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PD_4-complexes with free fundamental group

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Abstract. We show that PD_4-complexes with free fundamental group are determined by their intersection pairings and that every hermitian form on a finitely generated free module over the group ring of a free group is realized by some such complex.

1. Introduction

The purpose of this article is to show that some of the basic properties of PD_4-complexes with free fundamental group can be derived homologically, without reference to the topology of 4-manifolds or stabilization by connected sums, as used in [4, 8, 13]. We also avoid explicit calculations of obstructions, relying instead on the easily verified fact that the 3-skeletons of the complexes considered have sufficiently many self homotopy equivalences. In particular we give a new proof of the fact that such complexes are determined by their intersection pairings, and that every hermitian form on a finitely generated free module over the group ring of a free group is realized by some such complex. In the final section we consider briefly the classification (up to s-cobordism) of closed 4-manifolds with free fundamental group.

2. Modules over free groups

Let $F(r)$ be the free group with basis $\{x_1, \dots, x_r\}$, and let $\Gamma = \mathbf{Z}[F(r)]$. Let $w : F(r) \to \{\pm 1\}$ be a homomorphism and define an involution on Γ by $\bar{g} = w(g)g^{-1}$ for all $g \in F(r)$. If R is a right Γ-module let $\bar{R}$ be the corresponding left Γ-module with the conjugate structure given by $\gamma.r = r.\bar{\gamma}$, for all $g \in \Gamma$ and $r \in R$. In particular, if L is a left Γ-module let $L^\dagger = \overline{\mathrm{Hom}_\Gamma(L, \Gamma)}$ be the conjugate dual module. Let $\varepsilon : \Gamma \to \mathbf{Z}$ be the augmentation homomorphism, and let $\varepsilon_w : \Gamma \to \mathbf{Z}^w$ be the w-twisted augmentation, determined by $\varepsilon_w(g) = w(g)$ for all $g \in F(r)$. Let $\mathscr{I}_w = \mathrm{Ker}(\varepsilon_w)$.

Let $\partial : \Gamma^r \to \Gamma$ be the homomorphism given by $\partial(\gamma_1, \dots \gamma_r) = \Sigma\gamma_i(x_i - 1)$, with image the augmentation ideal. Let $\delta_w = \partial^\dagger : \Gamma \to \Gamma^r$ and let $E_w^1\mathbf{Z} =$

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$\mathrm{Coker}(\delta_w) = \overline{\mathrm{Ext}^1_\Gamma(\mathbf{Z}, \Gamma)}$. (We emphasize that the conjugation depends on w). If $w' : F(r) \to \{\pm 1\}$ is another homomorphism then $\mathrm{Hom}_\Gamma(E^1_w\mathbf{Z}, E^1_{w'}\mathbf{Z}) = \mathrm{Hom}_\Gamma(\mathbf{Z}^{w'}, \mathbf{Z}^w)$, which is $\mathbf{Z}$ if $w = w'$ and is 0 if $w \neq w'$. In particular, $\mathrm{End}_\Gamma(E^1_w\mathbf{Z}) = \mathbf{Z}$ and $E^1_w\mathbf{Z}$ is hopfian, that is, surjective endomorphisms of $E^1_w\mathbf{Z}$ are automorphisms.

Lemma 1. *If L is a finitely presentable left Γ-module then $L^\dagger$ is a free module.*

Proof. Let $F_1 \xrightarrow{p} F_0 \to L$ be a presentation for L. Dualizing gives an exact sequence $0 \to L^\dagger \to F_0^\dagger \xrightarrow{p^\dagger} F_1^\dagger$. Now $\mathrm{Coker}(p^\dagger)$ has a projective resolution of length at most 2, since Γ has global dimension 2. Hence $L^\dagger$ is projective, by Schanuel's Lemma, and therefore free [1]. □

3. The 3-skeleton

If X is a space with basepoint $*$ and fundamental group π let $\tilde{X}$ be the universal covering space and $c_X : X \to K(\pi, 1)$ the classifying map for the fundamental group, and let $f_X : X \to P_2(X)$ denote the second stage of the Postnikov tower for X. Let $E(X)$ and $E_*(X)$ denote the groups of self homotopy equivalences of the space X and the pair $(X, *)$, respectively, and let $E_\pi(X)$ be the subgroup of self homotopy equivalences which induce the identity on π. Then $E_*(K(\pi, 1)) \cong \mathrm{Aut}(\pi)$ and $E_\pi(X) = \mathrm{Ker}(E_*(c_X))$.

Let P be a PD_4-complex with fundamental group $\pi \cong F(r)$. We may assume that P is a finite complex, since projective Γ-modules are free [1], and we shall fix an isomorphism $\pi = F(r)$, once and for all. Let $w = w_1(P)$ be the orientation character, and let P^+ be the orientable covering space associated to $\pi^+ = \mathrm{Ker}(w)$. Let $C_* = C_*(P; \Gamma)$ be the cellular chain complex of $\tilde{P}$, with respect to the natural π-equivariant cell structure. This is a complex of free left Γ-modules. Let $B_q \leq Z_q$ denote the q-dimensional boundaries and q-cycles in C_q, respectively, and let $H_q = H_q(C_*) = Z_q/B_q$, for $q \geq 0$. Then $H_q = H_q(P; \Gamma)$ is isomorphic to $H_q(\tilde{P}; \mathbf{Z})$, with the left Γ-module structure given by the action of the covering group π on $\tilde{P}$. In particular, $H_2 \cong \Pi = \pi_2(P)$. The equivariant cohomology modules are defined by $H^q = H^q(P; \Gamma) = H^q(C^*)$, where C^* is the dual cochain complex, with $C^q = C_q^\dagger$.

We may assume that $P = P_\theta = P_o \cup_\theta D^4$, where P_o is a 3-complex with one 0-cell and $\theta \in \pi_3(P_o)$ is the attaching map for the 4-cell [18].

Theorem 2. *Π is free of rank $\beta = \beta_2(P)$ and $P_o \simeq \bigvee^r (S^1 \vee S^3) \vee (\bigvee^\beta S^2)$.*

Proof. There are exact sequences

$$0 \to B_0 \to C_0 \to \mathbf{Z} \to 0,$$

$$0 \to B_1 \to C_1 \to B_0 \to 0,$$

$$0 \to Z_2 \to C_2 \to B_1 \to 0,$$

$$0 \to Z_3 \to C_3 \to Z_2 \to \Pi \to 0,$$

and

$$0 \to H_4 \to C_4 \to Z_3 \to H_3 \to 0.$$

Since π is a free group the augmentation module $\mathbf{Z}$ has a short free resolution and so B_0 is stably free, by Schanuel's Lemma, and hence free [1]. Therefore the second and third of these sequences are split exact. In particular, B_0, B_1 and Z_2 are finitely generated free Γ-modules. Poincaré duality and the Universal Coefficient spectral sequence give $H_4 = H_4(\tilde{P}; \mathbf{Z}) = H^0(C^*) = 0$ and an isomorphism $\Pi \cong \Pi^\dagger$. Hence Π is also a free left Γ-module, by Lemma 1, and so the fourth sequence also splits. Therefore Z_3 is free and the complex C_* is chain homotopy equivalent to the sum of the three free complexes $B_0 \to C_0$, Π and $C_4 \to Z_3$ (with $C_0 \cong C_4 \cong \Gamma$, B_0 in degree 1, Π in degree 2 and the degrees otherwise given by the subscripts). Therefore $B_0 \cong Z_3 \cong \Gamma^r$, $\mathbf{Z} \otimes_\Gamma \Pi \cong H_2(\mathbf{Z} \otimes_\Gamma C_*) \cong H_2(P; \mathbf{Z}) \cong \mathbf{Z}^\beta$, and so $\Pi \cong \Gamma^\beta$.

If P is nonorientable then $\pi^+ \cong F(2r-1)$ and Π has rank 2β as a $\mathbf{Z}[\pi^+]$-module. It follows easily from Poincaré duality in P^+ that $\chi(P) = \chi(P^+)/|w(\pi)| = 2 - 2r + \beta$ (in all cases). Hence $\chi(P_o) = 1 - 2r + \beta$, and so $H_3(P_o; \Gamma) = Z_3 \cong \Gamma^r$.

The Hurewicz homomorphism in degree 3 for a 1-connected space is onto ([20], or see Section 6 below), and so we may represent a basis of $H_3(P_o; \Gamma)$ by elements of $\pi_3(P_o)$. Hence there is a map $j : (\bigvee^r S^1) \vee (\bigvee^\beta S^2) \vee (\bigvee^r S^3) \to P_o$ which induces isomorphisms $\pi_1(j), \pi_2(j)$ and $H_3(j; \Gamma)$, and which is therefore a homotopy equivalence. □

In [13] a stable factorization theorem is used to prove this result for the case when P is a closed 4-manifold. The fact that Π is free also follows from Theorem 3.12 of [9].

4. The case $\boldsymbol{\beta} = 0$

Let $i_o : Q_o = \bigvee^r (S^1 \vee S^3) \to P_o = \bigvee^r (S^1 \vee S^3) \vee (\bigvee^\beta S^2)$ be the natural inclusion and let $c : P_o \to Q_o$ be the retraction which collapses the 2-spheres to the basepoint. Let $c_\phi : P_\phi \to Q_\phi = Q_o \cup_{c\phi} D^4$ be the canonical extension of c. If P_ϕ is an orientable PD_4-complex then so is Q_ϕ, and c_ϕ is a 2-connected degree 1 map, and conversely, if Q_ϕ is a PD_4 complex with fundamental class $[Q_\phi]$ cap product with the corresponding class in $H_4(P_\phi; \mathbf{Z})$ induces duality isomorphisms on the (co)homology of $\widetilde{P_\phi}$, excepting perhaps in degree 2 [8]. Their argument extends immediately to the nonorientable case.

Theorem 3. *The complex Q_ϕ is a PD_4-complex with orientation character w if and only if $c\phi = \alpha\delta_w(1)$ for some automorphism $\alpha \in GL(r, \Gamma)$.*

Proof. We may identify $\pi_3(Q_o)$ with $H_3(\widetilde{Q_o}; \mathbf{Z}) \cong \Gamma^r$, by the Hurewicz Theorem for $\widetilde{Q_o}$. The equivariant cellular chain complex for $\widetilde{Q_\phi}$ is chain homotopy equivalent to the complex $C_4 \to Z_3 \to 0 \to B_0 \to C_0$, where $\partial_1 = \partial$, $Z_3 = \pi_3(Q_o) \cong \Gamma^r$, $C_4 \cong \Gamma$, and ∂_4 is given by $\partial_4(\gamma) = \gamma.c\phi$. If Q_ϕ is a PD_4-complex with orientation character w then C_* is chain homotopy equivalent to C^{4-*} and $H_3(C_*) \cong E^1_w\mathbf{Z}$. Therefore $\mathbf{Z} = H_0(C_*) \cong \mathrm{Coker}(\partial_4^\dagger)$, and so there is an automorphism $\mu \in GL(r, \Gamma)$ such that $\partial_4^\dagger\mu = \partial_1 = \partial : \Gamma^r \to \Gamma$. Therefore $c\phi = \alpha\delta_w(1)$, where $\alpha = u\mu^{\dagger -1}$ for some unit $u \in \Gamma$. The converse is clear. □

Let $S^1 \tilde{\times} S^3$ be the nonorientable S^3-bundle over S^1.

Corollary A. *If Q_ϕ is orientable then $Q_\phi \simeq \#^r(S^1 \times S^3)$; otherwise $Q_\phi \simeq (S^1 \tilde{\times} S^3)\#(\#^{r-1}(S^1 \times S^3))$.*

Proof. We may assume that $w(x_i) = 1$ for $2 \le i \le r$, as every automorphism of $F(r)$ may be realized by a basepoint-preserving self homotopy equivalence of Q_o. As these orientation characters are realized by the given 4-manifolds and as every automorphism of Γ^r may be realized by a self homotopy equivalence of Q_o which is the identity on the 1-skeleton $\bigvee^r S^1$, the result follows from the theorem. □

This argument is simpler than the Postnikov argument used in [10]. (The argument for this result in [3] appears to have a gap on page 242, since $\pi_3(\tilde{Y}^*)$ is not finitely generated, and so $\pi_3(F^*) \neq 0$).

Corollary B. *If P_ϕ and P_ψ are PD_4-complexes with the same 3-skeleton P_o and orientation character then $\psi \equiv \alpha\phi \bmod \Gamma_W(\pi_2)$ for some self homotopy equivalence $\alpha \in E(P_o)$.*

Proof. The boundary map $C_4 \to C_3$ in the cellular chain complex for $\tilde{P}_\phi$ is essentially determined by $\mathrm{hwz}(\phi)$, the image of the attaching map, and this may be identified with $c\phi$. □

The corresponding assertion in [8] does not take into account the role of self homotopy equivalences of P_o.

5. Homotopy equivalences of pairs

We wish to determine how $P_\phi = P_o \cup_\phi D^4$ depends on the attaching map $\phi \in \pi_3(P_o)$ and when it is a PD_4-complex. Let $j_\phi : P_o \to P_\phi$ be the inclusion. A homotopy equivalence $f : P_\phi \to P_\psi$ is *rel* P_o if $fj_\phi \simeq j_\psi$.

THEOREM 4. *Suppose that P_ϕ and P_ψ are PD_4-complexes. Then*
(i) *there is a homotopy equivalence of pairs $(P_\phi, P_o) \simeq (P_\psi, P_o)$ if and only if there is a self homotopy equivalence $\alpha \in E(P_o)$ such that $\alpha\phi = \psi$;*
(ii) *$P_\phi \simeq P_\psi$ rel P_o if and only if $\phi = \pm g.\psi$ for some $g \in \pi$.*

PROOF. If $f : (P_\phi, P_o) \simeq (P_\psi, P_o)$ is a map of pairs then $f\phi$ is nullhomotopic in P_ψ. Thus $f\phi \in \mathrm{Ker}(\pi_3(j_\psi)) = \langle\psi\rangle$, which is freely generated as a $\mathbf{Z}[\pi_1(P_\psi)]$-module by ψ, by the relative Hurewicz Theorem. If f is a homotopy equivalence of pairs with homotopy inverse f^{-1} we must also have $f^{-1}\psi \in \langle\phi\rangle$, and so $f\phi = u.\psi$ for some unit $u \in \mathbf{Z}[\pi_1(P_\psi)]^\times$. Since free groups are orderable their group rings have only trivial units and so $u = \pm g$ for some $g \in \pi_1(P_\psi)$. If α is the composite of $f|_{P_o}$ with a self homotopy equivalence of P_o which drags the basepoint around a loop representing g^{-1} then $\alpha\phi = \pm\psi$. We may adjust the sign by composition with a self homeomorphism of P_o which is the identity on the 2-skeleton and has degree -1 on each 3-sphere. Conversely, a self homotopy equivalence $\alpha \in E(P_o)$ induces a homotopy equivalence $(P_\phi, P_o) \simeq (P_{\alpha\phi}, P_o)$.

Since P_ϕ is a PD_4-complex $H_3(P_\phi; \Gamma) \cong H^1(P_\phi; \Gamma) \cong E^1_w\mathbf{Z}$. (Note that if $r = 0$ then P_ϕ is orientable and $H_3(P_\phi; \Gamma) = 0$). If $f : P_\phi \to P_\psi$ is a map such that $fj_\phi \sim j_\psi$ then $H_3(f; \Gamma)$ is an epimorphism. Therefore if P_ϕ and P_ψ are each PD_4-complexes they have the same orientation character ($f^*w_1(P_\psi) = w_1(P_\phi)$) and $H_3(f; \Gamma)$ is an isomorphism (since $E^1_w\mathbf{Z}$ is hopfian). Moreover $\langle\phi\rangle = \langle\psi\rangle$, so $\phi = \pm g.\psi$ for some $g \in \pi$, and there is a map $f' : P_\psi \to P_\phi$ such that $f'j_\psi \sim j_\phi$. Clearly $f'f \sim 1_{P_\phi}$ and $ff' \sim 1_{P_\psi}$ *rel* P_o. The converse is clear. (Here we may adjust the sign by composition with a degree -1 self homeomorphism of S^3). □

If $f : P_\phi \to P_\psi$ is a homotopy equivalence, is it homotopic to a cellular map F such that $F|_{P_o}$ is a homotopy equivalence?

6. Intersection pairings

If X is a 1-connected cell complex there is an exact sequence

$$H_4(X; \mathbf{Z}) \xrightarrow{b_4} \Gamma_W(\pi_2(X)) \longrightarrow \pi_3(X) \xrightarrow{\mathrm{hwz}} H_3(X, \mathbf{Z}) \to 0,$$

where $A \mapsto \Gamma_W(A)$ is the universal quadratic functor of Whitehead and hwz is the Hurewicz homomorphism [20]. (See Chapter 1 of [2] for a recent exposition of Whitehead's work, in particular, for a description of the "secondary boundary" b_4). The Whitehead sequence is functorial, and so the Whitehead sequence for $\widetilde{P_o}$ is an exact sequence of Γ-modules. If X has dimension at most 3 the group of automorphisms of $\pi_3(X)$ which preserve the Whitehead exact sequence for $\tilde{X}$ and restrict to the identity on $\Gamma_W(\Pi)$ is a semidirect product $\mathrm{Hom}_{\mathbf{Z}[\pi]}(H_3(X;\mathbf{Z}[\pi]),\Gamma_W(\Pi)) \rtimes \mathrm{Aut}_{\mathbf{Z}[\pi]}(H_3(X;\mathbf{Z}[\pi]))$.

We may use this observation to understand the action of $E_\pi(P_o)$ on $\pi_3(P_o)$. Let $V=\bigvee^r S^1 \vee (\bigvee^\beta S^2)$ be the 2-skeleton of P_o, $j_o: V\to P_o$ the inclusion and $d: P_o\to V$ be the retraction which collapses the 3-cells to the basepoint. Then $s=i_oc$ and $t=j_od$ induce complementary projections on $\pi_3(P_o)=\mathrm{Im}(s_*)\oplus\mathrm{Im}(t_*)\cong\Gamma^r\oplus\Gamma_W(\Pi)$, splitting the Whitehead sequence for $\widetilde{P_o}$. Since $H_3(P_o;\Gamma)$ is free of rank r we have $\mathrm{Hom}_\Gamma(H_3(P_o;\Gamma),\Gamma_W(\Pi))\cong\Gamma_W(\Pi)^r$ and it is easily seen that $E_\pi(P_o)$ has a subgroup $\Gamma_W(\Pi)^r\rtimes(GL(r,\Gamma)\times GL(\beta,\Gamma))$ which acts on $\pi_3(P_o)$ via $(\xi,M,N).(\gamma,v)=(M(\gamma),\Gamma_W(N)(v)+\gamma.\xi)$ for all $(\gamma,v)\in\Gamma^r\oplus\Gamma_W(\Pi)$, and thus generates the action of $E_\pi(P_o)$ on $\pi_3(P_o)$. (It can be shown that $E_\pi(V)\cong E_\pi(P_2(P_o))\cong GL(\beta,\Gamma)$ and $E_\pi(Q_o)\cong GL(r,\Gamma)$). If $\gamma=\delta_w(1)$, the orbits of the action correspond bijectively to the orbits of the induced action of $GL(\beta,\Gamma)$ on $\mathbf{Z}^w\otimes_\Gamma\Gamma_W(\Pi)$.

Tensoring the Whitehead sequence for $\widetilde{P_\phi}$ with the bimodule $\mathbf{Z}^w$ gives a homomorphism from $\mathrm{Tor}_1^\Gamma(\mathbf{Z}^w,H_3)$ to $\mathbf{Z}^w\otimes_\Gamma\Gamma_W(\Pi)$, while tensoring C_* with $\mathbf{Z}^w$ gives an isomorphism $H_4(P_\phi;\mathbf{Z}^w)\cong\mathrm{Tor}_1^\Gamma(\mathbf{Z}^w,H_3)$. If $(\varepsilon_w\otimes\mathrm{hwz})(\phi)=0$ $[\phi]=(\varepsilon_w\otimes 1)(\phi)$ is in the subgroup $\mathbf{Z}^w\otimes_\Gamma\Gamma_W(\Pi)$ of $\mathbf{Z}^w\otimes_\Gamma\pi_3(P_o)$. This is the case if P_ϕ is a PD_4-complex, by Theorem 3.

Let G be a group and $w: G\to\{\pm1\}$ a homomorphism, and denote the w-twisted involution on $\mathbf{Z}[G]$ by an overbar, as in Section 2. A *w-hermitian form* on a finitely generated projective $\mathbf{Z}[G]$-module N is a pairing $\lambda: N\times N\to\mathbf{Z}[G]$ which is $\mathbf{Z}[G]$-linear in the first variable, conjugate symmetric (i.e., $\lambda(n,n')=\overline{\lambda(n',n)}$, for all $n,n'\in N$) and such that $\lambda(gn,gn')=w(g)g\lambda(n,n')g^{-1}$, for all $n,n'\in N$ and $g\in G$. It is *nonsingular* if the adjoint map $\tilde{\lambda}: N\to N^\dagger$ given by $\tilde{\lambda}(n')(n)=\lambda(n,n')$ (for all $n,n'\in N$) is an isomorphism. The set $\mathrm{Her}_w(N)$ of w-hermitian forms on N is an abelian group with respect to addition of the values of the forms. If $N=M^\dagger$ is the dual of a finitely generated projective $\mathbf{Z}[G]$-module M let $ev(m)(n,n')=\overline{n(m)}n'(m)$, for all $m\in M$ and $n,n'\in M^\dagger$. Then $ev(m)(n,n')$ is $\mathbf{Z}$-quadratic in m and w-hermitian in n and n', and so ev determines a homomorphism B from $\mathbf{Z}^w\otimes_{\mathbf{Z}[G]}\Gamma_W(M)$ to $\mathrm{Her}_w(M^\dagger)$. (In our applications G shall be either trivial or π).

Lemma 5. *Let $M\cong\Gamma^r$. Then $B:\mathbf{Z}^w\otimes_\Gamma\Gamma_W(M)\to\mathrm{Her}_w(M^\dagger)$ is an isomorphism.*

Proof. Let $e_1,\dots,e_r$ be a basis for M as a free left Γ-module, and let $e_1^*,\dots,e_r^*$ be the dual basis for $M^\dagger$, determined by $e_i^*(e_i)=1$ and $e_i^*(e_j)=0$ if $i\neq j$.

If A is an abelian group the universal quadratic map $\gamma_A: A\to\Gamma_W(A)$ determines a map s from the symmetric product $A\circ_{\mathbf{Z}}A$ to $\Gamma_W(A)$ by $s(a\circ b)=\gamma(a+b)-\gamma(a)-\gamma(b)$, and there is an exact sequence

$$A \circ_{\mathbf{Z}} A \xrightarrow{s} \Gamma_W(A) \to A/2A \to 0,$$

where the right-hand map is induced by the projection of A onto $A/2A$ (which is quadratic) [2]. If A and B are abelian groups the inclusions into $A \oplus B$ induce a canonical splitting $\Gamma_W(A \oplus B) \cong \Gamma_W(A) \oplus \Gamma_W(B) \oplus (A \otimes B)$. Since $\Gamma(\mathbf{Z}) \cong \mathbf{Z}$ it follows by a finite induction that if $A \cong \mathbf{Z}^r$ then $\Gamma_W(\mathbf{Z}^r)$ is again finitely generated and free, and that s is injective. The latter conditions hold for A any free abelian group, since every finitely generated subgroup of such a group lies in a finitely generated direct summand.

In particular, as M is a free abelian group there is a short exact sequence

$$0 \to M \circ_{\mathbf{Z}} M \to \Gamma_W(M) \to M/2M \to 0,$$

and $\Gamma_W(M)$ is free as an abelian group. This is a sequence of Γ-modules and homomorphisms, if we define the action on $M \circ_{\mathbf{Z}} M$ by $g(m \circ n) = gm \circ gn$, for all $g \in \pi$ and $m, n \in M$.

The sequence

$$0 \to \mathbf{Z}^w \otimes_\Gamma (M \circ_{\mathbf{Z}} M) \to \mathbf{Z}^w \otimes_\Gamma \Gamma_W(M) \to \mathbf{F}_2 \otimes_\Gamma M \to 0$$

is also exact, since $\mathrm{Tor}_1^\Gamma(\mathbf{Z}^w, M/2M) = \mathrm{Ker}(2 : \mathbf{Z}^w \otimes_\Gamma M \to \mathbf{Z}^w \otimes_\Gamma M) = 0$. Let $\eta_M : M \to \mathbf{Z}^w \otimes_\Gamma \Gamma_W(M)$ be the composite of γ_M with the reduction from $\Gamma_W(M)$ to $\mathbf{Z}^w \otimes_\Gamma \Gamma_W(M)$. Then the composite of η_M with the projection to $\mathbf{F}_2 \otimes_\Gamma M$ is the canonical epimorphism.

Since $m \circ gn = g(g^{-1}m \circ n) = \bar{g}m \circ n$ in $\mathbf{Z}^w \otimes_\Gamma (M \circ_{\mathbf{Z}} M)$, the typical element of $\mathbf{Z}^w \otimes_\Gamma (M \circ_{\mathbf{Z}} M)$ may be expressed in the form $\mu = \sum_{i \le j} (r_{ij} e_i) \circ e_j$. For such an element $B(\mu)(e_k^*, e_l^*) = r_{kl}$, if $k < l$, and $= r_{kk} + \bar{r}_{kk}$, if $k = l$. In particular, $B(\mu)$ is *even*: if $\varepsilon_2 : \Gamma \to \mathbf{F}_2$ is the composite of the augmentation with reduction *mod* (2) then $\varepsilon_2(B(\mu)(n,n)) = 0$ for all $n \in M^\dagger$.

If $m \in M$ has nontrivial image in $\mathbf{F}_2 \otimes_\Gamma M$ then $\varepsilon_2(e_i^*(m)) \neq 0$ for some $i \le r$. Hence $B(\eta_M(m))$ is not even, and it follows easily that $\mathrm{Ker}(B) \le \mathbf{Z}^w \otimes_\Gamma (M \circ M)$. Suppose that $B(\mu) = 0$, for some $\mu = \sum_{i \le j} (r_{ij} e_i) \circ e_j$. Then $r_{kl} = 0$, if $k < l$, and $r_{ii} + \bar{r}_{ii} = 0$, for all i. Therefore $\mu = \sum (r_{ii} e_i) \circ e_i$, and $r_{ii} = \sum_{g \in F(i)} a_{ig}(g - \bar{g})$ where $F(i)$ is a finite subset of π, for $1 \le i \le r$. Since $((g - \bar{g})e_i) \circ e_i = 0$ it follows easily that $\mu = 0$. Hence B is injective.

To show that B is surjective it shall suffice to assume that M has rank 1 or 2, since h is determined by the values $h_{ij} = h(e_i^*, e_j^*)$. Let $\varepsilon_w[m, m']$ be the image of $m \circ m'$ in $\mathbf{Z}^w \otimes_{\mathbf{Z}[G]} \Gamma_W(M)$. Then $B(\varepsilon_w[m, m'])(n, n') = \overline{n(m)} n'(m') + \overline{n(m')} n'(m)$, for all $m, m' \in M$ and $n, n' \in M^\dagger$. Suppose first that M has rank 1. Since $h_{11} = \bar{h}_{11}$ and $\mathrm{Ker}(w)$ has no element of order 2 we may write $h_{11} = 2b + \delta + \sum_{g \in F}(g + \bar{g})$, where $b = \bar{b}$, $\delta = 1$ or 0 and F is a finite subset of $F(r)$. Let $\mu = \varepsilon_w[(b + \delta + \sum_{g \in F} g)e_1, e_1] + \delta \eta_M(e_1)$. Then $B(\mu)(e_1^*, e_1^*) = h_{11}$. If M has rank 2 and $h_{11} = h_{22} = 0$ let $\mu = \varepsilon_w[h_{12} e_1, e_2]$. Then $B(\mu)(e_i^*, e_j^*) = h_{ij}$. In each case $B(\mu) = h$, since each side of the equation is a w-hermitian form on $M^\dagger$. $\square$

If $H_4(P_\phi;\mathbf{Z}^w) = H_4(\mathbf{Z}^w \otimes_\Gamma C_*) \neq 0$ then it is infinite cyclic; fix a generator $[P_\phi]$. Cap product with $[P_\phi]$ defines a homomorphism from $\Pi^\dagger$ to Π, and hence determines a cohomology intersection pairing $\lambda^\phi : \Pi^\dagger \times \Pi^\dagger \to \Gamma$, given by $\lambda^\phi(u,v) = v(u \cap [P_\phi])$ for all $u, v \in \Pi^\dagger$. This pairing is a w-hermitian form, by Proposition 4.58 of [15]. Let $c_\phi : \Pi^\dagger \cong \overline{H^2} \to \Pi = H_2$ be the homomorphism determined by cap product: $c_\phi(u) = u \cap [P_\phi]$. Then $\tilde{\lambda}^\phi(v)(u) = c_\phi^\dagger(v)(u)$, for all $u, v \in \Pi^\dagger$, and so $\tilde{\lambda}^\phi = c_\phi^\dagger$. Hence λ^ϕ is nonsingular if and only if P_ϕ is a PD_4-complex. In the latter case we may use duality to define the equivalent homology intersection pairing λ_ϕ. (In general it may be shown that the cohomology intersection pairing of a PD_4-complex with fundamental group G is nonsingular if and only if $H^s(G;\mathbf{Z}[G]) = 0$ for $s = 2$ and 3).

We shall show that the two pairings just defined in terms of ϕ are equivalent, by using a Postnikov approximation to P_o. The space $P_2(P_o)$ is the total space of the $K(\Pi,2)$ fibration over $K(\pi,1)$ corresponding to the natural action of π on Π, which is uniquely determined since the k-invariant is in $H^3(\pi;\Pi) = 0$. We may construct $P_2(P_o)$ by adjoining cells of dimension ≥ 4 to P_o to kill the higher homotopy. For clarity of notation let $L = P_2(P_o)$ and let $j = f_{P_o} : P_o \to L$ be the (3-connected) map induced by the inclusion of P_o into L.

The following lemma is the crux of this section. Underlying this lemma is the fact that $\Gamma_W(\Pi)$ may be viewed homotopically as the subgroup of $\pi_3(P_o)$ generated from Π by Whitehead products and composition with the Hopf map and homologically as $H_4(P_2(P_o);\Gamma) = H_4(K(\Pi,2);\mathbf{Z})$.

LEMMA 6. *Let $\phi \in \pi_3(P_o)$ have image 0 in $H_3(P_o;\mathbf{Z}^w)$. Then $H_4(P_\phi;\mathbf{Z}^w) \cong \mathbf{Z}^w$ and $b_{\tilde{L}}$ induces an isomorphism $b' : H_4(L;\mathbf{Z}^w) \cong \mathbf{Z}^w \otimes_\Gamma \Gamma_W(\Pi)$ such that $b'(j_*[X_\phi]) = [\phi]$.*

PROOF. Let $\delta \in \pi_4(P, P_o)$ be the map of pairs corresponding to ϕ and let δ_L be its image in $\pi_4(L, L_o)$, corresponding to $j\phi$. Then $\phi = \partial_P \delta$ and $j\phi = \partial_L \delta_L$, where ∂_P and ∂_L are the connecting homomorphisms in the exact sequences of homotopy. Since P_o is 3-dimensional and ϕ has image 0 in $H_3(P_o;\mathbf{Z}^w)$ the inclusion of $H_4(P_\phi;\mathbf{Z}^w)$ into the relative group $H_4(P_\phi, P_o;\mathbf{Z}^w) \cong \mathbf{Z}^w$ is an isomorphism. The relative Hurewicz homomorphism $\pi_4(P_\phi, P_o) \cong H_4(P_\phi, P_o;\Gamma)$ is an isomorphism, and comparison of the exact sequences of homotopy and homology for the pair (P_ϕ, P_o) shows that $[P_\phi]$ and δ have the same image in $H_4(P_\phi, P_o;\mathbf{Z}^w)$. Hence $j_*[P_\phi]$ and δ_L have the same image in $H_4(L, L_o;\mathbf{Z}^w)$, by naturality of the Hurewicz homomorphism. Now $b_{\tilde{L}} : H_4(L;\Gamma) \to \Gamma_W(\Pi)$ is an isomorphism, since $\pi_3(L) = \pi_4(L) = 0$. Since Π is free and $H_3(L;\Gamma) = H_3(K(\Pi,2);\mathbf{Z}) = 0$ the Cartan-Leray spectral sequence for the projection $p : \tilde{L} \to L$ gives an isomorphism from $\mathbf{Z}^w \otimes_\Gamma H_4(L;\Gamma)$ to $H_4(L;\mathbf{Z}^w)$. Hence we obtain an isomorphism $b' : H_4(L;\mathbf{Z}^w) \cong \mathbf{Z}^w \otimes_\Gamma \Gamma_W(\Pi)$.

Choose $\xi \in H_4(L;\Gamma)$ with image $j_*[P_\phi]$ in $H_4(L;\mathbf{Z}^w)$, and let ξ_{rel} be its image in $H_4(L,L_o;\Gamma)$ under the monomorphism induced by the natural map. Then $\xi_{rel} - \delta_L$ has image 0 in $H_4(L,L_o;\mathbf{Z}^w) = \mathbf{Z}^w \otimes_\Gamma H_4(L,L_o;\Gamma)$, and so is in $\mathscr{I}_w H_4(L,L_o;\Gamma)$. Hence $\partial_L(h_{\tilde{L}}^{-1}(\xi_{rel}) - \delta_L)$ is in $\mathscr{I}_w \pi_3(L_o)$. Now $b_{\tilde{L}}(\xi) = \partial_L(h_{\tilde{L}}^{-1}(\xi_{rel}))$ (see [2]) and $\partial_L \delta_L = j\phi$. Therefore $b'(j_*[P_\phi]) = [\phi]$. □

Let G be a group and $w : G \to \{\pm 1\}$ a homomorphism as above and A a finitely generated free abelian group. Let $A^* = \mathrm{Hom}(A,\mathbf{Z})$ and $\hat{A} = A^* \otimes_{\mathbf{Z}} \mathbf{Z}[G]$. If $B : A^* \times A^* \to \mathbf{Z}$ is a symmetric bilinear pairing we may extend it to a w-hermitian pairing $\hat{B} : \hat{A} \times \hat{A} \to \mathbf{Z}[G]$ by setting $\hat{B}(\sum_{g\in G} a_g g, \sum_{h\in G} b_h h) = \sum_{g,h\in G} a_g b_h \bar{h} g$.

Lemma 7. *Let* $\beta_h = B(b_{(CP^\infty)^n}(h))$, *for* $h \in H_4((CP^\infty)^n;\mathbf{Z})$, *and let* G *be a group. Then* $\widehat{\beta_h}(u,v) = v(u \cap h)$ *for all* $u,v \in H^2((CP^\infty)^n;\mathbf{Z}[G]) \cong H^2((CP^\infty)^n;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[G]$ *and* $h \in H_4((CP^\infty)^n;\mathbf{Z})$.

Proof. As each side of the equation is linear in h and $H_4((CP^\infty)^n;\mathbf{Z})$ is generated by the images of homomorphisms induced by maps from CP^∞ or $(CP^\infty)^2$, it suffices to assume $n = 1$ or 2. Since moreover $\widehat{\beta_h}(u,v)$ and $v(u \cap h)$ are bilinear in u and v we may reduce to the case $G = 1$. As these functions have integral values and $2(x \otimes y) = (x+y) \otimes (x+y) - x \otimes x - y \otimes y$ in $H_4((CP^\infty)^2;\mathbf{Z})$, for all $x, y \in \Pi \cong Z^2$, we may reduce further to the case $n = 1$, which is easy. □

Lemma 8. *Let* $x \in H_4(L;\mathbf{Z}^w)$. *Then* $B(b'(x))(u,v) = v(u \cap x)$ *for all* $u, v \in \Pi^\dagger$.

Proof. Let $p : \tilde{L} \to L$ be the covering projection. Then $x = p_*(\tilde{x})$ for some $\tilde{x} \in H_4(L;\Gamma)$. Since Π is the union of its finitely generated free abelian subgroups and homology commutes with direct limits there is an $n > 0$ and a map $k : (CP^\infty)^n \to \tilde{L}$ such that $\tilde{x} = k_*(\xi)$ for some $\xi \in H_4((CP^\infty)^n;\mathbf{Z})$. As $v(u \cap (pk)_*\xi) = k^*v(k^*u \cap \xi)$, and $ev(k_*y)(u,v) = \widehat{ev(y)}(k^*u, k^*v)$, for all $u, v \in \Pi^\dagger$, $\xi \in H_4((CP^\infty)^n;\mathbf{Z})$ and $y \in H_2((CP^\infty)^n;\mathbf{Z})$, the lemma follows easily from Lemma 7. □

Theorem 9. *Let* $\phi \in \pi_3(P_o)$ *and let* $w : \pi \to \{\pm 1\}$ *be a homomorphism, and assume that* ϕ *has image* 0 *in* $H_3(P_o;\mathbf{Z}^w)$. *Then*

(i) $B([\phi]) \cong \lambda^\phi$;

(ii) $P_\phi = P_o \cup_\phi D^4$ *is a* PD_4*-complex with* $w_1(P_\phi) = w$ *if and only if* $\mathrm{hwz}(\phi) = \alpha\delta_w(1)$ *in* $H_3(P_o;\Gamma) \cong \Gamma^r$ *for some automorphism* $\alpha \in GL(r,\Gamma)$ *and* $B([\phi])$ *is nonsingular;*

(iii) *every nonsingular w-hermitian form on a finitely generated free Γ-module is the cohomology intersection pairing of some PD_4-complex P with $\pi_1(P) \cong F(r)$ and $w_1(P) = w$;*

(iv) *two such PD_4-complexes P_ϕ and P_ψ are homotopy equivalent via a map $f : P_\phi \to P_\psi$ such that $f_*([P_\phi]) = [P_\psi]$ and $c_{P_\psi} f \sim c_{P_\phi}$ if and only if $\lambda^\phi \cong \lambda^\psi$.*

Proof. The map $j : P_o \to L$ is 3-connected. Since $b'(j_*[P_\phi]) = [\phi]$, by Lemma 6, we have $B([\phi])(u, v) = v(u \cap j_*[P_\phi])$ for all $u, v \in \Pi^\dagger$, by Lemma 8, and so $B([\phi]) \cong \lambda^\phi$.

If P_ϕ is a PD_4-complex with $w_1(P_\phi) = w$ then $\mathrm{hwz}(\phi) = \alpha\delta_w(1)$, by Theorem 3, and λ^ϕ is nonsingular, so the conditions in part (ii) are necessary. Suppose that they hold. Then Q_ϕ is a PD_4-complex, by Theorem 3 and the assumption on $\mathrm{hwz}(\phi)$. Cap product with $[P_\phi]$ induces an isomorphism $H^2 \cong H_2$, by the assumption on λ^ϕ, and induces isomorphisms $H^q \cong H_{4-q}$ in all other degrees, by comparison with Q_ϕ. (In fact $H^0 = H_4 = H^3 = H_1 = 0$, so it is only necessary to check that $\bigcap[P_\phi] : H^1 \to H_3$ is an isomorphism).

The final assertions follow from the fact that B induces a bijection from the $E_\pi(P_o)$ orbits in $\pi_3(P_o)$ via the $GL(\beta, \Gamma)$ orbits in $\mathbf{Z}^w \otimes_\Gamma \Gamma_W(\Pi)$ to the equivalence classes of Γ-sequilinear pairings on $\Pi^\dagger \times \Pi^\dagger$. □

The notion of PD_4-polarized Postnikov 2-stage from [6] is used in [3] to prove that $P_\phi \simeq P_\psi$ if and only if $\lambda_\phi \cong \pm\lambda_\psi$, for oriented closed 4-manifolds with $\beta \neq 0$. In [4] it is asserted that the image of $[P_\phi]$ in $\mathbf{Z}^w \otimes_\Gamma \pi_3(P_o)$ may be identified with the homology intersection pairing λ_ϕ, via Poincaré duality.

7. 4-Manifolds

If $r = 0$ or 1 the fundamental group $\pi = F(r)$ is abelian, and so we may use topological surgery to classify 4-manifolds with fundamental group π.

Suppose first that $r = 0$. Then $\pi = 1$, $P_o \simeq \bigvee^\beta S^2$, $\pi_3(P_o) = \Gamma_W(\Pi)$ and $E_\pi(P_o) \cong GL(\beta, \mathbf{Z})$. The homomorphism B from $\Gamma_W(\Pi)$ to the set of symmetric forms on $\Pi^\dagger$ is an isomorphism which maps ϕ to λ^ϕ and the orbits of the natural action of $GL(\beta, \mathbf{Z})$ on $\Gamma_W(\Pi)$ correspond to the equivalence classes of such forms. Hence every nonsingular symmetric form over $\mathbf{Z}$ is the intersection form of a 1-connected PD_4-complex, which is well defined up to homotopy equivalence. Every such complex is homotopy equivalent to a closed 4-manifold, and two such manifolds are homeomorphic if and only if their intersection pairings are isomorphic and the KS smoothing obstructions agree. (If the intersection pairing is even the KS invariant is determined by the signature). See [5].

A similar result holds when $r = 1$ (i.e., $\pi = Z$). Every nonsingular hermitian form λ on a finitely generated free $\mathbf{Z}[Z]$-module is the equivariant intersection form of some closed oriented 4-manifold with fundamental group Z, and two such manifolds are homeomorphic if and only if their intersection pairings are isomorphic and the KS invariants agree. (The KS invariant is again determined by the signature in the even-dimensional case). See Chapter 10 of [5] (as corrected in [17]). The classification is extended to the non-orientable cases in [19]. Every such manifold is stably homeomorphic to the connected sum of $S^1 \times S^3$ or $S^1 \tilde{\times} S^3$ with a 1-connected 4-manifold [7, 19]. (However stabilization is necessary, as there are intersection forms over $\mathbf{Z}[Z]$ which are not extended from symmetric forms over $\mathbf{Z}$ [7]).

If $r > 1$, the most we can hope for at present is to obtain classifications up to s-cobordism or up to stabilization by connected sum with copies of $S^2 \times S^2$. Homotopy equivalences are simple, since $Wh(F(r)) = 0$ [16]. It follows from Lemma 6.9 of [9] that if M is a closed 4-manifold and $\pi_1(M)$ is a free group then homotopy equivalences $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ are s-cobordant if and only if they have the same normal invariants in $[M, G/TOP]$. The Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M; Z/2Z)$ is onto since $\pi_1(M)$ is free. Hence if moreover M is orientable and $w_2(\tilde{M}) = 0$ every normal invariant with surgery obstruction 0 is realizable by a self homotopy equivalence ([14]—see Lemma 6.5 of [Hi']), and so 4-manifolds homotopy equivalent to M are s-cobordant to M. In the remaining cases ($w_2(\widetilde{P_\phi}) \neq 0$) there are at most two s-cobordism classes in each homotopy type. In [11] it is shown that if M is orientable and $\pi_1(M)$ is free then M is s-cobordant to the connected sum of $\#^r(S^1 \times S^3)$ with a 1-connected 4-manifold if and only if the intersection form is extended from a form over $\mathbf{Z}$.

If M and N are h-cobordant closed 4-manifolds then $M\#(\#^k S^2 \times S^2)$ is homeomorphic to $N\#(\#^k S^2 \times S^2)$ for some $k \geq 0$. (See Chapter VII of [5]). In the spin case $w_2(M) = 0$ this is an elementary consequence of the existence of a well-indexed handle decomposition of the h-cobordism. Moreover, the KS invariant of a TOP 4-manifold M is 0 if and only if $M\#(\#^k S^2 \times S^2)$ is smoothable for some $k \geq 0$ [12].

If a nonsingular hermitian form on a finitely generated free Γ-module is even then it is stably equivalent to one extended from a form over $\mathbf{Z}$, since it may be equipped with a quadratic enhancement, and the inclusion of the trivial group induces an epimorphism of quadratic surgery groups: $L_4(1) \cong Z \to L_4(F(r), w) \cong Z$ or $Z/2Z$. In the odd case one needs to know the corresponding result for the Witt groups (symmetric surgery groups) $L^0(\pi)$ and that $T(\Gamma) = Z/2Z$, as in Section 3 of [7]. Thus every PD_4-complex P_ϕ with free fundamental group and $w_2(\widetilde{P_\phi}) = 0$ is stably homotopy equivalent to a connected sum of Q_ϕ with a 1-connected manifold [3, 13]. Is this so in the remaining cases? Is every such complex itself homotopy equivalent to a closed 4-manifold?

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