. Thus, by a celebrated result attributable to G. T. Whyburn [12], there exist two disjoint compact subsets of M, M_A and M_B, such that

$$A \subset M_A, \qquad B \subset M_B, \qquad M = M_A \cup M_B.$$

Then, the open set

$$\Omega := M_A + B_\eta(0,0)$$

provides us with an open isolating neighborhood of $\mathfrak{C}$ for any sufficiently small $\eta > 0$. Indeed, by construction,

$$A = \mathfrak{C} \subset M_A \subset \Omega.$$

Moreover, since $\operatorname{dist}(M_A, M_B) > 0$ and $M = M_A \cup M_B$, for any sufficiently small $\eta > 0$,

$$\partial\Omega \cap M = \emptyset.$$

Finally, since $M_A \subset \mathscr{U}$, for any sufficiently small $\eta > 0$ we have that $\partial\Omega \subset \mathscr{U}$, and, hence,

$$\emptyset = \partial\Omega \cap M = \partial\Omega \cap \bar{\mathscr{U}} \cap \mathfrak{S} = \partial\Omega \cap \mathfrak{S}.$$

Therefore, Ω is an open isolating neighborhood of $\mathfrak{C}$ in $\mathbf{R} \times U$. $\square$

Now, we will use Proposition 2.2 to obtain an important property satisfied by any isola of $\mathfrak{S}$. To state the corresponding result, we need to introduce some notations. For any subset $S \subset \mathbf{R} \times U$ and $\lambda \in \mathbf{R}$ we will denote

$$S_\lambda := \{u \in U : (\lambda, u) \in S\},$$

and $\mathscr{P}_\lambda$ will stand for the projection on the λ-component, i.e., $\mathscr{P}_\lambda(\lambda, u) := \lambda$ for each $(\lambda, u) \in \mathbf{R} \times U$.

PROPOSITION 2.3. *Suppose $\mathfrak{C} \subset \mathfrak{S}$ is an isola and let Ω be any open isolating neighborhood of $\mathfrak{C}$. Then, for each $\lambda \in \mathbf{R}$,*

$$\operatorname{Deg}(\mathfrak{F}(\lambda, \cdot), \Omega_\lambda) = 0, \tag{2.6}$$

where $\operatorname{Deg}(\mathfrak{F}(\lambda, \cdot), \Omega_\lambda)$ denotes the topological degree of $\mathfrak{F}(\lambda, \cdot)$ in $\Omega_\lambda \subset U$.

PROOF. Since $\mathfrak{C}$ is compact, there exist $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$, such that $\mathfrak{C} \subset$

$(\alpha, \beta) \times U$. Then, by the homotopy invariance of the topological degree, we have that, for each $\lambda \in \mathbf{R}$,

$$\mathrm{Deg}(\mathfrak{F}(\lambda, \cdot), \Omega_\lambda) = \mathrm{Deg}(\mathfrak{F}(\alpha, \cdot), \Omega_\alpha) = 0,$$

because $\partial\Omega \cap \mathfrak{S} = \emptyset$. □

The main result of this section reads as follows.

Theorem 2.4. *Suppose* $\mathfrak{C} \subset \mathfrak{S}$ *is an isola and set*

$$\lambda_* := \min\{\lambda \in \mathbf{R} : \mathfrak{C}_\lambda \neq \emptyset\}, \qquad \lambda^* := \max\{\lambda \in \mathbf{R} : \mathfrak{C}_\lambda \neq \emptyset\}.$$

Note that, necessarily, $\mathscr{P}_\lambda \mathfrak{C} = [\lambda_*, \lambda^*]$, *since* $\mathfrak{C}$ *is compact and connected. Moreover,* $\lambda_* \leq \lambda^*$*—the equality* $\lambda_* = \lambda^*$ *might occur.*

Suppose, in addition, that $\lambda_* < \lambda^*$ *and that there exist* $\lambda_1, \lambda_2 \in (\lambda_*, \lambda^*)$, $\lambda_1 < \lambda_2$, *and a continuous curve* $\gamma \in \mathscr{C}([\lambda_1, \lambda_2]; U)$ *such that*

(A1) *The graph of* γ,

$$\Gamma := \{(\lambda, \gamma(\lambda)) : \lambda \in [\lambda_1, \lambda_2]\},$$

is contained in the isola $\mathfrak{C}$ *and there exists* $\varepsilon_0 > 0$ *such that*

$$[\Gamma + B_{\varepsilon_0}(0,0)] \cap \mathfrak{S} \cap ([\lambda_1, \lambda_2] \times U) = \Gamma.$$

(A2) *There exists* $\lambda_0 \in [\lambda_1, \lambda_2]$ *such that*

$$\mathrm{Ind}(\gamma(\lambda_0), \mathfrak{F}(\lambda_0, \cdot)) \in \{-1, 1\}, \tag{2.7}$$

where Ind *denotes the index—local topological degree—of an isolated solution.*

Then, for each $\lambda \in [\lambda_1, \lambda_2]$,

$$(\mathfrak{C} \backslash \Gamma)_\lambda \neq \emptyset,$$

and, therefore, $\mathfrak{C}_\lambda$ *possesses, at least, two nontrivial solutions of* (2.2).

Assumption (A1) establishes that Γ is an *isolated* arc of curve within $\mathfrak{C}$. By the homotopy invariance of the topological degree, Assumption (A2) guarantees that the index of any solution along Γ equals either 1, or -1.

Proof of Theorem 2.4. As we have just commented,

$$\mathrm{Ind}(\gamma(\lambda), \mathfrak{F}(\lambda, \cdot)) = \mathrm{Ind}(\gamma(\lambda_0), \mathfrak{F}(\lambda_0, \cdot)) \in \{-1, 1\} \qquad \forall \lambda \in [\lambda_1, \lambda_2]. \tag{2.8}$$

Now, pick $\varepsilon \in (0, \varepsilon_0)$ and let Ω be any open isolating neighborhood $\Omega \subset \mathbf{R} \times U$ such that

$$\Omega \subset \mathfrak{C} + B_\varepsilon(0,0).$$

The existence of Ω is guaranteed by Proposition 2.2. Thanks to Proposition 2.3,

$$\operatorname{Deg}(\mathfrak{F}(\lambda,\cdot),\Omega_\lambda) = 0 \qquad \forall\lambda \in [\lambda_1,\lambda_2].$$

Moreover, due to the additivity property of the degree, for each $\lambda \in [\lambda_1,\lambda_2]$,

$$\operatorname{Deg}(\mathfrak{F}(\lambda,\cdot),\Omega_\lambda) = \operatorname{Deg}(\mathfrak{F}(\lambda,\cdot),\Omega_\lambda\backslash\bar{B}_\varepsilon(\gamma(\lambda))) + \operatorname{Ind}(\gamma(\lambda),\mathfrak{F}(\lambda,\cdot))$$

and, hence, thanks to (2.8),

$$\operatorname{Deg}(\mathfrak{F}(\lambda,\cdot),\Omega_\lambda\backslash\bar{B}_\varepsilon(\gamma(\lambda))) = -\operatorname{Ind}(\gamma(\lambda),\mathfrak{F}(\lambda,\cdot)) \tag{2.9}$$

equals either -1, or 1, for each $\lambda \in [\lambda_1,\lambda_2]$. Consequently, for each $\lambda \in [\lambda_1,\lambda_2]$ there exists $u_\lambda \in \Omega_\lambda\backslash\bar{B}_\varepsilon(\gamma(\lambda))$ such that

$$\mathfrak{F}(\lambda,u_\lambda) = 0,$$

though, in general, one cannot guarantee that $(\lambda,u_\lambda) \in \mathfrak{C}$. Thus, in order to prove $(\mathfrak{C}\backslash\Gamma)_\lambda \neq \varnothing$ a further argument is needed.

First, we show that for any compact interval $J := [\mu_1,\mu_2] \subset [\lambda_1,\lambda_2]$, $\mu_1 < \mu_2$, there exists $\mu_0 \in [\mu_1,\mu_2]$ such that

$$\mathfrak{C}_{\mu_0}\backslash\{\gamma(\mu_0)\} \neq \varnothing.$$

In order to prove this we proceed by contradiction. So, suppose there exists $J := [\mu_1,\mu_2] \subset [\lambda_1,\lambda_2]$, $\mu_1 < \mu_2$, such that

$$\mathfrak{C}_\lambda\backslash\{\gamma(\lambda)\} = \varnothing \qquad \forall\lambda \in [\mu_1,\mu_2]. \tag{2.10}$$

Pick

$$\varepsilon \in \left(0, \min\left\{\varepsilon_0, \frac{\mu_2-\mu_1}{3}\right\}\right)$$

and let Ω be any open isolating neighborhood of $\mathfrak{C}$ such that $\Omega \subset \mathfrak{C} + B_\varepsilon(0,0)$. Thanks to (2.10), for each $\lambda \in (\mu_1+\varepsilon, \mu_2-\varepsilon)$ one has that

$$\Omega_\lambda\backslash\bar{B}_\varepsilon(\gamma(\lambda)) = \varnothing$$

and, hence,

$$\operatorname{Deg}(\mathfrak{F}(\lambda,\cdot),\Omega_\lambda\backslash\bar{B}_\varepsilon(\gamma(\lambda))) = 0,$$

which is impossible, because of (2.9). This contradiction shows the validity of the claim above.

To complete the proof of the theorem, pick $\lambda \in [\lambda_1, \lambda_2]$ and consider a sequence of compact intervals of positive length, J_n, $n \geq 1$, such that

$$J_{n+1} \subset J_n \subset [\lambda_1, \lambda_2], \qquad n \geq 1,$$

and

$$\bigcap_{n=1}^{\infty} J_n = \{\lambda\}.$$

We already know that, for each $n \geq 1$, there exists $\rho_n \in J_n$ such that

$$\mathfrak{C}_{\rho_n} \backslash \{\gamma(\rho_n)\} \neq \emptyset.$$

Now, for each $n \geq 1$, pick $u_n \in \mathfrak{C}_{\rho_n} \backslash \{\gamma(\rho_n)\}$. Then, thanks to (A1), the sequence,

$$\{(\rho_n, u_n)\}_{n \geq 1} \subset \mathfrak{C} \backslash \Gamma$$

possesses a convergent subsequence, labeled again by n, to some point $(\lambda_\infty, u_\infty) \in \mathfrak{C} \backslash \Gamma$. Moreover, by construction,

$$\lim_{n \to \infty} \rho_n = \lambda,$$

and, hence, $\lambda_\infty = \lambda$. Therefore,

$$u_\infty \in \mathfrak{C}_\lambda \backslash \{\gamma(\lambda)\},$$

which concludes the proof. □

3. The existence of isolas in semilinear elliptic b.v.p's

In this section we construct a general class of semilinear weighted boundary value problems of the form

$$\begin{cases} \mathscr{L} u = \lambda V u - (A - \varepsilon B) u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

exhibiting an isola of positive solutions—in λ—for each sufficiently small $\varepsilon > 0$. To carry out the mathematical analysis of this problem we make the following assumptions:

A1. Ω is a bounded domain of $\mathbf{R}^N$, $N \geq 1$, of class $\mathscr{C}^{2+\nu}$, for some $\nu \in (0,1)$, $p > 1$, $\varepsilon \geq 0$, $\lambda \in \mathbf{R}$, $\mathscr{L}$ is a second order uniformly elliptic operator in $\bar{\Omega}$ of the form

$$\mathscr{L} := -\sum_{i,j=1}^{N} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i(x)\frac{\partial}{\partial x_i} + \alpha_0(x) \tag{3.2}$$

where $\alpha_{ij} = \alpha_{ji} \in \mathscr{C}^{\nu}(\bar{\Omega})$, $\alpha_i, \alpha_0 \in \mathscr{C}^{\nu}(\bar{\Omega})$, $1 \le i, j \le N$.

A2. $A, B \in \mathscr{C}^{\nu}(\bar{\Omega})$, are two positive functions with disjoint supports,

$$K_A^+ := \operatorname{supp} A, \qquad K_B^+ := \operatorname{supp} B,$$

such that

$$\Omega_A^0 := \Omega \backslash K_A^+, \qquad \Omega_B^0 := \Omega \backslash K_B^+,$$

are two proper open subsets of Ω of class $\mathscr{C}^{2+\nu}$ with a finite number of separated components.

A3. Either $N \in \{1, 2\}$, or else $N \ge 3$, $p < \frac{N+2}{N-2}$, and, for some constant $\gamma \ge \frac{2N}{N-2}$,

$$\frac{B}{[\operatorname{dist}(\cdot, \partial K_B^+)]^{\gamma}} \in \mathscr{C}(K_B^+; (0, \infty)).$$

A4. $V \in \mathscr{C}^{\nu}(\bar{\Omega})$ is a function changing of sign in Ω_A^0 and satisfying

$$\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega] < 0 < \max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega_A^0]. \tag{3.3}$$

Subsequently, given any elliptic operator L in a smooth domain D, we denote by $\sigma_1[L; D]$ the principal eigenvalue of L in D under homogeneous Dirichlet boundary conditions.

The existence of $\mathscr{L}, A, B$ and V satisfying (A1)–(A4) can be guaranteed as follows. Choose A satisfying the requirements of (A2), V changing of sign in Ω_A^0, and any elliptic operator $\mathscr{L}_0$ in $\bar{\Omega}$. Then,

$$\lim_{\lambda \to \pm\infty} \sigma_1[\mathscr{L}_0 - \lambda V; \Omega_A^0] = -\infty$$

and, due to the monotonicity of the principal eigenvalue with respect to the domain,

$$\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L}_0 - \lambda V; \Omega] < \max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L}_0 - \lambda V; \Omega_A^0].$$

Thus, the elliptic operator $\mathscr{L}$ defined by

$$\mathscr{L} := \mathscr{L}_0 - \frac{1}{2}\left\{\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L}_0 - \lambda V; \Omega] + \max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L}_0 - \lambda V; \Omega_A^0]\right\}$$

satisfies (A4). Finally, choose any B satisfying (A2) and (A3). For these choices, all requirements of this section are satisfied.

For any $D \in \{\Omega, \Omega_A^0\}$, consider the linear weighted boundary value problem

$$\begin{cases} \mathscr{L}\varphi = \lambda V \varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{3.4}$$

Thanks to (3.3), Problem (3.4) does not admit a principal eigenvalue λ if $D = \Omega$, while it possesses two, denoted by $\lambda_1^0 < \lambda_2^0$, if $D = \Omega_A^0$ (cf. [9] and the references there in). When $\sigma_1[\mathscr{L}; \Omega_A^0] > 0$, this result is attributable to P. Hess and T. Kato [6]. Within the setting of [6] one necessarily has $\lambda_1^0 < 0 < \lambda_2^0$, although in our general setting λ_1^0 and λ_2^0 can have the same sign. The following result is well known (cf. e.g., [5]).

Theorem 3.1. *Suppose* $\varepsilon = 0$. *Then,* (3.1) *possesses a positive solution if, and only if,*

$$\lambda_1^0 < \lambda < \lambda_2^0. \tag{3.5}$$

Moreover, it is unique and linearly asymptotically stable, if it exists, i.e., the principal eigenvalue of the linearization of (3.1) *around it is always positive. Actually, the positive solution attracts to all positive solutions of the parabolic counterpart of* (3.1). *Furthermore, if we denote the positive solution by* $\theta_{[\lambda,0]}$, *then*

$$\lim_{\lambda \downarrow \lambda_1^0} \|\theta_{[\lambda,0]}\|_{L_\infty(\Omega)} = \infty = \lim_{\lambda \uparrow \lambda_2^0} \|\theta_{[\lambda,0]}\|_{L_\infty(\Omega)},$$

and, therefore, λ_1^0 *and* λ_2^0 *are bifurcation values from infinity to positive solutions of* (3.1). *Note that, thanks to the first inequality of* (3.3), *zero is always linearly unstable.*

Figure 1(c) of [5] shows the bifurcation diagram of positive solutions of (3.1) in case $\varepsilon = 0$. The a priori bounds of [1] and [2] show that the curve of positive solutions $(\lambda, \theta_{[\lambda,0]})$, $\lambda_1^0 < \lambda < \lambda_2^0$, at $\varepsilon = 0$, becomes into an isola of solutions of (3.1) for each sufficiently small $\varepsilon > 0$. This is the main result of this section, which can be stated as follows.

Theorem 3.2. *Suppose* (A1)–(A4). *Then, for each*

$$\delta \in (0, (\lambda_2^0 - \lambda_1^0)/2) \tag{3.6}$$

there exists $\varepsilon_0 := \varepsilon_0(\delta) > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the problem* (3.1) *possesses an isola of positive solutions* $\mathfrak{C}_\varepsilon^+$, *satisfying the following properties:*

(a) *Setting*

$$\lambda_{*,\varepsilon} := \min_{\lambda \in \mathbf{R}} \mathscr{P}_\lambda(\mathfrak{C}_\varepsilon^+), \qquad \lambda^{*,\varepsilon} := \max_{\lambda \in \mathbf{R}} \mathscr{P}_\lambda(\mathfrak{C}_\varepsilon^+), \tag{3.7}$$

one has that

$$\lambda_1^0 < \lambda_{*,\varepsilon} < \lambda_1^0 + \delta < \lambda_2^0 - \delta < \lambda^{*,\varepsilon} < \lambda_2^0. \tag{3.8}$$

(b) *For each* $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$, $\mathfrak{C}_\varepsilon^+$ *possesses, at least, two positive solutions.*

(c) *For each* $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$, $\mathfrak{C}_\varepsilon^+$ *possesses, at least, one linearly asymptotically stable solution.*

Proof. Fix any constant

$$K > -\sigma_1[\mathscr{L};\Omega]$$

and consider, for each $(\lambda, \varepsilon) \in \mathbf{R}^2$, the operators

$$\mathfrak{L}(\lambda) := \mathscr{J}(\mathscr{L} + K)^{-1}[\lambda V + K] - I : \mathscr{C}_0^\nu(\bar{\Omega}) \to \mathscr{C}_0^\nu(\bar{\Omega}),$$

and

$$\mathfrak{N}(\lambda, \cdot, \varepsilon) := -\mathscr{J}(\mathscr{L} + K)^{-1}[(A - \varepsilon B)(\cdot)^p] : \mathscr{C}_0^\nu(\bar{\Omega}) \to \mathscr{C}_0^\nu(\bar{\Omega}),$$

where I is the identity map of $\mathscr{C}_0^\nu(\bar{\Omega})$ and $\mathscr{J}$ stands for the compact imbedding $\mathscr{C}_0^{2+\nu}(\bar{\Omega}) \hookrightarrow \mathscr{C}_0^\nu(\bar{\Omega})$. Then, by Schauder's theory, the positive solutions of (3.1) are given by the positive solutions of the nonlinear abstract equation

$$\mathfrak{F}(\lambda, u, \varepsilon) := \mathfrak{L}(\lambda)u + \mathfrak{N}(\lambda, u, \varepsilon) = 0, \qquad (\lambda, u, \varepsilon) \in X := \mathbf{R} \times \mathscr{C}_0^\nu(\bar{\Omega}) \times \mathbf{R}.$$

Note that, for each $\varepsilon \geq 0$, this equation fits into the abstract setting of Section 2. Moreover, since $p > 1$, $\mathfrak{F} \in \mathscr{C}^1(X; \mathscr{C}_0^\nu(\bar{\Omega}))$, and, given any solution $(\lambda, \theta, \varepsilon)$ of (3.1), $(\lambda, \theta, \varepsilon)$ is linearly asymptotically stable if, and only if,

$$\operatorname{spr}(I + D_u\mathfrak{F}(\lambda, \theta, \varepsilon)) < 1.$$

Note that the linearization

$$I + D_u\mathfrak{F}(\lambda, \theta, \varepsilon) = \mathscr{J}(\mathscr{L} + K)^{-1}\{[\lambda V + K - p(A - \varepsilon B)\theta^{p-1}]\cdot\}$$

is compact.

Let $\mathfrak{C}_0^+$ denote the component of positive solutions of (3.1) for $\varepsilon = 0$. Thanks to Theorem 3.1, $\mathfrak{C}_0^+$ is the $\mathscr{C}^1$ curve of X consisting of all points of the form $(\lambda, \theta_{[\lambda,0]}, 0)$ with $\lambda_1^0 < \lambda < \lambda_2^0$. Now, fix δ satisfying (3.6) and consider the compact arc of curve of class $\mathscr{C}^1$

$$\Gamma_0 := \{(\lambda, \theta_{[\lambda,0]}, 0) : \lambda_1^0 + \delta \leq \lambda \leq \lambda_2^0 - \delta\} \subset \mathfrak{C}_0^+.$$

Thanks to the implicit function theorem, there exist $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a map of class $\mathscr{C}^1$,

$$\Theta : [\lambda_1^0 + \delta, \lambda_2^0 - \delta] \times [-\varepsilon_0, \varepsilon_0] \subset \mathbf{R}^2 \to \mathscr{C}_0^\nu(\bar{\Omega}),$$

such that:

(C1) For each $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$, $\Theta(\lambda, 0) = \theta_{[\lambda,0]}$.

(C2) For each $(\lambda, \varepsilon) \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta] \times [-\varepsilon_0, \varepsilon_0]$, $\mathfrak{F}(\lambda, \Theta(\lambda, \varepsilon), \varepsilon) = 0$, i.e., $(\lambda, \Theta(\lambda, \varepsilon), \varepsilon)$ provides us with a solution of (3.1).

(C3) There exists $\rho > 0$ such that if $\mathfrak{F}(\lambda, u, \varepsilon) = 0$ for some $(\lambda, u, \varepsilon) \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta] \times \mathscr{C}_0^\nu(\bar{\Omega}) \times [-\varepsilon_0, \varepsilon_0]$ with

$$\|u - \theta_{[\lambda,0]}\|_{\mathscr{C}(\bar{\Omega})} \le \rho, \tag{3.9}$$

then $u = \Theta(\lambda, \varepsilon)$.

By elliptic regularity (3.9) implies an equivalent estimate in $\mathscr{C}_0^\nu(\bar{\Omega})$. Moreover, since

$$\lim_{\varepsilon \to 0} \|\Theta(\lambda, \varepsilon) - \theta_{[\lambda,0]}\|_{\mathscr{C}_0(\bar{\Omega})} = 0 \qquad \text{uniformly in } \lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta],$$

reducing ε_0, if necessary, we can assume that $(\lambda, \Theta(\lambda, \varepsilon), \varepsilon)$ is a positive solution of (3.1) for each $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Actually, by the continuity of the map

$$\varrho(\lambda, \varepsilon) := \operatorname{spr}(I + D_u \mathfrak{F}(\lambda, \Theta(\lambda, \varepsilon), \varepsilon)), \tag{3.10}$$

one can make a further reduction of ε_0, if necessary, so that

$$\varrho(\lambda, \varepsilon) \le \omega < 1, \qquad (\lambda, \varepsilon) \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta] \times [-\varepsilon_0, \varepsilon_0], \tag{3.11}$$

for some constant $\omega < 1$. Although the continuity of the map (3.10) can be obtained from the abstract theory of T. Kato [7], in our particular setting it is an easy consequence from the continuity of the principal eigenvalue of the linearization of (3.1) at $\Theta(\lambda, \varepsilon)$ with respect to the underlying potential. Subsequently, we assume that ε_0 has been chosen to satisfy all these requirements, set

$$\theta_{[\lambda,\varepsilon]} := \Theta(\lambda, \varepsilon)$$

and fix $\varepsilon \in (0, \varepsilon_0]$. Let $\mathfrak{C}_\varepsilon$ denote the component of the set of nontrivial solutions of $\mathfrak{F}(\lambda, u, \varepsilon) = 0$ containing $(\lambda, \theta_{[\lambda,\varepsilon]}, \varepsilon)$, $\lambda_1^0 + \delta \le \lambda \le \lambda_2^0 - \delta$, and let $\mathfrak{C}_\varepsilon^+$ be the subcomponent of $\mathfrak{C}_\varepsilon$ lying in the interior of the cone of positive functions. By the strong maximum principle, $\mathfrak{C}_\varepsilon = \mathfrak{C}_\varepsilon^+$, since the positive solutions of (3.1) cannot bifurcate from $(\lambda, 0, \varepsilon)$. To complete the proof of the theorem it suffices to show that $\mathscr{C}_\varepsilon^+$ is an isola satisfying the conditions (a), (b) and (c) of the statement.

Thanks to [1, Th. 3.3] and [2, Prop. 4.3], there exists $\eta > 0$ such that (3.1) does not admit a positive solution if

$$\lambda \in (-\infty, \lambda_1^0 + \eta) \cup (\lambda_2^0 - \eta, \infty).$$

Thus,

$$\mathscr{P}_\lambda(\mathfrak{C}_\varepsilon^+) \subset [\lambda_1^0 + \eta, \lambda_2^0 - \eta]. \tag{3.12}$$

Moreover, thanks to (A3), the positive solutions of (3.1) possess uniform L_∞-bounds in compact intervals of the parameter λ within $(\lambda_1^0, \lambda_2^0)$ (cf. e.g., [1] and [2]). Therefore, $\mathfrak{C}_\varepsilon^+$ is compact, and, consequently, $\mathfrak{C}_\varepsilon^+$ is an isola entirely filled in by positive solutions of (3.1)—with respect to the trivial state $(\lambda, 0, \varepsilon)$.

Now, let $\lambda_{*,\varepsilon}$ and $\lambda^{*,\varepsilon}$ be the quantities defined through (3.7). Since $\mathfrak{C}_\varepsilon^+$ is connected and $\mathscr{P}_\lambda$ is continuous,

$$\mathscr{P}_\lambda(\mathfrak{C}_\varepsilon^+) = [\lambda_{*,\varepsilon}, \lambda^{*,\varepsilon}]$$

and, hence, (3.12) implies

$$\lambda_1^0 < \lambda_{*,\varepsilon} \le \lambda_1^0 + \delta < \lambda_2^0 - \delta \le \lambda^{*,\varepsilon} < \lambda_2^0,$$

by construction. Actually, thanks to (3.11), we have that

$$\varrho(\lambda_1^0 + \delta, \varepsilon) < 1, \qquad \varrho(\lambda_2^0 - \delta, \varepsilon) < 1,$$

and, hence, the operators

$$D_u\mathfrak{F}(\lambda_1^0 + \delta, \theta_{[\lambda_1^0+\delta,\varepsilon]}, \varepsilon) \qquad \text{and} \qquad D_u\mathfrak{F}(\lambda_2^0 - \delta, \theta_{[\lambda_2^0-\delta,\varepsilon]}, \varepsilon)$$

are isomorphisms of $\mathscr{C}_0^\nu(\bar{\Omega})$. Therefore, thanks to the implicit function theorem, the compact arc of $\mathscr{C}^1$-curve of $\mathfrak{C}_\varepsilon^+$ defined by

$$\Gamma_\varepsilon := \{(\lambda, \theta_{[\lambda,\varepsilon]}, \varepsilon) : \lambda_1^0 + \delta \le \lambda \le \lambda_2^0 - \delta\}$$

extends to a compact arc of curve of class $\mathscr{C}^1$, say $\tilde{\Gamma}_\varepsilon$, such that

$$[\lambda_1^0 + \delta, \lambda_2^0 - \delta] \subset \operatorname{Int} \mathscr{P}_\lambda(\tilde{\Gamma}_\varepsilon).$$

As $\mathfrak{C}_\varepsilon^+$ is connected, necessarily $\tilde{\Gamma}_\varepsilon \subset \mathfrak{C}_\varepsilon^+$ and, hence,

$$[\lambda_1^0 + \delta, \lambda_2^0 - \delta] \subset \operatorname{Int} \mathscr{P}_\lambda(\mathfrak{C}_\varepsilon^+).$$

Therefore,

$$\lambda_{*,\varepsilon} < \lambda_1^0 + \delta < \lambda_2^0 - \delta < \lambda^{*,\varepsilon},$$

which shows (3.8) and completes the proof of Property (a).

As an immediate consequence from (3.10) and (3.11), for each $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$, $(\lambda, \theta_{[\lambda,\varepsilon]}, \varepsilon)$ is a linearly asymptotically stable solution of the parabolic counterpart of (3.1), and, consequently, Property (c) holds. Actually, thanks to Schauder's formula for the local index, for each $\lambda \in [\lambda_1^0 + \delta, \lambda_2^0 - \delta]$,

$$\mathrm{Ind}(\theta_{[\lambda,\varepsilon]}, \mathfrak{F}(\lambda, \cdot, \varepsilon)) = 1 \tag{3.13}$$

and, therefore, thanks to Theorem 2.4, $\mathfrak{C}_\varepsilon^+$ possesses, at least, two solutions. This shows Property (b) and concludes the proof of the theorem. □

4. Numerical experiments

In this section we treat numerically the following one-dimensional prototype model in $\Omega = (0,1)$:

$$\begin{cases} -u'' + \mu u = \lambda \sin(2\pi x)u - a(x)u^2, \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where

$$a(x) = \begin{cases} -0.2 \sin\left(\frac{\pi}{0.2}(0.2 - x)\right) & \text{if } 0 \le x \le 0.2, \\ \sin\left(\frac{\pi}{0.6}(x - 0.2)\right) & \text{if } 0.2 < x \le 0.8, \\ -0.2 \sin\left(\frac{\pi}{0.2}(x - 0.8)\right) & \text{if } 0.8 < x \le 1, \end{cases} \tag{4.2}$$

and $(\lambda, \mu) \in \mathbf{R}^2$ are regarded as two real parameters. Note that $a > 0$ in $(0.2, 0.8)$, $a < 0$ in $(0, 0.2) \cup (0.8, 1)$, and $a(0) = a(0.2) = a(0.8) = a(1) = 0$. For an adequate choice of the parameter μ, this problem fits into the abstract setting of Section 3 by choosing

$$\Omega := (0,1), \qquad \mathscr{L} := -\frac{d^2}{dx^2} + \mu, \qquad V := \sin(2\pi\cdot), \qquad p = 2,$$

and, e.g.,

$$A(x) := \begin{cases} 0 & \text{if } 0 \le x \le 0.2, \\ \sin\left(\frac{\pi}{0.6}(x - 0.2)\right) & \text{if } 0.2 < x \le 0.8, \\ 0 & \text{if } 0.8 < x \le 1, \end{cases}$$

$\varepsilon = 0.2$, and

$$B(x) = \begin{cases} \sin\left(\frac{\pi}{0.2}(0.2 - x)\right) & \text{if } 0 \le x \le 0.2, \\ 0 & \text{if } 0.2 < x \le 0.8, \\ \sin\left(\frac{\pi}{0.2}(x - 0.8)\right) & \text{if } 0.8 < x \le 1. \end{cases}$$

Indeed, since $N = 1$, $\Omega_A^0 = (0, 0.2) \cup (0.8, 1)$, $\Omega_B^0 = (0.2, 0.8)$, and

$$\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega] = \sigma_1\left[-\frac{d^2}{dx^2} + \mu; \Omega\right] = \pi^2 + \mu, \tag{4.3}$$

$$\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega_A^0] = \sigma_1\left[-\frac{d^2}{dx^2} + \mu; \Omega_A^0\right] = 25\pi^2 + \mu, \tag{4.4}$$

condition (3.3) is satisfied as soon as

$$-25\pi^2 < \mu < -\pi^2. \tag{4.5}$$

Moreover, since $V > 0$ in $(0, 0.2)$ and $V < 0$ in $(0.8, 1)$, V changes sign in Ω_A^0. The validity of (4.3) follows straight away from the fact that, for each $\lambda < 0$, any eigenfunction of $\sigma_1\left[-\frac{d^2}{dx^2} + \mu - \lambda \sin(2\pi\cdot); \Omega\right]$ must be the vertical reflection around 0.5 of a principal eigenfunction of $\sigma_1\left[-\frac{d^2}{dx^2} + \mu + \lambda \sin(2\pi\cdot); \Omega\right]$. Thus, the zeroes of the map

$$\lambda \mapsto \sigma_1\left[-\frac{d^2}{dx^2} + \mu - \lambda \sin(2\pi\cdot); \Omega\right]$$

always have opposite sign, if it has two. This entails that its maximum must be reached at $\lambda = 0$. A similar argument, whose details are omitted here, shows (4.4).

Actually, as a result of the symmetry of $a(x)$, if u is a positive solution of (4.1) for some $\lambda > 0$, then its vertical reflection around $x = 0.5$ provides us with a positive solution of (4.1) for $-\lambda$. Consequently, for each $\mu \in \mathbf{R}$, the set of values of λ for which (4.1) possesses a positive solution must be an interval centered at $\lambda = 0$. As a consequence, any bifurcation diagram of positive solutions where we represent the parameter λ versus the L_∞-norm of the solutions must be symmetric around $\lambda = 0$. These features have been confirmed by all our numerical computations.

Suppose

$$\mu \in (-\pi^2, 0]. \tag{4.6}$$

Then, $\sigma_1[\mathscr{L}; \Omega] > 0$ and, due to Kato-Hess's Theorem, there exist $\lambda_-(\mu) < 0 < \lambda_+(\mu)$ such that

$$\sigma_1[\mathscr{L} - \lambda_-(\mu)V; \Omega] = \sigma_1[\mathscr{L} - \lambda_+(\mu)V; \Omega] = 0.$$

Actually, these are the unique zeroes of $\lambda \mapsto \sigma_1[\mathscr{L} - \lambda V; \Omega]$ and, thanks to the symmetry properties of the problem, $\lambda_-(\mu) = -\lambda_+(\mu)$. As μ decreases from zero approaching the critical value $-\pi^2$, $\lambda_-(\mu)$ increases, and, hence, $\lambda_+(\mu)$ decreases, approaching 0, i.e.,

$$\lim_{\mu\downarrow-\pi^2} \lambda_-(\mu) = 0 = \lim_{\mu\downarrow-\pi^2} \lambda_+(\mu).$$

It should be noted that, if $\mu < -\pi^2$, then the map $\lambda \mapsto \sigma_1[\mathscr{L} - \lambda V; \Omega]$ is always negative. Under assumption (4.6), $\lambda_-(\mu)$ and $\lambda_+(\mu)$ are simple eigenvalues, in the sense of [4], and, hence, they are bifurcation values to positive solutions from the trivial state $(\lambda, u) = (\lambda, 0)$. Actually, these are the unique bifurcation values to positive solutions from $(\lambda, 0)$. Moreover, thanks to [2, Prop. 4.3], it is easy to see that (4.1) does not admit a positive solution if

$$\lambda \in (-\infty, \lambda_-^0(\mu)] \cup [\lambda_+^0(\mu), \infty),$$

where $\lambda_-^0(\mu)$ and $\lambda_+^0(\mu)$ are the unique zeroes of the map

$$\lambda \mapsto \sigma_1[\mathscr{L} - \lambda V; (0, 0.2)].$$

Note that $\lambda_-^0(\mu) = -\lambda_+^0(\mu) < 0$. As we have at our disposal L_∞ a priori bounds, uniform in compact intervals of the parameter λ, for all positive solutions of (4.1)—because $N = 1$—, it is apparent, from the abstract unilateral bifurcation results of [10], that there is a continuum of positive solutions of (4.1) connecting $(\lambda_-^0(\mu), 0)$ with $(\lambda_+^0(\mu), 0)$. The left plot of Figure 4.1 shows it for the special choice $\mu = 0$. In this case, $\lambda_+(0) \sim 28.0233$ and $\lambda_-(0) \sim -28.0233$. The central and right plots of Figure 4.1 show the per-

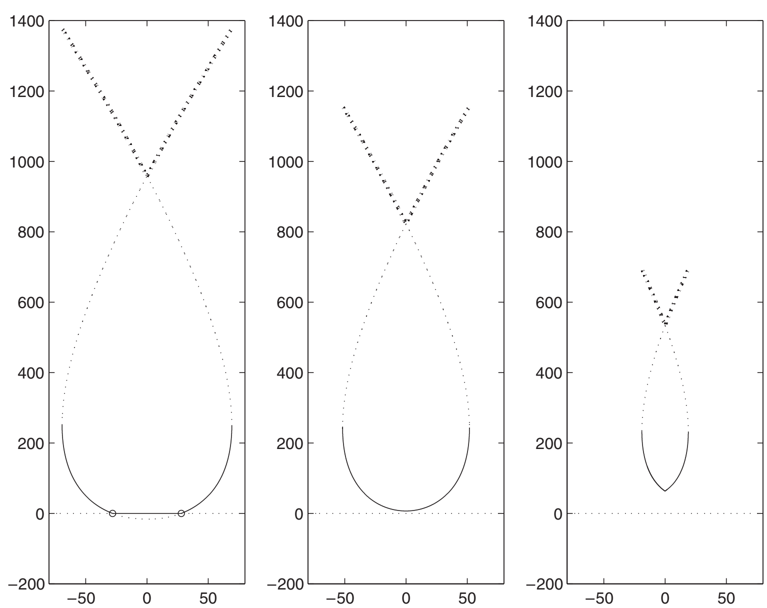

Fig. 4.1. Three components of positive solutions for $\mu = 0, -14, -40$, respectively.

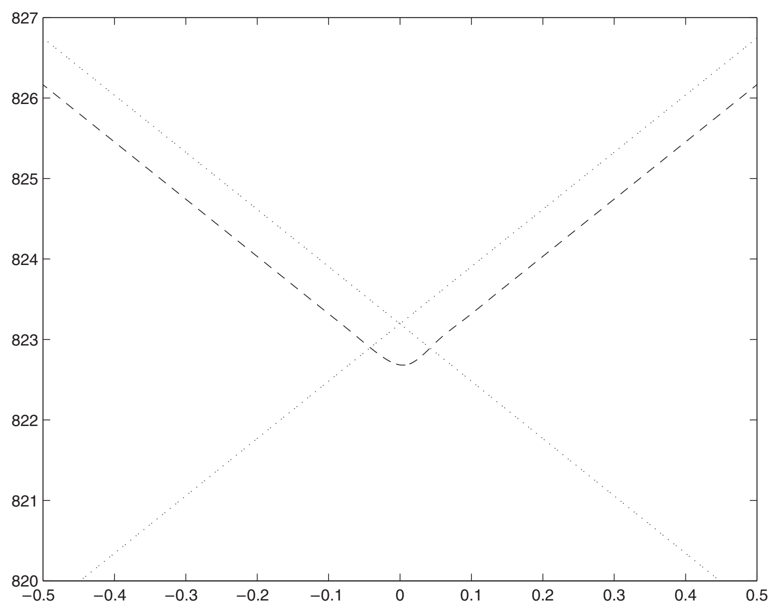

Fig. 4.2. Magnified piece of the bifurcation diagrams.

turbations of the positive solutions of the left plot as the secondary parameter μ decreases from zero up to reach the values $\mu = -14$ and $\mu = -40$, for which $(\lambda, 0)$ always is linearly unstable, and, consequently, no bifurcation value to positive solutions from it is available. Therefore, the perturbed continua of positive solutions must be isolas with respect to $(\lambda, 0)$.

In Figure 4.1 we are plotting the value of λ against the L_∞-norm of the corresponding positive solution. Stable solutions are indicated by solid lines, unstable by dotted lines. As there are some ranges of values of λ where the model possesses at least two solutions with very similar L_∞-norms, the plot did not allow us distinguishing them, but rather plotted twice these pieces. This is why the bifurcation diagrams exhibit a darker arc of curve. To explain in full detail what's going on, in Figure 4.2 we have magnified a very small area around the *apparent crossing point* of the curve of positive solution shown in the second plot of Figure 4.1.

The crossing point in Figure 4.2 represents two positive solutions for $\lambda = 0$ with the same L_∞-norm; each of them the vertical reflection around $x = 0.5$ of the other. The solutions on the main diagonals possess a one-dimensional unstable manifold, while the solutions on the $\bigvee$-shaped curve have two-dimensional unstable manifolds. Actually, in all cases the fine structure of the upper part of the global bifurcation diagram obeys the following general scheme

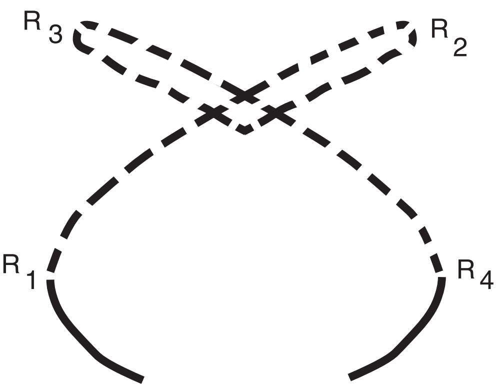

Fig. 4.3. Scheme of the upper parts of the bifurcation diagrams.

As we across the first turning point, R_1, the solutions become unstable with one-dimensional unstable manifold, until reaching the second turning point along the curve, R_2, where the dimension of the unstable manifold increases by one. Then, all solutions along the $\bigvee$-shaped arc of curve have two-dimensional unstable manifolds until reaching the third turning point, R_3, where such dimension reduces by one. All solutions along the arc of curve in between R_3 and R_4 have one-dimensional unstable manifold. Finally, when R_4 is crossed, all solutions become stable. It should be noted that, thanks to the symmetry properties of (4.1),

$$R_1 = -R_4, \qquad R_3 = -R_2.$$

To compute the plots of Figure 4.1 we have coupled a pure spectral method with collocation and a path continuation solver. In all our numerical computations we have used trigonometric modes and the collocation points have been taken to be equidistant; the number of modes equaling the number of collocation points. Let N denote the number of modes, set $x_0 = 0$, $x_{N+1} = 1$, $h = 1/(N+1)$, and

$$x_i = x_{i-1} + h, \qquad 1 \le i \le N,$$

the N collocation points. Then, the solutions $u(x)$ of (4.1) are approximated by the Fourier truncated series

$$u_N(x) = \sum_{j=1}^{N} c_j \sin(j\pi x),$$

where $C = (c_1, \ldots, c_N)$ is a solution of

$$(LC^T)_i = \lambda V(x_i)(JC^T)_i - a(x_i)((JC^T)_i)^2, \qquad 1 \le i \le N. \tag{4.7}$$

In (4.7) we have denoted

$$J := (\sin(j\pi x_i))_{1 \le i,j \le N}, \qquad L := ((j\pi)^2 \sin(j\pi x_i))_{1 \le i,j \le N}. \tag{4.8}$$

For this choice the zero solution of (4.1) is preserved and, as the number of modes increases, any compact arc of approximated non-trivial solution curve approximates the corresponding curve of the continuous problem (e.g., [11] and the references therein). Since u is a function of class $\mathscr{C}^2$, its jth Fourier coefficient c_j decays as $O(j^{-2})$ as $j \uparrow \infty$ (cf. C. Canuto et al. [3, pp. 35]) and, hence,

$$\max_{0 \le x \le 1} |u_{N+1}(x) - u_N(x)| = O(N^{-2})$$

as $N \uparrow \infty$. Due to these features we have used the following criterion to choose the number of modes in all our computations

$$\max_{0 \le j \le 10} |c_{N-j}| \le \frac{1}{2} 10^{-4}.$$

Actually, in order to respect it we have needed $80 \le N \le 132$ nodes in all our computations. The global continuation solvers we use to compute the solution curves and the dimensions of the unstable manifolds of all the solutions along them come from [8], [11], and the references therein.

The central and right plots of Figure 4.1 show two isolas of positive solutions of (4.1) for $\mu = -14$ and $\mu = -40$, respectively. To compute them we have taken the minimal positive solution of the first plot of Figure 4.1 for a value of λ greater and sufficiently close to $\lambda_+(0)$, and then we have fixed λ and used μ as the main continuation parameter to construct a positive solution of (4.1) for $\mu = -14$ at the fixed value of λ. Further, we fixed $\mu = -14$ and used λ as the main continuation parameter to compute the whole solution isola. A posterior continuation in μ, up to reach the value $\mu = -40$ provided us with the isola shown in the third plot of Figure 4.1. It should be noted that $\mu = -14$ and $\mu = -40$ satisfy (4.5). Therefore, they are within the range of values of μ for which no positive solution can bifurcate from $u = 0$, because of the absence of principal eigenvalues for the corresponding linearized problem. The turning points of the third diagram of Figure 4.1 are the following

$$R_1 \sim -18.9693, \qquad R_2 \sim 18.3983, \qquad R_3 \sim -18.3983, \qquad R_4 \sim 18.9693.$$

In the next figures we have plotted the profiles of some representative positive solutions along the third isola of Figure 4.1. We have chosen clock-wise orientation. Precisely, Figure 4.4 provides us with the plots of 6 solutions along the arc of curve joining R_4 with R_1. All those solutions are stable.

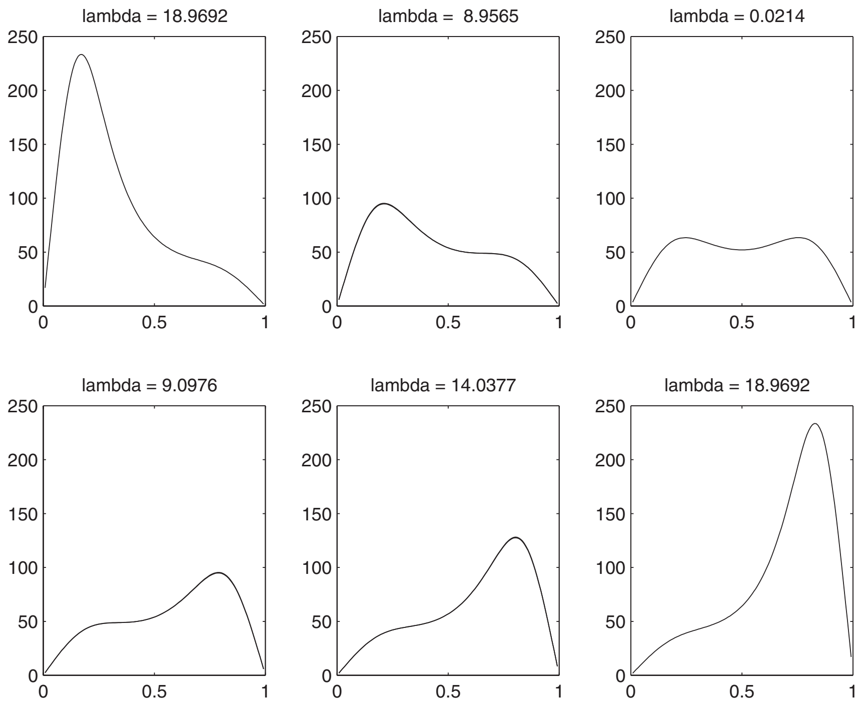

Fig. 4.4. Some positive solutions along the arc of curve $\widehat{R_4 R_1}$.

As predicted by the theory, the first and sixth plots of Figure 4.4 are vertical reflection around $x = 0.5$ of each other.

In Figure 4.5 we have represented the plots of 4 representative solutions along the arc of curve joining R_1 with R_2. The first and fourth solutions correspond with R_1 and R_2, respectively, and, hence, they are neutrally stable, whereas the two intermediate solutions are unstable with one-dimensional unstable manifold.

The solutions along the arc of curve $\widehat{R_1 R_2}$ exhibit a genuine super-linear behaviour at each of the intervals $(0, 0.2)$ and $(0.8, 1)$, where they have a pick around $x = 0.1$ and $x = 0.9$, respectively. It should be noted that 0.1 and 0.9 are the points where the absolute minimum of $a(x)$ is attained. Clearly, along this arc of curve the super-linear behaviour of the positive solutions is much more emphasized in the interval $(0.8, 1)$ than in $(0, 0.2)$.

In Figure 4.6 we have represented the plots of 6 representative solutions along the arc of curve joining R_2 with R_3. All these solutions are linearly unstable with two-dimensional unstable manifolds. Note that they exhibit a genuine super-linear behavior with two picks around each of the points $x = 0.1$ and $x = 0.9$. Along this arc of curve, the solutions of (4.1) for $\lambda < 0$ are

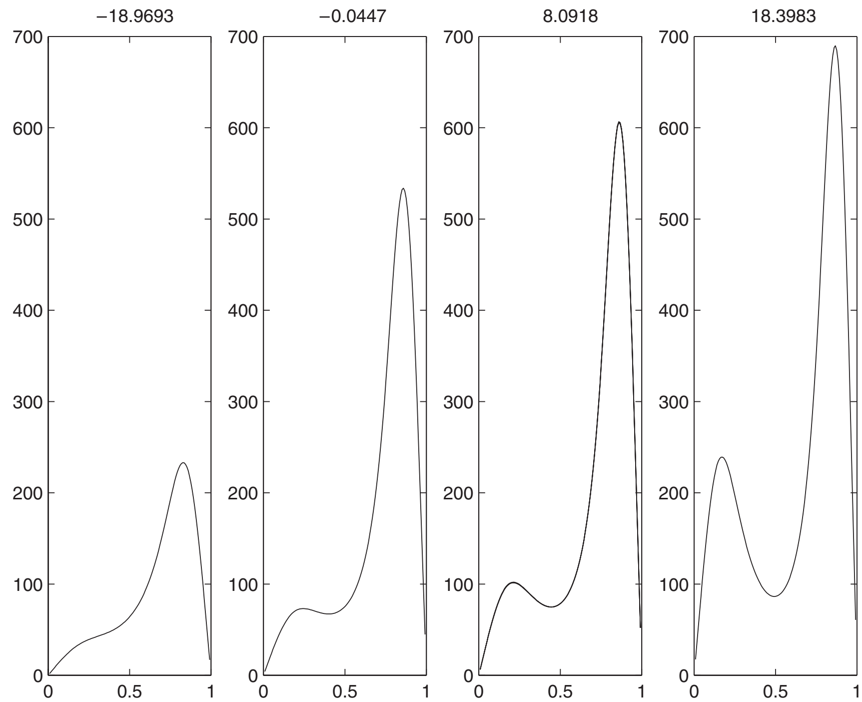

Fig. 4.5. Some positive solutions along the arc of curve $\widehat{R_1R_2}$.

obtained through vertical reflection around $x = 0.5$ from the corresponding solutions for $-\lambda > 0$.

The solutions of (4.1) along the arc of curve joining R_3 with R_4 are the vertical reflection around $x = 0.5$ of the corresponding solutions of (4.1) (substituting λ by $-\lambda$) along the arc of curve $\widehat{R_1R_2}$. Therefore, they can be easily reconstructed from the ones shown in Figure 4.5.

Finally, in Figure 4.8 we have super-imposed the profiles of some of the most representative solutions along each of the four arcs of curve delimited by the four turning points R_1, R_2, R_3 and R_4. It is a very suggestive picture because it reveals the symmetries exhibited by the solutions along $\widehat{R_4R_1}$ and $\widehat{R_3R_4}$. It also shows that the super-linear behaviour of the solutions along $\widehat{R_1R_2}$ is substantially stronger in $(0.8, 1)$ than in $(0, 0.2)$, while the super-linear behaviour of the solutions along $\widehat{R_3R_4}$ is substantially stronger in $(0, 0.2)$ than in $(0.8, 1)$.

When the parameter μ decreases up to reach the critical value $-25\pi^2$ the corresponding isolas shrink to a single point at $\mu = -25\pi^2$, while they disappear if $\mu < -25\pi^2$. Indeed, for this range of values of μ one has that

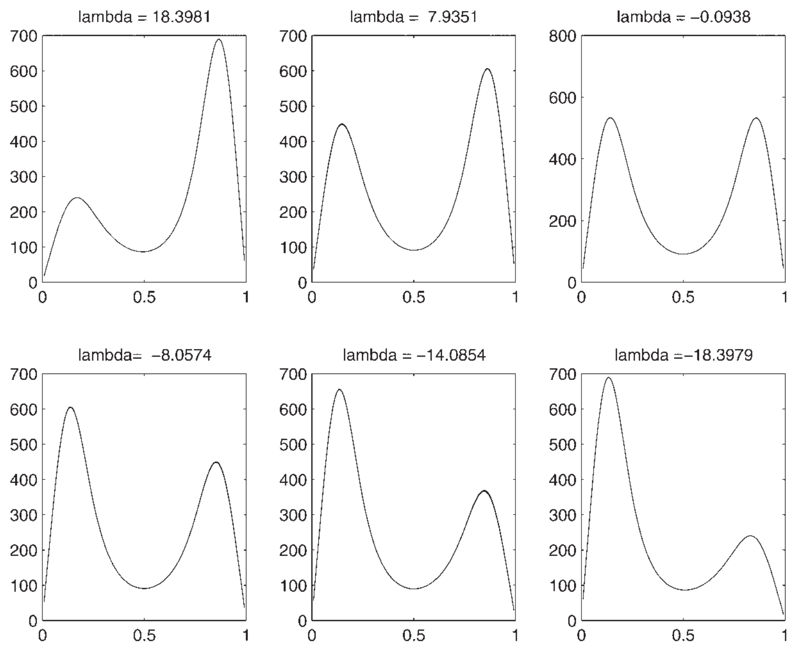

Fig. 4.6. Some positive solutions along the arc of curve $\widehat{R_2R_3}$.

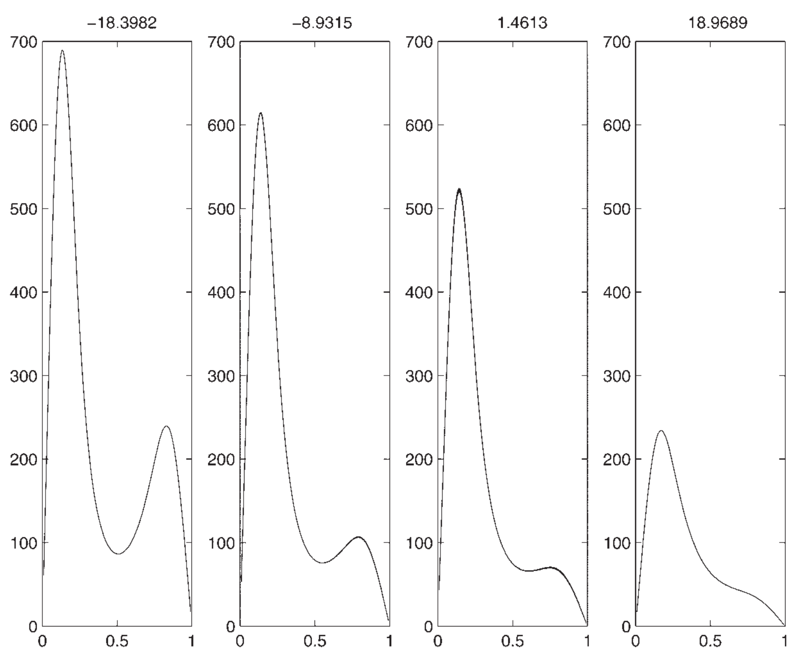

Fig. 4.7. Some positive solutions along the arc of curve $\widehat{R_3R_4}$.

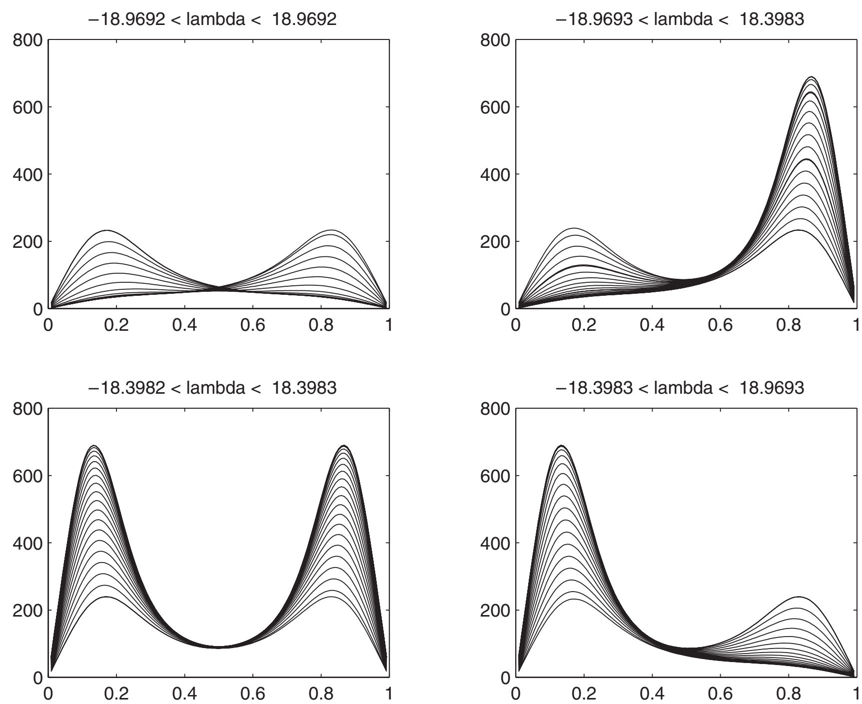

Fig. 4.8. Imposing all profile plots in each of the representative regimes.

$$\max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega] < \max_{\lambda \in \mathbf{R}} \sigma_1[\mathscr{L} - \lambda V; \Omega_A^0] < 0$$

and, hence, thanks to [2, Prop. 4.3], (4.1) cannot admit a positive solution.

Acknowledgements

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Santiago Cano-Casanova
Departamento de Matemática Aplicada y Computación
Universidad Pontificia Comillas de Madrid
28015-*Madrid, Spain*
e-mail: scano@dmc.icai.upco.es

Julián López-Gómez
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040-*Madrid, Spain*
e-mail: Lopez_Gomez@mat.ucm.es

Marcela Molina-Meyer
Departamento de Matemáticas
Universidad Carlos III de Madrid
28911-*Leganés, Madrid, Spain*
e-mail: mmolinam@math.uc3m.es