

Stable maps between 2-spheres with a connected fold curve

Shin-ichi DEMOTO

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ABSTRACT. Stable maps between 2-dimensional spheres, whose fold curve is connected and its image is simple with minimal number of cusps, are classified for every degree $d \geq 2$.

1. Introduction

We deal with the following problem.

Let M and N be connected surfaces and $f : M \rightarrow N$ a smooth map. Then is there a map $h : M \rightarrow N$ which satisfies the following conditions?

1. *h is a stable map.*
2. *h is homotopic to f .*
3. *h has a connected fold curve.*
4. *The set of critical values of h has the smallest possible number of singular points.*

Furthermore, how many such maps are there?

By Pignoni [8] the form for the set of critical values of such a map h is determined when $N = \mathbf{R}^2$. In this paper we determine the form for the maps $f : S^2 \rightarrow S^2$ with $\deg f = d \geq 2$. More precisely, we show that the set of critical values of such a map has $2d$ cusps and no self-intersections. Furthermore we give the number of their right-left equivalence classes. All such stable maps are right-left equivalent in the case of $d = 2$, but not in the case of $d \geq 3$.

The paper is organized as follows. In §2 we define some notions, the *apparent contour*, the *irreducible contour* and so on. We quote a theorem of Quine [9] which will be used in §3 and §4. In §3 we study the form for the maps $f : S^2 \rightarrow S^2$ with $\deg f = 2$. Then we prove that all these maps are right-left equivalent by using the result of [2] and the argument of [3]. In §4 we generalize the argument to the case of $\deg f = d \geq 3$.

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2. Preliminaries

Let M and N be compact, oriented and connected surfaces without boundary. We denote by $C^\infty(M, N)$ the set of the smooth maps of M into N with the Whitney C^∞ topology. For a smooth map $f : M \rightarrow N$, S_f denotes the singular set of f , i.e., S_f is the set of points in M where the rank of the differential df is less than two. A smooth map $f : M \rightarrow N$ is *stable* if there exists an open neighborhood $N(f)$ of f in $C^\infty(M, N)$ such that every h in $N(f)$ is *right-left equivalent* to f , i.e., there exist diffeomorphisms $\phi : M \rightarrow M$ and $\psi : N \rightarrow N$ satisfying $h = \psi \circ f \circ \phi^{-1}$.

Let $f : M \rightarrow N$ be a stable map. Then for each point x in M , there exist local coordinates (x_1, x_2) centered at x and (y_1, y_2) centered at $f(x)$ such that f is given by one of the following local normal forms (see [10]):

- (1) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1, x_2)$ x : regular point,
- (2) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1^2, x_2)$ x : fold point,
- (3) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1 x_2 - x_1^3, x_2)$ x : cusp point.

In this case S_f is called a *fold curve* and is divided into $S_f = F_f \cup C_f$, where

$$F_f = \{p \in S_f \mid p \text{ is a fold point}\},$$

$$C_f = \{p \in S_f \mid p \text{ is a cusp point}\}.$$

A stable map between surfaces is also known as *Whitney's plane map* or *excellent map*. The following theorem is well-known.

THEOREM 2.1. *A smooth map $f : M \rightarrow N$ is stable if and only if the following four conditions are satisfied:*

1. $S_f = F_f \cup C_f$.
2. $f|_{F_f}$ is an immersion with normal crossings.
3. $f(F_f) \cap f(C_f) = \emptyset$.
4. If $p, q \in C_f$ and $p \neq q$ then $f(p) \neq f(q)$.

DEFINITION 2.2. The set of critical values of a stable map f forms a union of finite closed curves and is called the *apparent contour* of f .

Let γ be the apparent contour of a stable map $h : M \rightarrow N$. A point $q \in \gamma$ is said to be a *self-intersection* if there exist $p_\alpha \in S_h$ and $p_\beta \in S_h$ such that

$p_\alpha \neq p_\beta$ and $h(p_\alpha) = h(p_\beta) = q$. Furthermore $q \in \gamma$ is said to be a *cusps* if there exists $p \in C_h$ such that $h(p) = q$.

DEFINITION 2.3. If the fold curve S_h is connected, γ is called an *irreducible contour*.

Let $h : M \rightarrow N$ be a stable map which has an irreducible contour γ . As usual, we shall indicate with c the number of cusps, while n shall denote the number of self-intersections.

DEFINITION 2.4. Let f be a smooth map. By THEOREM 4.8 of [1], there exist stable maps which have irreducible contours and are homotopic to f . If h has the smallest possible value $c + n$ among such stable maps, then the irreducible contour of h will be called a *minimal contour* of f .

Let $f : M \rightarrow N$ be a stable map. Let M^+ be the closure of the set of regular points whose neighborhoods are orientation-preserved by f and M^- be the one whose neighborhoods are orientation-reversed by f . We notice that M^+ and M^- are compact surfaces with boundary and satisfy the following conditions,

$$\partial M^+ = \partial M^- = M^+ \cap M^- = S_f \quad \text{and} \quad M^+ \cup M^- = M.$$

Now we define the sign for any cusp q_k as in Figure 1:

$$\text{sign}(q_k) = \begin{cases} +1 & \text{(if } q_k \text{ is a positive cusp)} \\ -1 & \text{(if } q_k \text{ is a negative cusp).} \end{cases}$$

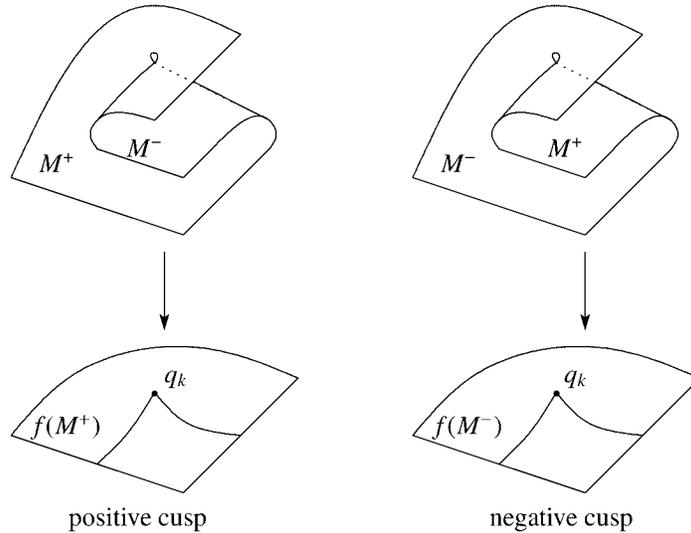


Fig. 1. Sign of a cusp

Then the following formula is proved by J. R. Quine.

THEOREM 2.5 ([9]).

$$\chi(M) - 2\chi(M^-) + \sum_{q_k \in f(C_f)} \text{sign}(q_k) = (\text{deg } f)\chi(N)$$

where χ is the Euler characteristic, deg is the topological degree and C_f is the set of cusp points.

Now, we assume that f is a stable map of a closed, oriented and connected surface M into the 2-sphere S^2 , and that S_f is non-empty and connected. Then THEOREM 2.5 becomes

$$\sum_{q_k \in f(C_f)} \text{sign}(q_k) - 2 \text{deg } f = 2\chi(M^-) - \chi(M).$$

Since M^+ and M^- are surfaces having the same circle as boundary by the assumption that S_f is non-empty and connected, we get

$$\chi(M^+) \leq 1 \quad \text{and} \quad \chi(M^-) \leq 1.$$

Furthermore, since $M^+ \cup M^- = M$, we get

$$\begin{aligned} \chi(M) &= \chi(M^+) + \chi(M^-) - \chi(S^1) \\ &= \chi(M^+) + \chi(M^-). \end{aligned}$$

Then

$$\begin{aligned} \chi(M^-) &= \chi(M) - \chi(M^+) \\ &\geq \chi(M) - 1. \end{aligned}$$

Now we have

$$\chi(M) - 1 \leq \chi(M^-) \leq 1.$$

Combining this with THEOREM 2.5, we get

$$\begin{aligned} \sum_{q_k \in f(C_f)} \text{sign}(q_k) - 2 \text{deg } f &\leq 2 \cdot 1 - \chi(M) \\ &= 2 - \chi(M) \end{aligned}$$

and

$$\begin{aligned} \sum_{q_k \in f(C_f)} \text{sign}(q_k) - 2 \deg f &\geq 2(\chi(M) - 1) - \chi(M) \\ &= \chi(M) - 2. \end{aligned}$$

So, we get the following corollary.

COROLLARY 2.6. *Let $f : M \rightarrow S^2$ be a stable map of a closed, oriented and connected surface M . If S_f is non-empty and connected, then*

$$\left| \sum_{q_k \in f(C_f)} \text{sign}(q_k) - 2 \deg f \right| \leq 2 - \chi(M).$$

When the target of a smooth map is S^2 , the following theorem is known. (For example, see [7, §7].)

THEOREM 2.7 (HOPF). *Let M be an oriented, connected, and closed surface and $f, g \in C^\infty(M, S^2)$. Then f and g are homotopic if and only if $\deg f = \deg g$.*

3. Minimal contour of the map between spheres with degree two

The purpose of this section is to investigate the minimal contour of a map $f : S^2 \rightarrow S^2$ with $\deg f = 2$.

THEOREM 3.1. *Let $f : S^2 \rightarrow S^2$ be a smooth map with $\deg f = 2$. The minimal contour of f has exactly four cusps and no self-intersections.*

PROOF. Let h be a stable map which is homotopic to f and has a connected fold curve. Since $S_h \neq \emptyset$, we have

$$\left| \sum_{q_k \in h(C_h)} \text{sign}(q_k) - 2 \cdot 2 \right| \leq 0$$

by **COROLLARY 2.6**. Then,

$$\sum_{q_k \in h(C_h)} \text{sign}(q_k) = 4.$$

So, the map which gives the minimal contour of f must have at least four positive cusps. Then if we can construct a stable map $h_0 : S^2 \rightarrow S^2$ with $\deg h_0 = 2$ whose apparent contour has exactly four cusps and no self-intersections, h_0 realizes the minimal contour.

We can construct such an h_0 as in Figure 2, where $+$ indicates the positive orientation of S^2 and $-$ the negative one.

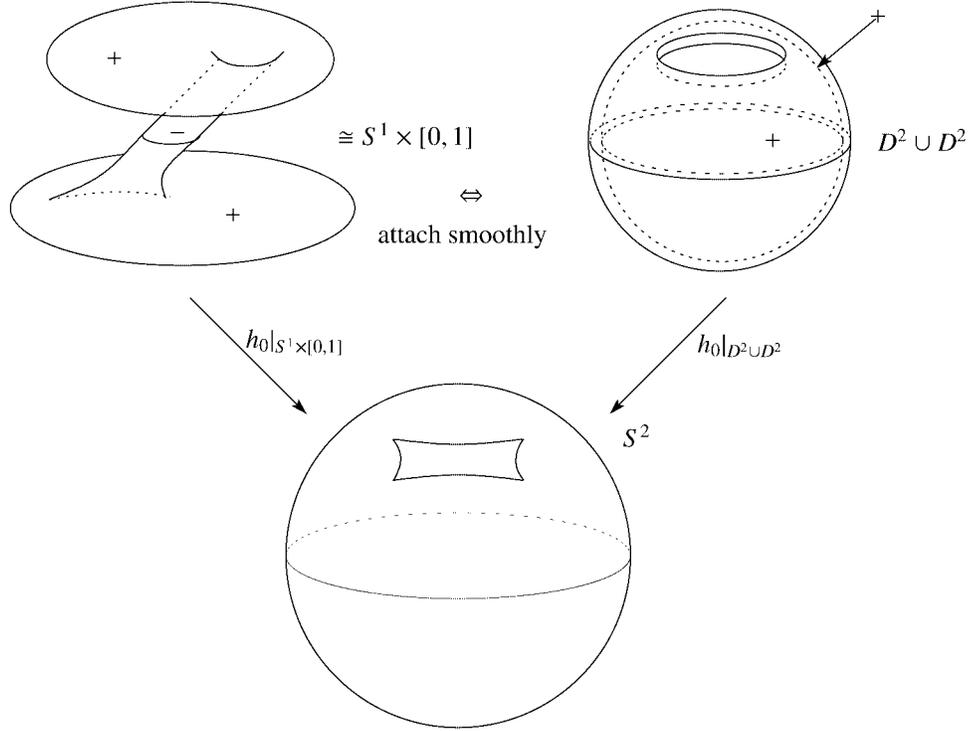


Fig. 2. Construction of h_0

□

Moreover, we have the following theorem.

THEOREM 3.2. *Let $f, h : S^2 \rightarrow S^2$ be two stable maps which realize the minimal contour. If $\deg f = \deg h = 2$, then f and h are right-left equivalent.*

To prove this theorem, we use the result of [2]. Since we do not use the argument of the branch point, we may apply the argument of [2] more easily.

Let g be an immersion with normal crossings of a finite set of oriented circles into $\mathbf{R}^2 = S^2 - \{\infty\}$. Such a g is called a *normal family of curves* and any double point $q \in \mathbf{R}^2$ is called a *node*. Note that the set of nodes on a normal family is a finite set.

Let g be an immersion of S^1 into \mathbf{R}^2 . We denote by $\tau(g)$ the degree of the following map:

$$S^1 \ni \theta \mapsto \frac{g'(\theta)}{\|g'(\theta)\|} \in S^1.$$

We shall call it the *turning number* of g . If the source of g is a union of finite

circles S_i^1 ($1 \leq i \leq n$), $\tau(g)$ is the sum of the turning numbers of the restricted maps $g|_{S_i^1}$.

Now let q be a node of a normal family g in \mathbf{R}^2 , and let g^q denote the normal family obtained from g by *cutting through* q . That is, we exchange the two exiting subarcs of g at q as in Figure 3. Note that $\tau(g)$ and $\tau(g^q)$ are equal to each other.

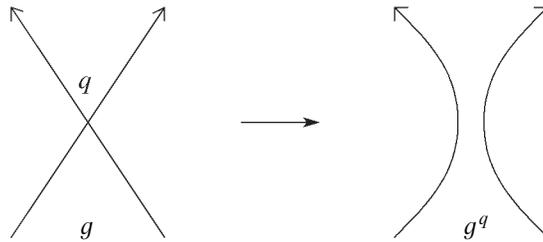


Fig. 3. Cutting through q

If every node of a normal family is cut through, the new curve family g^\oplus consists of a finite collection of mutually non-crossing closed oriented Jordan curves, and $\tau(g) = \tau(g^\oplus)$. We shall call these curves the *Gaussian circles* of g . They are also known as *Seifert circuits* in knot theory.

Let α be a finite family of *rays* on S^2 concurrent at the point ∞ . That is, α is the image of a finite set of closed radii of an embedding of a disk into S^2 , with center going to ∞ (see Figure 4).

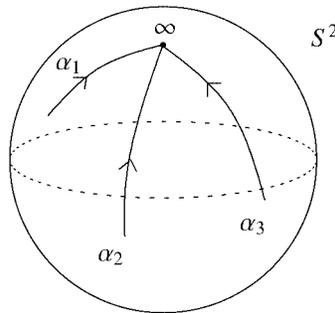


Fig. 4. A family of rays concurrent at ∞

Such an α is called a *raying* for a normal family g , provided that they lie in general position. That is, if x is a common point of g and α , then

- (1) $x \neq \infty$,
- (2) x is not a node of g ,

- (3) x is not the initial point of the ray through x ,
- (4) g intersects α transversely at x .

Any raying α of g is said to be *sufficient* if there is at least one crossing on each member curve of g , and at least one crossing on each Gaussian circle in g^\oplus which is negatively (= clockwise) oriented in the plane $S^2 - \{\infty\}$.

Let $X(g, \alpha)$ denote the set of crossings of g and α . For $x \in X(g, \alpha)$, α_x denotes the ray through x , and g_x the closed curve in the family g that passes through x . We say x is *positive* if g_x crosses from the right-hand side to the left-hand side of α_x oriented to ∞ at x , and we say x is *negative* otherwise (Figure 5).

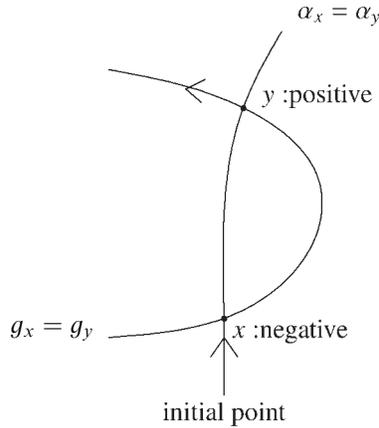


Fig. 5. A pair (x, y)

Among the permutations on $X = X(g, \alpha)$, by a *pair* on X we shall mean a transposition (x, y) that interchanges two crossings of opposite sign on the same ray with the condition that the negative crossing separates the positive crossing from the initial point (Figure 5).

Let S denote the *successor permutation* that takes each crossing x into the next succeeding crossing x^S on g_x in the orientation of g_x . (In the example of Figure 6, $S = (a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3)$, where a_1, a_2, a_3 are positive crossings and $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are corresponding negative ones. The succeeding crossings are denoted as $a_1^S = a_2, a_2^S = a_3$ and so on.)

We say a permutation P on X is a *pairing* if P is a product of disjoint pairs. (In Figure 6, $P = (a_1, \bar{a}_1)(a_2, \bar{a}_2)(a_3, \bar{a}_3)$ is such an example.) Let P_x denote the transposition which is a part of P including x . Let us call a pairing P on $X(g, \alpha)$ *effective* if it has exactly as many pairs as the negative crossings.

Let $P = P_1 P_2$ be the product of the two permutations P_1 and P_2 . That is,

$$\text{if } y = x^{P_1} \quad \text{and} \quad z = y^{P_2} \quad \text{then } z = x^P.$$

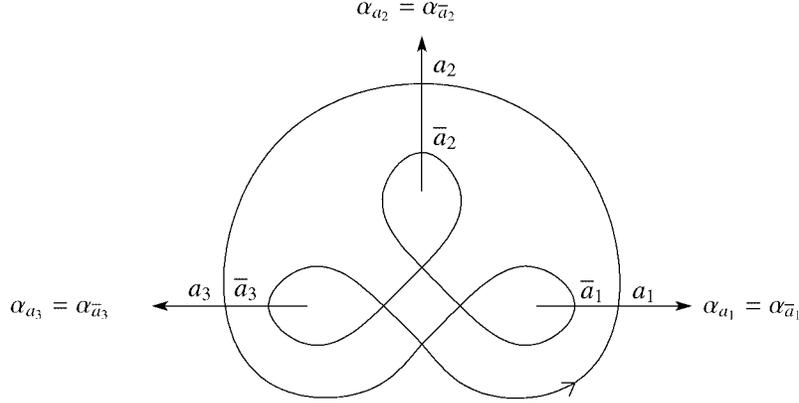


Fig. 6. An example

When we decompose S into disjoint cyclic permutations, each of them acts transitively on the crossings on some connected component of g . If the operation of the permutation P is transitive on the set of connected components of g , we say that P is *transitive* on S .

PROPOSITION 3.3 ([2, PROPOSITION 2]). *Let g be a normal family of closed curves, α a sufficient raying and*

- ρ : number of closed curves in g ,
- τ : turning number of g ,
- v : number of negative crossings in $X(g, \alpha)$,
- S : successor permutation,
- P : pairing which is effective and transitive on S ,
- ζ : number of the disjoint cyclic permutations in $R = SP$, and
- $\mu = v + \tau - \zeta$: expected number of the branched points.

If $\mu = 0$, then there exists an immersion $G_P : M_P \rightarrow \mathbf{R}^2 = S^2 - \{\infty\}$ of a compact, oriented and connected surface M_P of genus $(2 + v - \rho - \zeta)/2$ such that $G_P|_{\partial M_P} = g$, where the orientation of ∂M_P is induced from that of M_P .

REMARK. Let ζ' be the number of the disjoint cyclic permutations in $R' = PS$. Then, $\zeta' = \zeta$ because R and R' are conjugate. So, we will identify ζ' and ζ from now on.

Now we consider an immersion $G : M \rightarrow S^2 - \{\infty\}$ of a compact, oriented, connected, bordered surface M . Suppose that $g = G|_{\partial M}$ forms a normal family. We get a pairing P by the following construction. Let α be a sufficient raying for g . Note that for each negative crossing $\bar{a} \in X(g, \alpha)$, there exists a unique lift $\hat{\bar{a}} \in G^{-1}(\alpha) \cap \partial M$ of the point. The point $\hat{\bar{a}} \in G^{-1}(\alpha) \cap \partial M$ must terminate at a unique border point $\hat{a} \in \partial M$ for a positive crossing $a =$

$G(\hat{a})$. Thus G uniquely determines an effective pairing P such that $P_{\bar{a}} = (\bar{a}, a)$ for each negative crossing \bar{a} , when a sufficient raying α for g is given.

PROPOSITION 3.4 ([2, PROPOSITION 4]). *Let G be an immersion into $S^2 - \{\infty\}$ of a compact, oriented, connected, bordered surface M such that $g = G|_{\partial M}$ is a normal family of ρ closed curves with turning number τ . Let α be a sufficient raying for g with v negative crossings in $X(g, \alpha)$. Let S be the successor permutation induced by (g, α) and P the effective pairing induced by G . Then P is transitive on S and there is an orientation-preserving diffeomorphism $H : M_P \rightarrow M$ such that $G_P = G \circ H$, where M_P and the immersion $G_P : M_P \rightarrow \mathbf{R}^2 = S^2 - \{\infty\}$ are those of PROPOSITION 3.3. Moreover, $\mu = v + \tau - \zeta$ must be equal to 0.*

Now, let us prove THEOREM 3.2.

PROOF OF THEOREM 3.2. Let $f : S^2 \rightarrow S^2$ be a stable map with $\deg f = 2$ such that f has a connected fold curve, and that the apparent contour γ of f has exactly four positive cusps, but no negative cusps nor self-intersections. By the structure of cusps, the number of points in the preimage increases by $+2$ at each crossing as in Figure 7 and hence the apparent contour as in Figure 7 does not occur. So, the apparent contour of f must be as in Figure 8, where we take ∞ outside the star-like quadrilateral whose boundary curve is the apparent contour.

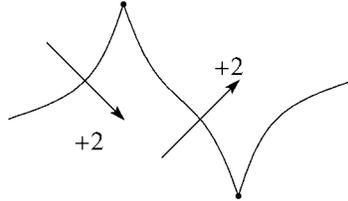


Fig. 7. Contradicting two cusps



Fig. 8. Apparent contour on $S^2 - \{\infty\}$

Now let U_∞ be a sufficiently small open neighborhood of ∞ such that $f(S_f) \cap \bar{U}_\infty = \emptyset$, where \bar{U}_∞ is the closure of U_∞ . Since $\deg f = 2$, $f^{-1}(U_\infty)$ consists of:

- Case (1) two open discs which are orientation-preserved by f , or
- Case (2) n open discs which are orientation-preserved by f and $n - 2$ open discs which are orientation-reversed by f ($n \geq 3$).

CASE (1): We divide the source sphere of f into some parts. Since $f^{-1}(U_\infty)$ consists of two open discs, f is divided as in Figure 9. Now we look at a neighborhood of the fold curve S_f and the corresponding neighborhood of the apparent contour γ . Since the four positive cusps are connected as in Figure 8, the behavior of f around these neighborhoods should be as in Figure 10. Dividing f in this way, we get four parts of f as in Figure 11, where the source of the second part from left is a surface with three border circles. In fact, the simple closed curve S_f in the annulus in the source sphere of Figure 9 is isotopic either to one as in Figure 11 or to one parallel to the boundary. But we see that the latter case does not occur by considering the orientation of the parts.

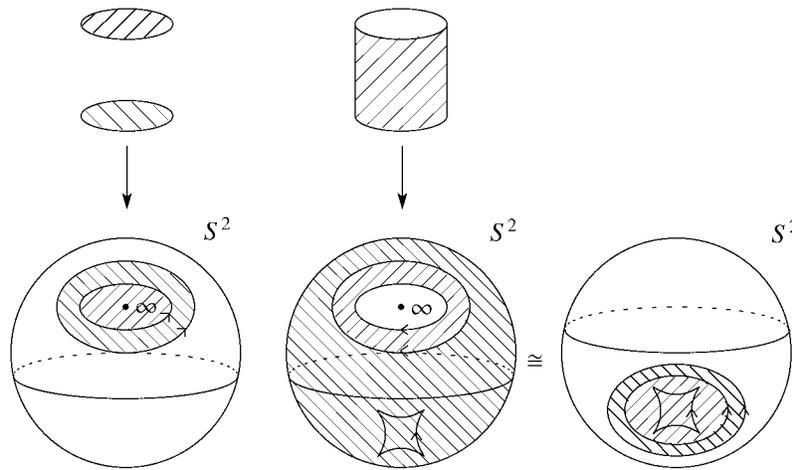


Fig. 9. The division of f

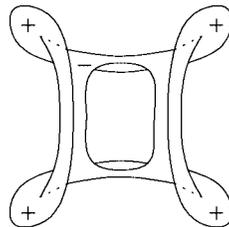


Fig. 10. The neighborhood of four cusps

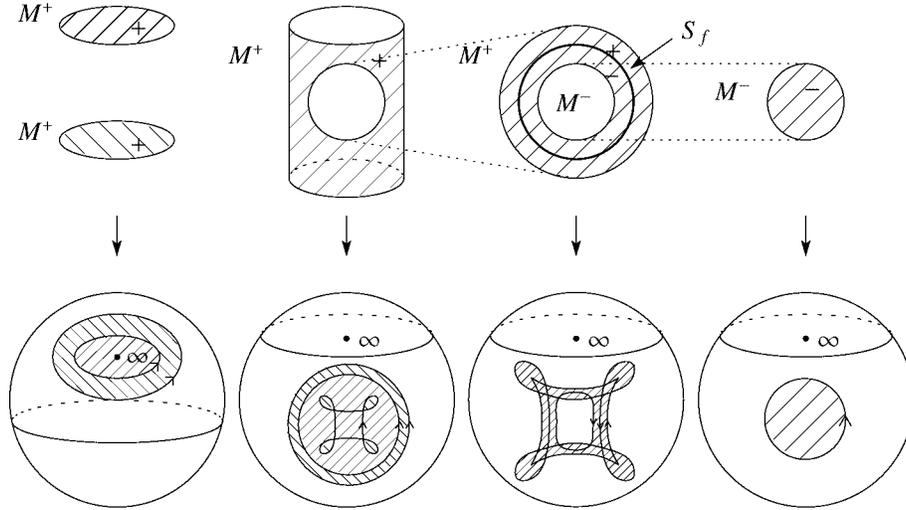


Fig. 11. The subdivision of f into four parts

Among the four parts of Figure 11, we have only to consider the second part from left because the other three parts are clearly understood. To apply PROPOSITION 3.3, let us regard the image of the boundary closed curves of this part as a normal family g . The orientation of g is given such that the number of the inverse points increases from right-hand side to left-hand side. Let α be a sufficient raying as in Figure 12 and define the crossing points,

$$X(g, \alpha) = \{a, b, c, d, a', b', c', d', \bar{a}, \bar{b}, \bar{c}, \bar{d}\}.$$

Note that $\bar{a}, \bar{b}, \bar{c}$ and \bar{d} are negative crossings and the other points are positive.

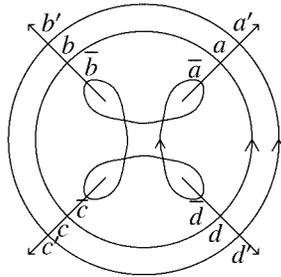


Fig. 12. g and α

The successor permutation is given as

$$S = (a, b, c, d)(a', b', c', d')(\bar{a}, \bar{b}, \bar{c}, \bar{d}).$$

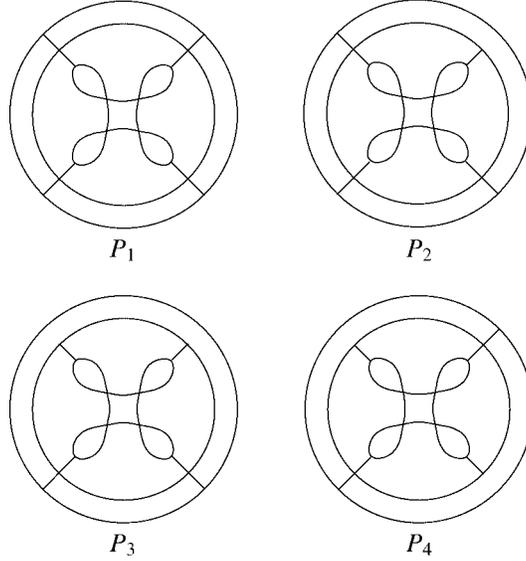


Fig. 13. Effective pairings P_1 , P_2 , P_3 , and P_4

The numbers in PROPOSITION 3.3 are given as

$$v = 4, \quad \rho = 3 \quad \text{and} \quad \tau = -1.$$

Since we can change g by diffeomorphisms of its source and target, we have only to consider the following four effective pairing types as in Figure 13:

$$P_1 = (a', \bar{a})(b', \bar{b})(c', \bar{c})(d', \bar{d}),$$

$$P_2 = (a, \bar{a})(b', \bar{b})(c', \bar{c})(d', \bar{d}),$$

$$P_3 = (a, \bar{a})(b, \bar{b})(c', \bar{c})(d', \bar{d}), \quad \text{and}$$

$$P_4 = (a', \bar{a})(b, \bar{b})(c', \bar{c})(d, \bar{d}).$$

a) Type 1 (P_1): The pairing P_1 is not transitive on S . So this type cannot be realized by an immersion of a connected surface by PROPOSITION 3.4.

b) Type 2 (P_2): Since $R' = P_2S = (a', b', \bar{c}, d', \bar{a}, b, c, d, a, \bar{b}, c', \bar{d})$, the number of disjoint cyclic permutations in R' is $\zeta' = 1$. Then

$$\mu = v + \tau - \zeta' = 4 - 1 - 1 = 2 \neq 0.$$

So, there is no immersion by PROPOSITION 3.4.

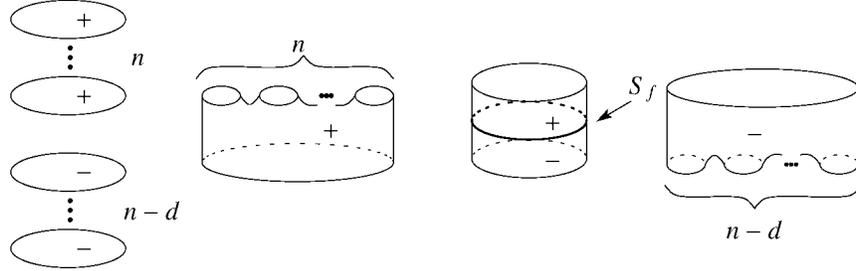


Fig. 14. The four parts

c) Type 3 (P_3): Since $R' = P_3S = (a', b', c', \bar{d})(a, \bar{b}, c, d)(\bar{a}, b, \bar{c}, d')$, we get $\zeta' = 3$. Then,

$$\mu = \nu + \tau - \zeta' = 4 - 1 - 3 = 0.$$

By PROPOSITIONS 3.3 and 3.4, we get a uniquely determined immersion $G : M_{P_3} \rightarrow \mathbf{R}^2$, where M_{P_3} is of genus $(2 + \nu - \rho - \zeta')/2 = 0$ and has $\rho = 3$ border circles so that M_{P_3} and the source of the second part in Figure 11 are diffeomorphic.

d) Type 4 (P_4): Since $R' = P_4S = (a', \bar{b}, c, d, \bar{a}, b', c', \bar{d}, a, b, \bar{c}, d')$, we get $\zeta' = 1$. Then $\mu = 2 \neq 0$. So there is no immersion by PROPOSITION 3.4 as for Type 2.

As a result, we get an immersion $G : M_{P_3} \rightarrow \mathbf{R}^2$ only for Type 3. Note that this immersion $G : M_{P_3} \rightarrow \mathbf{R}^2$ can be identified with the map $h_0|_{S^1 \times [0,1]}$ in Figure 2. In addition, by PROPOSITION 3.4 all such maps are right equivalent to each other, i.e., for any such map $G' : M_{P'_3} \rightarrow \mathbf{R}^2$, there exist an orientation-preserving diffeomorphism $\psi : M_{P'_3} \rightarrow M_{P_3}$ satisfying $G' = G \circ \psi$. Therefore, the original map $f : S^2 \rightarrow S^2$ is reconstructed by attaching smoothly and uniquely these four parts and we see that such maps are all right-left equivalent to each other.

CASE (2): We will exclude this case. We can divide the source 2-sphere of f in four parts. We see that the position of S_f is as in Figure 14 for $d = 2$ by the orientational reasoning. As we saw in the case (1) we have only to consider the immersion of the second part from left in Figure 14 (which corresponds to Figure 11 with the number of the surrounding circles increased by $n - 2$). Let g be the image of the family of boundary closed curves.

To get a sufficient raying of g , we modify Figure 12 by adding $(n - 2)$ more surrounding circles. Immediately, we get

$$\rho = n + 1, \quad \tau = n - 3 \quad \text{and} \quad \nu = 4.$$

In order to have an immersion, $\mu = \nu + \tau - \zeta$ must be equal to 0 by PROPOSITION 3.4, that is, $\zeta = \nu + \tau = n + 1 = \rho$.

We consider the positive crossings on each surrounding circle. Because P is transitive on S , at least one positive crossing should be interchanged with some negative crossing on the innermost circle by P . In order that the number of disjoint cyclic permutations in $R' = PS$ is equal to $\zeta' = \zeta = n + 1 = \rho$, at least one more crossing on the same circle should be interchanged by P , because any cyclic permutation in S rotates the crossings along the same surrounding circle or the innermost circle. So, at least two crossings on each surrounding circle are in P . Hence, $2n \leq \nu = 4$, that is, $n \leq 2$. This means that case (2) does not occur. \square

We may get this conclusion more easily: If $n \geq 3$, there is no immersion of the rightmost surface in Figure 14 such that the bordered circles are mapped to parallel $n - 1$ circles in $\mathbf{R}^2 = S^2 - \{\infty\}$ with the same orientation, because there is no effective pairing in this case. But the above argument will be a good exercise to understand the next section.

4. The case of $\deg f \geq 3$

We generalise THEOREMS 3.1 and 3.2 for the case of $\deg f \geq 3$.

THEOREM 4.1. *Let $f : S^2 \rightarrow S^2$ be a smooth map with $\deg f = d \geq 2$. The minimal contour of f has exactly $2d$ cusps and no self-intersections.*

Let $f : S^2 \rightarrow S^2$ be a stable map with $\deg f = d \geq 2$. Since $d \geq 2$, we get $S_f \neq \emptyset$ from the theory of covering spaces. So, assuming that S_f is connected, we see

$$\left| \sum_{q_k \in f(C_f)} \text{sign}(q_k) - 2 \cdot d \right| \leq 0$$

by COROLLARY 2.6. Then we get

$$\sum_{q_k \in f(C_f)} \text{sign}(q_k) = 2d.$$

Hence we have at least $2d$ positive cusps. We will construct a stable map $f : S^2 \rightarrow S^2$ with $\deg f = d$ whose apparent contour has exactly $2d$ cusps and no self-intersections so that f realizes a minimal contour. As in the case of $\deg f = 2$, a minimal contour γ can be assumed as in Figure 15, and we take ∞ outside the star-like $2d$ -gon bounded by γ .



Fig. 15. Minimal contour γ with $2d$ cusps in $S^2 - \{\infty\}$

Now, let U_∞ be a sufficiently small open neighborhood of ∞ such that $f(S_f) \cap \bar{U}_\infty = \emptyset$, where \bar{U}_∞ is the closure of U_∞ . Since $\deg f = d$, $f^{-1}(U_\infty)$ consists of n open discs which are orientation-preserved by f and $n - d$ open discs which are orientation-reversed by f ($n \geq d$). By using an argument similar to the proof of THEOREM 3.2, we divide the source 2-sphere of f into four parts as in Figure 14.

In the four parts in Figure 14 except the second part from left the map can be clearly defined in the case $n = d$. Let g be a normal family of closed curves and α a sufficient raying as in Figure 16 for $n \geq d$. We assume now that there is an immersion of the second part from left such that the bordered circles are g . We will show $n = d$ and get further conditions.

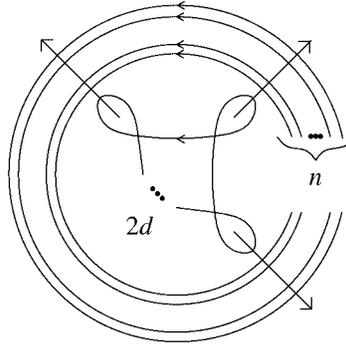


Fig. 16. g and α

In Figure 16, we get

$$\rho = 1 + n,$$

$$\tau = n - 2d + 1,$$

$$v = 2d.$$

Here, $\mu = v + \tau - \zeta$ must be equal to 0 by PROPOSITION 3.4, that is,

$$\zeta = v + \tau = n + 1.$$

So, in particular $\rho = \zeta = n + 1$. By using an argument similar to the proof of THEOREM 3.2, we see that there must be at least two pairs in P for each of the n surrounding circles, because P must be transitive. Then we get

$$2n \leq 2d,$$

and hence

$$d \leq n \leq d, \quad \text{that is, } n = d.$$

The argument shows moreover that there are exactly two pairs in P for each surrounding circle.

Hereafter we treat only the case of $n = d$. Hence g has $d + 1$ closed curves and $R' = PS$ should have $d + 1$ cyclic permutations. To simplify the argument we regard the neighborhood of the $2d$ cusps as a regular $2d$ -gon or a circular $2d$ -gon as in Figure 17.

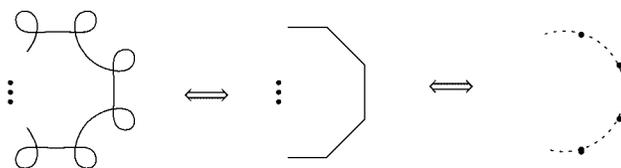


Fig. 17. $2d$ -gon

For a pairing P , we connect two vertices of the $2d$ -gon by an arc (or by a line segment for the circular $2d$ -gon), if the two corresponding negative crossings on the innermost circle are transposed to crossings on the same circle by P as in Figure 18. Note that a change of order of the d surrounding circles corresponds to a diffeomorphism of the source manifold and we can ignore it.

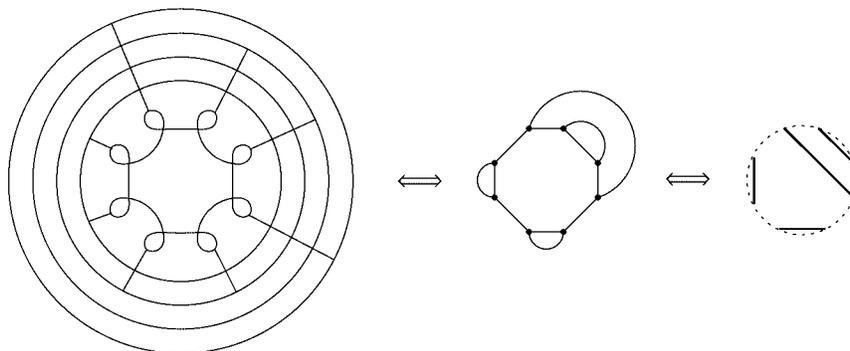


Fig. 18. Expressing a pairing by connecting the vertices of a $2d$ -gon ($d = 4$)

DEFINITION 4.2. Any crossing of two line segments connecting vertices on the circular $2d$ -gon is called a *bad crossing*. See Figure 19.

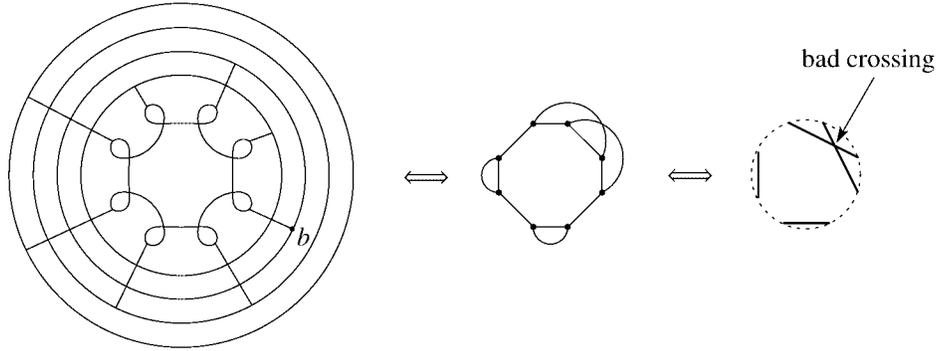


Fig. 19. Example with a bad crossing

Let g be a normal family of closed curves and α a sufficient raying as in Figure 16 with $n = d$ surrounding circles. Then we get the following lemma.

LEMMA 4.3. *Let P be an effective and transitive pairing on the successor permutation S determined by (g, α) . Then, there is an immersion $G_P : M_P \rightarrow \mathbf{R}^2 = S^2 - \{\infty\}$ with $G_P|_{\partial M_P} = g$ if and only if the circular $2d$ -gon diagram has no bad crossings.*

PROOF. Let us consider the cyclic permutations for $R' = PS$. Take an initial starting crossing point on a surrounding circle and apply R' on it repeatedly. Note that one application of R' gives rise to the $2d$ -th positive rotation on the rays. Only in the case that the starting crossing in each step is in a pair of P , the crossing on the innermost circle and that on the relevant surrounding circle are interchanged before the $2d$ -th rotation. In order that $\zeta = d + 1$, we must go back to the initial starting point within one full rotation. If we do not have bad crossings, then it is easy to check that $\zeta = d + 1$ and there is a desired immersion bounded by g by Proposition 3.3.

If we have some bad crossings, then we first consider only the lines with bad crossings on the circular $2d$ -gon. We can find at least one region bounded by a circular arc and two lines with bad crossings. Then as an initial starting point we take the crossing on the surrounding circle corresponding to the initial point of the circular arc on the boundary of the selected region. (See b in Figure 19 for example.) Since we have just two pairs in P on each surrounding circle, it is impossible for that crossing to go back to the initial starting point within one full rotation. So, $\zeta < d + 1$ and there is no immersion bounded by g . \square

PROOF OF THEOREM 4.1. We already know that the minimal contour γ is as in Figure 15. We divided the source manifold S^2 into four parts and we saw that in case $n = d$ we have only to consider the second part from left in Figure 14. Since we have a circular $2d$ -gon with d disjoint lines which connect two vertices for each d , we get a desired immersion and get a stable map $f : S^2 \rightarrow S^2$ with degree d that has γ as its apparent contour. This completes the proof of THEOREM 4.1. \square

By the proof of THEOREM 4.1, we obtain that the minimal contour of a smooth map $f : S^2 \rightarrow S^2$ of degree $d \geq 2$ is as in Figure 15. Furthermore, the right-left equivalence class of a stable map that gives the minimal contour corresponds to the pattern obtained by connecting vertices of the circular $2d$ -gon (with a fixed radius) without bad crossings, where we admit revolving and reflecting the $2d$ -gon. Note that a rotation or a reflection of the regular $2d$ -gon corresponds to a diffeomorphism of the target manifold. Complete lists of the patterns without bad crossings modulo congruence in the case $d = 2, 3, 4$ and 5 are given in Figure 20.

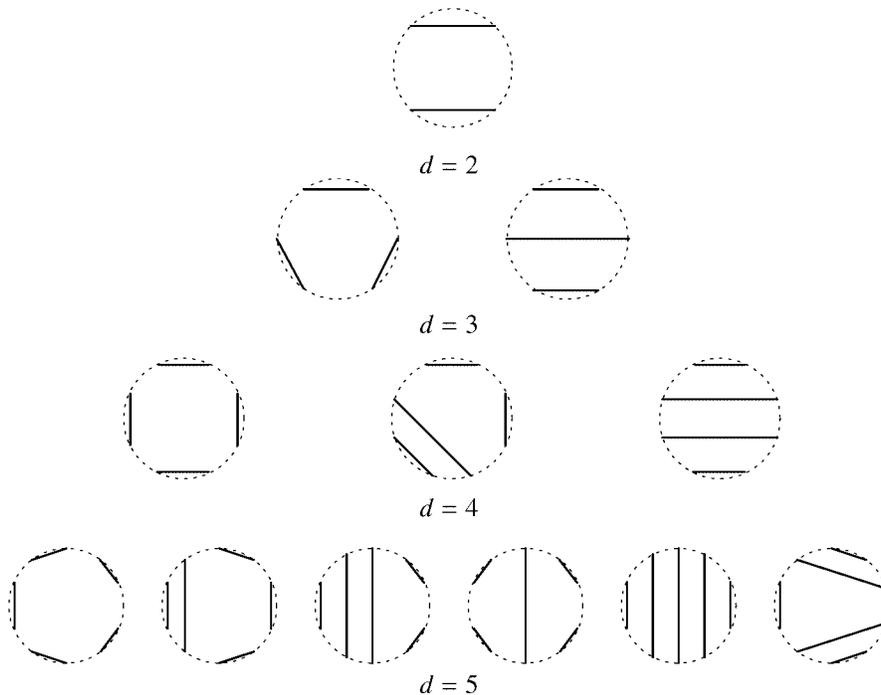


Fig. 20. Lists of the patterns without bad crossings

THEOREM 4.4. *Let $f : S^2 \rightarrow S^2$ be a smooth map with $\deg f = d \geq 2$. Let $n(d)$ be the number of congruence classes of the plane figure patterns, where a plane figure pattern is a unit circular $2d$ -gon with d disjoint line segments which connect two vertices. Then the number of the right-left equivalence classes of the stable maps homotopic to f which realize the minimal contour as in Figure 15 is equal to $n(d)$.*

Furthermore, for a given pattern as above, let us consider each area in the unit circle divided by the line segments to be a vertex, and connect two vertices if the corresponding regions are adjacent. Then we get a plane tree.

In fact, (1) any plane tree determines a patterns unique up to rotation and (2) two patterns are congruent if and only if the corresponding plane trees are plane-equivalent or mirror-plane-equivalent. So, by using the theory of plane trees, we can get a method to calculate $n(d)$. If the reflection of a plane tree G with respect to a line in the plane is plane-equivalent to G , then G is said to be *achiral*. Let us define the following functions:

$$\left\{ \begin{array}{l} g(x) = \frac{1 - (1 - 4x)^{1/2}}{2}, \\ p(x) = xg(x), \\ a(x) = \frac{x^4}{x^2 - x^3 - p(x^2)}, \\ i(x) = \frac{p(x)^2 - p(x^2)}{2x^2}, \\ t(x) = \left(x \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k|n} \phi(k) g(x^k)^{n/k} \right) - i(x), \end{array} \right.$$

where $\phi(k)$ is the ‘‘Euler ϕ -function’’, the number of positive integers less than k and relatively prime to k , with $\phi(1) = 1$. Note that $t(x)$ is known to be the generating function for plane trees (Theorems 2 and 3 of [5], formula (15.14) of [4]), and $a(x)$ is that for achiral plane trees (formulas (6) and (26) of [6]). Then, we get the following proposition by which we can calculate $n(d)$ numerically. Note that our $t(x)$ is denoted by $b(x)$ and not by $t(x)$ in [6].

PROPOSITION 4.5. *The number $n(d)$ is equal to the coefficient of x^{d+1} in the Taylor expansion of $(a(x) + t(x))/2$ around $x = 0$.*

For examples, $n(d)$ is equal to 1, 2, 3, 6, 12, 27, 65, 175, 490, 1473, 4588, 14782 and 48678 for $d = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ and 14 respectively.

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*Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan*