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Lindelöf theorems for monotone Sobolev functions on uniform domains

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ABSTRACT. This paper deals with Lindelöf type theorems for monotone Sobolev functions on a uniform domain.

1. Introduction

A continuous function u on an open set D in the n-dimensional Euclidean space $\mathbf{R}^n$, $n \geq 2$, is called monotone in the sense of Lebesgue (see [6]) if the equalities

$$\max_{\bar{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\bar{G}} u = \min_{\partial G} u$$

hold whenever G is a domain with compact closure $\bar{G} \subset D$. If u is a monotone Sobolev function on D and $p > n - 1$, then

$$|u(x) - u(x')| \leq C(n,p) r^{1-n/p} \left(\int_{B(z,r)} |\nabla u(y)|^p dy \right)^{1/p} \tag{1.1}$$

whenever $x, x' \in B(z, r/2)$ with $B(z,r) \subset D$, where $B(z,r)$ is the open ball centered at z with radius r and $C(n,p)$ is a positive constant depending only on n and p (see [11, Chapter 8] and [13, Section 16]). Using this inequality (1.1), we proved Lindelöf theorems for monotone Sobolev functions on the half space of $\mathbf{R}^n$ in [1]. For related results, see Koskela-Manfredi-Villamor [5], Manfredi-Villamor [7, 8] and Mizuta [10]. In this paper we will generalize this result to a uniform domain in a metric space.

Let X be a metric space with a metric d and μ be a Borel measure on X which is positive and finite on balls. We denote by $B(x,r)$ the open ball centered at $x \in X$ with radius $r > 0$ and set $\lambda B = B(x, \lambda r)$ for each ball $B = B(x,r)$ and $\lambda > 0$. A domain D in X with $\partial D \neq \emptyset$ is a uniform domain if there exists a constant $A \geq 1$ such that each pair of points $x, y \in D$ can be joined by a curve γ in D for which

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$$\ell(\gamma) \le Ad(x, y), \tag{1.2}$$

$$\delta_D(z) \ge A^{-1} \min\{\ell(\gamma(x,z)), \ell(\gamma(y,z))\} \qquad \text{for all } z \in \gamma, \tag{1.3}$$

where $\ell(\gamma), \delta_D(z)$ and $\gamma(x,z)$ denote the length of γ, the distance from z to ∂D and the subarc of γ connecting x and z, respectively (see [9] and [12]). Here a curve means simple curve.

Our first aim in this paper is to deal with Lindelöf type theorems for functions u on a uniform domain D for which there exist a nonnegative Borel function $g \in L^p_{loc}(D;\mu)$, $p > 1$, constants $M > 0$ and $0 < \lambda \le 1$ such that

$$|u(x) - u(x')| \le Mr\left(⨍_{B(z,r)} g(y)^p d\mu(y)\right)^{1/p} \tag{1.4}$$

whenever $x, x' \in B(z, \lambda r)$ with $B(z,r) \subset D$ and

$$\int_D g(y)^p \delta_D(y)^\alpha d\mu(y) < \infty \tag{1.5}$$

for some real number α. Here we used the standard notation

$$u_F = ⨍_F u\, d\mu = \frac{1}{\mu(F)} \int_F u\, d\mu$$

for a measurable set F with $0 < \mu(F) < \infty$. For this purpose we assume that there exists a constant $C_1 \ge 1$ such that

$$\mu(2B) \le C_1 \mu(B) \tag{1.6}$$

for all balls B. We further assume that there exist constants $Q \ge 1$ and $C_2 > 0$ such that

$$\frac{\mu(B)}{\mu(B_0)} \ge C_2 \left(\frac{\operatorname{diam} B}{\operatorname{diam} B_0}\right)^Q \tag{1.7}$$

for all balls B and B_0 with $B \subset B_0$. For $\xi \in \partial D$ and $a > 1$, consider the set

$$\Gamma_D(\xi; a) = \{x \in D : d(x, \xi) < a\delta_D(x)\}.$$

A function u defined on D is said to have a nontangential limit L at $\xi \in \partial D$ if for every $a > 1$, $\lim_{z \to \xi, z \in \Gamma_D(\xi;a)} u(z) = L$. The main result of this paper is the following theorem.

Theorem 1. *Let D be a uniform domain in X. Let u be a function on D with $g \ge 0$ satisfying* (1.4) *and* (1.5). *Suppose $p > Q + \alpha - 1$ and set*

$$E = \left\{\xi \in \partial D : \limsup_{r \to 0} \frac{r^{p-\alpha}}{\mu(B(\xi, r))} \int_{B(\xi,r) \cap D} g(y)^p \delta_D(y)^\alpha d\mu(y) > 0\right\}.$$

If $\xi \in \partial D \setminus E$ and there exists a curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

Remark 1. If g satisfies (1.5), then $\mathscr{H}^{Q+\alpha-p}(E) = 0$, where $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure.

Corollary 1. *Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying*

$$\int_D |\nabla u(y)|^p \delta_D(y)^\alpha dy < \infty,$$

where $p > \max\{n-1, n-1+\alpha\}$. Set

$$E' = \left\{ \xi \in \partial D : \limsup_{r \to 0} r^{p-\alpha-n} \int_{B(\xi,r)\cap D} |\nabla u(y)|^p \delta_D(y)^\alpha dy > 0 \right\}.$$

If $\xi \in \partial D \setminus E'$ and there exists a curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

2. Proof of Theorem 1

Throughout this paper, let M denote various constants independent of the variables in question.

For a proof of Theorem 1, we need the following Lemmas.

Lemma 1. *Let D be a uniform domain. Then, for each $\xi \in \partial D$, there exists a curve γ_ξ in D ending at ξ such that*

$$\delta_D(z) \geq A_1^{-1} \ell(\gamma_\xi(\xi, z)) \qquad \text{for all } z \in \gamma_\xi, \tag{2.1}$$

where $A_1 = 2^5 A^3$.

Proof. Fix $\xi \in \partial D$. For each j sufficiently large (say $j \geq j_0$), take a point $w_j \in D \cap \partial B(\xi, 2^{-j})$. Further, take a curve γ_j in D joining w_{j-1} and w_{j+1} satisfying (1.2) and (1.3), and take a point $z_j \in \gamma_j \cap \partial B(\xi, 2^{-j})$. Since $\ell(\gamma_j(w_{j+1}, z_j)) \geq 2^{-j-1}$ and $\ell(\gamma_j(w_{j-1}, z_j)) \geq 2^{-j}$, we have by (1.3)

$$\delta_D(z_j) \geq A^{-1} 2^{-j-1}.$$

Let $\hat{\gamma}_j$ be a curve in D joining z_j and z_{j+1} satisfying (1.2) and (1.3). Then $\ell(\hat{\gamma}_j) \leq A 2^{-j+1}$ and $\delta_D(z) \geq A^{-2} 2^{-j-3}$ for all $z \in \hat{\gamma}_j$. Set

$$\hat{\gamma}_\xi = \hat{\gamma}_{j_0} + \hat{\gamma}_{j_0+1} + \hat{\gamma}_{j_0+2} + \cdots$$

Then it is not difficult to construct a simple curve γ_ξ from $\hat{\gamma}_\xi$ satisfying (2.1) with $A_1 = 2^5 A^3$. □

For each $\tau \in \mathbf{R}$, consider the function

$$\kappa_\tau(r_1, r_2) = \left(\int_{r_1}^{r_2} t^{(1-\tau-Q)/(p-1)}\, dt\right)^{1-1/p}$$

for $0 \le r_1 < r_2$.

Lemma 2 (cf. [2, Lemma 3]). *Let u be a function on D with $g \ge 0$ satisfying (1.4) and $\tau \in \mathbf{R}$. Then*

$$|u(x) - u(y)| \le M\kappa_\tau(\delta_D(x), 8A \max_{z \in \gamma} \delta_D(z)) \left(\frac{r^Q}{\mu(B(w,r))} \int_{\mathscr{B}(\gamma)} g(z)^p \delta_D(z)^\tau d\mu(z)\right)^{1/p}$$

whenever x and y can be joined by a rectifiable curve γ in D such that

$$\delta_D(z) \ge A^{-1}\ell(\gamma(x,z)) \qquad \text{for all } z \in \gamma \tag{2.2}$$

and $\mathscr{B}(\gamma) = \bigcup_{z \in \gamma} B(z, \delta_D(z)/2) \subset B(w,r)$, where M is a positive constant independent of x, y, γ and $B(w,r)$.

Proof. Let γ be a curve in D joining x and y satisfying (2.2) and $\mathscr{B}(\gamma) \subset B(w,r)$. We can take a finite chain of balls $B_0, B_1, \dots, B_N$ with the following properties:

(i) $B_j = B(z_j, \delta_D(z_j)/2)$ with $z_j \in \gamma$, $z_0 = x$ and $y \in \lambda B_N$;
(ii) $\lambda B_j \cap \lambda B_{j+1} \ne \varnothing$ for all $0 \le j < N$;
(iii) For small $t > 0$, the number of z_j such that $t < \delta_D(z_j) \le 2t$ is bounded by $(2A + \lambda)/\lambda$;
(iv) $\sum_j \chi_{B_j} \le C_3$, where C_3 is a positive constant depending only on C_1 and λ;

see [1, Proof of Theorem 1] and [2, Lemma 2.2].

Pick $x_{j+1} \in \lambda B_j \cap \lambda B_{j+1}$ for $0 \le j < N$: set $x_0 = x$ and $x_{N+1} = y$. By (1.4), we see that

$$|u(x_j) - u(x_{j+1})| \le M\delta_D(z_j)\left(⨍_{B_j} g(z)^p d\mu(z)\right)^{1/p}$$

for $0 \le j \le N$. Then we have by (1.7), Hölder's inequality and (iv)

$$\begin{aligned}
&|u(x) - u(y)| \\
&\le M\mu(B(w,r))^{-1/p} \sum_{j=0}^{N} \delta_D(z_j)^{1-\tau/p} \left(\frac{\mu(B(w,r))}{\mu(B_j)}\right)^{1/p} \left(\int_{B_j} g(z)^p \delta_D(z)^\tau d\mu(z)\right)^{1/p} \\
&\le M\mu(B(w,r))^{-1/p} \sum_{j=0}^{N} \delta_D(z_j)^{1-\tau/p} \left(\frac{r}{\delta_D(z_j)}\right)^{Q/p} \left(\int_{B_j} g(z)^p \delta_D(z)^\tau d\mu(z)\right)^{1/p} \\
&\le Mr^{Q/p}\mu(B(w,r))^{-1/p} \left(\sum_{j=0}^{N} \delta_D(z_j)^{(p-\tau-Q)/(p-1)}\right)^{1-1/p} \left(\int_{\mathscr{B}(\gamma)} g(z)^p \delta_D(z)^\tau d\mu(z)\right)^{1/p}.
\end{aligned}$$

Further, since $(2A)^{-1}\delta_D(x) \le \delta_D(z_j) \le \max_{z\in\gamma}\delta_D(z)$, we see from (iii) that

$$\sum_{j=0}^{N}\delta_D(z_j)^{(p-\tau-Q)/(p-1)} \le M\left(\kappa_\tau\left(\delta_D(x), 8A \max_{z\in\gamma}\delta_D(z)\right)\right)^{p/(p-1)}.$$

Thus the proof is completed. □

A sequence $\{x_j\}$ is called regular at ξ if $x_j \to \xi$ and

$$d(\xi, x_{j+1}) \le d(\xi, x_j) \le cd(\xi, x_{j+1})$$

for some constant $c > 1$.

Lemma 3 (cf. [1, Lemma 1]). *Let u, g, D and E be as in Theorem 1. Suppose there exists a regular sequence $\{x_j\}$ at $\xi \in \partial D\setminus E$ such that $x_j \in \gamma_\xi$ and $\lim_{j\to\infty} u(x_j) = L$, where γ_ξ is as in Lemma 1. Then u has a nontangential limit L at ξ.*

Proof. Set $r_j = d(\xi, x_j)$. Since $\{x_j\}$ is regular at ξ, there exists a constant $c > 1$ such that $r_{j+1} \le r_j \le cr_{j+1}$. Fix $x \in \Gamma_D(\xi; a) \cap B(\xi, r_1)$. Then there exists an integer j such that $r_j \le d(\xi, x) < r_{j-1}$. Let γ be a curve in D joining x and x_j with (1.2) and (1.3), and take $y \in \gamma$ such that $\ell(\gamma(x, y)) = \ell(\gamma(x_j, y))$; Set $\gamma_1 = \gamma(x, y)$ and $\gamma_2 = \gamma(x_j, y)$. Then γ_i satisfies (2.2) for $i = 1, 2$ and $d(\xi, z) \le c_1 r_j$ for all $z \in \gamma$, where $c_1 = (c+1)A + 1$. Since $\delta_D(x) \ge a^{-1}r_j$, $\delta_D(x_j) \ge A_1^{-1}r_j$ and $\mathscr{B}(\gamma_i) \subset B(\xi, 2c_1 r_j) \cap D$, we see from Lemma 2 with $\tau = \alpha$ that

$$\begin{aligned}
|u(x) - u(x_j)| &\le |u(x) - u(y)| + |u(y) - u(x_j)| \\
&\le M\kappa_\alpha(a^{-1}r_j, 8Ac_1 r_j)\left(\frac{(2c_1 r_j)^Q}{\mu(B(\xi, 2c_1 r_j))}\int_{\mathscr{B}(\gamma_1)} g(z)^p \delta_D(z)^\alpha d\mu(z)\right)^{1/p} \\
&\quad + M\kappa_\alpha(A_1^{-1}r_j, 8Ac_1 r_j)\left(\frac{(2c_1 r_j)^Q}{\mu(B(\xi, 2c_1 r_j))}\int_{\mathscr{B}(\gamma_2)} g(z)^p \delta_D(z)^\alpha d\mu(z)\right)^{1/p} \\
&\le M\left(\frac{r_j^{p-\alpha}}{\mu(B(\xi, 2c_1 r_j))}\int_{B(\xi, 2c_1 r_j)\cap D} g(z)^p \delta_D(z)^\alpha d\mu(z)\right)^{1/p}.
\end{aligned}$$

Since $\xi \notin E$, this implies that u has a nontangential limit L at ξ. □

Now we can prove Theorem 1.

Proof of Theorem 1. Suppose $u(z)$ tends to L as $z \to \xi$ along γ. Let γ_ξ be as in Lemma 1. For $r > 0$ sufficiently small, take $x_1(r) \in \gamma \cap \partial B(\xi, r)$ and

$x_2(r) \in \gamma_\xi \cap \partial B(\xi, r)$. Then $x_1(r)$ and $x_2(r)$ can be connected by a curve γ_0 in D with (1.2) and (1.3). Set $\gamma_1 = \gamma_0(x_1(r), y(r))$ and $\gamma_2 = \gamma_0(x_2(r), y(r))$ with a point $y(r) \in \gamma_0$ such that $\ell(\gamma_1) = \ell(\gamma_2)$. Then

$$\delta_D(z) \geq A^{-1}\ell(\gamma_i(x_i(r), z)) \qquad \text{for all } z \in \gamma_i, \ i = 1, 2.$$

Note that $\delta_D(x_2(r)) \geq A_1^{-1} r$, $d(\xi, z) \leq c_2 r$ for all $z \in \gamma_0$ and

$$|r - d(\xi, z)| \leq d(z, x_1(r)) \leq c_2 \delta_D(z)$$

for all $z \in \mathscr{B}(\gamma_1)$, where $c_2 = 2A + 1$. By Lemma 2 with $\tau = \alpha$, we see that

$$\begin{aligned}
|u(x_2(r)) - u(y(r))| &\leq M\kappa_\alpha(A_1^{-1}r, 8Ac_2 r)\left(\frac{(2c_2 r)^Q}{\mu(B(\xi, 2c_2 r))}\int_{\mathscr{B}(\gamma_2)} g(z)^p \delta_D(z)^\alpha d\mu\right)^{1/p} \\
&\leq M\left(\frac{r^{p-\alpha}}{\mu(B(\xi, 2c_2 r))}\int_{B(\xi, 2c_2 r)\cap D} g(z)^p \delta_D(z)^\alpha d\mu\right)^{1/p}.
\end{aligned}$$

Since $p > Q + \alpha - 1$ by our assumption, there exists $\beta > 0$ such that $Q + \alpha - p < \beta < 1$. We have by Lemma 2 with $\tau = \alpha - \beta$

$$\begin{aligned}
&|u(x_1(r)) - u(y(r))| \\
&\quad \leq M\kappa_{\alpha-\beta}(0, 8Ac_2 r)\left(\frac{(2c_2 r)^Q}{\mu(B(\xi, 2c_2 r))}\int_{\mathscr{B}(\gamma_1)} g(z)^p \delta_D(z)^{\alpha-\beta} d\mu(z)\right)^{1/p} \\
&\quad \leq M\left(\frac{r^{p-\alpha+\beta}}{\mu(B(\xi, 2c_2 r))}\int_{B(\xi, c_2 r)\cap D} g(z)^p \delta_D(z)^\alpha |r - d(\xi, z)|^{-\beta} d\mu(z)\right)^{1/p}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
(2.3) \quad &|u(x_1(r)) - u(x_2(r))|^p \\
&\quad \leq M\frac{r^{p-\alpha+\beta}}{\mu(B(\xi, 2c_2 r))}\int_{B(\xi, 2c_2 r)\cap D} g(z)^p \delta_D(z)^\alpha |r - d(\xi, z)|^{-\beta} d\mu(z).
\end{aligned}$$

Moreover, since $0 < \beta < 1$, we see that

$$\int_{2^{-j-1}}^{2^{-j}} |r - d(\xi, z)|^{-\beta} dr \leq M 2^{-j(1-\beta)}. \tag{2.4}$$

Hence it follows from (2.3) and (2.4) that

$$\begin{aligned}
&\inf_{2^{-j-1} \le r \le 2^{-j}} |u(x_1(r)) - u(x_2(r))|^p \\
&\le M \int_{2^{-j-1}}^{2^{-j}} \left(\frac{r^{p-\alpha+\beta}}{\mu(B(\xi, 2c_2 r))} \int_{B(\xi, 2c_2 r) \cap D} g(z)^p \delta_D(z)^\alpha |r - d(\xi, z)|^{-\beta} d\mu(z) \right) \frac{dr}{r} \\
&\le M \frac{2^{-j(p-\alpha+\beta-1)}}{\mu(B(\xi, c_2 2^{-j}))} \int_{B(\xi, c_2 2^{-j+1}) \cap D} g(z)^p \delta_D(z)^\alpha \left(\int_{2^{-j-1}}^{2^{-j}} |r - d(\xi, z)|^{-\beta} dr \right) d\mu(z) \\
&\le M \frac{2^{-j(p-\alpha)}}{\mu(B(\xi, c_2 2^{-j}))} \int_{B(\xi, c_2 2^{-j+1}) \cap D} g(z)^p \delta_D(z)^\alpha d\mu(z).
\end{aligned}$$

Since $\xi \notin E$, we can find a sequence $\{r_j\}$ such that $2^{-j-1} < r_j \le 2^{-j}$ and

$$\lim_{j \to \infty} u(x_2(r_j)) = L.$$

Thus u has a nontangential limit L at ξ by Lemma 3. □

3. A_q weights

Let w be a Muckenhoupt A_q weight, that is, a nonnegative measurable functions on $\mathbf{R}^n$ satisfying

$$\sup \left(⨍_B w(x)dx \right) \left(⨍_B w(x)^{1/(1-q)} dx \right)^{q-1} < \infty, \tag{3.1}$$

where the supremum is taken over all balls B in $\mathbf{R}^n$ (see [4]). Let u be a monotone function on a uniform domain D in $\mathbf{R}^n$ in the sense of Lebesgue which satisfies

$$\int_D |\nabla u(x)|^p w(x) dx < \infty. \tag{3.2}$$

Suppose $1 \le q < p/(n-1)$. Since $p_1 = p/q > n-1$, then

$$|u(x) - u(x')| \le M r^{1-p_1/n} \left(\int_{B(z,r)} |\nabla u(y)|^{p_1} dy \right)^{1/p_1}$$

whenever $x, x' \in B(z, r/2)$ with $B(z, r) \subset D$.

Hence we derive the following extension of a result by Manfredi-Villamor [8] to a uniform domain (see also [1]).

COROLLARY 2. *Let $1 \le q < p/(n-1)$ and w be a Muckenhoupt A_q weight. Suppose u is a monotone function on a uniform domain D in $\mathbf{R}^n$ satisfying* (3.2). *Set*

$$E_1 = \left\{ \xi \in \partial D : \limsup_{r \to 0} \frac{r^p}{w(B(\xi, r))} \int_{B(\xi, r) \cap D} |\nabla u(y)|^p w(y) dy > 0 \right\},$$

where $w(B(\xi, r)) = \int_{B(\xi, r)} w(y) dy$. *If* $\xi \in \partial D \setminus E_1$ *and there exists a curve* γ *in* D *tending to* ξ *along which* u *has a finite limit* L, *then* u *has a nontangential limit* L *at* ξ.

Proof. Set

$$E_2 = \left\{ \xi \in \partial D : \limsup_{r \to 0} r^{p_1 - n} \int_{B(\xi, r) \cap D} |\nabla u(y)|^{p_1} dy > 0 \right\},$$

where $p_1 = p/q$. Using Hölder inequality and (3.1), we see that $E_2 \subset E_1$. Thus Corollary 2 follows from Theorem 1 with p and μ replaced by p_1 and the n-dimensional Lebesgue measure. □

4. Generalizations of Lindelöf theorems

In this section, we give a generalization of Theorem 1 in case $X = \mathbf{R}^n$. Let m be an integer such that $1 \le m < n$. We say that Γ is an m-approach set at ξ with $\lambda_1 > 1$ and $\lambda_2 > 0$, if there exist a sequence of positive numbers $\{r_j\}$ tending to zero and a sequence of contraction maps P_j from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that $r_{j+1} \le r_j \le \lambda_1 r_{j+1}$ and

$$\mathscr{H}^m(P_j(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1})))) \ge \lambda_2 r_j^m. \tag{4.1}$$

Theorem 2. *Let* D *be a uniform domain in* $\mathbf{R}^n$. *Let* u *be a function on* D *with* $g \ge 0$ *satisfying* (1.4) *and* (1.5). *Suppose* $p > Q + \alpha - m$ *and set*

$$E = \left\{ \xi \in \partial D : \limsup_{r \to 0} \frac{r^{p-\alpha}}{\mu(B(\xi, r))} \int_{B(\xi, r) \cap D} g(y)^p \delta_D(y)^\alpha d\mu(y) > 0 \right\}.$$

If $\xi \in \partial D \setminus E$ *and there exists an* m*-approach set* $\Gamma \subset D$ *at* ξ *along which* u *has a finite limit* L *at* ξ, *then* u *has a nontangential limit* L *at* ξ.

Proof. Let r_j, P_j, λ_1 and λ_2 be retained from the definition of m-approach set Γ at ξ, and set

$$G_j = \Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1})).$$

For $\omega \in P_j(G_j)$, take $x_1(\omega) \in G_j$ and set $r = |\xi - x_1(\omega)|$. Let γ_ξ be as in Lemma 1 and take $x_2(\omega) \in \gamma_\xi \cap \partial B(\xi, r)$. By our assumption, we can take $\beta > 0$ such that $Q + \alpha - p < \beta < m$. Since $|P_j(z) - \omega| \le |z - x_1(\omega)|$, in view of the estimate (2.3) in the proof of Theorem 1, we obtain

$$|u(x_1(\omega)) - u(x_2(\omega))|^p \\ \le M \frac{r^{p-\alpha+\beta}}{\mu(B(\xi, 2c_2 r))} \int_{B(\xi, 2c_2 r)\cap D} g(z)^p \delta_D(z)^\alpha |P_j(z) - \omega|^{-\beta} d\mu(z).$$

Further, since $P_j(G_j) \subset B(P_j(\xi), r_j) (\subset \mathbf{R}^m)$ and $0 < \beta < m$, we see that

$$\int_{P_j(G_j)} |P_j(z) - \omega|^{-\beta} d\mathscr{H}^m(\omega) \le \int_{B(P_j(\xi), r_j)} |P_j(z) - \omega|^{-\beta} d\mathscr{H}^m(\omega) \le M r_j^{m-\beta}.$$

Hence we have by (4.1)

$$\inf_{\omega \in P_j(G_j)} |u(x_1(\omega)) - u(x_2(\omega))|^p \le M \frac{r_j^{p-\alpha}}{\mu(B(\xi, 2c_2\lambda_1^{-1} r_j))} \int_{B(\xi, 2c_2 r_j)\cap D} g(z)^p \delta_D(z)^\alpha d\mu(z).$$

From $\xi \notin E$, we can find a sequence $\{\omega_j\}$ such that $\omega_j \in P_j(G_j)$ and

$$\lim_{j\to\infty} u(x_2(\omega_j)) = L.$$

Since $\{x_2(\omega_j)\}$ is regular at ξ, we can show that u has a nontangential limit L at ξ by Lemma 3. □

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