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Almost sure invariance principle for dynamical systems with stretched exponential mixing rates

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Abstract. We prove the almost sure invariance principle for a class of abstract dynamical systems including dynamical systems with stretched exponential mixing rates. The result can be applied to chaotic billiards and hyperbolic attractors with Markov sieves as well as expanding maps of the interval and Axiom A diffeomorphisms.

1. Introduction

Let T be a measure preserving transformation on a probability space $(M, \mathfrak{B}, \mu)$ and F an element of $L^2(M, \mathfrak{B}, \mu)$. We are interested in the limiting behavior of the random process $\{S_N\}_{N=1}^{\infty}$ on $(M, \mathfrak{B}, \mu)$ defined by $S_N = \sum_{i=0}^{N-1} F \circ T^i$. Especially the central limit theorem, the weak invariance principle, the almost sure invariance principle, and the law of the iterated logarithm are our main concern. It is well known that the almost sure invariance principle implies the other limit theorems above. Therefore we shall devote ourselves to the almost sure invariance principle in the sequel. For the sake of simplicity we say that the almost sure invariance principle holds for F (with $\lambda \in (0, 1/2)$) if the process $\{S_N\}_{N=1}^{\infty}$ satisfies the following property.

Without changing the distribution, we can redefine the random process $\{S_N\}_{N=1}^{\infty}$ on a richer probability space together with a Brownian motion $\{B(t)\}_{t \in [0, \infty)}$ such that

$$S_N - E[S_N] = B(\sigma_F^2 N) + O(N^{\frac{1}{2}-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{1.1}$$

holds for some positive number $\lambda \in (0, 1/2)$.

Here σ_F^2 denotes the limiting variance defined by $\sigma_F^2 = C_F(0) + 2\sum_{n=1}^{\infty} C_F(n)$, where $C_F(n)$ are the autocorrelation coefficients of F given by the formula

$$C_F(n) = \int_M F(x)F(T^n x)d\mu(x) - (E[F])^2 \qquad n = 0, 1, 2, \dots. \tag{1.2}$$

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In [8, Theorem 7.1], Philipp and Stout give a sufficient condition for the almost sure invariance principle for mixing random processes in a quite general setup. This theorem is known to be applicable to the process $\{S_N\}_{N=1}^{\infty}$ in the following cases. (1) T is a uniformly expanding transformation on the unit interval $[0,1]$ (L-Y map) and F is of bounded p-variation with some $p \geqq 1$ and (2) T is an Axiom A diffeomorphism on a compact manifold and F is Hölder continuous. The reason why the Philipp-Stout theorem works well in these cases is that these dynamical systems have nice measurable partitions such as the generating partition for T in the case (1) and the Markov partition for T in the case (2) (see [6] and [2]). More precisely, one can apply the Philipp-Stout theorem if M is a separable metric space and there exists a finite or countable measurable partition $\mathscr{A}$ having the following properties.

(i) There exist positive constants C_1 and κ_1 with $0 < \kappa_1 < 1$ such that

$$\operatorname{diam}\left(\bigvee_{i=0}^{N} T^{-i}\mathscr{A}\right) \leqq C_1\kappa_1^N \tag{1.3}$$

holds for any nonnegative integer N, or T is invertible and T^{-1} is also measurable and

$$\operatorname{diam}\left(\bigvee_{i=-N}^{N} T^{-i}\mathscr{A}\right) \leqq C_1\kappa_1^N \tag{1.4}$$

holds for any nonnegative integer N.

(ii) There exist positive constants C_2 and κ_2 with $0 < \kappa_2 < 1$ such that

$$\beta\left(\bigvee_{i=0}^{k} T^{-i}\mathscr{A}, \bigvee_{i=k+n}^{k+n+l} T^{-i}\mathscr{A}\right) \leqq C_2\kappa_2^n \tag{1.5}$$

holds for any nonnegative integers k, l and n, where

$$\beta(\mathscr{A}_1, \mathscr{A}_2) = \sum_{A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2} |\mu(A_1 \cap A_2) - \mu(A_1)\mu(A_2)| \tag{1.6}$$

for finite or countable measurable partitions $\mathscr{A}_1$ and $\mathscr{A}_2$.

On the other hand many researchers have been interested in the stochastic behavior of the dynamical systems such as the hyperbolic billiards and the hyperbolic attractors. But in the case of the hyperbolic billiard it seems hard to prove the almost sure invariance principle by the direct application of the Philipp-Stout theorem since we do not have a measurable partition satisfying the above conditions. It is remarkable that Chernov succeeded in proving the weak invariance principle for the dynamical systems having stretched exponential mixing rates in [4]. We recall that a dynamical system T on a metric space M is said to have stretched exponential mixing rates if it satisfies the following.

There exists a constant $\theta \in (0,1]$ such that for any $a \in (0,1]$ there exists a sequence $\{\mathscr{A}^{(N,a)}\}_{N=1}^{\infty}$ of finite or countable measurable partitions of M satisfying;

(i) there exist positive constants C_1 and λ_1 with $0 < \lambda_1 < 1$ such that

$$\mathrm{diam}(\mathscr{A}^{(N,a)}) \leqq C_1 \lambda_1^{N^{a\theta}} \tag{1.7}$$

holds for any positive integer N;

(ii) there exist positive constants C_2 and λ_2 with $0 < \lambda_2 < 1$ such that

$$\beta_{\mathscr{A}^{(N,a)}}(N, [N^a]) \leqq C_2 \lambda_2^{N^{a\theta}} \tag{1.8}$$

holds for any positive integer N, where

$$\beta_{\mathscr{A}}(N,n) = \sup_{0 \leqq k \leqq N-n} \beta\left(\bigvee_{i=0}^{k} T^{-i}\mathscr{A}, \bigvee_{i=k+n}^{N} T^{-i}\mathscr{A}\right) \tag{1.9}$$

for a finite or countable measurable partition $\mathscr{A}$ and nonnegative integers N and n with $n \leqq N$.

The constants C_1, C_2, λ_1 and λ_2 in the above may depend on a but not on N.

We note that the hyperbolic billiards and hyperbolic attractors are typical examples which have stretched exponential mixing rates (see section 7 of [4], c.f. [1], [3]).

In this paper, we aim to establish the almost sure invariance principle for the dynamical systems with stretched exponential mixing rates inspired by Chernov's results in [4]. To this end, we first prove a slightly abstract result for the dynamical system which has a special family of measurable partitions (see Theorem 2.1). Afterward, it is shown that the dynamical systems with stretched exponential mixing rates has such a family. As a consequence we show the following theorem (see Remark just after Corollary 2.3).

Theorem 1.1. *Assume that a dynamical system $(M, \mathfrak{B}, \mu, T)$ have stretched exponential mixing rates. Let F be a Hölder continuous function belonging to $L^{2+\delta}(M, \mathfrak{B}, \mu)$ for some δ with $0 < \delta < 2$. Then the limiting variance σ_F^2 exists and if it is positive, the almost sure invariance principle holds for F with any positive number $\lambda < \frac{\delta}{8+6\delta}$.*

The organization of this paper is as follows. In Section 2, we first introduce some definitions and notations. Next, we give the statement of the main theorem (Theorem 2.1), which is more or less abstract. The almost sure invariance principle for dynamical systems with stretched mixing rates will also be given as corollaries to the theorem. In Section 3, we mention about two examples to which our results can be applied. Finally, Section 4 is devoted to the proof of our results.

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2. Preliminaries and statement of results

Let T be a measure preserving transformation on a probability space $(M, \mathfrak{B}, \mu)$. We call the quartet $(M, \mathfrak{B}, \mu, T)$ a measure preserving dynamical system. Throughout the paper all functions are assumed to be real valued.

First of all, we define autocorrelation coefficients of the stationary process $\{F \circ T^i\}_{i=0}^{\infty}$ by

$$C_F(n) = \int_M F(x)F(T^n x)d\mu(x) - (E[F])^2 \qquad (n = 0, 1, 2, 3, \ldots). \tag{2.1}$$

We note that if

$$\sum_{n=1}^{\infty} |C_F(n)| < \infty \tag{2.2}$$

is satisfied, then we have

$$\begin{aligned} \left| \left(C_F(0) + 2\sum_{n=1}^{\infty} C_F(n) \right) - \frac{V[S_N]}{N} \right| &= \left| 2\sum_{n=1}^{N} \frac{n}{N} C_F(n) + 2\sum_{n=N+1}^{\infty} C_F(n) \right| \\ &\leqq 2\sum_{n=1}^{N} \frac{n}{N} |C_F(n)| + 2\sum_{n=N+1}^{\infty} |C_F(n)| \\ &\leqq 2\sum_{n=1}^{[\sqrt{N}]} \frac{[\sqrt{N}]}{N} |C_F(n)| + 2\sum_{n=[\sqrt{N}]+1}^{\infty} |C_F(n)| \\ &\leqq \frac{2}{\sqrt{N}} \sum_{n=1}^{\infty} |C_F(n)| + 2\sum_{n=[\sqrt{N}]+1}^{\infty} |C_F(n)| \\ &\to 0 \qquad (N \to \infty). \end{aligned} \tag{2.3}$$

Therefore, we obtain

$$\lim_{N\to\infty} \frac{V[S_N]}{N} = C_F(0) + 2\sum_{n=1}^{\infty} C_F(n) = \sigma_F^2. \tag{2.4}$$

Next we introduce some definitions and notations. We say a finite or countable family of measurable sets $\mathscr{A} = \{A_i\}_{i \in I}$ ($I = \mathbf{N}$ or $I = \{1, 2, 3, \dots, l\}$ $(l \in \mathbf{N})$) a measurable partition of the probability space $(M, \mathfrak{B}, \mu)$ if $\mu(A_i \cap A_j) = 0$ for any i, j with $i \neq j$ and $M = \bigcup_{i \in I} A_i$ up to μ-null set. For a measurable partition $\mathscr{A} = \{A_i\}_{i \in I}$ and a nonnegative integer n we define a new measurable partition $T^{-n}\mathscr{A}$ by $T^{-n}\mathscr{A} = \{T^{-n}A_i\}_{i \in I}$. For two measurable partitions $\mathscr{A} = \{A_i\}_{i \in I}$ and $\mathscr{A}' = \{A'_j\}_{j \in J}$ we denote the new measurable partition $\{A_i \cap A'_j\}_{i \in I, j \in J}$ by $\mathscr{A} \vee \mathscr{A}'$. Besides, for finitely many measurable partitions $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_N$ we denote the measurable partition $\mathscr{A}_1 \vee \mathscr{A}_2 \vee \cdots \vee \mathscr{A}_N$ by $\bigvee_{k=1}^{N} \mathscr{A}_k$.

For measurable partitions $\mathscr{A} = \{A_i\}_{i \in I}$ and $\mathscr{A}' = \{A'_j\}_{j \in J}$ we define a measure of their independence $\beta(\mathscr{A}, \mathscr{A}')$ by

$$\beta(\mathscr{A}, \mathscr{A}') = \sum_{i \in I, j \in J} |\mu(A_i \cap A'_j) - \mu(A_i)\mu(A'_j)|. \tag{2.5}$$

For a measurable partition $\mathscr{A}$ and integers n, N with $0 \leqq n \leqq N$ we define

$$\beta_{\mathscr{A}}(N, n) = \sup_{0 \leqq l \leqq N-n} \beta\left(\bigvee_{k=0}^{l} T^{-k}\mathscr{A}, \bigvee_{k=l+n}^{N} T^{-k}\mathscr{A}\right). \tag{2.6}$$

We denote by $\sigma(\mathscr{A})$ the σ algebra generated by a family $\mathscr{A}$ of subsets of M. We also denote by $\sigma(X_1, X_2, \dots, X_N)$ and $\sigma(X_1, X_2, \dots)$, the σ algebras generated by random variables $\{X_k\}_{k=1}^{N}$ and $\{X_k\}_{k=1}^{\infty}$, respectively.

If the space M is endowed with a metric d_M, the diameter of a measurable partition $\mathscr{A} = \{A_i\}_{i \in I}$ is defined by

$$\operatorname{diam}(\mathscr{A}) = \sup_{i \in I} \sup_{x, y \in A_i} d_M(x, y). \tag{2.7}$$

Moreover if the metric space M is separable and $\mathfrak{B}$ is the topological Borel σ algebra of M, for $F \in L^2(M, \mathfrak{B}, \mu)$ and any positive number d we put

$$\mathscr{H}_F(d) = \sup_{\substack{\mathscr{A}:\text{measurable partition} \\ \operatorname{diam}(\mathscr{A}) \leqq d}} \|F - E[F \mid \sigma(\mathscr{A})]\|_2. \tag{2.8}$$

In the above and also in what follows, we regard $\frac{1}{p}$ as 0 when $p = \infty$. Now we are in a position to state our results.

Theorem 2.1. *Let $(M, \mathfrak{B}, \mu, T)$ be a measure preserving dynamical system, δ be a positive constant, and ρ be a constant with $0 \leqq \rho < 1$. Assume that a function $F \in L^{2+\delta}(M, \mathfrak{B}, \mu)$ satisfies*

$$\sum_{n=1}^{\infty}|C_F(n)| < \infty \tag{2.9}$$

and

$$\sum_{n=1}^{N} nC_F(n) + \sum_{n=N+1}^{\infty} NC_F(n) = O(N^{\rho}) \qquad (N \to \infty). \tag{2.10}$$

Furthermore we assume that there exist constants s, γ *with* $0 < s < \min\left\{\frac{\delta}{2+2\delta}, \frac{1}{3}\right\}$, $\gamma > \max\left\{\frac{2(2+\delta)(1-s)}{\delta-(2+2\delta)s}, \frac{1-s}{s}\right\}$ *and a sequence of measurable partitions* $\{\mathscr{A}^{(N)}\}_{N\in\mathbf{N}}$ *satisfying the following properties.*

(i)

$$\beta_{\mathscr{A}^{(N)}}(N, [N^s]) = O(N^{-\gamma s}) \qquad (N \to \infty). \tag{2.11}$$

(ii) *There are constants* p, τ *with* $1 \leqq p \leqq \infty$, $\tau > \frac{5}{2} + \frac{1}{p} + \left(1 + \frac{1}{p}\right)\gamma s$ *and it holds that*

$$\|F - E[F \,|\, \sigma(\mathscr{A}^{(N)})]\|_p = O(N^{-\tau}) \qquad (N \to \infty). \tag{2.12}$$

Then the almost sure invariance principle holds for F *provided* $\sigma_F \neq 0$.

REMARK. It is easy to see from (2.4) that the weak invariance principle and the law of iterated logarithm follow from the almost sure invariance principle when the conditions $\sum_{n=1}^{\infty}|C_F(n)| < \infty$ and $\sigma_F \neq 0$ are satisfied. On the other hand the central limit theorem always follows from the weak invariance principle. Consequently if the conditions of Theorem 2.1 are satisfied, all the limit theorems that we mentioned in Introduction are valid.

From Theorem 2.1 and its proof we obtain the following corollaries.

COROLLARY 2.2. *Let* $(M, \mathfrak{B}, \mu, T)$ *have stretched exponential mixing rates and let* F *be a member of* $F \in L^{2+\delta}(M, \mathfrak{B}, \mu)$ *for a positive number* δ. *Assume that there exists a number* $v > \frac{24+15\delta}{\theta\delta}$ *such that*

$$\mathscr{H}_F(d) = O\left(\frac{1}{|\log d|^v}\right) \qquad (d \downarrow 0). \tag{2.13}$$

Then the almost sure invariance principle holds for F *provided* $\sigma_F \neq 0$.

COROLLARY 2.3. *Let* $(M, \mathfrak{B}, \mu, T)$ *have stretched exponential mixing rates and let* F *be a member of* $F \in L^{2+\delta}(M, \mathfrak{B}, \mu)$ *for a positive number* δ *with* $0 < \delta \leqq 2$. *Assume that for any positive number* v *the function* F *satisfies the condition*

$$\mathscr{H}_F(d) = O\left(\frac{1}{|\log d|^v}\right) \qquad (d \downarrow 0).$$

Then the almost sure invariance principle holds for F with any positive number $\lambda < \frac{\delta}{8+6\delta}$ provided $\sigma_F \neq 0$.

Remark. If the function F is Hölder continuous, it satisfies the condition

$$\mathscr{H}_F(d) = O\left(\frac{1}{|\log d|^v}\right) \qquad (d \downarrow 0)$$

for any positive number v. Hence Theorem 1.1 immediately follows from Corollary 2.3.

Remark. We must discuss the condition $\sigma_F \neq 0$. Suppose that

$$\sum_{n=1}^{\infty} n|C_F(n)| < \infty. \tag{2.14}$$

Then we see that $\sigma_F = 0$ is equivalent to $\overline{\lim}_{N\to\infty} V[S_N] < \infty$ by the expansion

$$V[S_N] = \sigma_F^2 N - 2\sum_{n=1}^{N} nC_F(n) - 2\sum_{n=N+1}^{\infty} NC_F(n).$$

On the other hand, under the condition $\lim_{n\to\infty} C_F(n) = 0$, it is easy to see that $\overline{\lim}_{N\to\infty} V[S_N] < \infty$ holds if and only if there exists a function $G \in L^2(M, \mathfrak{B}, \mu)$ such that $F = G - G \circ T + E[F]$ holds (see [7 Theorem 18.2.2]). Consequently under the assumption (2.14) we can conclude that $\sigma_F = 0$ if and only if there exists a function $G \in L^2(M, \mathfrak{B}, \mu)$ such that $F = G - G \circ T + E[F]$. But it is not easy to see whether there exists the function G such that $F = G - G \circ T + E[F]$ for a given function F. Therefore it is a troublesome problem to check the condition $\sigma_F \neq 0$ for a given function F. So it is remarkable that if F is the first collision time of the two dimensional hyperbolic billiard with finite horizon, then F satisfies $\sigma_F \neq 0$ as well as the condition (2.13) (see [3, Section 7]). It provide us with a non trivial and interesting example to which our result is applicable.

3. Examples

In this section we give two examples with Markov sieve. To such dynamical systems not only Chernov's results in [4] but also ours are applicable.

(1) Two dimensional hyperbolic billiards.

Let Q be a compact closed domain on a plane or 2-torus. We assume that the boundary ∂Q consists of finitely many smooth (of class C^3) components Γ_i ($1 \leqq i \leqq d$) each of whixh satisfies the following conditions.

(a) Γ_i is strictly convex as seen from the inside of Q.

(b) Γ_i is a rectilinear segment.

(c) Γ_i is a convex (as seen from the outside of Q) incomplete arc of a circle whose complement to complete the circle do not intersect the other components of ∂Q.

Now we put $M_0 = \{(q, v) \in Q \times S^1 \mid q \in \partial Q \setminus \bigcup_{1 \leqq i < j \leqq d} (\Gamma_i \cap \Gamma_j), \langle n(q), v \rangle > 0\}$, where $n(q)$ is the unit interior normal vector of ∂Q at the point q and $\langle \cdot, \cdot \rangle$ represents the ordinal inner product in the Euclidean space. We denote the closure of M_0 in $Q \times S^1$ by M, then we have $M \subset \partial Q \times S^1$. We define a map T from M to itself by the following way.

Let (q, v) be a point of M. We suppose that a point particle runs into ∂Q at q with velocity v and next runs into ∂Q again at q' with velocity v' after the elastic reflection (reflection such that the angle of incidence equals the angle of reflection) at q and motion of constant velocity in Q. Then we define $T(q, v) = (q', v')$. But when q is an end point of Γ_i, we define $T(q, v) = (q, v)$ since we can not define $T(q, v)$ as above. In the case (b) or (c), since we can not define T as above for the point (q, v) of M such that $q \in \Gamma_i$ and $\langle n(q), v \rangle = 0$, we need some idea to define T well. We omit details.

We denote the parameter representing length of ∂Q by r and the parameter representing angle of $v \in S^1$ by φ. Then r and φ make natural coordinates of $\partial Q \times S^1$ and we think $\partial Q \times S^1$ is a metric space by the coordinates. In what follows we think M is a metric space as a subset of $\partial Q \times S^1$. Now we define a probability measure μ on $(M, \mathfrak{B})$ ($\mathfrak{B}$ is the topological Borel σ algebra of the metric space M) by $d\mu = c_\mu \cos \varphi \, dr \times d\varphi$, where c_μ is a normalizing factor. Then it is known that $(M, \mathfrak{B}, \mu, T)$ is a measure preserving dynamical system.

In [3] and [4], it is shown that a generic class of hyperbolic billiards admits a family $\{\mathscr{R}_{N,m}\}_{1 \leqq m < N, N \in \mathbf{N}}$ of family of measurable subsets of M satisfying following conditions.

(i) Each $\mathscr{R}_{N,m}$ consists finitely many measurable subsets $R_1^{(N,m)}, \dots, R_l^{(N,m)}$ of M and when $i \neq j$, it holds that $R_i^{(N,m)} \cap R_j^{(N,m)} = \phi$.

(ii) There are positive constants K_1, α_1 independent of N, m such that $0 < \alpha_1 < 1$ and it holds that

$$\max_{1 \leqq i \leqq l} \sup_{x, y \in R_i^{(N,m)}} d(x, y) \leqq K_1 \alpha_1^m \tag{3.1}$$

for any positive integers N, m with $m < N$, where d is the metric of M.

(iii) There are positive constants K_2, α_2 independent of N, m such that

$0 < \alpha_2 < 1$ and it holds that $\mu(R_0^{(N,m)}) \leqq K_2 \alpha_2^m$ for any positive integers N, m with $m < N$, where $R_0^{(N,m)} = M \setminus \bigcup_{i=1}^{l} R_i^{(N,m)}$.

(iv) There are positive constants K_3, α_3 independent of N, m such that $0 < \alpha_3 < 1$ and it holds for any positive integer $n \leqq N$ and $(i_0, \dots, i_n) \in \{1, \dots, l\}^{n+1}$ that $|\Delta| \leqq K_3$, where Δ is the real number such that

$$\mu(T^{-n} R_{i_n}^{(N,m)} \mid T^{-(n-1)} R_{i_{n-1}}^{(N,m)} \cap \cdots \cap R_{i_0}^{(N,m)}) = \mu(T^{-1} R_{i_n}^{(N,m)} \mid R_{i_{n-1}}^{(N,m)})(1 + \Delta). \tag{3.2}$$

(v) There are positive constants K_4, α_4, g_0, g_1 independent of N, m such that $0 < \alpha_4 < 1$ and it holds that for any positive integer k with $k \geqq [g_0 m]$ that

$$\sum_{\substack{1 \leqq i \leqq l \\ \sum_{j \in S(i)} \mu(R_j^{(N,m)}) > 1 - K_4 N \alpha_4^m}} \mu(R_i^{(N,m)}) > 1 - K_4 N \alpha_4^m, \tag{3.3}$$

where $S(i) = \{ j \in \mathbf{N} \mid 1 \leqq j \leqq l, \mu(T^{-k} R_j^{(N,m)} \mid R_i^{(N,m)}) \geqq g_1 \mu(R_j^{(N,m)}) \}$.

Note that such a family $\{\mathscr{R}_{N,m}\}_{1 \leqq m < N, N \in \mathbf{N}}$ is called a Markov sieve. We can show the dynamical system $(M, \mathfrak{B}, \mu, T)$ has stretched exponential mixing rates by using Markov sieve (see [4]). Thus we can apply Corollary 2.2 and Corollary 2.3 to this system.

(2) Two dimensional hyperbolic attractors.

Let M be a smooth two dimensional Riemannian manifold, U be an open connected subset of M with compact closure and Γ be a closed subset of U. We assume that the set $S^+ = \Gamma \cup \partial U$ consists of a finite number of compact smooth curves. Let $T : U \setminus \Gamma \to U$ be a C^2-diffeomorphism from the open set $U \setminus \Gamma$ onto its image $T(U \setminus \Gamma)$. We assume that T is differentiable on $U \setminus \Gamma$ up to its boundary $\partial(U \setminus \Gamma) = S^+$. Also we assume that T^{-1} is differentiable on $T(U \setminus \Gamma)$ up to its boundary $\partial(T(U \setminus \Gamma))$.

Denote $U^+ = \{x \in U \mid T^n x \in U \setminus \Gamma$ for any nonnegative integer $n\}$ and $D = \bigcap_{n=0}^{\infty} T^n(U^+)$. The set D is invariant for both T and T^{-1}. Its closure $\Lambda = \overline{D}$ is called the attractor for T.

We define the cone $C(z, P, \alpha)$ for $z \in U$, a line P through the origin in the tangent space $T_z M$ and a positive number α by $C(z, \alpha, P) = \{v \in T_z M \mid \angle(P, v) \leqq \alpha\}$. An attractor Λ is called a generalized hyperbolic attractor if for each $z \in U \backslash \Gamma$ there exist two cone $C^u(z) = C(z, \alpha^u(z), P^u(z))$ and $C^s(z) = C(z, \alpha^s(z), P^s(z))$ having the following three properties:

(1)

$$\inf_{z \in U \backslash \Gamma} \inf_{\substack{v_1 \in C^u(z) \\ v_2 \in C^s(z)}} \angle(v_1, v_2) > 0;$$

(2) $DT(C^u(z)) \subset C^u(Tz)$ for any $z \in U \backslash \Gamma$ and $DT^{-1}(C^s(z)) \subset C^s(T^{-1}z)$ for any $z \in T(U \backslash \Gamma)$;

(3) there exist a positive constant C and a constant λ with $0 < \lambda < 1$ such that for any positive integer

(a) if $z \in U^+$ and if $v \in C^u(z)$, then $\|DT^n v\| \geqq C\lambda^{-n}\|v\|$;

(b) if $z \in T^n(U^+)$ and if $v \in C^s(z)$ then $\|DT^{-n} v\| \geqq C\lambda^{-n}\|v\|$.

If we assume some generic conditions on the singularity set of a generalized hyperbolic attractor Λ, then there exist subsets Λ_i $(i = 0, 1, 2, \ldots)$ of Λ and Gibbs u-measures (the definition is found in [1]) μ_i $(i = 1, 2, 3, \ldots)$, which are T-invariant probability measures on $(\Lambda, \mathfrak{B})$ ($\mathfrak{B}$ is the topological Borel field of Λ), satisfying:

(1) $\Lambda = \bigcup_{i \geqq 0} \Lambda_i$ and $\Lambda_i \cap \Lambda_l = \phi$ when $i \neq j$;

(2) for $i \geqq 1$ $\Lambda_i \subset D$, $T(\Lambda_i) = \Lambda_i$, $\mu_i(\Lambda_i) = 1$ and $T|_{\Lambda_i}$ is ergodic with respect to μ_i;

(3) for $i \geqq 1$ there exists a decomposition of Λ_i to its subsets $\Lambda_i = \bigcup_{j=1}^{r_i} \Lambda_{i,j}$ such that $\Lambda_{i,j} \cap \Lambda_{i,j'} = \phi$ if $j \neq j'$, $T(\Lambda_{i,j}) = \Lambda_{i,j+1}$ if $1 \leqq j \leqq r_i - 1$, $T(\Lambda_{i,r_i}) = \Lambda_{i,1}$ and $T^{r_i}|_{\Lambda_{i,1}}$ has the Bernoulli property.

Now we choose i with $1 \leqq i$ and j with $1 \leqq j \leqq r_i$ arbitrarily, and write $\Lambda_* = \Lambda_{i,j}$, $T_* = T^{r_i}|_{\Lambda_*}$ and $\mu_* = \frac{\mu_i|_{\mathfrak{B}_*}}{\mu_i(\Lambda_*)}$ ($\mathfrak{B}_*$ is the topological Borel σ algebra of Λ_*). We consider the measure preserving dynamical system $(\Lambda_*, \mathfrak{B}_*, \mu_*, T_*)$. In [1] Markov sieves have been constructed for this system. Note that the definition of Markov sieve for a generalized hyperbolic attractor is slightly different from that for hyperbolic billiards in the above. One has to replace the condition (v) in the above by the following one:

(v′) there are positive constants g_0, g_1 independent of N, m such that for every integer $k \geqq [g_0 m]$ and any pair of integer i, j with $1 \leqq i, j \leqq l$ one has

$$\frac{1}{2}\sum_{h=1}^{l} |\mu_*(T_*^{-k} R_h^{(N,m)} \mid R_i^{(N,m)}) - \mu_*(T_*^{-k} R_h^{(N,m)} \mid R_j^{(N,m)})| < 1 - g_1. \tag{3.4}$$

We can show the dynamical system $(\Lambda_*, \mathfrak{B}_*, \mu_*, T_*)$ has stretched exponential mixing rates by using Markov sieve (see [4]). Thus we can apply Corollary 2.2 and Corollary 2.3 to this system.

4. Proof of results

In what follows, we may assume $E[F] = 0$, without loss of generality. First of all we recall Theorem 4.3 in [9], which is a martingale version of the Skorokhod representation theorem. It plays an important role in the proof of Theorem 2.1.

THEOREM 4.1 (Theorem 4.3 in [9]). *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of random variables on a probability space $(\Omega, \mathfrak{F}, P)$ satisfying:*

(i) $E[Y_1] = 0$ *and* $E[Y_1^2] < \infty$

(ii) $E[Y_i^2 \mid \sigma(Y_1, \dots, Y_{i-1})]$ *is defined and* $E[Y_i \mid \sigma(Y_1, \dots, Y_{i-1})] = 0$ *P-a.s. for any* $i \geqq 2$.

Then there exists a sequence of random variables $\{\tilde{Y}_i\}_{i=1}^{\infty}$ and a Brownian motion $\{B(t)\}_{t \in [0,\infty)}$ together with a sequence of nonnegative random variables $\{T_i\}_{i=1}^{\infty}$ on an appropriate probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ with the following properties.

(1) $\{Y_i\}_{i=1}^{\infty}$ *and* $\{\tilde{Y}_i\}_{i=1}^{\infty}$ *have the same distribution.*

(2)

$$\sum_{i=1}^{n} \tilde{Y}_i = B\left(\sum_{i=1}^{n} T_i\right) \qquad \tilde{P}\text{-a.s.} \tag{4.1}$$

for any $n \in \mathbf{N}$.

(3) T_n *is* $\tilde{\mathfrak{F}}_n$*-measurable and*

$$E[T_n|\tilde{\mathfrak{F}}_{n-1}] = E[\tilde{Y}_n^2|\tilde{\mathfrak{F}}_{n-1}] \qquad \tilde{P}\text{-a.s.} \qquad n = 1, 2, 3, \dots \tag{4.2}$$

where $\tilde{\mathfrak{F}}_0 = \{\phi, \tilde{\Omega}\}$ *and* $\tilde{\mathfrak{F}}_n$ *defined as the* σ *algebra generated by* $\tilde{Y}_1, \dots, \tilde{Y}_n$ *and* $\{B(t)\}_{0 \leqq t \leqq \sum_{i=1}^{n} T_i}$ *for* $n \geqq 1$.

(4) *If* $E[|Y_1|^{2k}] < \infty$ *for a real number* $k > 1$, *then one has*

$$E[T_1^k] \leqq D_k E[|\tilde{Y}_1|^{2k}]. \tag{4.3}$$

In addition if the conditional expectation $E[|Y_n|^{2k} \mid \sigma(Y_1, \dots, Y_{n-1})]$ *is defined for an integer* $n \geqq 2$ *and, then* $E[T_n^k|\tilde{\mathfrak{F}}_{n-1}]$ *is also defined and*

$$\begin{aligned} E[T_n^k|\tilde{\mathfrak{F}}_{n-1}] &\leqq D_k E[|\tilde{Y}_n|^{2k}|\tilde{\mathfrak{F}}_{n-1}] \qquad \tilde{P}\text{-a.s.} \\ &= D_k E[|\tilde{Y}_n|^{2k} \mid \sigma(\tilde{Y}_1, \dots, \tilde{Y}_{n-1})] \qquad \tilde{P}\text{-a.s.}, \end{aligned} \tag{4.4}$$

where D_k *are constants depending only on* k.

We can expect that one can prove our result with the help of Theorem 4.1 if one succeed in showing that the sequence $\{F \circ T^i\}_{i=0}^{\infty}$ is approximated by a martingale difference sequence. If the sequence of measurable partitions $\{\mathscr{A}^{(N)}\}_{N=1}^{\infty}$ in Theorem 2.1 is increasing in N, we can directly make a martingale difference sequence approximating the sequence $\{F \circ T^i\}_{i=0}^{\infty}$ by using

$\{\mathscr{A}^{(N)}\}_{N=1}^{\infty}$. But unfortunately we can not expect such a situation in general. Therefore we have to construct a new increasing sequence of measurable partitions $\{\mathscr{U}^{(N)}\}_{N=1}^{\infty}$ which enjoy a nice properties with respect to the function F. The next proposition plays a crucial role in the construction of the desired partitions.

PROPOSITION 4.2. *Let λ, γ_0 be positive numbers. Suppose that there exist $1 \leqq p \leqq \infty$ and τ such that $\tau > \frac{5}{2} + \frac{1}{p} + \left(1 + \frac{1}{p}\right)\gamma_0 + \lambda$ and*

$$\|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p = O(N^{-\tau}) \qquad (N \to \infty) \tag{4.5}$$

hold. Then, there exist constants $0 \leqq r_0 < 1$ and C_0 such that the following holds.

If we put

$$U_k^{(N)} = \{x \in M \mid r_0 + k \cdot 2^{-[(1/2+\lambda)\log_2 N]} \leqq F(x) < r_0 + (k+1) \cdot 2^{-[(1/2+\lambda)\log_2 N]}\},$$

then the family $\mathscr{U}^{(N)} = \{U_k^{(N)}\}_{k \in \mathbf{Z}}$ of subset of M becomes a measurable partition satisfying

$$\beta_{\mathscr{U}^{(N)}}(N, n) \leqq \beta_{\mathscr{A}^{(N)}}(N, n) + C_0 N^{-\gamma_0} \tag{4.6}$$

for any pair of positive integers $n \leqq N$.

PROOF. We have only to prove in the case when $1 \leqq p < \infty$. We choose a real number a with $\frac{2+\gamma_0}{\tau + \frac{1}{p} - \frac{1}{2} - \lambda} < a < \frac{p}{p+1}$. This is possible because

$$\tau > \frac{5}{2} + \frac{1}{p} + \left(1 + \frac{1}{p}\right)\gamma_0 + \lambda \Leftrightarrow \frac{2 + \gamma_0}{\tau + \frac{1}{p} - \frac{1}{2} - \lambda} < \frac{p}{p+1}. \tag{4.7}$$

For $N \in \mathbf{N}$, $0 \leqq r < 1$, and $k \in \mathbf{Z}$, we define the set $U_{r,k}^{(N)}$ by

$$U_{r,k}^{(N)} = \{x \in M \mid r + k \cdot 2^{-[(1/2+\lambda)\log_2 N]} \leqq F(x) < r + (k+1) \cdot 2^{-[(1/2+\lambda)\log_2 N]}\}. \tag{4.8}$$

Writing $\mathscr{A}^{(N)}$ as $\{A_i^{(N)}\}_{i \in I_N}$, for $N \in \mathbf{N}$ and $i \in I_N$, we set

$$b_{N,i} = \begin{cases} 0 & \text{if } \mu(A_i^{(N)}) = 0, \\ \frac{1}{\mu(A_i^{(N)})} \int_{A_i^{(N)}} F \, d\mu & \text{if } \mu(A_i^{(N)}) > 0. \end{cases} \tag{4.9}$$

Next, for $N \in \mathbf{N}$, $i \in I_N$ and $0 \leqq r < 1$, we select the integer $k(N, i, r)$ satisfying

$$r + k(N, i, r) \cdot 2^{-[(1/2+\lambda)\log_2 N]} \leqq b_{N,i} < r + (k(N, i, r) + 1) \cdot 2^{-[(1/2+\lambda)\log_2 N]} \tag{4.10}$$

Take a positive integer N and fix it for a while. For $r \in [0, 1)$ and $i \in I_N$, we set

$$h(N,i,r) = \min\{b_{N,i} - r - k(N,i,r)\cdot 2^{-[(1/2+\lambda)\log_2 N]},$$
$$r + (k(N,i,r)+1)\cdot 2^{-[(1/2+\lambda)\log_2 N]} - b_{N,i}\}. \tag{4.11}$$

It is easy to see that all number $r \in [0,1)$ except for countable set satisfy $h(N,i,r) \neq 0$ for all $i \in I_N$. For such an r we obtain

$$\begin{aligned}
&\sum_{i\in I_N} \mu(A_i^{(N)} \backslash U_{r,k(N,i,r)}^{(N)}) \\
&\quad \leqq \sum_{i\in I_N} \mu(\{x \in A_i^{(N)} \mid |F - b_{N,i}| \geqq h(N,i,r)\}) \\
&\quad \leqq \sum_{i\in I_N} \frac{\int_{A_i^{(N)}} |F - b_{N,i}|^a d\mu}{(h(N,i,r))^a} \\
&\quad \leqq \sum_{i\in I_N} \frac{(\int_{A_i^{(N)}} |F - b_{N,i}|^p d\mu)^{a/p} (\mu(A_i^{(N)}))^{1-a/p}}{(h(N,i,r))^a} \\
&\quad \leqq \left(\sum_{i\in I_N} \int_{A_i^{(N)}} |F - b_{N,i}|^p d\mu\right)^{a/p} \left(\sum_{i\in I_N} \frac{\mu(A_i^{(N)})}{(h(N,i,r))^{pa/(p-a)}}\right)^{1-a/p} \\
&\quad = \|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p^a \left(\sum_{i\in I_N} \frac{\mu(A_i^{(N)})}{(h(N,i,r))^{pa/(p-a)}}\right)^{1-a/p}.
\end{aligned} \tag{4.12}$$

Noting that $\frac{pa}{p-a} < 1$ holds from the choice of a, the estimate above yields

$$\begin{aligned}
&\int_0^1 \left(\sum_{i\in I_N} \mu(A_i^{(N)} \backslash U_{r,k(N,i,r)}^{(N)})\right)^{p/(p-a)} dr \\
&\quad \leqq \|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p^{pa/(p-a)} \int_0^1 \left(\sum_{i\in I_N} \frac{\mu(A_i^{(N)})}{(h(N,i,r))^{pa/(p-a)}}\right) dr \\
&\quad = \|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p^{pa/(p-a)} \\
&\qquad \times \sum_{i\in I_N} \left(\mu(A_i^{(N)}) \cdot 2^{[(1/2+\lambda)\log_2 N]} \int_{-2^{-[(1/2+\lambda)\log_2 N]-1}}^{2^{-[(1/2+\lambda)\log_2 N]-1}} \frac{1}{|t|^{pa/(p-a)}}\, dt\right) \\
&\quad = \frac{1}{1-\frac{pa}{p-a}} \|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p^{pa/(p-a)} \cdot 2^{(pa/(p-a))([(1/2+\lambda)\log_2 N]+1)} \\
&\quad \leqq \frac{2}{1-\frac{pa}{p-a}} \|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_p^{pa/(p-a)} N^{(pa/(p-a))(1/2+\lambda)}.
\end{aligned} \tag{4.13}$$

Therefore, we conclude by (4.5) that

$$m\left(\left\{r \in [0,1) \mid \sum_{i \in I_N} \mu(A_i^{(N)} \setminus U_{r,k(N,i,r)}^{(N)}) \geqq N^{-\gamma_0 - 1}\right\}\right)$$
$$= O(N^{-\tau(pa/(p-a)) + (pa/(p-a))(1/2+\lambda) + (p/(p-a))(\gamma_0+1)}) \qquad (N \to \infty),$$

where m is the one dimensional Lebesgue measure. Since

$$-\tau \frac{pa}{p-a} + \frac{pa}{p-a}\left(\frac{1}{2} + \lambda\right) + \frac{p}{p-a}(\gamma_0 + 1) < -1 \Leftrightarrow \frac{2+\gamma_0}{\tau + \frac{1}{p} - \frac{1}{2} - \lambda} < a,$$

the choice of a implies

$$\sum_{N=1}^{\infty} m\left(\left\{r \in [0,1) \mid \sum_{i \in I_N} \mu(A_i^{(N)} \setminus U_{r,k(N,i,r)}^{(N)}) \geqq N^{-\gamma_0 - 1}\right\}\right) < \infty. \tag{4.14}$$

In virtue of the Borel Cantelli lemma, for almost all $r \in [0,1)$ there exists a positive constant $C(r)$ such that

$$\sum_{i \in I_N} \mu(A_i^{(N)} \setminus U_{r,k(N,i,r)}^{(N)}) < C(r) N^{-\gamma_0 - 1} \tag{4.15}$$

for all $N \in \mathbf{N}$. We chose one of such r and denote it by r_0.

If we set $G_k^{(N)} = \bigcup_{\substack{i \in I_N \\ k(N,i,r_0)=k}} A_i^{(N)}$ and $\mathscr{G}^{(N)} = \{G_k^{(N)}\}_{k \in \mathbf{Z}}$ for any $N \in \mathbf{N}$, then $\mathscr{G}^{(N)}$ is a measurable partition of M and $\mathscr{A}^{(N)}$ is refinement of $\mathscr{G}^{(N)}$ and

$$\begin{aligned}\sum_{k \in \mathbf{Z}} \mu(G_k^{(N)} \setminus U_k^{(N)}) &= \sum_{k \in \mathbf{Z}} \sum_{\substack{i \in I_N \\ k(N,i,r_0)=k}} \mu(A_i^{(N)} \setminus U_k^{(N)}) \\ &= \sum_{i \in I_N} \mu(A_i^{(N)} \setminus U_{k(N,i,r_0)}^{(N)}) \\ &\leqq C(r_0) N^{-\gamma_0 - 1}. \end{aligned} \tag{4.16}$$

The last inequality follows from (4.15) with $r = r_0$. Hence, by using Lemma 4.3 and Lemma 4.4 below, we have

$$\begin{aligned}&\sum_{k_0,\ldots,k_{l_1},k_{l_2},\ldots,k_N \in \mathbf{Z}} |\mu(U_{k_0}^{(N)} \cap \cdots \cap T^{-l_1} U_{k_{l_1}}^{(N)} \cap T^{-l_2} U_{k_{l_2}}^{(N)} \cap \cdots \cap T^{-N} U_{k_N}^{(N)}) \\ &\qquad - \mu(U_{k_0}^{(N)} \cap \cdots \cap T^{-l_1} U_{k_{l_1}}^{(N)}) \mu(T^{-l_2} U_{k_{l_2}}^{(N)} \cap \cdots \cap T^{-N} U_{k_N}^{(N)})| \\ &\leqq 4(N - l_2 + l_1 + 1) C(r_0) N^{-\gamma_0 - 1}\end{aligned}$$

$$+\sum_{k_0,\dots,k_{l_1},k_{l_2},\dots,k_N\in\mathbf{Z}}|\mu(G^{(N)}_{k_0}\cap\cdots\cap T^{-l_1}G^{(N)}_{k_{l_1}}\cap T^{-l_2}G^{(N)}_{k_{l_2}}\cap\cdots\cap T^{-N}G^{(N)}_{k_N})$$

$$-\mu(G^{(N)}_{k_0}\cap\cdots\cap T^{-l_1}G^{(N)}_{k_{l_1}})\mu(T^{-l_2}G^{(N)}_{k_{l_2}}\cap\cdots\cap T^{-N}G^{(N)}_{k_N})|$$

for any integers l_1, l_2, N such that $0\leqq l_1<l_2\leqq N$. Consequently, for any pair of positive integers $n\leqq N$, we obtain

$$\beta_{\mathscr{U}^{(N)}}(N,n)\leqq\beta_{\mathscr{G}^{(N)}}(N,n)+4(N+1-n)C(r_0)N^{-\gamma_0-1} \tag{4.17}$$

Combining this and that $\mathscr{A}^{(N)}$ is a refinement of $\mathscr{G}^{(N)}$, we conclude that

$$\beta_{\mathscr{U}^{(N)}}(N,n)\leqq\beta_{\mathscr{A}^{(N)}}(N,n)+4C(r_0)N^{-\gamma_0}. \tag{4.18}$$

This completes the proof with setting $C_0=4C(r_0)$. ■

In the above, we have used the following well known facts. We just summarize them as Lemma 4.3 and Lemma 4.4 for the sake of convenience.

Lemma 4.3. *Let N be a positive integer and $\{\{A^{(n)}_i\}_{i\in I_n}\}_{n=1,\dots,N}$, $\{\{B^{(n)}_i\}_{i\in I_n}\}_{n=1,\dots,N}$ be finite sequences of measurable partitions satisfying for any positive integer $n\leqq N$*

$$\sum_{i\in I_n}\mu(A^{(n)}_i\backslash B^{(n)}_i)\leqq\varepsilon. \tag{4.19}$$

Then one has

$$\sum_{(i_1,\dots,i_N)\in I_1\times\cdots\times I_N}|\mu(A^{(1)}_{i_1}\cap\cdots\cap A^{(N)}_{i_N})-\mu(B^{(1)}_{i_1}\cap\cdots\cap B^{(N)}_{i_N})|\leqq 2N\varepsilon. \tag{4.20}$$

Lemma 4.4. *Let $\{A^{(1)}_i\}_{i\in I_1},\{B^{(1)}_i\}_{i\in I_1},\{A^{(2)}_i\}_{i\in I_2},\{B^{(2)}_i\}_{i\in I_2}$ be measurable partitions with*

$$\sum_{i\in I_1}|\mu(A^{(1)}_i)-\mu(B^{(1)}_i)|\leqq\varepsilon_1,\qquad\sum_{i\in I_2}|\mu(A^{(2)}_i)-\mu(B^{(2)}_i)|\leqq\varepsilon_2. \tag{4.21}$$

Then one has

$$\sum_{\substack{i\in I_1\\ j\in I_2}}|\mu(A^{(1)}_i)\mu(A^{(2)}_j)-\mu(B^{(1)}_i)\mu(B^{(2)}_j)|\leqq\varepsilon_1+\varepsilon_2. \tag{4.22}$$

Before proceeding further we specify the choice of λ in Theorem 2.1. The constants δ, s, γ, ρ and τ below are the same as in Theorem 2.1. First we have $\frac{s}{1-s}<\frac{1}{2}$ by the assumption on s. By the assumption on γ

$$\frac{\delta}{2+\delta}-\frac{2}{\gamma}>\frac{\delta}{2+\delta}-\frac{\delta-(2+2\delta)s}{(2+\delta)(1-s)}=\frac{s}{1-s} \tag{4.23}$$

holds. Next, we choose a real number α so that

$$\frac{s}{1-s}<\alpha<\min\left\{\frac{1}{2},\frac{\delta}{2+\delta}-\frac{2}{\gamma}\right\}. \tag{4.24}$$

We notice that the number α chosen above satisfies $s(1+\alpha)<\alpha$.

By $\gamma>\frac{1-s}{s}$ and $\frac{s}{1-s}<\alpha<\frac{\delta}{2+\delta}-\frac{2}{\gamma}<\frac{\delta}{2+\delta}$, we obtain

$$\frac{s\gamma\delta}{4(2+\delta)}-\frac{\alpha}{4(1+\alpha)}>\frac{(1-s)\delta}{4(2+\delta)}-\frac{\alpha}{4(1+\alpha)}>\frac{\delta-\alpha(2+\delta)}{4(1+\alpha)(2+\delta)}>0. \tag{4.25}$$

Therefore, by the choice of α and the assumptions on ρ and τ, we can choose positive constants λ,λ' so that

$$\lambda<\lambda'<\min\left\{\frac{\delta-(2+\delta)\left(\alpha+\frac{2}{\gamma}\right)}{2(1+\alpha)(2+\delta)},\frac{1-2\alpha}{4(1+\alpha)},\frac{(1-s)\alpha-s}{2(1+\alpha)},\frac{(1-\rho)\alpha}{2(1+\alpha)},\right.$$

$$\left.\frac{s\gamma\delta}{4(2+\delta)}-\frac{\alpha}{4(1+\alpha)},\tau-\frac{5}{2}-\frac{1}{p}-\left(1+\frac{1}{p}\right)\gamma s\right\}. \tag{4.26}$$

We will prove Theorem 2.1 with λ chosen above.

From now on, we can employ the methods similar to those that used in the proof of Theorem 7.1 of [8].

We define two sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=1}^{\infty}$ by

$$L_0=0,\quad L_j=\sum_{i=1}^{j}[i^{\alpha}]+\sum_{i=2}^{j}[i^{s(1+\alpha)}]\qquad(j=1,2,\dots)$$

and

$$M_1=0,\quad M_j=\sum_{i=1}^{j-1}[i^{\alpha}]+\sum_{i=2}^{j}[i^{s(1+\alpha)}]\qquad(j=2,3,\dots)$$

and we also define two sequences of random variables $\{y_j\}_{j=1}^{\infty}$ and $\{z_j\}_{j=2}^{\infty}$ by

$$y_j=\sum_{i=M_j}^{L_j-1}F\circ T^i\qquad(j=1,2,\dots)$$

$$z_j=\sum_{i=L_{j-1}}^{M_j-1}F\circ T^i\qquad(j=2,3,\dots).$$

For $N \in \mathbf{N}$, let $j(N)$ denote the positive integer j with $L_{j-1} < N \leqq L_j$. Then we have

$$\sum_{i=0}^{N-1} F \circ T^i = y_1 + z_2 + y_2 + \cdots + z_{j(N)-1} + y_{j(N)-1} + \sum_{i=L_{j(N)-1}}^{N-1} F \circ T^i$$

$$= \sum_{i=1}^{j(N)-1} y_i + \sum_{i=2}^{j(N)-1} z_i + \sum_{i=L_{j(N)-1}}^{N-1} F \circ T^i. \tag{4.27}$$

The next lemma asserts that the last term in the right hand side of (4.27) has the appropriate growth rate as N tends to infinity almost surely.

Lemma 4.5.

$$\sum_{i=0}^{N-1} F \circ T^i = \sum_{i=1}^{j(N)-1} y_i + \sum_{i=2}^{j(N)-1} z_i + O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s.$$

Proof. Put $\eta_j = \sum_{i=L_{j-1}}^{L_j - 1} |F \circ T^i|$ $(j = 1, 2, 3, \ldots)$. It is enough to show that

$$\eta_{j(N)} = O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.28}$$

For each $j \in \mathbf{N}$ we have

$$\begin{aligned} &\mu(\{x \in M \mid \eta_j(x) \geqq j^{(1+\alpha)(1/2-\lambda)}\}) \\ &\leqq \frac{\|\eta_j\|_{2+\delta}^{2+\delta}}{j^{(1+\alpha)(1/2-\lambda)(2+\delta)}} \\ &\leqq \frac{\|F\|_{2+\delta}^{2+\delta}(j^{\alpha} + j^{s(1+\alpha)})^{2+\delta}}{j^{(1+\alpha)(1/2-\lambda)(2+\delta)}} \\ &= O(j^{\alpha(2+\delta)-(1+\alpha)(1/2-\lambda)(2+\delta)}) \qquad (j \to \infty). \end{aligned} \tag{4.29}$$

From the choice of λ (4.26) we have $\lambda < \frac{\delta - (2+\delta)\left(\alpha + \frac{2}{\gamma}\right)}{2(1+\alpha)(2+\delta)}$, therefore, we obtain

$$\alpha(2+\delta) - (1+\alpha)\left(\frac{1}{2} - \lambda\right)(2+\delta) < -1. \tag{4.30}$$

Thus it follows that

$$\sum_{j=1}^{\infty} \mu(\{x \in M \mid \eta_j(x) \geqq j^{(1+\alpha)(1/2-\lambda)}\}) < \infty. \tag{4.31}$$

Therefore the Borel-Cantelli lemma implies that

$$\eta_j = O(j^{(1+\alpha)(1/2-\lambda)}) \quad (j \to \infty) \qquad \mu\text{-}a.s. \tag{4.32}$$

On the other hand we have

$$j(N) = O(N^{1/(1+\alpha)}) \qquad (N \to \infty) \tag{4.33}$$

by definition of $j(N)$. It is not hard to see that (4.28) follows from (4.32) and (4.33). ■

Next we investigate the asymptotic behavior of the second term in the right hand side of (4.27) as N tends to infinity.

LEMMA 4.6.

$$\sum_{i=2}^{j(N)-1} z_i = O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.34}$$

In order to prove the lemma we employ the Gaal-Koksma strong law of large numbers in [8, Theorem A.1 of Appendix 1] as Lemma 4.7.

LEMMA 4.7. *Let* $\{X_k\}_{k=1}^{\infty}$ *be a sequence of random variables on a probability space* $(\Omega, \mathfrak{F}, P)$ *whose expectations are* 0. *Suppose that there exist positive constants* σ *and* C *such that*

$$E\left[\left(\sum_{k=n+1}^{n+m} X_k\right)^2\right] \leqq C((n+m)^{\sigma} - n^{\sigma}) \tag{4.35}$$

for all nonnegative integer n *and all positive integer* m. *Then*

$$\sum_{k=1}^{N} X_k = O(N^{\sigma/2}(\log N)^{2+\varepsilon}) \quad (N \to \infty) \qquad P\text{-}a.s. \tag{4.36}$$

holds with any positive number ε.

PROOF OF LEMMA 4.6. If n and m are natural numbers, we have

$$\begin{aligned}\int_M \left(\sum_{j=n+1}^{n+m} z_j\right)^2 d\mu &\leqq \sum_{j=n+1}^{n+m} \sum_{k=L_{j-1}}^{M_j-1} \left(C_F(0) + 2\sum_{v=1}^{\infty} |C_F(v)|\right) \\ &= \left(C_F(0) + 2\sum_{v=1}^{\infty} |C_F(v)|\right) \sum_{j=n+1}^{n+m} [j^{s(1+\alpha)}] \\ &\leqq c_0((n+m)^{s(1+\alpha)+1} - n^{s(1+\alpha)+1}),\end{aligned}$$

where c_0 is a positive constant independent of n and m. Hence we can apply Lemma 4.7. Note that $\left(\frac{1}{2} - \lambda\right)(1+\alpha) > \frac{s(1+\alpha)+1}{2}$ is valid since $\lambda < \frac{(1-s)\alpha - s}{2(1+\alpha)}$ holds by (4.26). Thus we conclude that

$$\sum_{i=1}^{j} z_i = O(j^{(1/2-\lambda)(1+\alpha)}) \quad (j \to \infty) \qquad \mu\text{-}a.s. \tag{4.37}$$

Combining this and (4.33), we obtain the lemma. ■

Now we are in a position to apply Proposition 4.2. By (2.12) and (4.26) all the hypotheses of Proposition 4.2 are satisfied with λ' instead of λ and with $\gamma_0 = \gamma s$. We choose the sequence of measurable partitions $\{\mathscr{U}^{(N)}\}_{N=1}^{\infty}$ as in Proposition 4.2. We can see that

$$\beta_{\mathscr{U}^{(N)}}(N, [N^s]) \leqq c_1 N^{-\gamma s}, \tag{4.38}$$

for any $N \in \mathbf{N}$, where c_1 a positive constant independent of N.

Next putting

$$\bar{y}_j = \sum_{i=M_j}^{L_j-1} E[F \mid \sigma(\mathscr{U}^{(i+1)})] \circ T^i \qquad (j = 1, 2, \ldots),$$

we obtain the following lemma.

Lemma 4.8.

$$\sum_{i=1}^{j(N)-1} y_i = \sum_{i=1}^{j(N)-1} \bar{y}_i + O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.39}$$

Proof. Since λ and λ' are chosen so that $\lambda < \lambda'$, we have

$$|F - E[F \mid \sigma(\mathscr{U}^{(n)})]| \leqq 2^{-[(1/2+\lambda')\log_2 n]} \leqq 2^{-[(1/2+\lambda)\log_2 n]} \qquad \mu\text{-}a.s. \tag{4.40}$$

for any $n \in \mathbf{N}$ by the definition of $\mathscr{U}^{(n)}$. Thus, almost surely with respect to μ we have

$$\begin{aligned}\left|\sum_{i=1}^{j(N)-1} y_i - \sum_{i=1}^{j(N)-1} \bar{y}_i\right| &\leqq \sum_{i=1}^{j(N)-1} \sum_{n=M_i}^{L_i-1} |F \circ T^n - E[F \mid \sigma(\mathscr{U}^{(n+1)})] \circ T^n| \\ &\leqq \sum_{n=0}^{N-1} |F \circ T^n - E[F \mid \sigma(\mathscr{U}^{(n+1)})] \circ T^n| \\ &\leqq \sum_{n=1}^{N} 2^{-(1/2+\lambda)\log_2 n+1} \\ &\leqq 2\sum_{n=1}^{N} n^{-(1/2+\lambda)} \\ &= O(N^{1/2-\lambda}) \qquad (N \to \infty).\end{aligned}$$

We note that the last equality follows from $0 < \lambda < \frac{1-2\alpha}{4(1+\alpha)} < \frac{1}{2}$ which is a consequence of the choice of λ (4.26). Therefore, the lemma is proved. ■

From Lemma 4.5, Lemma 4.6 and Lemma 4.8, we get

$$\sum_{i=0}^{N-1} F \circ T^i = \sum_{i=1}^{j(N)-1} \bar{y}_i + O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-a.s.} \tag{4.41}$$

Next we have to show that the sequence of random variables $\{\bar{y}_i\}_{i=1}^{\infty}$ is approximated by a martingale difference sequence. To this end we need the following lemma.

Lemma 4.9. *Let q be a positive number with $\frac{1}{\gamma} + \frac{1}{2+\delta} < \frac{1}{q} \leqq 1$. Then, for each $j = 2, 3, 4, \ldots$, the sequence of functions*

$$\left\{ \sum_{i=j}^{j+m} E\left[\bar{y}_i \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)} \right) \right] \right\}_{m=0}^{\infty}$$

converges in $L^q\left(M, \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)} \right), \mu\Big|_{\sigma(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)})}\right)$. The limit functions u_j satisfies

$$\|u_j\|_q = O(j^{\alpha - s(1+\alpha)\gamma(1/q - 1/(2+\delta))}) \qquad (j \to \infty). \tag{4.42}$$

We need the following to prove Lemma 4.9.

Lemma 4.10. *Let $\mathscr{A} = \{A_i\}_{i \in I}$ and $\mathscr{A}' = \{B_j\}_{j \in J}$ be measurable partitions of $(M, \mathfrak{B}, \mu)$ such that*

$$\sum_{i \in I} \sum_{j \in J} |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)| \leqq \beta. \tag{4.43}$$

Suppose that G is a member of $L^{q_0}(M, \mathfrak{B}, \mu)$ for some q_0 with $1 < q_0 \leqq \infty$ which is $\sigma(\mathscr{A})$-measurable, and $E[G] = 0$. Then for any q with $1 \leqq q < q_0$, we have the estimation

$$\|E[G \mid \sigma(\mathscr{A}')]\|_q \leqq 2\|G\|_{q_0} \beta^{1/q - 1/q_0}. \tag{4.44}$$

Proof. We have only to prove in the case when $q_0 < \infty$. We assume $G(x) = G_i$ μ-a.s. $x \in A_i$ $(i \in I)$. Then for any $j \in J$ with $\mu(B_j) \neq 0$ and for almost all $x \in B_j$, we have

$$E[G \mid \sigma(\mathscr{A}')](x) = \frac{1}{\mu(B_j)} \sum_{i \in I} G_i \mu(A_i \cap B_j)$$
$$= \frac{1}{\mu(B_j)} \sum_{i \in I} G_i(\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)).$$

Here the second inequality follows from the equation $\sum_{i \in I} G_i \mu(A_i)\mu(B_j) = \mu(B_j)E[F] = 0$. Therefore, we obtain

$$\begin{aligned}
&|E[G \mid \sigma(\mathscr{A}')](x)|^q \\
&\quad \leqq \left(\frac{1}{\mu(B_j)} \sum_{i \in I} |G_i|\, |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)| \right)^q \\
&\quad = \left(\sum_{i \in I} \frac{|G_i|\, |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|^{1/q}}{(\mu(B_j))^{1/q}} \frac{|\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|^{1-1/q}}{(\mu(B_j))^{1-1/q}} \right)^q \\
&\quad \leqq \left(\sum_{i \in I} \frac{|G_i|^q |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|}{\mu(B_j)} \right) \left(\sum_{i \in I} \frac{|\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|}{\mu(B_j)} \right)^{q-1} \\
&\quad \leqq 2^{q-1} \sum_{i \in I} \frac{|G_i|^q |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|}{\mu(B_j)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\int_M |E[G \mid \sigma(\mathscr{A}')]|^q d\mu \\
&\leqq \sum_{\substack{j \in J \\ \mu(B_j) \neq 0}} \mu(B_j) \left(2^{q-1} \sum_{i \in I} \frac{|G_i|^q |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|}{\mu(B_j)} \right) \\
&\leqq 2^{q-1} \sum_{j \in J} \sum_{i \in I} |G_i|^q |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)| \\
&= 2^{q-1} \sum_{j \in J} \sum_{i \in I} (|G_i|^q |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|^{q/q_0}) |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|^{1-q/q_0} \\
&\leqq 2^{q-1} \left(2 \sum_{i \in I} (|G_i|^{q_0} \mu(A_i) \right)^{q/q_0} \beta^{1-q/q_0} \\
&\leqq 2^q \|G\|_{q_0}^q \beta^{1-q/q_0}.
\end{aligned}$$

Consequently, we obtain $\|E[G \mid \sigma(\mathscr{A}')]\|_q \leqq 2\|G\|_{q_0} \beta^{1/q - 1/q_0}$. ■

Proof of Lemma 4.9. First, we estimate $\left\|E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right\|_q$ for any natural number $j \geqq 2$. There exists a natural number N_j such that $[N_j^s] = M_j - (L_{j-1} - 1) = [j^{s(1+\alpha)}] + 1$. It can be checked easily by the definition of L_j that

$$L_j \leqq j^{1+\alpha} \qquad (j = j_0, j_0 + 1, j_0 + 2, \dots) \tag{4.45}$$

for j_0 large enough. Therefore $L_j < N_j$ holds for $j_0 \leqq j$ by the definition of N_j. Thus if $j_0 \leqq j$, we have

$$\begin{aligned}
&\beta\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)}, \bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)}\right) \\
&\quad \leqq \beta\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(N_j)}, \bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(N_j)}\right) \\
&\quad \leqq \beta_{\mathscr{U}^{(N_j)}}(N_j, M_j - (L_{j-1} - 1)) \\
&\quad = \beta_{\mathscr{U}^{(N_j)}}(N_j, [N_j^s]) \\
&\quad \leqq c_1 N_j^{-\gamma s} \\
&\quad \leqq c_1([j^{s(1+\alpha)}] + 1)^{-\gamma} \\
&\quad \leqq c_1 j^{-\gamma s(1+\alpha)}
\end{aligned}$$

by using the inequality (4.38) and the choice of N_j. Here the first inequality follows from the fact that $\mathscr{U}^{(N_j)}$ is a refinement of $\mathscr{U}^{(k)}$ for any $k < L_j$ by $L_j < N_j$. Noting that $\bar{y}_j$ is $\sigma\left(\bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)}\right)$-measurable, we conclude from Lemma 4.10 and the above estimation that

$$\left\|E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right\|_q \leqq 2\|\bar{y}_j\|_{2+\delta} \cdot c_1^{1/q-1/(2+\delta)} j^{-\gamma s(1+\alpha)(1/q-1/(2+\delta))},$$

when $j_0 \leqq j$. Thus we can pick a positive constant c_2 so that

$$\left\|E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right\|_q \leqq c_2 j^{\alpha-\gamma s(1+\alpha)(1/q-1/(2+\delta))} \tag{4.46}$$

holds for any $j \geqq 2$.

Next, we estimate $\left\| E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)} \right) \right] \right\|_q$ for any $j \geqq 3$ and any l with $1 \geqq l \leqq j-2$. There exists a natural number $N_{j,l}$ such that $[N_{j,l}^s] = M_j - (L_l - 1)$. Then it is obvious that $L_j < N_j < N_{j,l}$ if $j \geqq j_0$. Thus if $j \geqq j_0$, we have

$$\begin{aligned}
&\beta\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}, \bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)} \right) \\
&\quad \leqq \beta\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(N_{j,l})}, \bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(N_{j,l})} \right) \\
&\quad \leqq \beta_{\mathscr{U}^{(N_{j,l})}}(N_j, M_j - (L_l - 1)) \\
&\quad = \beta_{\mathscr{U}^{(N_{j,l})}}(N_{j,l}, [N_{j,l}^s]) \\
&\quad \leqq c_1 N_{j,l}^{-\gamma s} \\
&\quad \leqq c_1 (M_j + 1 - L_l)^{-\gamma}
\end{aligned} \tag{4.47}$$

by using the inequality (4.38) and the choice of N_j. Here the first inequality follows from the fact that $\mathscr{U}^{(N_{j,l})}$ is a refinement of $\mathscr{U}^{(k)}$ for any $k < L_j$ by $L_j < N_{j,l}$. Noting that $\bar{y}_j$ is $\sigma\left(\bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)} \right)$-measurable, we conclude from Lemma 4.10 and the estimation above that

$$\begin{aligned}
&\left\| E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)} \right) \right] \right\|_q \\
&\qquad \leqq 2\|\bar{y}_j\|_{2+\delta} (c_1 (M_j + 1 - L_l)^{-\gamma})^{1/q - 1/(2+\delta)}.
\end{aligned} \tag{4.48}$$

Thus, by the definition of M_j and L_l, we can take a positive constant c_3 so that

$$\left\| E\left[\bar{y}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)} \right) \right] \right\|_q \leqq c_3 j^{\alpha - \alpha\gamma(1/q - 1/(2+\delta))} (j - 1 - l)^{-\gamma(1/q - 1/(2+\delta))} \tag{4.49}$$

holds for any $j \geqq 3$ and any l with $1 \leqq l \leqq j - 2$. By (4.46) and (4.49), we obtain

$$\sum_{i=j}^{\infty}\left\|E\left[\bar{y}_i \middle| \sigma\left(\bigvee_{k=0}^{L_{j-1}-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right\|_q$$
$$\leqq c_2 j^{\alpha-\gamma s(1+\alpha)(1/q-1/(2+\delta))} + \sum_{m=1}^{\infty} c_3(j+m)^{\alpha-\alpha\gamma(1/q-1/(2+\delta))} m^{-\gamma(1/q-1/(2+\delta))}$$
$$\leqq c_2 j^{\alpha-\gamma s(1+\alpha)(1/q-1/(2+\delta))} + c_3\sum_{m=1}^{\infty} j^{\alpha-\alpha\gamma(1/q-1/(2+\delta))} m^{-\gamma(1/q-1/(2+\delta))}$$
$$\leqq \left(c_2 + c_3\sum_{m=1}^{\infty} m^{-\gamma(1/q-1/(2+\delta))}\right) j^{\alpha-\gamma s(1+\alpha)(1/q-1/(2+\delta))}.$$

In the above, the second inequality follows from $\alpha - \alpha\gamma\left(\frac{1}{q} - \frac{1}{2+\delta}\right) < 0$ since $\frac{1}{\gamma} + \frac{1}{2+\delta} < \frac{1}{q}$ and the third inequality follows from $s(1+\alpha) < \alpha$ (see (4.24)). In addition we notice that

$$\sum_{m=1}^{\infty} m^{-\gamma(1/q-1/(2+\delta))} < \infty$$

holds since $\frac{1}{\gamma} + \frac{1}{2+\delta} < \frac{1}{q}$. We have thus proved the lemma. ∎

We note that $\frac{1}{\gamma} + \frac{1}{2+\delta} < \frac{1}{2}$ by the assumption on γ. Let u_j $(j = 2, 3, 4, \dots)$ be as defined in Lemma 4.9 and $u_1 = 0$. If we define a sequence of functions $\{Y_j\}_{j=1}^{\infty}$ by

$$Y_j = \bar{y}_j - u_j + u_{j+1},$$

then it is obvious by definition that $\{Y_j\}_{j=1}^{\infty}$ is a martingale difference sequence. The following lemma gives the desired fact that $\{\bar{y}_i\}_{i=1}^{\infty}$ is approximated by a martingale difference sequence.

LEMMA 4.11.

$$\sum_{i=1}^{j(N)-1} \bar{y}_i = \sum_{i=1}^{j(N)-1} Y_i + O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.50}$$

PROOF. First, we notice that

$$\sum_{i=1}^{j} Y_i - \sum_{i=1}^{j} \bar{y}_i = \sum_{i=1}^{j}(u_{i+1} - u_i) = u_{j+1} - u_1 = u_{j+1} \tag{4.51}$$

holds for any natural number j. By $\gamma > \frac{1-s}{s}$ and $\frac{s}{1-s} < \alpha$, we have

$$\frac{1}{2}+s-\frac{1}{1+\alpha}\left(\alpha+\frac{1}{\gamma}+\frac{1}{2+\delta}\right)>\frac{\delta-\alpha(2+\delta)}{2(2+\delta)(1+\alpha)}. \tag{4.52}$$

Combining this estimation with the inequality (4.26) we obtain $\lambda<\frac{1}{2}+s-\frac{1}{1+\alpha}\left(\alpha+\frac{1}{\gamma}+\frac{1}{2+\delta}\right)$. Thus, noting that $\frac{1}{\gamma}+\frac{1}{2+\delta}<\frac{1}{2}$, we can choose $q\geqq 1$ so that $\frac{1}{\gamma}+\frac{1}{2+\delta}<\frac{1}{q}<\left(\frac{1}{2}-\lambda+s\right)(1+\alpha)-\alpha$ holds.

By (4.42) we obtain

$$\begin{aligned}&\mu(\{x\in M\,|\,|u_j(x)|\geqq j^{(1/2-\lambda)(1+\alpha)}\})\\ &\quad=O(j^{(\alpha-s(1+\alpha)\gamma(1/q-1/(2+\delta)))q-(1/2-\lambda)(1+\alpha)q})\qquad(j\to\infty).\end{aligned} \tag{4.53}$$

The exponent in the right hand side is less than -1. Indeed,

$$\begin{aligned}&\left(\alpha-s(1+\alpha)\gamma\left(\frac{1}{q}-\frac{1}{2+\delta}\right)\right)q-\left(\frac{1}{2}-\lambda\right)(1+\alpha)q\\ &\quad<(\alpha-s(1+\alpha))q-\left(\frac{1}{2}-\lambda\right)(1+\alpha)q\\ &\quad<-1\end{aligned}$$

is valid since $\frac{1}{\gamma}+\frac{1}{2+\delta}<\frac{1}{q}$ and $\frac{1}{q}<\left(\frac{1}{2}-\lambda+s\right)(1+\alpha)-\alpha$ hold. Therefore, we obtain

$$\sum_{j=1}^{\infty}\mu(\{x\in M\,|\,|u_j(x)|\geqq j^{(1/2-\lambda)(1+\alpha)}\})<\infty. \tag{4.54}$$

Hence by the Borel-Cantelli lemma we conclude that

$$u_j=O(j^{(1/2-\lambda)(1+\alpha)})\quad(j\to\infty)\qquad\mu\text{-}a.s. \tag{4.55}$$

This with (4.33) and (4.51) completes the proof. ■

From Lemma 4.11 and the equality (4.41) we get

$$\sum_{k=0}^{N-1}F\circ T^k=\sum_{i=1}^{j(N)-1}Y_i+O(N^{1/2-\lambda})\quad(N\to\infty)\qquad\mu\text{-}a.s. \tag{4.56}$$

On the other hand we can apply Theorem 4.1 to $\{Y_j\}_{j=1}^{\infty}$ since $\{Y_j\}_{j=1}^{\infty}$ is a martingale difference sequence. In what follows, $(\tilde{\Omega},\tilde{\mathfrak{F}},\tilde{P}),\{\tilde{Y}_i\}_{i=1}^{\infty},\{T_i\}_{i=1}^{\infty}$ and $\{B(t)\}_{t\in[0,\infty)}$ are as in Theorem 4.1. It remains to show that

$$B\left(\sum_{i=1}^{j(N)-1}T_i\right)=B(\sigma_F^2N)+O(N^{1/2-\lambda})\quad(N\to\infty)\qquad\tilde{P}\text{-}a.s. \tag{4.57}$$

To this end we prove the following.

Lemma 4.12.

$$E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right] = \sigma_F^2 N + O(N^{1-2\lambda'}) \qquad (N \to \infty).$$

Proof. For any positive integer N we have

$$\begin{aligned} E\left[\sum_{i=1}^{j(N)-1} y_i^2\right] &= \sum_{i=1}^{j(N)-1} \left([i^{\alpha}]C_F(0) + 2\sum_{v=1}^{[i^{\alpha}]-1}([i^{\alpha}] - v)C_F(v)\right) \\ &= N\sigma_F^2 - \sigma_F^2 \sum_{i=2}^{j(N)-1} [i^{s(1+\alpha)}] - \sigma_F^2(N - L_{j(N)-1}) \\ &\quad - 2\sum_{i=1}^{j(N)-1}\left(\sum_{v=1}^{[i^{\alpha}]} vC_F(v) + [i^{\alpha}]\sum_{v=[i^{\alpha}]+1}^{\infty} C_F(v)\right). \end{aligned} \tag{4.58}$$

By $\lambda' < \frac{\alpha - s(1+\alpha)}{2(1+\alpha)}$ we obtain

$$s(1+\alpha) + 1 < (1+\alpha)(1-2\lambda') \tag{4.59}$$

This implies

$$\sum_{i=2}^{j(N)-1} [i^{s(1+\alpha)}] \leqq \int_0^{j(N)} t^{s(1+\alpha)}\, dt = O(j(N)^{(1+\alpha)(1-2\lambda')}) \qquad (N \to \infty). \tag{4.60}$$

Since $\alpha < 1 < s(1+\alpha) + 1 < (1+\alpha)(1-2\lambda')$ holds from (4.59), we obtain

$$N - L_{j(N)-1} \leqq [j(N)^{\alpha}] + [j(N)^{s(1+\alpha)}] = O(j(N)^{(1+\alpha)(1-2\lambda')}) \qquad (N \to \infty) \tag{4.61}$$

It follows form the assumption (2.10) that

$$\begin{aligned} &2\sum_{i=1}^{j(N)-1}\left(\sum_{v=1}^{[i^{\alpha}]-1} vC_F(v) + [i^{\alpha}]\sum_{v=[i^{\alpha}]}^{\infty} C_F(v)\right) \\ &\leqq 2c_4 \int_0^{j(N)} t^{\alpha\rho}\, dt \\ &= O(j(N)^{\alpha\rho+1}) \qquad (N \to \infty) \\ &= O(j(N)^{(1+\alpha)(1-2\lambda')}) \qquad (N \to \infty), \end{aligned} \tag{4.62}$$

where c_4 is a constant independent of N. Here we used the assumption that $\lambda' < \frac{\alpha(1-\rho)}{2(1+\alpha)}$. Thus, from (4.58), (4.60), (4.61), (4.62) and (4.33) we conclude that

$$E\left[\sum_{i=1}^{j(N)-1} y_i^2\right] = \sigma_F^2 N + O(N^{1-2\lambda'}) \qquad (N \to \infty). \tag{4.63}$$

Next, by the definitions of $\bar{y}_i$ and $\mathscr{U}^{(k)}$ we have

$$\begin{aligned}
&\left|E\left[\sum_{i=1}^{j(N)-1} y_i^2\right] - E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right]\right| \\
&\quad \leqq 2\sum_{i=1}^{j(N)-1} \|y_i\|_1 \sum_{k=M_i}^{L_i-1} \|F - E[F \mid \sigma(\mathscr{U}^{(k+1)})]\|_\infty \\
&\qquad + \sum_{i=1}^{j(N)-1} \left(\sum_{k=M_i}^{L_i-1} \|F - E[F \mid \sigma(\mathscr{U}^{(k+1)})]\|_\infty\right)^2 \\
&\quad \leqq 2\sum_{i=1}^{j(N)-1} [i^\alpha]\|F\|_1 \cdot [i^\alpha] 2^{-[(1/2+\lambda')\log_2(M_i+1)]} + \sum_{i=1}^{j(N)-1} [i^\alpha]^2 2^{-2[(1/2+\lambda')\log_2(M_i+1)]} \\
&\quad = O(j(N)^{2\alpha-(1+\alpha)(1/2+\lambda')}) \qquad (N \to \infty).
\end{aligned}$$

Since $\lambda' < \frac{1-2\alpha}{4(1+\alpha)} < \frac{1-\alpha}{2(1+\alpha)}$ holds from (4.26), the above investigation implies that

$$\left|E\left[\sum_{i=1}^{j(N)-1} y_i^2\right] - E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right]\right| = O(j(N)^{(1+\alpha)(1-2\lambda')}) \qquad (N \to \infty). \tag{4.64}$$

Combining this, (4.63) and (4.33), we have proved the lemma. ■

Lemma 4.12 means that the difference of $E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right]$ from $\sigma_F^2 N$ has the appropriate growth rate. Next, we prove that difference of $\sum_{i=1}^{j(N)-1} \bar{y}_i^2$ from $E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right]$ also has the appropriate growth rate in a.s. sense.

Lemma 4.13.

$$\sum_{i=1}^{j(N)-1} \bar{y}_i^2 = E\left[\sum_{i=1}^{j(N)-1} \bar{y}_i^2\right] + O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \mu\text{-a.s.} \tag{4.65}$$

Proof. First we prove in the case when $0 < \delta \leqq 2$. Since $\lambda' < \frac{\delta-(2+\delta)\left(\alpha+\frac{2}{\gamma}\right)}{2(1+\alpha)(2+\delta)} < \frac{\delta-\alpha(2+\delta)}{2(1+\alpha)(2+\delta)}$ holds from (4.26), we can select a real number ζ such that $2\alpha + \frac{2}{2+\delta} < \zeta$ and

$$(1-2\lambda')(1+\alpha) < \frac{\zeta}{2}\left(1-\frac{\delta}{4}\right) + \alpha\left(1+\frac{\delta}{4}\right) - \frac{\delta}{4(2+\delta)} + \frac{1}{2}. \tag{4.66}$$

We define a sequence $\{w_j\}_{j=1}^{\infty}$ of functions on M by

$$w_j(x) = \min\{(\bar{y}_j(x))^2, j^{\zeta}\} \qquad x \in M \ \ (j = 1, 2, 3, \ldots)$$

Then for any positive integer j we have

$$\mu(\{x \in M \mid w_j(x) \neq (\bar{y}_j(x))^2\}) \leqq \frac{\|\bar{y}_j\|_{2+\delta}^{2+\delta}}{j^{\zeta(1+\delta/2)}} \leqq \|F\|_{2+\delta}^{2+\delta} j^{(2+\delta)(\alpha-\zeta/2)}. \tag{4.67}$$

Since $(2+\delta)\left(\alpha - \frac{\zeta}{2}\right) < -1$ holds from $2\alpha + \frac{2}{2+\delta} < \zeta$, we obtain

$$\sum_{j=1}^{\infty} \mu(\{x \in M \mid w_j(x) \neq (\bar{y}_j(x))^2\}) < \infty. \tag{4.68}$$

Thus, by the Borel-Cantelli lemma, for almost all $x \in M$ there exist only finitely many positive integers j such that $w_j(x) \neq (\bar{y}_j(x))^2$. Therefore it follows that

$$\sum_{i=1}^{j(N)-1} \bar{y}_i^2 = \sum_{i=1}^{j(N)-1} w_i + O(1) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.69}$$

Next, for any positive integer j we have

$$\begin{aligned}
|E[w_j] - E[\bar{y}_j^2]| &= \int_M \bar{y}_j^2 \chi_{\{x \in M \mid (\bar{y}_j(x))^2 \geqq j^{\zeta}\}}\, d\mu \\
&\leqq \|\bar{y}_j\|_{2+\delta}^2 (\mu(\{x \in M \mid (\bar{y}_j(x))^2 \geqq j^{\zeta}\}))^{\delta/(2+\delta)} \\
&\leqq \|F\|_{2+\delta}^{2+\delta} j^{(2+\delta)\alpha - \delta\zeta/2},
\end{aligned}$$

where χ_A is the indicator function of A. We note that

$$(2+\delta)\alpha - \frac{\delta\zeta}{2} < (1-2\lambda')(1+\alpha) - 1 \Leftrightarrow \lambda' < \frac{1}{4(1+\alpha)}(\delta\zeta - (2+2\delta)\alpha) \tag{4.70}$$

and

$$\lambda' < \frac{\delta - (2+\delta)\left(\alpha + \frac{2}{\gamma}\right)}{2(1+\alpha)(2+\delta)} < \frac{1}{4(1+\alpha)}(\delta\zeta - (2+2\delta)\alpha) \tag{4.71}$$

from (4.26) and $2\alpha + \frac{2}{2+\delta} < \zeta$. Therefore it follows that

$$|E[w_j] - E[\bar{y}_j^2]| \leqq \|F\|_{2+\delta}^{2+\delta} j^{(1-2\lambda')(1+\alpha)-1}. \tag{4.72}$$

Hence we obtain

$$\sum_{i=1}^{j} E[\bar{y}_i^2] = \sum_{i=1}^{j} E[w_i] + O(j^{(1-2\lambda')(1+\alpha)}) \qquad (j \to \infty). \tag{4.73}$$

By (4.33) this implies

$$\sum_{i=1}^{j(N)-1} E[\bar{y}_i^2] = \sum_{i=1}^{j(N)-1} E[w_i] + O(N^{1-2\lambda'}) \qquad (N \to \infty). \tag{4.74}$$

From this and (4.69), it suffices to show

$$\sum_{i=1}^{j(N)-1} \overline{w}_i = O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \mu\text{-}a.s., \tag{4.75}$$

where $\overline{w}_j = w_j - E[w_j]$ $(j = 1, 2, \ldots)$. To this end, we estimate $E\left[\left(\sum_{j=n+1}^{n+m} \overline{w}_j\right)^2\right]$ for any nonnegative integer n and any positive integer m.

Noting that $2\alpha + \frac{2}{2+\delta} < \zeta$, we have

$$\begin{aligned} E[\overline{w}_j^2] &\leqq \|\overline{w}_j\|_\infty^{1-\delta/2} \|\overline{w}_j\|_{1+\delta/2}^{1+\delta/2} \\ &\leqq 4\|w_j\|_\infty^{1-\delta/2} \|w_j\|_{1+\delta/2}^{1+\delta/2} \\ &\leqq 4 j^{\zeta(1-\delta/2)} \|F\|_{2+\delta}^{2+\delta} j^{\alpha(2+\delta)} \\ &\leqq 4\|F\|_{2+\delta}^{2+\delta} j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)} \end{aligned} \tag{4.76}$$

and

$$\begin{aligned} |E[\overline{w}_j \overline{w}_{j+1}]| &\leqq \|[\overline{w}_j\|_2 \|\overline{w}_{j+1}\|_2 \\ &\leqq 4\|F\|_{2+\delta}^{2+\delta} (j+1)^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)} \end{aligned} \tag{4.77}$$

for any positive integer j. Next for $j \geqq 3$ and l with $1 \leqq l \leqq j-2$, we estimate $|E[\overline{w}_j \overline{w}_l]|$. We notice that $\overline{w}_j$ is $\sigma\left(\bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)}\right)$-measurable and $\overline{w}_l$ is $\sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)$-measurable. Thus by using Lemma 4.10 we obtain

$$
\begin{aligned}
|E[\overline{w}_j\overline{w}_l]| &= \left|E\left[E\left[\overline{w}_j\overline{w}_l \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right]\right| \\
&= \left|E\left[E\left[\overline{w}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\cdot \overline{w}_l\right]\right| \\
&\leqq \left\|E\left[\overline{w}_j \,\middle|\, \sigma\left(\bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right]\right\|_2 \|\overline{w}_l\|_2 \\
&\leqq 2\|\overline{w}_j\|_{2+\delta}\left(\beta\left(\bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)}, \bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right)^{1/2-1/(2+\delta)}\cdot\|\overline{w}_l\|_2 \\
&\leqq 8\|w_j\|_{2+\delta}\|w_l\|_2\left(\beta\left(\bigvee_{k=M_j}^{L_j-1} T^{-k}\mathscr{U}^{(k+1)}, \bigvee_{k=0}^{L_l-1} T^{-k}\mathscr{U}^{(k+1)}\right)\right)^{1/2-1/(2+\delta)}.
\end{aligned}
$$

On the other hand there exists a positive integer j_0 such that (4.47) holds for any $j \geqq j_0$, as we saw in the proof of Lemma 4.9. Therefore for $j \geqq j_0$ we obtain

$$
|E[\overline{w}_j\overline{w}_l]| \leqq 8c_1^{1/2-1/(2+\delta)}\|w_j\|_{2+\delta}\|w_l\|_2(M_j+1-L_l)^{-\gamma(1/2-1/(2+\delta))}. \quad (4.78)
$$

Hence, by the definition of M_j and L_l, there exists a positive constant c_5 such that

$$
|E[\overline{w}_j\overline{w}_l]| \leqq c_5\|w_j\|_{2+\delta}\|w_l\|_2 j^{-\alpha\gamma(1/2-1/(2+\delta))}(j-l-1)^{-\gamma(1/2-1/(2+\delta))} \quad (4.79)
$$

holds for any $j \geqq 3$ and any l with $1 \leqq l \leqq j-2$. Now we have the estimations

$$
\|w_j\|_{2+\delta} \leqq \|w_j\|_\infty^{1/2}\|w_j\|_{1+\delta/2}^{1/2} \leqq j^{\zeta/2}\|\overline{y}_j^2\|_{1+\delta/2}^{1/2} \leqq j^{\zeta/2}\|y_j\|_{2+\delta} \leqq \|F\|_{2+\delta} j^{\zeta/2+\alpha} \quad (4.80)
$$

and

$$
\|w_l\|_2 \leqq \|w_l\|_\infty^{1/2-\delta/4}\|w_l\|_{1+\delta/2}^{1/2+\delta/4} \leqq \|F\|_{2+\delta}^{1+\delta/2} l^{\zeta(1/2-\delta/4)+\alpha(1+\delta/2)}. \quad (4.81)
$$

Thus, by (4.79) and $l < j$ we have

$$
\begin{aligned}
|E[\overline{w}_j\overline{w}_l]| &\leqq c_5\|F\|_{2+\delta}^{2+\delta/2} j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\alpha\gamma(1/2-1/(2+\delta))}(j-l-1)^{-\gamma(1/2-1/(2+\delta))} \\
&\leqq c_5\|F\|_{2+\delta}^{2+\delta/2} j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)}(j-l-1)^{-\gamma(1/2-1/(2+\delta))}. \quad (4.82)
\end{aligned}
$$

In the above the second inequality follows from the fact that $\gamma > \frac{1-s}{s} > \frac{1}{\alpha}$ holds by the assumption on γ and (4.24).

By using (4.76), (4.77) and (4.82) we obtain

$$
\begin{aligned}
&E\left[\left(\sum_{j=n+1}^{n+m}\overline{w}_j\right)^2\right]\\
&\leqq \sum_{j=n+1}^{n+m}E[\overline{w}_j^2]+2\sum_{j=n+2}^{n+m}|E[\overline{w}_j\overline{w}_{j-1}]|+2\sum_{j=n+3}^{n+m}\sum_{l=1}^{j-2}|E[\overline{w}_j\overline{w}_l]|\\
&\leqq \sum_{j=n+1}^{n+m}4\|F\|_{2+\delta}^{2+\delta}j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)}+2\sum_{j=n+2}^{n+m}4\|F\|_{2+\delta}^{2+\delta}j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)}\\
&\quad+2\sum_{j=n+3}^{n+m}\sum_{l=1}^{j-2}c_5\|F\|_{2+\delta}^{2+\delta/2}j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)}(j-l-1)^{-\gamma(1/2-1/(2+\delta))}\\
&\leqq \left(12\|F\|_{2+\delta}^{2+\delta}+2c_5\|F\|_{2+\delta}^{2+\delta/2}\sum_{i=1}^{\infty}i^{-\gamma(1/2-1/(2+\delta))}\right)\sum_{j=n+1}^{n+m}j^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)}.
\end{aligned}
$$

By the assumption on γ we have

$$
\begin{aligned}
-\gamma\left(\frac{1}{2}-\frac{1}{2+\delta}\right)&<-\frac{2(2+\delta)(1-s)}{\delta-(2+2\delta)s}\,\frac{\delta}{2(2+\delta)}\\
&=-1-\frac{s(2+\delta)}{\delta-(2+2\delta)s}<-1.
\end{aligned}
\tag{4.83}
$$

This implies

$$
\sum_{i=1}^{\infty}i^{-\gamma(1/2-1/(2+\delta))}<\infty. \tag{4.84}
$$

Therefore there exists a positive constant c_6 such that

$$
\begin{aligned}
E\left[\left(\sum_{j=n+1}^{n+m}\overline{w}_j\right)^2\right]&\leqq c_6((n+m)^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)+1}\\
&\quad-n^{\zeta(1-\delta/4)+\alpha(2+\delta/2)-\delta/2(2+\delta)+1}
\end{aligned}
\tag{4.85}
$$

holds for any positive integer m and any nonnegative integer n. Thus by Lemma 4.7 we have

$$\sum_{i=1}^{j} \overline{w}_i = O(j^{(\zeta/2)(1-\delta/4)+\alpha(1+\delta/4)-\delta/4(2+\delta)+1/2}(\log j)^{2+\varepsilon}) \quad (j \to \infty) \qquad \mu\text{-}a.s. \tag{4.86}$$

for any positive number ε. Hence from (4.66) we obtain

$$\sum_{i=1}^{j} \overline{w}_i = O(j^{(1-2\lambda')(1+\alpha)}) \quad (j \to \infty) \qquad \mu\text{-}a.s. \tag{4.87}$$

Combining this with (4.33), we conclude that (4.75) holds.

In the case when $2 < \delta$, we can select a real number ζ such that $2\alpha + \frac{1}{2} < \zeta$ and

$$(1 - 2\lambda')(1+\alpha) < \frac{\zeta}{4} + \frac{3}{2}\alpha - \frac{1}{8} + \frac{1}{2}, \tag{4.88}$$

since $\lambda' < \frac{1-2\alpha}{4(1+\alpha)}$ holds from (4.26). One can show the lemma by the same way with the ζ selected above as in the case when $0 < \delta \leqq 2$. We omit the details. ∎

From Lemma 4.12 and Lemma 4.13 we have

$$\sum_{i=1}^{j(N)-1} \overline{y}_i^2 = \sigma_F^2 N + O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.89}$$

Next we show that $\overline{y}_i$ can be replaced by Y_i in (4.89).

Lemma 4.14.

$$\sum_{i=1}^{j(N)-1} Y_i^2 = \sum_{i=1}^{j(N)-1} \overline{y}_i^2 + O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.90}$$

Proof. By definition, for each positive integer N we have

$$\left|\sum_{i=1}^{j(N)-1} Y_i^2 - \sum_{i=1}^{j(N)-1} \overline{y}_i^2\right| \leqq 2\left|\sum_{i=1}^{j(N)-1} (u_{i+1} - u_i)\overline{y}_i\right| + \sum_{i=1}^{j(N)-1} (u_{i+1} - u_i)^2$$

$$\leqq \left(\sum_{i=1}^{j(N)-1} (u_{i+1} - u_i)^2\right)^{1/2} \left(\sum_{i=1}^{j(N)-1} \overline{y}_i^2\right)^{1/2} + \sum_{i=1}^{j(N)-1} (u_{i+1} - u_i)^2.$$

Since

$$\sum_{i=1}^{j(N)-1} \bar{y}_i^2 = O(N) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.91}$$

holds from (4.89), it suffices to show

$$\sum_{i=1}^{j(N)-1} (u_{i+1} - u_i)^2 = O(N^{1-4\lambda'}) \quad (N \to \infty) \qquad \mu\text{-}a.s. \tag{4.92}$$

First, since $\lambda' < \frac{s\gamma\delta}{4(2+\delta)} - \frac{\alpha}{4(1+\alpha)}$ holds from (4.26), we can choose a real number ζ' such that $\max\left\{-\frac{1}{2}, \alpha - s(1+\alpha)\gamma\frac{\delta}{2(2+\delta)}\right\} < \zeta' < \left(\frac{1}{2} - 2\lambda'\right)(1+\alpha) - \frac{1}{2}$. Then by Lemma 4.9 we have

$$\|u_j\|_2 = O(j^{\zeta'}) \qquad (j \to \infty). \tag{4.93}$$

This implies

$$\begin{aligned}\left\|\sum_{i=1}^{j}(u_{i+1} - u_i)^2\right\|_1 &\leqq \sum_{i=1}^{j}(\|u_{i+1}\|_2^2 + 2\|u_i\|_2\|u_{i+1}\|_2 + \|u_i\|_2^2) \\ &= O(j^{1+2\zeta'}) \qquad (j \to \infty).\end{aligned} \tag{4.94}$$

In the above we used the assumption that $\zeta' > -\frac{1}{2}$. From this investigation we obtain

$$\begin{aligned}&\mu\left(\left\{x \in M \mid \sum_{i=1}^{2^k}(u_{i+1}(x) - u_i(x))^2 \geqq 2^{k(1-4\lambda')(1+\alpha)}\right\}\right) \\ &\quad = O(2^{k(2\zeta'+1-(1-4\lambda')(1+\alpha))}) \qquad (k \to \infty).\end{aligned} \tag{4.95}$$

Since $\zeta' < \left(\frac{1}{2} - 2\lambda'\right)(1+\alpha) - \frac{1}{2}$ holds by the choice of ζ', it follows that

$$\sum_{k=0}^{\infty}\mu\left(\left\{x \in M \mid \sum_{i=1}^{2^k}(u_{i+1}(x) - u_i(x))^2 \geqq 2^{k(1-4\lambda')(1+\alpha)}\right\}\right) < \infty. \tag{4.96}$$

By the Borel-Cantelli lemma, for almost all $x \in M$ there exists a positive number K_x such that

$$\sum_{i=1}^{2^k}(u_{i+1}(x) - u_i(x))^2 \leqq K_x \cdot 2^{k(1-4\lambda')(1+\alpha)} \tag{4.97}$$

for any nonnegative integer k.

Now, for a positive integer j we take a positive integer so that $2^{k-1} \leqq j < 2^k$ holds. Then we have

$$\begin{aligned}\sum_{i=1}^{j}(u_{i+1}(x)-u_i(x))^2 &\leqq \sum_{i=1}^{2^k}(u_{i+1}(x)-u_i(x))^2\\ &\leqq K_x\cdot 2^{k(1-4\lambda')(1+\alpha)}\\ &\leqq 2^{(1-4\lambda')(1+\alpha)}K_x j^{(1-4\lambda')(1+\alpha)}\end{aligned}$$

for almost all $x \in M$. Therefore we obtain

$$\sum_{i=1}^{j}(u_{i+1}-u_i)^2 = O(j^{(1-4\lambda')(1+\alpha)}) \quad (j\to\infty) \qquad \mu\text{-}a.s. \tag{4.98}$$

This with (4.33) implies (4.92). ■

From (4.89) and the above lemma we have

$$\sum_{i=1}^{j(N)-1} Y_i^2 = \sigma_F^2 N + O(N^{1-2\lambda'}) \quad (N\to\infty) \qquad \mu\text{-}a.s. \tag{4.99}$$

Since $\{Y_i\}_{i=1}^{\infty}$ and $\{\tilde{Y}_i\}_{i=1}^{\infty}$ have the same distribution, we also have

$$\sum_{i=1}^{j(N)-1} \tilde{Y}_i^2 = \sigma_F^2 N + O(N^{1-2\lambda'}) \quad (N\to\infty) \qquad \tilde{P}\text{-}a.s. \tag{4.100}$$

Finally we prove the following lemma which asserts that the difference of $\sum_{i=1}^{j(N)-1} T_i$ from $\sum_{i=1}^{j(N)-1} \tilde{Y}_i^2$ has the appropriate growth rate.

Lemma 4.15.

$$\sum_{i=1}^{j(N)-1} T_i = \sum_{i=1}^{j(N)-1} \tilde{Y}_i^2 + O(N^{1-2\lambda'}) \quad (N\to\infty) \qquad \tilde{P}\text{-}a.s. \tag{4.101}$$

We use the Corollary 5 of Chow [5] to show the above lemma. So we recall its statement below.

Lemma 4.16 (Corollary 5 [4]). *Let $\{s_n, \mathfrak{F}_n\}_{n=1}^{\infty}$ be a martingale sequence on a probability space $(\Omega, \mathfrak{F}, P)$ such that $E[|s_n|] < \infty$ for any positive integer n. Let p be a real number with $1 \leqq p \leqq 2$. Then the sequence of functions $\{s_n\}_{n=1}^{\infty}$ converges almost surely on the set where*

$$\sum_{n=2}^{\infty} E[|s_n - s_{n-1}|^p \mid \mathfrak{F}_{n-1}](\omega) < \infty$$

holds.

Proof of Lemma 4.15. For any positive integer N we have

$$\sum_{i=1}^{j(N)-1} T_i - \sum_{i=1}^{j(N)-1} \tilde{Y}_i^2$$
$$= \sum_{i=1}^{j(N)-1} (T_i - E[T_i|\tilde{\mathfrak{F}}_{i-1}]) + \sum_{i=1}^{j(N)-1} (E[T_i|\tilde{\mathfrak{F}}_{i-1}] - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}])$$
$$+ \sum_{i=1}^{j(N)-1} (E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}] - \tilde{Y}_i^2).$$

Since for any positive integer i

$$E[T_i|\tilde{\mathfrak{F}}_{i-1}] - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}] = 0 \qquad \tilde{P}\text{-}a.s. \tag{4.102}$$

holds from (4.2), it suffices to show

$$\sum_{i=1}^{j(N)-1} (T_i - E[T_i|\tilde{\mathfrak{F}}_{i-1}]) = O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \tilde{P}\text{-}a.s. \tag{4.103}$$

and

$$\sum_{i=1}^{j(N)-1} (\tilde{Y}_i^2 - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}]) = O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \tilde{P}\text{-}a.s. \tag{4.104}$$

Since $\lambda' < \min\left\{\frac{\delta-(2+\delta)\left(\alpha+\frac{2}{\gamma}\right)}{2(1+\alpha)(2+\delta)}, \frac{1-2\alpha}{4(1+\alpha)}\right\}$ holds from (4.26), we can choose a real number $\bar{\zeta}$ so that $\frac{1}{1-\alpha-2\lambda'(1+\alpha)} < \bar{\zeta} < \min\left\{2, \frac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{2+\delta}\right)^{-1}\right\}$.

Next, for any positive integer i we define the function R_i on $\tilde{\Omega}$ by

$$R_i = i^{-(1+\alpha)(1-2\lambda')}(\tilde{Y}_i^2 - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}]).$$

Then it is obvious that

$$E[R_i|\tilde{\mathfrak{F}}_{i-1}] = 0 \qquad \tilde{P}\text{-}a.s. \qquad (i = 2, 3, 4, \ldots)$$

holds. Noting that $2\bar{\zeta} < \left(\frac{1}{\gamma} + \frac{1}{2+\delta}\right)^{-1} < 2 + \delta$, we obtain

$$E[|R_i|^{\bar{\zeta}}] \leqq 2^{\bar{\zeta}} i^{-\bar{\zeta}(1+\alpha)(1-2\lambda')} E[|\tilde{Y}_i|^{2\bar{\zeta}}]$$
$$\leqq 2^{\bar{\zeta}} i^{-\bar{\zeta}(1+\alpha)(1-2\lambda')} (\|\bar{y}_i\|_{2\bar{\zeta}} + \|u_i\|_{2\bar{\zeta}} + \|u_{i+1}\|_{2\bar{\zeta}})^{2\bar{\zeta}}. \tag{4.105}$$

Now we have

$$\|\bar{y}_i\|_{2\bar{\zeta}} \leqq \|y_i\|_{2\bar{\zeta}} \leqq i^{\alpha}\|F\|_{2\bar{\zeta}}. \tag{4.106}$$

On the other hand, from Lemma 4.9 and the assumption $\frac{1}{\gamma}+\frac{1}{2+\delta}<\frac{1}{2\bar{\zeta}}$ we have

$$\|u_i\|_{2\bar{\zeta}} = O(i^{\alpha-s(1+\alpha)\gamma(1/2\bar{\zeta}-1/(2+\delta))}) \qquad (i \to \infty). \tag{4.107}$$

Thus it follows that

$$E[|R_i|^{\bar{\zeta}}] = O(i^{2\bar{\zeta}\alpha-\bar{\zeta}(1+\alpha)(1-2\lambda')}) \qquad (i \to \infty). \tag{4.108}$$

Since

$$2\bar{\zeta}\alpha - \bar{\zeta}(1+\alpha)(1-2\lambda') = -\bar{\zeta}(1-\alpha-2\lambda'(1+\alpha)) < -1 \tag{4.109}$$

holds by the choice of $\bar{\zeta}$, we obtain

$$\sum_{i=1}^{\infty} E[|R_i|^{\bar{\zeta}}] < \infty. \tag{4.110}$$

This implies that

$$\sum_{i=1}^{\infty} E[|R_i|^{\bar{\zeta}} | \tilde{\mathfrak{F}}_{i-1}] < \infty \qquad \tilde{P}\text{-}a.s. \tag{4.111}$$

Therefore we can apply Lemma 4.16 to $s_n = \sum_{i=1}^{n} R_i$ with $p = \bar{\zeta}$. Hence the series $\sum_{i=1}^{\infty} R_i(\omega) = \sum_{i=1}^{\infty} i^{-(1+\alpha)(1-2\lambda')}((\tilde{Y}_i(\omega))^2 - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}](\omega))$ converge for almost all $\omega \in \tilde{\Omega}$. By Kronecker's lemma this implies that

$$\frac{1}{j^{(1+\alpha)(1-2\lambda')}} \sum_{i=1}^{j} ((\tilde{Y}_i(\omega))^2 - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}](\omega)) \to 0 \qquad (j \to \infty) \tag{4.112}$$

for almost all $\omega \in \tilde{\Omega}$. Thus it follows that

$$\sum_{i=1}^{j} (\tilde{Y}_i^2 - E[\tilde{Y}_i^2|\tilde{\mathfrak{F}}_{i-1}]) = O(j^{(1+\alpha)(1-2\lambda')}) \quad (j \to \infty) \qquad \tilde{P}\text{-}a.s. \tag{4.113}$$

From this and (4.33) we obtain (4.104).

Next for any positive integer i we define the function R_i' on $\tilde{\Omega}$ by

$$R_i' = i^{-(1+\alpha)(1-2\lambda')}(T_i - E[T_i|\tilde{\mathfrak{F}}_{i-1}]).$$

Then it is obvious that

$$E[R_i'|\tilde{\mathfrak{F}}_{i-1}] = 0 \qquad \tilde{P}\text{-}a.s. \qquad (i = 2, 3, 4, \ldots).$$

On the other hand, from (4.3) and (4.4) we have

$$E[|R_i'|^{\bar{\zeta}}] \leqq 2^{\bar{\zeta}} i^{-\bar{\zeta}(1+\alpha)(1-2\lambda')} E[|T_i|^{\bar{\zeta}}] \leqq 2^{\bar{\zeta}} D_{\bar{\zeta}} i^{-\bar{\zeta}(1+\alpha)(1-2\lambda')} E[|\overline{Y}_i|^{2\bar{\zeta}}]. \quad (4.114)$$

Thus one can show (4.103) by the same way as (4.104). ■

The final step of the proof of Theorem 2.1. From (4.100) and Lemma 4.15 we get

$$\sum_{i=1}^{j(N)-1} T_i = \sigma_F^2 N + O(N^{1-2\lambda'}) \quad (N \to \infty) \qquad \tilde{P}\text{-}a.s. \quad (4.115)$$

By a property of Brownian motion it is easy to show that this implies that

$$B\left(\sum_{i=1}^{j(N)-1} T_i\right) = B(\sigma_F^2 N) + O(N^{1/2-\lambda}) \quad (N \to \infty) \qquad \tilde{P}\text{-}a.s. \quad (4.116)$$

Consequently we have just proved Theorem 2.1. ■

In the rest of the paper we prove Corollary 2.2 and Corollary 2.3.

Proof of Corollary 2.2. It suffices to show that the assumptions of Theorem 2.1 are satisfied. First, we have

$$\begin{aligned}|C_F(N)| \leqq & \left|\int_M E[F \mid \sigma(\mathscr{A}^{(N,1)})] \cdot (E[F \mid \sigma(\mathscr{A}^{(N,1)})] \circ T^N) d\mu\right| \\ & + 2\|F\|_2 \|F - E[F \mid \sigma(\mathscr{A}^{(N,1)})]\|_2 + \|F - E[F \mid \sigma(\mathscr{A}^{(N,1)})]\|_2^2. \quad (4.117)\end{aligned}$$

Now we have

$$\begin{aligned}&\left|\int_M E[F \mid \sigma(\mathscr{A}^{(N,1)})] \cdot (E[F \mid \sigma(\mathscr{A}^{(N,1)})] \circ T^N) d\mu\right| \\ &\quad = \left|\int_M E[F \mid \sigma(\mathscr{A}^{(N,1)})] \cdot E[E[F \mid \sigma(\mathscr{A}^{(N,1)})] \circ T^N \mid \sigma(\mathscr{A}^{(N,1)})] d\mu\right| \\ &\quad \leqq \|E[F \mid \sigma(\mathscr{A}^{(N,1)})]\|_2 \|E[E[F \mid \sigma(\mathscr{A}^{(N,1)})] \circ T^N \mid \sigma(\mathscr{A}^{(N,1)})]\|_2 \\ &\quad \leqq 2\|F\|_2 \|F\|_{2+\delta} (\beta_{\mathscr{A}^{(N,1)}}(N,N))^{1/2-1/(2+\delta)} \\ &\quad \leqq 2\|F\|_2 \|F\|_{2+\delta} C_2^{1/2-1/(2+\delta)} \lambda_2^{(1/2-1/(2+\delta))N^\theta} \\ &\quad = O(N^{-3}) \qquad (N \to \infty).\end{aligned}$$

Note that the second inequality in the above follows from Lemma 4.10. On the other hand we have

$$\begin{aligned}\|F - E[F \mid \sigma(\mathscr{A}^{(N,1)})]\|_2 &\leqq \mathscr{H}_F(C_1\lambda_1^{N^\theta}) \\ &= O\left(\frac{1}{|\log C_1 + N^\theta \log \lambda_1|^v}\right) \qquad (N \to \infty) \\ &= O(N^{-\theta v}) \qquad (N \to \infty).\end{aligned}$$

Thus, since $\theta v > \frac{24+15\delta}{\delta} > 15$ holds from the assumption on v, we obtain

$$|C_F(N)| = O(N^{-3}) \qquad (N \to \infty). \tag{4.118}$$

Hence it follows that

$$\sum_{n=1}^{\infty} n|C_F(n)| < \infty.$$

Next we set $s = \frac{\delta}{4+3\delta}$. We notice that

$$\frac{3}{s} + \frac{3}{2}\frac{4(2+\delta)}{\delta} = \frac{24+15\delta}{\delta} < \theta v \tag{4.119}$$

holds from the assumption on v. Thus we can choose γ so that

$$\frac{2(2+\delta)(1-s)}{\delta - 2s(1+\delta)} = \frac{4(2+\delta)}{\delta} < \gamma \tag{4.120}$$

and

$$\theta v > \frac{3}{s} + \frac{3}{2}\gamma \tag{4.121}$$

hold. Then

$$s < \min\left\{\frac{\delta}{2+2\delta}, 3\right\}$$

and

$$\gamma > \frac{4+2\delta}{\delta} = \frac{1-s}{s}$$

are valid. Now, for any positive integer N we set $\mathscr{A}^{(N)} = \mathscr{A}^{(N,s)}$. Then we obtain

$$\beta_{\mathscr{A}^{(N)}}(N, [N^s]) = O(N^{-s\gamma}) \qquad (N \to \infty)$$

since

$$\beta_{\mathscr{A}^{(N)}}(N, [N^s]) \leqq C_2 \lambda_2^{N^{\theta s}}$$

holds.

On the other hand we have

$$\begin{aligned}\|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_2 &\leqq \mathscr{H}_F(C_1 \lambda_1^{N^{s\theta}}) \\ &= O\left(\frac{1}{|\log C_1 + N^{s\theta} \log \lambda_1|^v}\right) \qquad (N \to \infty) \\ &= O(N^{-s\theta v}) \qquad (N \to \infty).\end{aligned}$$

Therefore, since $s\theta v > 3 + \frac{3}{2}\gamma s > \left(\frac{5}{2} + \frac{1}{2}\right) + \left(1 + \frac{1}{2}\right)s\gamma$ holds from the choice of γ, (2.12) is valid with $\tau = s\theta v$. Consequently all assumptions of Theorem 2.4 are satisfied. ■

Proof of Corollary 2.3. We can show that

$$\sum_{n=1}^{\infty} n|C_F(n)| < \infty$$

by the same way as in the proof of Corollary 2.2. We take arbitrary λ with $\lambda < \frac{\delta}{8+6\delta}$ and choose s such that $0 < s < \frac{\delta}{2+2\delta}$ and

$$\lambda < \frac{\delta - (2 + 2\delta)s}{8 + 6\delta} \tag{4.122}$$

hold. Now we set $\alpha = \frac{\delta + s(2+\delta)}{(2-s)(2+\delta)}$. Then we have $\alpha > \frac{s}{1-s}$ and

$$\frac{\delta}{2+\delta} - \alpha = \frac{(2-s)\delta - \delta - s(2+\delta)}{(2-s)(2+\delta)} = \frac{\delta - (2+2\delta)s}{(2-s)(2+\delta)} > 0. \tag{4.123}$$

From the last equality we obtain

$$\alpha < \frac{\delta}{2+\delta} - \frac{2}{\gamma} \tag{4.124}$$

for γ large enough. Therefore, from the assumption that $0 < \delta \leqq 2$, (4.24) in the proof of Theorem 2.1 is satisfied.

Next, we have

$$\frac{\alpha(1-s) - s}{2(1+\alpha)} = \frac{(\delta + s(2+\delta))(1-s) - s(2-s)(2+\delta)}{8 + 6\delta} = \frac{\delta - (2+2\delta)s}{8+6\delta} \tag{4.125}$$

$$\frac{\delta - (2+\delta)\left(\alpha + \frac{2}{\gamma}\right)}{2(2+\delta)(1+\alpha)} = \frac{\delta - s(2+2\delta)}{8+6\delta} - \frac{1}{\gamma(1+\alpha)} \tag{4.126}$$

and

$$\frac{1-\alpha}{4(1+\alpha)} > \frac{1-\alpha}{2(1+\alpha)} - \frac{1}{4(1+\alpha)} \geqq \frac{1-\alpha}{2(1+\alpha)} - \frac{2}{2(2+\delta)(1+\alpha)}$$
$$= \frac{\delta - \alpha(2+\delta)}{2(2+\delta)(1+\alpha)} \tag{4.127}$$

from the assumption that $0 < \delta \leqq 2$. Thus, noting (4.122) we conclude that for sufficiently large γ

$$\lambda < \min\left\{\frac{\delta - (2+\delta)\left(\alpha + \frac{2}{\gamma}\right)}{2(2+\delta)(1+\alpha)}, \frac{1-\alpha}{4(1+\alpha)}, \frac{\alpha(1-s)-s}{2(1+\alpha)}, \frac{s\gamma\delta}{4(2+\delta)} - \frac{\alpha}{4(1+\alpha)}\right\} \tag{4.128}$$

and $\gamma > \max\left\{\frac{2(2+\delta)(1-s)}{\delta-(2+2\delta)s}, \frac{1-s}{s}\right\}$ hold. Now we choose v so that

$$s\theta v > 3 + \frac{3}{2}\gamma s + \lambda = \left(\frac{5}{2} + \frac{1}{2}\right) + \left(1 + \frac{1}{2}\right)s\gamma + \lambda, \tag{4.129}$$

and set $\mathscr{A}^{(N)} = \mathscr{A}^{(N,s)}$. Then we have

$$\beta_{\mathscr{A}^{(N)}}(N, [N^s]) \leqq C_2 \lambda_2^{N^{\theta s}} = O(N^{-s\gamma}) \qquad (N \to \infty) \tag{4.130}$$

and

$$\|F - E[F \mid \sigma(\mathscr{A}^{(N)})]\|_2 = O(N^{-s\theta v}) \qquad (N \to \infty). \tag{4.131}$$

Therefore all assumptions are satisfied with $p = 2$, $\tau = s\theta v$ and $\rho = 0$. As we saw in the proof of Theorem 2.1, the almost sure invariance principle holds for F with any λ satisfying (4.26). On the other hand, by (4.128) and (4.129), any λ with $\lambda < \frac{\delta}{8+6\delta}$ satisfies (4.26) for the constants s, α, γ, τ, and ρ, p chosen above. Consequently the corollary has been proven. ■

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