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Cobordism group of Morse functions on manifolds

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ABSTRACT. The n-dimensional cobordism group of Morse functions on manifolds is defined by using maps into $\mathbf{R} \times [0,1]$ with only fold singularities. In this paper, we show that in the un-oriented case it is a direct sum of the n-dimensional cobordism group and a certain number of infinite cyclic groups. In the oriented case a finite cyclic group $\mathbf{Z}_2$ is further added when $n = 4k + 1$.

1. Introduction

The n-dimensional oriented cobordism group $\mathscr{M}_n$ of Morse functions was introduced in Ikegami–Saeki [2], where we used a different notation. The purpose of this paper is to determine the structures of $\mathscr{M}_n$ and the n-dimensional un-oriented cobordism group $\mathscr{N}_n$ of Morse functions. We use "elimination of cusps" [4] and "semi-characteristics" [5]. Note that we showed in [2] that $\mathscr{M}_2$ is an infinite cyclic group, by a different method.

For the cobordism theory of smooth maps, Thom [8] showed that the cobordism group of embeddings is isomorphic to a homotopy group of a certain Thom complex by using the Pontrjagin-Thom construction. Wells [10] studied the cobordism group of immersions in a similar way. Rimányi and Szűcs [6] extended these results to the cobordism group of maps with singularities by using the notion of a τ-map.

Usually a cobordism group is computed by using the method of algebraic topology as a certain homotopy group of a Thom complex. However, in this paper the cobordism group of Morse functions is completely determined in a geometric way.

Recently Saeki [7] considered another kind of n-dimensional cobordism groups of special Morse functions and got a relation with the h-cobordism group of homotopy n-spheres for $n \geq 6$.

The paper is organized as follows. In § 2 we recall the precise definition of the n-dimensional cobordism group of Morse functions and state our main theorem. In § 3 defining fold and cusp points of a smooth map into $\mathbf{R}^2$ to-

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gether with their absolute indices, we review Levine's method for the elimination of cusps. In §4 we define cobordism invariants of Morse functions and prove our main theorem for the un-oriented case by using the elimination of cusps. In §5 we complete the proof of our main theorem in the oriented case by using a method similar to the un-oriented case and especially the semi-characteristics for $n = 4k + 1$.

Throughout the paper, all manifolds and maps are of class C^∞. The symbol "$\cong$" denotes an appropriate isomorphism between algebraic objects.

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2. Statement of the main theorems

A smooth real-valued function on a smooth manifold is called a *Morse function* if its critical points are all non-degenerate. For a positive integer n, we denote by $M(n)$ and $N(n)$ the sets of all Morse functions on closed, possibly disconnected, oriented and un-oriented n-dimensional manifolds respectively. We adopt the convention that the function on the empty set $\emptyset$ is an element of both $M(n)$ and $N(n)$ for all n.

Definition 2.1. Two Morse functions $f_0 : M_0 \to \mathbf{R}$ and $f_1 : M_1 \to \mathbf{R}$ in $M(n)$ are said to be *oriented cobordant* (or more precisely *oriented fold cobordant*) if there exist a compact oriented $(n+1)$-dimensional manifold X and a smooth map $F : X \to \mathbf{R} \times [0,1]$ which has only fold points as its singularities (for the definition of a fold point, see Definition 3.1) such that

(1) the boundary ∂X of X is the disjoint union $M_0 \amalg (-M_1)$, where $-M_1$ denotes the manifold M_1 with the orientation reversed, and
(2) we have

$$F|_{M_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)} : M_0 \times [0,\varepsilon) \to \mathbf{R} \times [0,\varepsilon), \qquad \text{and}$$

$$F|_{M_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]} : M_1 \times (1-\varepsilon,1] \to \mathbf{R} \times (1-\varepsilon,1]$$

for some sufficiently small $\varepsilon > 0$, where we identify the collar neighborhoods of M_0 and M_1 in X with $M_0 \times [0,\varepsilon)$ and $M_1 \times (1-\varepsilon,1]$ respectively.

In this case, we call F an *oriented cobordism* between f_0 and f_1.

If a Morse function in $M(n)$ is oriented cobordant to the function on the empty set, then we say that it is *oriented null-cobordant*.

It is easy to show that the above relation defines an equivalence relation on the set $M(n)$ for each n. Furthermore, it is easy to see that the set of all equivalence classes forms an additive group under the disjoint union: the neutral element is the class corresponding to oriented null-cobordant Morse functions, and the inverse of a class represented by a Morse function $f : M \to \mathbf{R}$ is given by the class of $-f : -M \to \mathbf{R}$, where $(-f)(x) = -f(x)$ for all $x \in M$. We denote by $\mathscr{M}_n$ the additive group of all oriented cobordism classes of elements of $M(n)$ and call it the *cobordism group of Morse functions on oriented manifolds of dimension n*, or the *n-dimensional oriented cobordism group of Morse functions*.

For the set $N(n)$ of Morse functions on closed un-oriented manifolds, we can define the relation of un-oriented cobordism by ignoring the orientations of the manifolds in the above definition. We denote by $\mathscr{N}_n$ the additive group of all un-oriented cobordism classes of elements of $N(n)$ and call it the *cobordism group of Morse functions on un-oriented manifolds of dimension n*, or the *n-dimensional un-oriented cobordism group of Morse functions*.

For a Morse function $f : M \to \mathbf{R}$ on an n-dimensional closed manifold M, we denote by $C_\lambda(f)$ the number of its critical points of index λ $(0 \le \lambda \le n)$.

Definition 2.2. For $0 \le \lambda \le n$, we define the maps $\varphi_\lambda : \mathscr{N}_n \to \mathbf{Z}$ and $\tilde{\varphi}_\lambda : \mathscr{M}_n \to \mathbf{Z}$ by

$$\varphi_\lambda([f]) = C_\lambda(f) - C_{n-\lambda}(f) \in \mathbf{Z}$$

and $\tilde{\varphi}_\lambda = \varphi_\lambda \circ (\textit{natural map} : \mathscr{M}_n \to \mathscr{N}_n)$. Here, $[f]$ denotes the cobordism class of $f \in N(n)$.

Note that φ_λ and $\tilde{\varphi}_\lambda$ are well-defined by Lemma 4.1 which will be proved in §4, and that φ_λ and $\tilde{\varphi}_\lambda$ are homomorphisms, since we define the sum $[f] + [g]$ as $[f \amalg g]$, where $f \amalg g : M \amalg N \to \mathbf{R}$ is defined by

$$(f \amalg g)(x) = \begin{cases} f(x) & (x \in M) \\ g(x) & (x \in N), \end{cases}$$

for $f : M \to \mathbf{R}$ and $g : N \to \mathbf{R}$ in $N(n)$ (resp. in $M(n)$).

Definition 2.3. We define the maps $\Phi : \mathscr{N}_n \to \mathbf{Z}^{\lfloor n/2 \rfloor}$ and $\tilde{\Phi} : \mathscr{M}_n \to \mathbf{Z}^{\lfloor n/2 \rfloor}$ by

$$\Phi([f]) = (\varphi_{\lfloor (n+3)/2 \rfloor}([f]), \varphi_{\lfloor (n+3)/2 \rfloor + 1}([f]), \dots, \varphi_n([f])) \in \mathbf{Z}^{\lfloor n/2 \rfloor}$$

and

$$\tilde{\Phi}([f]) = (\tilde{\varphi}_{\lfloor (n+3)/2 \rfloor}([f]), \tilde{\varphi}_{\lfloor (n+3)/2 \rfloor + 1}([f]), \dots, \tilde{\varphi}_n([f])) \in \mathbf{Z}^{\lfloor n/2 \rfloor}$$

respectively, where $\lfloor x \rfloor$ means the greatest integer less than or equal to x for a real number x, and $[f]$ is the cobordism class of f.

Definition 2.4. Let $\mathfrak{N}_n$ and Ω_n be the usual n-dimensional un-oriented and oriented cobordism groups of manifolds respectively (see [8] and [9]). We define $\Psi : \mathcal{N}_n \to \mathfrak{N}_n$ and $\tilde{\Psi} : \mathcal{M}_n \to \Omega_n$ by

$$\Psi([f : M \to \mathbf{R}]) = [M]_2 \in \mathfrak{N}_n$$

and

$$\tilde{\Psi}([f : M \to \mathbf{R}]) = [M] \in \Omega_n$$

respectively, where $[M]_2$ or $[M]$ is the un-oriented (resp. oriented) cobordism class of M. Here, $[f : M \to \mathbf{R}]$ is the cobordism class of f.

It is clear that Ψ and $\tilde{\Psi}$ are well-defined homomorphisms. Moreover, they are surjective, since there exist at least one Morse function on any manifold.

We prepare some definitions which will be used in Theorem 2.9.

Definition 2.5. Let M be a closed oriented $(4k+1)$-dimensional manifold, and K a coefficient field (for example, $\mathbf{Z}_2$ or $\mathbf{Q}$). The *semi-characteristic* $\sigma(M;K) \in \mathbf{Z}_2$ of M with respect to the coefficient field K is defined as follows:

$$\sigma(M;K) = \sum_{i=0}^{2k} \dim H_i(M;K) \pmod 2 \in \mathbf{Z}_2.$$

Definition 2.6. Let $f : M \to \mathbf{R}$ be a Morse function on a closed oriented $(4k+1)$-dimensional manifold. Then we define $\sigma(f) \in \mathbf{Z}_2$ as follows:

$$\sigma(f) = \sum_{\lambda=0}^{2k} C_\lambda(f) \pmod 2 \in \mathbf{Z}_2.$$

Furthermore, we define the map $\Lambda : \mathcal{M}_{4k+1} \to \mathbf{Z}_2$ by

$$\Lambda([f : M \to \mathbf{R}]) = \sigma(f) - \sigma(M;\mathbf{Q}) \in \mathbf{Z}_2.$$

Note that the map Λ is a well-defined homomorphism by Lemma 5.3 which will be proved in §5. See also Remark 5.4.

Main results of this paper are the following three theorems.

Theorem 2.7 (The un-oriented case). *The n-dimensional un-oriented cobordism group $\mathcal{N}_n$ of Morse functions is isomorphic to the direct sum of the n-dimensional un-oriented cobordism group $\mathfrak{N}_n$ and $\lfloor n/2 \rfloor$ copies of the infinite cyclic group. More precisely, the map*

$$\Psi \oplus \Phi : \mathcal{N}_n \to \mathfrak{N}_n \oplus \mathbf{Z}^{\lfloor n/2 \rfloor}$$

defined by

$$(\Psi \oplus \Phi)([f : M \to \mathbf{R}]) = ([M]_2, \varphi_{\lfloor (n+3)/2 \rfloor}([f]), \varphi_{\lfloor (n+3)/2 \rfloor + 1}([f]) \dots, \varphi_n([f]))$$

is an isomorphism. Here $\lfloor x \rfloor$ *means the gratest integer less than or equal to* x *for a real number* x.

The oriented case is divided into the following two cases.

Theorem 2.8 (The oriented case with $n \neq 4k+1$). *For* $n \not\equiv 1 \pmod 4$ *the* n*-dimensional oriented cobordism group* $\mathcal{M}_n$ *of Morse functions is isomorphic to the direct sum of the* n*-dimensional oriented cobordism group* Ω_n *and* $\lfloor n/2 \rfloor$ *copies of the infinite cyclic group. More precisely, the map*

$$\tilde{\Psi} \oplus \tilde{\Phi} : \mathcal{M}_n \to \Omega_n \oplus \mathbf{Z}^{\lfloor n/2 \rfloor}$$

defined by

$$(\tilde{\Psi} \oplus \tilde{\Phi})([f : M \to \mathbf{R}]) = ([M], \tilde{\varphi}_{\lfloor (n+3)/2 \rfloor}([f]), \tilde{\varphi}_{\lfloor (n+3)/2 \rfloor + 1}([f]), \dots, \tilde{\varphi}_n([f]))$$

is an isomorphism.

Theorem 2.9 (The oriented case with $n = 4k+1$). *For* $n \equiv 1 \pmod 4$, *the* n*-dimensional oriented cobordism group* $\mathcal{M}_n$ *of Morse functions is isomorphic to the direct sum of the* n*-dimensional oriented cobordism group* Ω_n, $\lfloor n/2 \rfloor$ *copies of the infinite cyclic group, and the finite cyclic group of order two. More precisely, the map*

$$\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda : \mathcal{M}_n \to \Omega_n \oplus \mathbf{Z}^{\lfloor n/2 \rfloor} \oplus \mathbf{Z}_2$$

defined by

$$(\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda)([f : M \to \mathbf{R}]) = ([M], \tilde{\varphi}_{\lfloor (n+3)/2 \rfloor}([f]), \dots, \tilde{\varphi}_n([f]), \sigma(f) - \sigma(M; \mathbf{Q}))$$

is an isomorphism.

3. Elimination of cusps

In this section, we review some results of [4] which will be used in the proof of the main theorems.

Definition 3.1. Let $F : W \to \mathbf{R}^2$ be a smooth map of an m-dimensional manifold with $m \geq 2$. A singular point $p \in W$ of F is a *fold point* if we can choose coordinates $(u, z_1, \dots, z_{m-1})$ centered at p and (U, Y) centered at $F(p)$ so that we can express F as

$$\begin{cases} U = u, \\ Y = -\sum_{k=1}^{\lambda} z_k^2 + \sum_{k=\lambda+1}^{m-1} z_k^2 \end{cases} \tag{3.1}$$

for some $0 \le \lambda \le m-1$ in a neighborhood of p. We set $\tau(p) = \max\{\lambda, \mathrm{m} - 1 - \lambda\}$ and call it the *absolute index* of the fold point p.

We say that p is a *cusp point* if we can choose coordinates $(u, x, z_1, \dots, z_{m-2})$ centered at p and (U, Y) centered at $F(p)$ so that we can express F as

$$\begin{cases} U = u, \\ Y = ux + x^3 - \sum_{k=1}^{\lambda} z_k^2 + \sum_{k=\lambda+1}^{m-2} z_k^2 \end{cases} \tag{3.2}$$

for some $0 \le \lambda \le m-2$ in a neighborhood of p. We set $\tau(p) = \max\{\lambda, m-2-\lambda\}$ and call it the *absolute index* of the cusp point p. Note that $\tau(p)$ is well-defined.

It is well-known that any smooth map of an m-dimensional manifold W $(m \ge 2)$ into $\mathbf{R}^2$ can be approximated arbitrarily well by a C^∞-map $F : W \to \mathbf{R}^2$ which has the following properties (1), (2), (3), (4) and (5) (see, for example, [4]).

(1) The rank of the differential of F is never zero.

(2) If $S_1(F)$ denotes the set of points in the domain of F at which the differential of F has rank one, then $S_1(F)$ consists of smooth non-intersecting curves.

(3) Let $S_1^2(F) \subset S_1(F)$ be the set of points where $F|_{S_1(F)}$ has zero differential. Then $S_1^2(F)$ is a discrete set.

(4) If $p \in S_1(F) - S_1^2(F)$, then p is a fold point.

(5) If $p \in S_1^2(F)$, then p is a cusp point.

We say that a smooth map of an m-dimensional manifold into $\mathbf{R}^2$ is *generic* if it has properties (1) through (5) above. If W is compact and $F : W \to \mathbf{R}^2$ is generic, then $S_1(F)$ is a compact 1-dimensional regular submanifold of W and the number of cusps is finite.

In a neighborhood of a cusp point, the absolute index varies as follows. First of all we note that the absolute index is constant on every component of $S_1(F) - S_1^2(F)$. For a cusp point p of absolute index i, the absolute indices of the nearby fold points are as follows. If $i > (m-2)/2$ then they are as depicted in Fig. 1, and if $i = (m-2)/2$ then they are as depicted in Fig. 2.

Let $F : W \to \mathbf{R}^2$ be a generic map and G be the cokernel bundle of

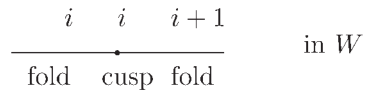

Fig. 1

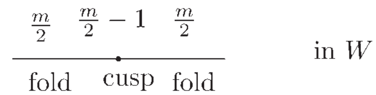

Fig. 2

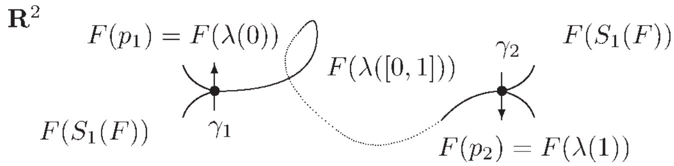

Fig. 3

$$dF : TW|_{S_1(F)} \to (F^*T\mathbf{R}^2)|_{S_1(F)}.$$

For a point $p \in S_1(F)$, the index $i(p,\gamma)$ of p is defined to be $\tau(p)$ or $m - 1 - \tau(p)$ if p is a fold point and $\tau(p)$ or $m - 2 - \tau(p)$ if p is a cusp point depending on the given orientation γ of the fiber G_p over p (for details, see [4]). Intuitively, the orientation γ determines a positive direction of the Y-axis with respect to the coordinates giving the normal form as in (3.1) or (3.2), and $i(p,\gamma)$ counts the number of minus signs appearing in the expression of Y.

Let $F : W \to \mathbf{R}^2$ be a generic map, where W is an m-dimensional connected manifold with $m \geq 3$. Suppose that F has two cusp points p_1 and p_2. If they satisfy certain conditions, then we can eliminate them by homotopy of F as follows (for details, see [4]). Let $\lambda : [0,1] \to W$ be a smooth embedding such that $\lambda(0) = p_1$, $\lambda(1) = p_2$, $\lambda([0,1]) \cap S_1(F) = \{p_1, p_2\}$, that $\lambda'(0)$ (or $\lambda'(1)$) points upward (resp. downward) in the sense of [4, p. 284] and that $F \circ \lambda$ is an immersion (see [4, (4.4)]). Note that such a joining curve λ always exists, since W is connected. Note also that $(F \circ \lambda)^*T\mathbf{R}^2$ is orientable and we fix an orientation. Let γ_j be an orientation of G_{p_j}, $j = 1, 2$, such that $(F \circ \lambda)'(0) \wedge \gamma_1$ and $-(F \circ \lambda)'(1) \wedge \gamma_2$ are consistent with the orientation of $(F \circ \lambda)^*T\mathbf{R}^2$ as in Fig. 3. Let $i(p_j, \gamma_j)$ be the index of the cusp point p_j with respect to γ_j, $j = 1, 2$. Then we say that the pair of cusp points p_1 and p_2 is a *matching pair* if $i(p_1, \gamma_1) + i(p_2, \gamma_2) = m - 2$. In this case we can eliminate the cusp points p_1 and p_2 by a homotopy of F whose support is contained in a small neighborhood of $\lambda([0,1])$.

Although Levine [4] assumes that the source manifolds are orientable, the method is applicable also for non-orientable source manifolds.

By [4] every generic map $F : W \to \mathbf{R}^2$ of a closed connected m-dimensional manifold W with $m \geq 3$ into $\mathbf{R}^2$ is homotopic to a generic map without cusp points if the Euler characteristic $\chi(W)$ is even, and to a generic map with exactly one cusp point if $\chi(W)$ is odd.

When $m = 2$, we cannot always find a joining curve for a given pair of cusps. So, one cannot directly apply Levine's method, but can instead apply Kálmán's method for the elimination of a pair of cusps [3, Lemma 1.4].

4. Proof of the main theorem (un-oriented case)

In this section we prove Theorem 2.7. Let M, N be n-dimensional un-oriented closed manifolds, $f : M \to \mathbf{R}$, $g : N \to \mathbf{R}$ be Morse functions, and $F : X \to \mathbf{R} \times [0,1]$ a cobordism between f and g. For a Morse function f, $S(f)$ denotes the set of critical points of f.

For $t \in [0,1]$ put

$$\begin{cases} M_t = F^{-1}(\mathbf{R} \times \{t\}), \\ f_t = F|_{M_t} : M_t \to \mathbf{R} \times \{t\}. \end{cases}$$

Let $\pi : \mathbf{R} \times [0,1] \to [0,1]$ be the projection to the second factor. Take a regular value $t \in [0,1]$ of $\pi \circ F|_{S_1(F)} : S_1(F) \to [0,1]$. Then M_t is a smooth manifold of dimension n and f_t is a Morse function. Moreover, we have $S(f_t) = S_1(F) \cap M_t$. For each point $p \in S(f_t) = S_1(F) \cap M_t$, let $\tau(p)$ be the absolute index of the fold point p with respect to F, and let $i(p)$ be the index of the critical point p with respect to the Morse function f_t. Then we have

$$\tau(p) = \begin{cases} i(p) & (i(p) \geq \lfloor n/2 \rfloor), \\ n - i(p) & (i(p) < \lfloor n/2 \rfloor). \end{cases}$$

Here, F is considered to be a restriction of a map of $M \times (-\varepsilon, 0] \cup X \cup N \times [1, 1+\varepsilon)$ for some $\varepsilon > 0$.

Lemma 4.1. *If two Morse functions f and g are cobordant, then we have*

$$C_\lambda(f) - C_{n-\lambda}(f) = C_\lambda(g) - C_{n-\lambda}(g)$$

for all λ.

Proof. Let us consider

$$\pi \circ F|_{S_1(F)} : S_1(F) \to [0,1]. \tag{4.1}$$

By slightly perturbing F if necessary, we may assume that (4.1) is a Morse function whose critical values are all distinct.

Let us study the change of $C_\lambda(f_t)$ when t varies from 0 to 1. We see easily that $C_\lambda(f_t)$ does not change if t does not cross a critical value of (4.1). Let T be a critical value of (4.1) and we denote its corresponding critical point by $p \in S_1(F)$. For a sufficiently small $\varepsilon > 0$, set $t_0 = T - \varepsilon$ and $t_1 = T + \varepsilon$. Let A be an arc neighborhood of p in $S_1(F)$ so that $F(A)$ is as depicted in Fig. 4; $M_{t_0} \cap A$ (or $M_{t_1} \cap A$) consists of two points, say p_1 and p_2.

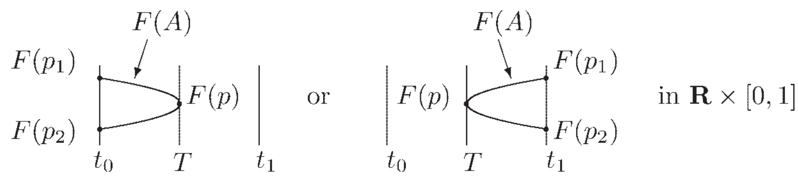

Fig. 4

Let G be the cokernel bundle of

$$dF : TW|_{S_1(F)} \to F^*T(\mathbf{R} \times [0,1])|_{S_1(F)}.$$

Then $G|_A$ is a trivial line bundle and we fix an orientation. Since the normal direction of $F(A)$ gives the opposite orientations at $F(p_1)$ and $F(p_2)$, compared with the natural orientation of $\mathbf{R} \times \{t_0\}$ (or $\mathbf{R} \times \{t_1\}$), we see that $i(p_1) + i(p_2) = n$, since $\tau(p_1) = \tau(p_2)$. Hence, for $\tau = i(p_1)$, we have

$$C_\tau(f_{t_0}) - C_{n-\tau}(f_{t_0}) = C_\tau(f_{t_1}) - C_{n-\tau}(f_{t_1}),$$

since one of the following sets of equations holds:

$$\begin{cases} C_\tau(f_{t_0}) - C_\tau(f_{t_1}) = +1, \\ C_{n-\tau}(f_{t_0}) - C_{n-\tau}(f_{t_1}) = +1 \end{cases}$$

or

$$\begin{cases} C_\tau(f_{t_0}) - C_\tau(f_{t_1}) = -1, \\ C_{n-\tau}(f_{t_0}) - C_{n-\tau}(f_{t_1}) = -1. \end{cases}$$

For $\lambda \neq \tau, n - \tau$, we see easily that $C_\lambda(f_{t_0}) = C_\lambda(f_{t_1})$ and hence we have

$$C_\lambda(f_{t_0}) - C_{n-\lambda}(f_{t_0}) = C_\lambda(f_{t_1}) - C_{n-\lambda}(f_{t_1})$$

for all λ. Therefore, we have the required conclusion. $\square$

Now we can use the notation $\varphi_\lambda([f])$ for $C_\lambda(f) - C_{n-\lambda}(f)$ as in Definition 2.2.

Proof of Theorem 2.7. We have only to show that the homomorphism $\Psi \oplus \Phi$ is bijective.

Let us first show that it is surjective. We will show that for arbitrary

$$([M]_2, a_{\lfloor (n+3)/2 \rfloor}, a_{\lfloor (n+3)/2 \rfloor + 1}, \dots, a_n) \in \mathfrak{N}_n \oplus \mathbf{Z}^{\lfloor n/2 \rfloor},$$

there exists a Morse function $f : M \to \mathbf{R}$ such that

$$(\Psi \oplus \Phi)([f]) = ([M]_2, a_{\lfloor (n+3)/2 \rfloor}, a_{\lfloor (n+3)/2 \rfloor + 1}, \dots, a_n).$$

We may assume that there is a Morse function $g_0 : M \to \mathbf{R}$ with

$$(\Psi \oplus \Phi)([g_0]) = ([M]_2, b_{\lfloor (n+3)/2 \rfloor}, b_{\lfloor (n+3)/2 \rfloor + 1}, \dots, b_n).$$

Recall that $b_\lambda = \varphi_\lambda([g_0]) = C_\lambda(g_0) - C_{n-\lambda}(g_0)$ and a pair of critical points of indices $\lambda + 1$ and λ can be created for $0 \le \lambda \le n-1$. First we increase the number of critical points of indices $n, n-1$ if $a_n > b_n$ or $0, 1$ if $a_n < b_n$ so that we have $C_n(g_1) - C_0(g_1) = a_n$, where g_1 is the new Morse function. Similarly by increasing the number of critical points of indices $n-1, n-2$ or $1, 2$, we get a Morse function g_2 such that $C_{n-1}(g_2) - C_1(g_2) = a_{n-1}$. Repeating this procedure, we can change b_λ to a_λ inductively for $n \ge \lambda \ge \lfloor (n+3)/2 \rfloor$. Then the resulting Morse function f satisfies the required property.

Now let us consider the injectivity. Since the injectivity is equivalent to $\mathrm{Ker}(\Psi \oplus \Phi) = 0$, we assume that $[f : M \to \mathbf{R}] \in \mathrm{Ker}(\Psi \oplus \Phi)$ and show that $[f : M \to \mathbf{R}] = 0$ in $\mathcal{N}_n$. By the definition of Ψ, we have $[M]_2 = 0 \in \mathfrak{N}_n$.

Lemma 4.2. *We have $\varphi_\lambda([f]) = 0$, that is, $C_\lambda(f) = C_{n-\lambda}(f)$ for all $0 \le \lambda \le n$.*

Proof. When n is even, we put $n = 2k$. By assumption, we have $\varphi_{k+1}([f]) = \varphi_{k+2}([f]) = \cdots = \varphi_{2k}([f]) = 0$. Since $\varphi_\lambda([f]) = 0$ is equivalent to $\varphi_{2k-\lambda}([f]) = 0$ for each λ, we have $\varphi_{k-1}([f]) = \varphi_{k-2}([f]) = \cdots = \varphi_0([f]) = 0$ as well. Furthermore, we clearly have $\varphi_k([f]) = C_k([f]) - C_k([f]) = 0$.

When n is odd, we put $n = 2k+1$. Similarly to the above we have $\varphi_\lambda([f]) = 0$ for $\lambda \ne k, k+1$ and $\varphi_k([f]) = -\varphi_{k+1}([f])$. Since the Euler characteristic $\chi(M)$ of M satisfies

$$\chi(M) = \sum_{\lambda=0}^{k} (-1)^\lambda \varphi_\lambda([f]) = 0,$$

we have $\varphi_k([f]) = 0$. Therefore we have the desired conclusion. $\square$

Since M is null-cobordant, there exists a compact $(n+1)$-dimensional manifold W with $\partial W = M$. Then there exists a smooth map $F : W \to \mathbf{R} \times [0,1]$ which satisfies the following properties.

(1) F is a generic map, i.e. F has only fold points and cusp points as its singularities.

(2) $F|_{M\times[0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)}$ for some sufficiently small $\varepsilon > 0$, where $M \times [0, \varepsilon)$ is a collar neighborhood of M in W.
(3) For any cusp point $p \in S_1^2(F)$, we have $F^{-1}(F(p)) \cap S_1(F) = \{p\}$.
(4) $F|_{(S_1(F)-S_1^2(F))}$ is an immersion with normal crossings.
(5) $S_1(F)$ is a compact 1-dimensional manifold which is properly embedded in W.

Such a map F exists, since $\mathbf{R} \times [0,1]$ is contractible and any smooth map satisfying (2) can be approximated by a generic map satisfying items (1), (3), (4) and (5).

If F has no cusps, then the proof is finished. When F has cusps, let us remove the cusps of F. By composing a diffeomorphism of $\mathbf{R} \times [0,1]$ if necessary, we may assume that the F-images of all the cusps lie on the line $\mathbf{R} \times \{1/2 + \delta\}$ for some small $\delta > 0$ and that the F-images of their neighborhoods in $S_1(F)$ are arranged as depicted in Fig. 5. Furthermore, we may also assume that the map $\pi \circ F : W \to [0,1]$ has no singular values on $[1/2, 1/2 + \delta]$.

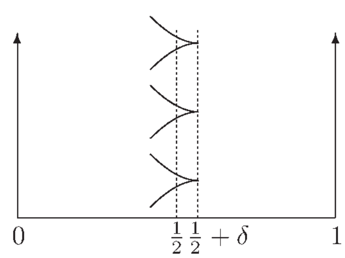

Fig. 5

Set $N = F^{-1}(\mathbf{R} \times \{1/2\})$ and $g = F|_N : N \to \mathbf{R} \times \{1/2\}$. Note that N is a smooth closed n-dimensional manifold and that g is a Morse function. Since there is no cusps of F on $F^{-1}(\mathbf{R} \times [0, 1/2])$, we see that f and g are fold cobordant as Morse functions. Therefore, we have only to show that $[g : N \to \mathbf{R}]$ is the neutral element in $\mathcal{N}_n$.

Set $V = F^{-1}(\mathbf{R} \times [1/2, 1])$, which is a compact $(n+1)$-dimensional manifold such that $\partial V = N$. If V is not connected, then we take two points p_1 and p_2 from distinct components of V such that they are definite fold points of $F|_{\mathrm{Int}\,V}$. (Such points p_1 and p_2 always exist.) We remove small open disk neighborhoods of p_1 and p_2 from V and attach $S^n \times [-1,1]$ along the sphere boundaries. Then we can construct a generic map from the resulting manifold into $\mathbf{R} \times [1/2, 1]$ appropriately by modifying $F|_V$ (see Fig. 6).

Since the number of connected components of the resulting manifold is smaller than that of V by one, we may assume that V is connected by repeating this procedure.

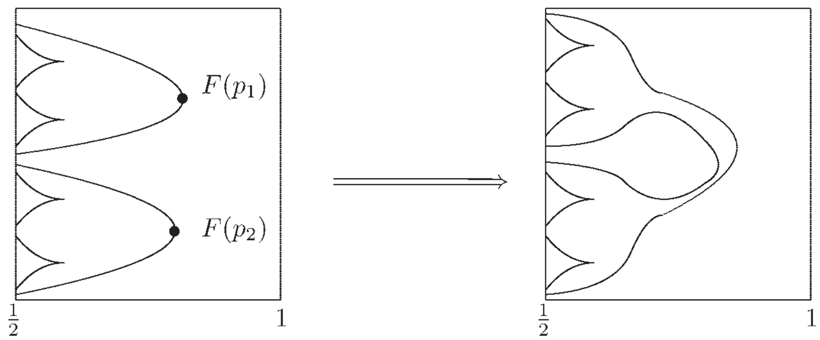

Fig. 6

Note that there are two kinds of components of $S_1(F|_V)$, i.e., arcs and circles. The end points of the arc components lie on N and they form the set of critical points of g. Furthermore by the special construction above of V each arc component has at most one cusp point. Take an arc component of $S_1(F|_V)$ without cusp points. Then the indices of its end points as critical points of g are of the form $(\lambda, n-\lambda)$ for some $0 \le \lambda \le n$ by the property of a regular plane arc, as is explained around Fig. 4.

Lemma 4.3. *Take an arc component of $S_1(F|_V)$ which contains a cusp point as in Fig. 5. Then the indices of its end points as critical points of g are of the form $(\lambda+1, \lambda)$ for some $0 \le \lambda \le n-1$, where the value of the critical point of index $\lambda+1$ is greater than that of index λ.*

Proof. Let us consider the normal form of a cusp point as follows:

$$\begin{cases} U = u, \\ Y = ux + x^3 - \displaystyle\sum_{k=1}^{\lambda} z_k^2 + \sum_{k=\lambda+1}^{n-1} z_k^2. \end{cases}$$

Then we have $S_1(F) = \{u = -3x^2, z_1 = 0, z_2 = 0, \dots, z_{n-1} = 0\}$ and $F(S_1(F)) = \{(-3x^2, -2x^3)\}$. For the proof of the lemma, we may assume that $\mathbf{R} \times \{1/2\}$ corresponds to $U = -3\varepsilon^2$ for some small $\varepsilon > 0$ and that g corresponds to Y. Then the corresponding critical points are $p_\pm = (-3\varepsilon^2, \pm\varepsilon, 0, \dots, 0)$ and the Hessians are given by the diagonal matrices whose diagonal entries are

$$(\pm 6\varepsilon, \underbrace{-2, \dots, -2}_{\lambda}, \underbrace{2, \dots, 2}_{n-\lambda-1}).$$

Therefore, the index of p_+ (or p_-) with respect to the Morse function g is equal to λ (resp. $\lambda+1$). Finally we have $g(p_-) > g(p_+)$ and the conclusion follows. □

By Lemma 4.2 we have

$$C_\lambda(f) = C_{n-\lambda}(f)$$

for all $0 \le \lambda \le n$. Since the Morse functions f and g are cobordant we have

$$C_\lambda(g) = C_{n-\lambda}(g)$$

for all $0 \le \lambda \le n$ by Lemma 4.1. Let C'_λ be the number of critical points of g of index λ which are end points of arc components of $S_1(F|_V)$ containing cusps. Then, we see that $C'_\lambda = C'_{n-\lambda}$ for all $0 \le \lambda \le n$, since $C''_\lambda = C''_{n-\lambda}$ for the number C''_λ of critical points of g of index λ which are end points of arc components of $S_1(F|_V)$ without cusps. Let μ be the greatest integer λ with $C'_\lambda \neq 0$. Then we have $C'_\mu = C'_{n-\mu} \neq 0$ and $C'_\lambda = 0$ for all $\lambda > \mu$ and for all $\lambda < n - \mu$. Note also that $\tau(p) \le \mu - 1$ for each cusp point p of $F|_V$.

(A) When n is even.

Suppose that $\mu - 1 > n - \mu$. We take two distinct arc components of $S_1(F|_V)$ which contain cusps and have end points of critical points with indices μ and $n - \mu$ respectively. Let p_1 and p_2 be the corresponding cusp points as in Fig. 7, where $\mu, \mu - 1, n - \mu + 1$ and $n - \mu$ indicate the indices of the corresponding critical points of g.

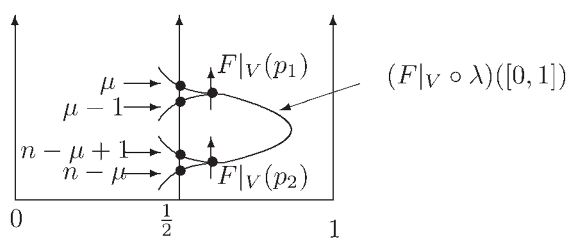

Fig. 7

If we take the orientation γ_i of the fiber G_{p_i} over p_i $(i = 1, 2)$ parallel to the orientation of $\mathbf{R} \times \{1/2\}$ as in Fig. 7, then we see that the indices of the cusp points p_1 and p_2 are given by $i(p_1, \gamma_1) = \mu - 1$ and $i(p_2, \gamma_2) = n - \mu$ respectively. Let $\lambda : [0,1] \to \operatorname{Int} V$ be a joining curve connecting p_1 and p_2. Then $(F|_V \circ \lambda)'(0) \wedge \gamma_1$ and $-(F|_V \circ \lambda)'(1) \wedge \gamma_2$ are consistent with an orientation of $(F|_V \circ \lambda)^* T(\mathbf{R} \times [0,1])$, as is depicted in Fig. 7. Therefore, the pair of cusp points p_1 and p_2 is a matching pair in the sense of Levine [4], since we have $i(p_1, \gamma_1) + i(p_2, \gamma_2) = (n+1) - 2$. Thus, we can eliminate the pair of two cusps by a homotopy of $F|_V$ by using Levine's method, since $n + 1 \ge 3$.

Repeating this procedure, we can eliminate all the cusp points p with $\tau(p) = \mu - 1$ if $\mu - 1 > n - \mu$. Since the absolute index of a cusp point is in

$\{n/2, n/2+1, \dots, n-1\}$, we see that we can eliminate all the cusp points in this case.

(B) When n is odd.

Repeating the procedure in (A) we finally reach the case where $\mu - 1 = n - \mu$. Now the absolute indices of the two end points of the relevant arc component are both equal to $\mu = (n+1)/2$, and hence the absolute index of the relevant cusp point is equal to $(n-1)/2$. So, any joining curve connecting two cusp points gives a matching pair at least when $n \geq 3$. Therefore if the number of cusp points of absolute index $(n-1)/2$ of $F|_V$ is even, then we can eliminate all such cusp points by a homotopy of $F|_V$ by using Levine's method [4] when $n \geq 3$ and Kálmán's method [3, Lemma 1.4] when $n = 1$. If the number of such cusp points is odd, then we modify $F|_V$ as follows.

Since n is odd, we can take a closed non-orientable $(n+1)$-dimensional manifold Y with odd Euler characteristic. (For example $Y = \mathbf{R}P^{n+1}$.) Then there exists a generic map $F_1 : Y \to \mathbf{R} \times (1/2, 1)$ which has a unique cusp point by [4]. This cusp point has absolute index $(n-1)/2$. Now we apply the argument that we used in order to make V connected: we combine $F|_V$ and F_1 to get a new generic map of the connected sum $V \# Y$ into $\mathbf{R} \times [1/2, 1]$. Then, the number of cusp points of absolute index $(n-1)/2$ of the resulting map is even and hence we can eliminate them by pairs by homotopy.

Therefore, we have removed all the cusps in both cases (A) and (B). This completes the proof of the injectivity of $\Psi \oplus \Phi$, and hence the proof of Theorem 2.7. □

5. Proof of the main theorem (oriented case)

In this section, we prove Theorems 2.8 and 2.9.

Proof of Theorem 2.8. We can prove that $\tilde{\Psi} \oplus \tilde{\Phi}$ is well-defined and surjective as in the un-oriented case.

The injectivity can also be proved as in the un-oriented case when n is even. So, we will prove that $\tilde{\Psi} \oplus \tilde{\Phi}$ is injective when $n = 4k - 1$.

Let $[f : M \to \mathbf{R}]$ be an arbitrary element in $\mathrm{Ker}(\tilde{\Psi} \oplus \tilde{\Phi})$. Since the $(4k-1)$-dimensional manifold M is oriented null-cobordant, there exists a compact oriented $4k$-dimensional manifold W with $\partial W = M$. Then we construct a generic map $F : W \to \mathbf{R} \times [0, 1]$ of a special kind as in the un-oriented case and set $N = F^{-1}(\mathbf{R} \times \{1/2\})$, $g = F|_N : N \to \mathbf{R} \times \{1/2\}$ and $V = F^{-1}(\mathbf{R} \times [1/2, 1])$. We may assume that V is connected.

If the number of cusps of the generic map $F|_V : V \to \mathbf{R} \times [1/2, 1]$ is odd, then we use a generic map of $Y = \mathbf{C}P^{2k}$ into $\mathbf{R} \times (1/2, 1)$ in order to modify $F|_V$. Note that $\mathbf{C}P^{2k}$ is an oriented closed manifold of dimension $4k = n + 1$

and that the Euler characteristic $\chi(\mathbf{C}P^{2k})$ is odd. Thus, as in the un-oriented case, we may assume that the number of cusps is even. Then we can remove all the cusp points by homotopy by using the theory of matching pairs. This completes the proof. $\square$

In order to prove Theorem 2.9, we need the following.

Lemma 5.1. *Let W be a compact oriented $(4k+2)$-dimensional manifold with $\partial W = M$. Then we have*

$$\sigma(M; \mathbf{Q}) \equiv \chi(W) \pmod 2,$$

where $\chi(W)$ denotes the Euler characteristic of W.

Proof. We consider the exact sequence of homology with $\mathbf{Q}$-coefficients of the pair (W, M):

$$\begin{aligned} 0 = H_{4k+2}(M) &\to H_{4k+2}(W) \to H_{4k+2}(W, M) \\ &\to H_{4k+1}(M) \to H_{4k+1}(W) \to H_{4k+1}(W, M) \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\to H_{2k+2}(M) \to H_{2k+2}(W) \to H_{2k+2}(W, M) \\ &\to H_{2k+1}(M) \to H_{2k+1}(W) \xrightarrow{j} H_{2k+1}(W, M). \end{aligned}$$

By Poincaré-Lefschetz duality and the universal coefficient theorem, we have

$$H_{(4k+2)-q}(W, M) \cong H^q(W) \cong \mathrm{Hom}(H_q(W), \mathbf{Q})$$

for every q. Furthermore, we have $\mathrm{Hom}(H_q(W), \mathbf{Q}) \cong H_q(W)$ and hence

$$b_{(4k+2)-q}(W, M) = b_q(W),$$

where b_i denotes the dimension of the i-th homology. By Poincaré duality we also have $b_{(4k+1)-q}(M) = b_q(M)$.

Then by the above exact sequence, we have

$$\begin{aligned} &(b_0(M) + b_1(M) + \cdots + b_{2k}(M)) + (b_0(W) + b_1(W) + \cdots + b_{2k}(W) + \mathrm{rank}\ j) \\ &\qquad \equiv b_{4k+2}(W) + b_{4k+1}(W) + \cdots + b_{2k+1}(W) \pmod 2. \end{aligned}$$

Therefore we have

$$\sigma(M; \mathbf{Q}) - \chi(W) \equiv \mathrm{rank}\ j \pmod 2.$$

Hence, we have only to show that rank j is even.

Since we have $H_{2k+1}(W, M) \cong H^{2k+1}(W) \cong \mathrm{Hom}(H_{2k+1}(W), \mathbf{Q})$ by Poincaré-Lefschetz duality and the universal coefficient theorem,

$$j : H_{2k+1}(W) \to \mathrm{Hom}(H_{2k+1}(W), \mathbf{Q})$$

can be identified with the intersection form

$$I : H_{2k+1}(W) \times H_{2k+1}(W) \to \mathbf{Q}$$

of W. Note that this is a skew-symmetric bilinear form. Let Q be a skew-symmetric matrix representing the intersection form I. It is easy to see that $\mathrm{rank}\, j = \mathrm{rank}\, Q$. So we have only to prove the following.

Lemma 5.2. *The rank of a skew-symmetric matrix Q whose components are complex numbers is always even.*

Proof. The charactristic polynomial $\Delta_Q(t)$ satisfies the following:

$$\begin{aligned} \Delta_Q(t) &= \det(tE - Q) = \det((tE - Q)^t) = \det(tE + Q) \\ &= (-1)^r \det(-tE - Q) = (-1)^r \Delta_Q(-t), \end{aligned}$$

where r is the size of Q and E is the unit matrix of degree r. Hence, if λ is a non-zero eigenvalue of Q, then $-\lambda$ is also a non-zero eigenvalue and their multiplicities coincide. Therefore counting the number of non-zero eigenvalues, we get the result. □

This completes the proof of Lemma 5.1. □

Lemma 5.3. *If two Morse functions $f : M \to \mathbf{R}$ and $g : N \to \mathbf{R}$ of closed oriented $(4k+1)$-dimensional manifolds are oriented cobordant, then we have*

$$\sigma(f) - \sigma(M; \mathbf{Q}) = \sigma(g) - \sigma(N; \mathbf{Q}).$$

Proof. Let $F : W \to \mathbf{R} \times [0, 1]$ be an oriented cobordism between f and g. Then by Lemma 5.1 we have

$$\sigma(M; \mathbf{Q}) + \sigma(N; \mathbf{Q}) \equiv \chi(W) \pmod 2.$$

Consequently we have

$$\sigma(M; \mathbf{Q}) - \sigma(N; \mathbf{Q}) \equiv \chi(W) \pmod 2. \tag{5.1}$$

Let $\pi : \mathbf{R} \times [0, 1] \to [0, 1]$ be the projection to the second factor, and we consider the composite function $h = \pi \circ F : W \to \mathbf{R}$. Slightly perturbing F if necessary, we may assume that the composite function $h : W \to \mathbf{R}$ is a Morse function. As in the un-oriented case we may assume that W is connected, although we have to be careful about the orientation of the components of W.

The components of $S_1(F)$ are divided into four types as follows.

(1) An arc joining two points, say p_1 and p_2, of M. Such an arc contains an odd number of critical points of h. The indices of p_1 and p_2 as critical points of f are of the form λ, $4k+1-\lambda$ for some $0 \le \lambda \le 4k+1$.

(2) An arc joining two points, say q_1 and q_2, of N. Such an arc contains an odd number of critical points of h. The indices of q_1 and q_2 as critical points of g are of the form λ, $4k+1-\lambda$ for some $0 \le \lambda \le 4k+1$.

(3) An arc joining a point, say p_3, of M and a point, say q_3, of N. Such an arc contains an even number of critical points of h. The indices of $p_3 \in M$ and $q_3 \in N$ as critical points of f and g respectively are equal to each other.

(4) A circle. Such a component contains an even number of critical points of h.

Therefore we have

$$\sigma(f) - \sigma(g) \equiv c \pmod 2, \tag{5.2}$$

where c is the number of critical points of h. Hence, we have $\chi(W) \equiv c \pmod 2$, since $\chi(M) = 0$. Therefore by (5.1) and (5.2) we have

$$\sigma(f) - \sigma(g) \equiv \sigma(M; \mathbf{Q}) - \sigma(N; \mathbf{Q}) \pmod 2.$$

This completes the proof of Lemma 5.3. □

Proof of Theorem 2.9. It is clear that the map $\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda$ is a well-defined homomorphism by Lemmas 4.1 and 5.3.

Let us show that $\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda$ is surjective. For any element $([M], a_{2k+2}, \dots, a_{4k+1}, l) \in \Omega_n \oplus \mathbf{Z}^{2k} \oplus \mathbf{Z}_2$, we can prove, by the argument similar to that in the proof of Theorem 2.7, that there exists a Morse function $f : M \to \mathbf{R}$ such that $(\tilde{\Psi} \oplus \tilde{\Phi})([f]) = ([M], a_{2k+2}, \dots, a_{4k+1})$. By creating a pair of critical points of indices $2k$ and $2k+1$, we can change $\sigma(f)$ without changing $(\tilde{\Psi} \oplus \tilde{\Phi})([f])$. Therefore, we can arrange so that $\Lambda([f]) = \sigma(f) - \sigma(M; \mathbf{Q}) = l$.

Now we have only to show that $\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda$ is injective. Let $[f : M \to \mathbf{R}]$ be an arbitrary element of $\mathrm{Ker}(\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda)$. Since the $(4k+1)$-dimensional manifold M is oriented null-cobordant, there exists a compact oriented $(4k+2)$-dimensional manifold W with $\partial W = M$. Then we construct a generic map $F : W \to \mathbf{R} \times [0,1]$ such that the F-images of the cusp neighborhoods are as depicted in Fig. 5 as in the un-oriented case, and set $N = F^{-1}(\mathbf{R} \times \{1/2\})$, $g = F|_N : N \to \mathbf{R} \times \{1/2\}$ and $V = F^{-1}(\mathbf{R} \times [1/2, 1])$. Note that the Morse function g is oriented cobordant to f. Therefore, $[g]$ is an element of $\mathrm{Ker}(\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda)$, and it satisfies

$$\sigma(g) - \sigma(N; \mathbf{Q}) = 0 \in \mathbf{Z}_2$$

by Lemma 5.3. Then we have

$$\sigma(N; \mathbf{Q}) \equiv \chi(V) \pmod 2$$

for the semi-characteristic of N with respect to the coefficient field $\mathbf{Q}$ by Lemma 5.1, and so

$$\sigma(g) \equiv \chi(V) \pmod 2.$$

Let $\pi : \mathbf{R} \times [0,1] \to [0,1]$ be the projection to the second factor. We may assume that $\pi \circ F$ is a Morse function. Then we have that

$$\chi(V) \equiv c \pmod 2$$

as in the proof of Lemma 5.3, and hence

$$\sigma(g) \equiv c \pmod 2, \tag{5.3}$$

where c denotes the number of critical points of $\pi \circ F$.

As in the un-oriented case we may assume that V is connected, although we have to be careful about the orientations of the components of V. Furthermore, we can eliminate all the cusps whose absolute indices are different from $2k$ by using Levine's method [4].

Let us show that then the number of cusps is even assuming that all the remaining cusps are of absolute index $2k$. Take an arc component α of $S_1(F|_V)$. We have two cases: α contains a cusp or no cusps. In both cases the indices of the end points of α as critical points of g are equal to λ and $(4k+1) - \lambda$ for some λ. Hence, the number $\sigma(g)$ of critical points of g whose indices are between 0 and $2k$ is equal to the number of arc components of $S_1(F|_V)$ (see Fig. 8). Note that an arc component α contains an odd number of critical points of $\pi \circ F$ if it contains no cusps, and α contains no critical points of $\pi \circ F$ if it contains a cusp. By (5.3), we have that the number of arc components of $S_1(F|_V)$ which contain a cusp point is even. Hence, the number of cusp points is even.

Finally we can remove all the cusps by using the theory of matching pairs by [4] for $n \neq 1$ and by [3] for $n = 1$. This completes the proof of Theorem 2.9. □

Remark 5.4. In the definition of $\Lambda : \mathscr{M}_{4k+1} \to \mathbf{Z}_2$ (Definition 2.6), we used the field of rational numbers $\mathbf{Q}$. In fact, we could as well use any other field K, for example $\mathbf{Z}_2$, since the semi-characteristic $\sigma(M; K)$ is independent of K for an orientable $(4k+1)$-dimensional manifold M which is null-cobordant [5].

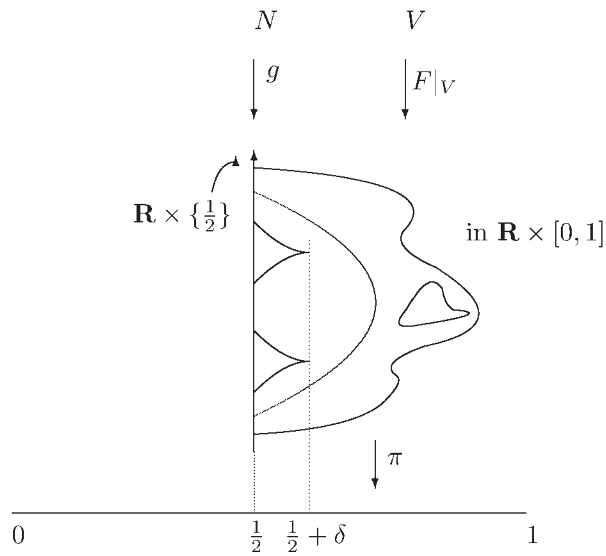

Fig. 8

In fact, in [5] it is shown that

$$\sigma(M; \mathbf{Z}_2) - \sigma(M; \mathbf{Q}) = W_2 W_{4k-1}[M],$$

where M is a closed orientable $(4k+1)$-dimensional manifold and $W_2 W_{4k-1}[M]$ denotes the Stiefel-Whitney number. So, we have another isomorphism

$$\tilde{\Psi} \oplus \tilde{\Phi} \oplus \Lambda' : \mathscr{M}_n \to \Omega_n \oplus \mathbf{Z}^{\lfloor n/2 \rfloor} \oplus \mathbf{Z}_2,$$

where $\Lambda' : \mathscr{M}_n \to \mathbf{Z}_2$ is the homomorphism defined by $\Lambda'([f : M \to \mathbf{R}]) = \sigma(f) - \sigma(M; \mathbf{Z}_2) \in \mathbf{Z}_2$. Since there exists an M with $W_2 W_{4k-1}[M] \neq 0$ (for example, the Dold manifold [1], or see [5, Remark 4]), this isomorphism is different from the one obtained in Theorem 2.9.

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