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Homotopy types of m-twisted $\mathbf{CP}^4$'s

Dedicated to the memory of Prof. Masahiro Sugawara

Kohhei Yamaguchi

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Abstract. We study the homotopy type classification problem of n dimensional m-twisted complex projective spaces for the case $n = 4$. In particular, we determine the number of homotopy types of m-twisted $\mathbf{CP}^4$'s when $m \geq 1$ is an odd integer.

1. Introduction

Let $n \geq 2$ be an integer and let M be a simply-connected $2n$ dimensional finite Poincaré complex. For an integer $m \geq 0$, M is called an *m-twisted* $\mathbf{CP}^n$ if there is an isomorphism $H_*(M, \mathbf{Z}) \cong H_*(\mathbf{CP}^n, \mathbf{Z})$ with the condition $x_2 \cdot x_2 = m x_4$, where $x_{2k} \in H^{2k}(M, \mathbf{Z}) \cong \mathbf{Z}$ denotes the corresponding generator $(k = 1, 2)$. Any m-twisted $\mathbf{CP}^n$ is homotoy equivalent to a CW complex of the form

$$M \simeq S^2 \cup_{m\eta_2} e^4 \cup e^6 \cup \cdots \cup e^{2n-2} \cup e^{2n} \qquad \text{(homotopy equivalence)},$$

and it has the homotopy type of $2n$ dimensional closed topological manifolds ([6]). Let $\mathscr{M}_m^n$ be the set consisting of all the homotopy equivalence classes of m-twisted $\mathbf{CP}^n$'s. For example, when $n = 2$, $\mathscr{M}_1^2 = \{[\mathbf{CP}^2]\}$ and $\mathscr{M}_m^2 = \varnothing$ if $m \neq 1$. When $n = 3$, the following result is known:

Theorem 1.1 ([11] (cf. [4])). *If $m \geq 0$ is an integer,*

$$\mathrm{card}(\mathscr{M}_m^3) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 2, \\ 3 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

where $\mathrm{card}(V)$ *denotes the cardinal number of a set* V.

In general, it is known that $\mathscr{M}_m^{2k+1} \neq \varnothing$ for any $m, k \geq 2$ (cf. [2]), and we have infinitely many non-trivial examples of m-twisted $\mathbf{CP}^{2k+1}$'s. On the other

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hand, it is even not known whether $\mathcal{M}_m^{2k}$ is an empty set or not if $m \neq 1$ and $k \geq 2$. As the first step of this question, we would like to study the set $\mathcal{M}_m^n$ for the case $n = 4$. Then if (a, b) denotes the greatest common divisor of positive integers a and b, the following result has been known.

THEOREM 1.2 ([6]).
(i) *If* $m = 0$, $3 \leq \mathrm{card}(\mathcal{M}_m^4) \leq 2^7 \cdot 3^2$.
(ii) *If* $m \equiv 1 \pmod 2$, $1 \leq \mathrm{card}(\mathcal{M}_m^4) \leq m \cdot (m, 3)$.
(iii) *If* $m \equiv 0 \pmod 8$ *and* $m \neq 0$, $3 \leq \mathrm{card}(\mathcal{M}_m^4) \leq 2^5 \cdot 3 \cdot m \cdot (m, 3)$.
(iv) *If* $m \equiv 0 \pmod 2$ *and* $m \not\equiv 0 \pmod 8$, $\mathcal{M}_m^4 = \varnothing$.

In this paper we shall investigate the set $\mathcal{M}_m^4$ when $m \equiv 1 \pmod 2$, and our main results are stated as follows:

THEOREM 1.3 (The main Theorem). *Let* $m \geq 1$ *be an odd integer, and let* M_0, M_1, M_{-1} *denote the m-twisted* $\mathbf{C}\mathrm{P}^4$*'s defined in Definition 4.*
(i) *If* $m \not\equiv 0 \pmod 3$, $\mathcal{M}_m^4 = \{[\mathrm{M}_0]\}$.
(ii) *If* $m \equiv 0 \pmod 3$, $\mathcal{M}_m^4 = \{[\mathrm{M}_0], [\mathrm{M}_1], [\mathrm{M}_{-1}]\}$, *such that the first* mod 3 *reduced power operation* $\mathscr{P}^1 : H^4(\mathrm{M}_\varepsilon, \mathbf{Z}/3) \to H^8(\mathrm{M}_\varepsilon, \mathbf{Z}/3)$ *is an isomorphism if* $\varepsilon = \pm 1$ *and is trivial if* $\varepsilon = 0$.

COROLLARY 1.4. *If* $m \geq 1$ *is an odd integer,*

$$\mathrm{card}(\mathcal{M}_m^4) = (m, 3) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 3, \\ 3 & \text{if } m \equiv 0 \pmod 3. \end{cases}$$

REMARK. Let $m \geq 3$ be an odd integer with $m \equiv 0 \pmod 3$, and let $\mathscr{A}_p$ denote the mod p Steenrod algebra. Although M_1 and M_{-1} are not homotopy equivalent, there are isomorphisms

$$\begin{cases} H^*(\mathrm{M}_1, \mathbf{Z}) \cong H^*(\mathrm{M}_{-1}, \mathbf{Z}) & \text{(as graded rings)} \\ H^*(\mathrm{M}_1, \mathbf{Z}/p) \cong H^*(\mathrm{M}_{-1}, \mathbf{Z}/p) & \text{(as } \mathscr{A}_p\text{-modules for any prime } p \geq 2). \end{cases}$$

This paper is organized as follows. In §2, we compute some Whitehead products and in §3, we consider the group of self-homotopy equivalences $\mathscr{E}(X_m)$ of the 6-skeleton X_m of m-twisted $\mathbf{C}\mathrm{P}^4$'s. In §4, we study the left $\mathscr{E}(X_m)$-action on $\pi_7(X_m)$ given by composite of maps, which is the key point for determining the homotopy types of m-twisted $\mathbf{C}\mathrm{P}^4$'s. In particular, we determine the set $\mathcal{M}_m^4$ explicitly when $(m, 6) = 1$. Finally, in §5, we compute the $\mathscr{E}(X_m)$-action on $\pi_7(X_m)$ explicitly and determine the set $\mathcal{M}_m^4$ when $(m, 6) = 3$.

2. Whitehead products

For an integer $m \geq 1$, let L_m and $\mathrm{P}^4(m)$ denote the CW complexes defined by $L_m = S^2 \cup_{m\eta_2} e^4$ and $\mathrm{P}^4(m) = S^3 \cup_{m\iota_3} e^4$, respectively. If $q : \tilde{L}_m \to L_m$

denotes the 2-connective covering of L_m, it is known that there is a homotopy equivalence $\tilde{L}_m \simeq \mathrm{P}^4(m) \vee S^5$ ([13]). If we identify $\tilde{L}_m = \mathrm{P}^5(m) \vee S^5$, then the map q is also identified with the map

$$q = (f_m, b_m) : \mathrm{P}^4(m) \vee S^5 \to L_m \qquad \text{(up to homotopy)}. \tag{1}$$

It follows from [[6], Lemma 3.3] that there is a homotopy commutative diagram

$$\begin{array}{ccccccc} S^3 & \xrightarrow{m\iota_3} & S^3 & \xrightarrow{i'} & \mathrm{P}^4(m) & \xrightarrow{q'_m} & S^4 \\ \| & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle f_m} & & \| \\ S^3 & \xrightarrow{m\eta_2} & S^2 & \xrightarrow{i} & L_m & \xrightarrow{q_m} & S^4, \end{array} \tag{2}$$

where two horizontal sequences are cofiber sequences.

Let $\omega \in \pi_6(S^3) \cong \mathbf{Z}/12$ and $\alpha_1(3) = 4\omega \in \pi_6(S^3)_{(3)} \cong \mathbf{Z}/3$ denote the generators. If $m \equiv 0 \pmod 3$, we denote by $\widetilde{\alpha_1}(3) \in \pi_7(\mathrm{P}^4(m))$ the coextension of $\alpha_1(3)$ which satisfies the condition $q'_m \circ \widetilde{\alpha_1}(3) = E\alpha_1(3)$.

Lemma 2.1 ([6], [11]). *If $m \geq 1$ is an odd integer, there are isomorphisms*

$$\begin{cases} \pi_5(L_m) = \mathbf{Z} \cdot b_m, \quad \pi_5(\mathrm{P}^4(m)) = 0, \\ \pi_6(L_m) = \mathbf{Z}/(m,3) \cdot i_*(\eta_2 \circ \omega) \oplus \mathbf{Z}/m \cdot f_m \circ \sigma \oplus \mathbf{Z}/2 \cdot b_m \circ \eta_5, \\ \pi_6(\mathrm{P}^4(m)) = \mathbf{Z}/(m,3) \cdot i' \circ \omega \oplus \mathbf{Z}/m \cdot \sigma, \\ \pi_7(L_m) = \mathbf{Z}/(m,3) \cdot f_m \circ \omega_m \oplus \mathbf{Z}/2 \cdot b_m \circ \eta_5^2 \oplus \mathbf{Z}/m \cdot [b_m, i_*(\eta_2)], \\ \pi_7(\mathrm{P}^4(m)) = \mathbf{Z}/(m,3) \cdot \omega_m, \end{cases}$$

where we can take $\omega_m = \widetilde{\alpha_1}(3) \in \{i', m\iota_3, \alpha_1(3)\}$ if $m \equiv 0 \pmod 3$.

Proof. The assertions follow from [6] except the last equality. If $m \equiv 0 \pmod 3$, by the proof of [[6], Proposition 2.9], the induced homomorphism

$$\mathbf{Z}/3 \cdot \omega_m = \pi_7(\mathrm{P}^4(m)) \xrightarrow{i'_{1*}} \pi_7(\mathrm{P}^4(m), S^3) \underset{\cong}{\xleftarrow{\alpha_{m*}}} \pi_7(D^4, S^3) \cong \mathbf{Z}/12$$

is injective, where $\alpha_m \in \pi_4(\mathrm{P}^4(m), S^3) \cong \mathbf{Z}$ denotes the characteristic map of the top cell e^4 in $\mathrm{P}^4(m)$. Because $\Delta'(E\omega) = m(i_3 \circ \omega)$ in the sequence (5) of [6], we have $q'_m \circ \omega_m = E\alpha_1(3)$ and the condition $\omega_m = \widetilde{\alpha_1}(3) \in \{i', m\iota_3, \alpha_1(3)\}$ is also satisfied. □

Definition 1. For an integer $m \geq 1$, let X_m be the space defined by $X_m = L_m \cup_{mb_m} e^6$. There is a cofiber sequence, $S^5 \xrightarrow{mb_m} L_m \xrightarrow{j} X_m \longrightarrow S^6$.

Lemma 2.2 ([6]). *If $m \geq 1$ is an odd integer, there are isomorphisms*

$$\begin{cases} \pi_6(X_m) = \mathbf{Z}/(m,3) \cdot j_*(i_*(\eta_2 \circ \omega)) \oplus \mathbf{Z}/m \cdot j_*(f_m \circ \sigma), \\ \pi_7(X_m) = \mathbf{Z} \cdot \varphi_m \oplus \mathbf{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbf{Z}/m \cdot j_*([b_m, i_*(\eta_2)]), \end{cases}$$

and the following equality holds:

$$j_{1*}(\varphi_m) = [\beta_m, i]_r + \beta_m \circ \eta'_5. \tag{3}$$

Here $\beta_m \in \pi_6(X_m, L_m) \cong \mathbf{Z}$ *denotes the characteristic map of the top cell* e^6 *in* X_m, $[\ ,\ ]_r$ *a relative Whitehead product,* $\eta'_k \in \pi_{k+2}(D^{k+1}, S^k) \cong \mathbf{Z}/2$ *the generator* $(k \geq 3)$, $j_1 : (X_m, *) \to (X_m, L_m)$ *is the inclusion and* $j_{1*} : \pi_7(X_m) \to \pi_7(X_m, L_m) = \mathbf{Z} \cdot [\beta_m, i]_r \oplus \mathbf{Z}/2 \cdot \beta_m \circ \eta'_5$ *the induced homomorphism.*

LEMMA 2.3. *If* $m \equiv 1 \pmod 2$, $[b_m, i_*(\eta_2)] = [[b_m, i], i] \in \pi_7(L_m)$.

PROOF. It follows from the Jacobi identity ([[10], Corollary 7.14]) that

$$[[b_m, i], i] + [[i, i], b_m] + [[i, b_m], i] = 0.$$

Because $[i, b_m] = [b_m, i]$, we have $2[[b_m, i], i] + [[i, i], b_m] = 0$. Then using $[i, i] = i \circ [\iota_2, \iota_2] = i_*(2\eta_2) = 2i_*(\eta_2)$, we have

$$2[[b_m, i], i] + 2[i_*(\eta_2), b_m] = 2[[b_m, i], i] - 2[b_m, i_*(\eta_2)] = 0.$$

Since the order of $[b_m, i_*(\eta_2)]$ is just m (by [[6], Corollary 3.5]) and $m \equiv 1 \pmod 2$, $[[b_m, i], i] - [b_m, i_*(\eta_2)] = 0$. □

LEMMA 2.4. *If* $m \equiv 1 \pmod 2$, $f_m \circ \sigma = [b_m, i] + b_m \circ \eta_5 \in \pi_6(L_m)$.

PROOF. It follows from [[6], Proposition 5.1] that there is a unit $x_m \in (\mathbf{Z}/m)^\times$ such that $[b_m, i] = x_m \cdot f_m \circ \sigma + b_m \circ \eta_5$. Since the order of $\sigma \in \pi_6(\mathrm{P}^4(m))$ is m ([8]), by changing the generator $\sigma \mapsto x_m^{-1}\sigma$, we may assume that $x_m = 1$ and the assertion follows. □

COROLLARY 2.5. *If* $m \equiv 1 \pmod 2$, $[f_m \circ \sigma, i] = [b_m, i_*(\eta_2)]$.

PROOF. It follows from Lemmas 2.3 and 2.4 that

$$[f_m \circ \sigma, i] = [b_m, i_*(\eta_2)] + [b_m \circ \eta_5, i]. \tag{4}$$

Since the orders of $[b_m, i_*(\eta_2)]$ and σ are m, we see $[b_m \circ \eta_5, i] = 0$ and the assertion follows. □

Now we remark the following general fact concerning m-twisted $\mathbf{CP}^4$'s.

LEMMA 2.6. *Let* $m \geq 0$ *be an integer and* M *an* m*-twisted* $\mathbf{CP}^4$. *Then if* $m \not\equiv 0 \pmod 3$, $\mathscr{P}^1 : H^4(M, \mathbf{Z}/3) \xrightarrow{\cong} H^8(M, \mathbf{Z}/3)$ *is an isomorphism.*

PROOF. If $y_{2l} \in H^{2l}(M, \mathbf{Z}/3) \cong \mathbf{Z}/3$ denotes the mod 3 generator $(1 \leq l \leq 4)$, $(y_2)^2 = my_4$, $y_2 \cdot y_4 = my_6$ and $(y_4)^2 = y_2 \cdot y_6 = y_8$ by [[6], (0.2)]. Hence, $\mathscr{P}^1(y_2) = (y_2)^3 = (my_4) \cdot y_2 = m^2 y_6 = \pm y_6$ and

$$m \cdot \mathscr{P}^1(y_4) = \mathscr{P}^1(my_4) = \mathscr{P}^1((y_2)^2) = 2y_2 \cdot \mathscr{P}^1(y_2) = \pm 2y_2 \cdot y_6 = \mp y_8.$$

Because $m \not\equiv 0 \pmod 3$, this implies that $\mathscr{P}^1(y_4) = \pm y_8$. □

3. Groups of self-homotopy equivalences

For a connected space X, we denote by $\mathscr{E}(X)$ the set consisting of all based homotopy classes of based self-homotopy equivalences of X, which becomes a group whose multiplication is induced from composite of maps. The group $\mathscr{E}(X)$ is called the group of self-homotopy equivalences of X.

Definition 2. If K is a CW complex and $X = K \cup_f e^n$ with $\dim K \leq n-2$, we define the homomorphism $\lambda : \tilde{j}_*(\pi_n(K)) \to \mathscr{E}(X)$ by

$$\lambda(\tilde{j} \circ g) = \nabla \circ (1 \vee \tilde{j} \circ g) \circ \mu' : X \xrightarrow{\mu'} X \vee S^n \xrightarrow{1 \vee \tilde{j} \circ g} X \vee X \xrightarrow{\nabla} X$$

for $g \in \pi_n(K)$, where $\tilde{j} : K \to X$ denotes an inclusion, $\mu' : X \to X \vee S^n$ the co-action map given by pinching the hemisphere of the top cell e^n and ∇ is a folding map.

If $\theta : X \xrightarrow{\simeq} X$ is a homotopy equivalence, it follows from the cellular approximation Theorem that the restriction $\theta|_K$ also defines a self-homotopy equivalence on K. So we can define the homomorphsim $\phi : \mathscr{E}(X) \to \mathscr{E}(K)$ by the restriction $\phi(\theta) = \theta|_K$ for $\theta \in \mathscr{E}(X)$.

Proposition 3.1. *If $m \geq 1$ is an odd integer, there is an exact sequence*

$$\pi_6(X_m) \xrightarrow{\lambda} \mathscr{E}(X_m) \xrightarrow{\phi} \mathscr{E}(L_m) \to 1,$$

where we take $\mathbf{Z}_2 = \{\pm 1\}$ and $\mathscr{E}(L_m) \cong \mathbf{Z}_2$.

Proof. Because $j_*(\pi_6(L_m)) = \pi_6(X_m)$, the assertion easily follows from the Barcus-Barratt Theorem [[1], Theorem 6.1] and [[11], Corollary 4.8]. □

4. An action of $\mathscr{E}(X_m)$ on $\pi_7(X_m)$

For CW complexes X and Y, we write $X \simeq Y$ if there is a homotopy equivalence $X \xrightarrow{\simeq} Y$. Let $\mathrm{M}(\varphi)$ denote the mapping cone defined by

$$\mathrm{M}(\varphi) = X_m \cup_\varphi e^8 \qquad \text{for } \varphi \in \pi_7(X_m). \tag{5}$$

Recall the following well-known result.

Lemma 4.1 (Homotopy Theorem). *Let K be a simply-connected CW complex and let X, Y denote the CW complexes defined by $X = K \cup_f e^n$ and $Y = K \cup_g e^n$, where $\dim K \leq n-2$, $n \geq 4$ and $f, g \in \pi_{n-1}(K)$. Then there is a homotopy equivalence $X \simeq Y$ if and only if there is a homotopy equivalence $\theta \in \mathscr{E}(K)$ such that $\theta \circ f = \pm g$.*

THEOREM 4.2 ([6]). *Let $m \geq 1$ be an odd integer. Then* M *is an m-twisted* $\mathbf{CP}^4$ *if and only if there is some element* $\gamma \in \pi_7(L_m)$ *such that* $\mathrm{M} \simeq \mathrm{M}(\varphi) = X_m \cup_\varphi e^8$, *where* $\varphi = \pm\varphi_m + j_*(\gamma)$.

PROOF. This follows from [[6], Theorem 4.5] and the homotopy exact sequence of the pair (X_m, L_m). □

So it is useful to consider the left $\mathscr{E}(X_m)$ action on $\pi_7(X_m)$ given by the composite of maps, $\mathscr{E}(X_m) \times \pi_7(X_m) \ni (\theta, \varphi) \mapsto \theta \circ \varphi \in \pi_7(X_m)$.

LEMMA 4.3. *Let $m \geq 1$ be an integer and $\varphi \in \pi_7(X_m)$ be an element such that $j_{1*}(\varphi) = a \cdot [\beta_m, i]_r + \varepsilon \cdot \beta_m \circ \eta_5'$ for some $(a, \varepsilon) \in \mathbf{Z} \times \mathbf{Z}/2$. Then*

$$\mu_*'(\varphi) = j_X \circ \varphi + a[j_6, j_X \circ j \circ i] + \varepsilon \cdot j_6 \circ \eta_6,$$

where $X_m \xrightarrow{j_X} X_m \vee S^6 \xleftarrow{j_6} S^6$ denote the corresponding inclusions, $\mu' : X_m \to X_m \vee S^6$ is a co-action map and $\mu_*' : \pi_7(X_m) \to \pi_7(X_m \vee S^6)$ is the induced homomorphism.

PROOF. This follows from [[12], Lemma 2.2]. □

COROLLARY 4.4. *If $m \geq 1$ be an odd integer, the following equalities hold:*
(i) $\mu_*'(\varphi_m) = j_X \circ \varphi_m + [j_6, j_X \circ j \circ i] + j_6 \circ \eta_6$.
(ii) $\mu_*'(j_*([b_m, i_*(\eta_2)])) = j_X \circ j_*([b_m, i_*(\eta_2)])$.
(iii) *If $m \equiv 0 \pmod 3$,* $\mu_*'(j_*(i_*(\eta_2 \circ \omega))) = j_X \circ j_*(i_*(\eta_2 \circ \omega))$.

PROOF. The assertions follows from (3) and Lemma 4.3. □

DEFINITION 3. Let $m \geq 1$ be an odd integer and let $\lambda : j_*(\pi_6(L_m)) \to \mathscr{E}(X_m)$ denote the homomorphism defined in Definition 2. Then define the homotopy equivalence $\theta_k \in \mathscr{E}(X_m)$ by

$$\theta_k = \lambda((k \cdot j_*(f_m \circ \sigma))) \qquad \text{for each } k \in \mathbf{Z}/m. \tag{6}$$

Similarly, when $m \equiv 0 \pmod 3$, we define $\theta_l' \in \mathscr{E}(X_m)$ by

$$\theta_l' = \lambda(l \cdot j_*(i_*(\eta_2 \circ \omega))) \qquad \text{for each } l \in \mathbf{Z}/3. \tag{7}$$

PROPOSITION 4.5. *If $m \geq 1$ is an odd integer, the following equalities hold for any $(k, l) \in \mathbf{Z}/m \times \mathbf{Z}/3$:*

(i) $$\begin{cases} \theta_k \circ \varphi_m = \varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]), \\ \theta_k \circ j_*([b_m, i_*(\eta_2)]) = j_*([b_m, i_*(\eta_2)]). \end{cases}$$

(ii) *If $m \equiv 0 \pmod 3$,*

$$\begin{cases} \theta_l' \circ \varphi_m = \varphi_m, \quad \theta_l' \circ j_*([b_m, i_*(\eta_2)]) = j_*([b_m, i_*(\eta_2)]), \\ \theta_l' \circ j_*(f_m \circ \omega_m) = \theta_k \circ j_*(f_m \circ \omega_m) = j_*(f_m \circ \omega_m). \end{cases}$$

Proof. (i) It follows from Corollary 4.4 that we have

$$\begin{aligned}\theta_k \circ j_*([b_m, i_*(\eta_2)] &= \nabla \circ (1 \vee (k \cdot j_*(f_m \circ \sigma))) \circ \mu'_*([b_m, i_*(\eta_2)]) \\ &= \nabla \circ (1 \vee (k \cdot j \circ f_m \circ \sigma)) \circ j_X \circ j_*([b_m, i_*(\eta_2)]) \\ &= j_*([b_m, i_*(\eta_2)]).\end{aligned}$$

Since $f_m \circ \sigma \circ \eta_6 = 0$ (by [6]), we also obtain

$$\begin{aligned}\theta_k \circ \varphi_m &= \nabla \circ (1 \vee (k \cdot j_*(f_m \circ \sigma))) \circ \mu'_*(\varphi_m) \\ &= \nabla \circ (1 \vee (k \cdot j_*(f_m \circ \sigma))) \circ (j_X \circ \varphi_m + [j_6, j_X \circ j \circ i] + j_6 \circ \eta_6) \\ &= \varphi_m + [k \cdot j_*(f_m \circ \sigma), j \circ i] + k \cdot j_*(f_m \circ \sigma \circ \eta_6) \\ &= \varphi_m + k \cdot j_*([f_m \circ \sigma, i]) \\ &= \varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]) \qquad \text{(by Corollary 2.5).}\end{aligned}$$

(ii) Because the proof is similar to that of (i), we only give the proof of the first equality. This follows from

$$\begin{aligned}\theta'_l \circ \varphi_m &= \nabla \circ (1 \vee (l \cdot j_*(i_*(\eta_2 \circ \omega))) \circ \mu'_*(\varphi_m) \\ &= \nabla \circ (1 \vee (l \cdot j_*(i_*(\eta_2 \circ \omega))) \circ (j_X \circ \varphi_m + [j_6, j_X \circ j \circ i] + j_6 \circ \eta_6) \\ &= \varphi_m + [l \cdot j_*(i_*(\eta_2 \circ \omega)), j \circ i] + l \cdot j_*(i_*(\eta_2 \circ \omega \circ \eta_6)) \\ &= \varphi_m + l \cdot j_*(i_*([\eta_2 \circ \omega, \iota_2])) \qquad \text{(by } i_*(\eta_2 \circ \omega \circ \eta_6) = 0) \\ &= \varphi_m \qquad \text{(by } [\eta_2 \circ \omega, \iota_2] = 0 \text{ (by [3])).}\end{aligned}$$

□

Definition 4. Let $m \geq 1$ be an odd integer. Then we denote by M_0 the m-twisted $\mathbf{CP}^4$ defined by

$$\mathrm{M}_0 = \mathrm{M}(\varphi_m) = X_m \cup_{\varphi_m} e^8. \tag{8}$$

Moreover, when $m \equiv 0 \pmod 3$, we denote by M_1 and M_{-1} the m-twisted $\mathbf{CP}^4$'s defined by

$$\mathrm{M}_\varepsilon = \mathrm{M}(\varphi_m + \varepsilon \cdot j_*(f_m \circ \omega_m)) = X_m \cup_{\varphi_m + \varepsilon \cdot j_*(f_m \circ \omega_m)} e^8 \qquad (\text{for } \varepsilon = \pm 1). \tag{9}$$

Theorem 4.6. *If* $m \equiv 1 \pmod 2$ *and* $m \not\equiv 0 \pmod 3$, $\mathscr{M}_m^4 = \{[\mathrm{M}_0]\}$.

Proof. We note that $\mathscr{M}_m^4 \neq \emptyset$ by Theorem 1.2. Now let M be any m-twisted $\mathbf{CP}^4$. It suffices to show that $M \simeq \mathrm{M}_0$. Since $\mathrm{M}(-\varphi) \simeq \mathrm{M}(\varphi)$ by Lemma 4.1, it follows from Theorem 4.2 that there exists some $k \in \mathbf{Z}/m$ such that $M \simeq \mathrm{M}(\varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]))$. Then because

$$\theta_k \circ \varphi_m = \varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]) \qquad \text{(by Proposition 4.5)},$$

$$\mathrm{M}_0 = \mathrm{M}(\varphi_m) \simeq \mathrm{M}(\varphi_m + k \cdot j_*([b_m, i_*(\eta_2)])) \simeq M. \qquad \square$$

COROLLARY 4.7. *Let $m \geq 1$ be an odd integer.*

(i) *If $m \not\equiv 0 \pmod 3$, there is an exact sequence*

$$0 \to \pi_6(X_m) \xrightarrow{\lambda} \mathscr{E}(X_m) \xrightarrow{\phi} \mathbf{Z}_2 \to 1,$$

where $\pi_6(X_m) \cong \mathbf{Z}/m$.

(ii) *If $m \equiv 0 \pmod 3$, there is an exact sequence*

$$0 \to \mathbf{Z}/m \oplus G_m \xrightarrow{\lambda} \mathscr{E}(X_m) \xrightarrow{\phi} \mathbf{Z}_2 \to 1,$$

where $G_m = \mathbf{Z}/3$ or $G_m = 0$.

PROOF. (i) It suffices to show that $\lambda : \mathbf{Z}/m \cdot j_*(f_m \circ \sigma) = \pi_6(X_m) \to \mathscr{E}(X_m)$ is injective. If we write $\theta_k = \lambda(k \cdot j_*(f_m \circ \sigma))$ (as in (6)), it follows from Proposition 4.5 that $\theta_k \circ \varphi_m \neq \theta_l \circ \varphi$ if $k \neq l \in \mathbf{Z}/m$. Hence, $\theta_k \neq \theta_l$ if $k \neq l \in \mathbf{Z}/m$, and λ is injective.

(ii) The same proof as that of (i) shows that $\lambda|_{\mathbf{Z}/m \cdot j_*(f_m \circ \sigma)} : \mathbf{Z}/m \cdot j_*(f_m \circ \sigma) \to \mathscr{E}(X_m)$ is injective. Because $\pi_6(X_m) = \mathbf{Z}/m \cdot j_*(f_m \circ \sigma) \oplus \mathbf{Z}/3 \cdot j_*(i_*(\eta_2 \circ \omega))$, $\mathrm{Ker}\,\lambda = \mathbf{Z}/3 \cdot j_*(i_*(\eta_2 \circ \omega))$ or $\mathrm{Ker}\,\lambda = 0$. Then (ii) follows from Proposition 3.1. $\square$

PROPOSITION 4.8. *If $m \geq 3$ is an odd integer with $m \equiv 0 \pmod 3$ and M is an m-twisted $\mathbf{C}\mathrm{P}^4$, then $M \simeq \mathrm{M}_0$ or $M \simeq \mathrm{M}_1$ or $M \simeq \mathrm{M}_{-1}$ holds.*

PROOF. Because $\pi_7(X_m) = \mathbf{Z} \cdot \varphi_m \oplus \mathbf{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbf{Z}/3 \cdot j_*(f_m \circ \omega)$, by Theorem 4.2, there is a homotopy equivalence

$$M \simeq \mathrm{M}(\varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]) + l \cdot j_*(f_m \circ \omega_m)) = M_{k,l}$$

for some $k \in \mathbf{Z}/m$ and $l \in \mathbf{Z}/3$. Then by Proposition 4.5,

$$\begin{aligned}\theta_k \circ (\varphi_m + l \cdot j_*(f_m \circ \omega_m)) &= \theta_k \circ \varphi_m + l \cdot \theta_k \circ j_*(f_m \circ \omega_m) \\ &= \varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]) + l \cdot j_*(f_m \circ \omega_m).\end{aligned}$$

Hence, there is a homotopy equivalence $M_{k,l} \simeq M_{0,l}$. Then because

$$M_{0,l} = \begin{cases} \mathrm{M}_0 & \text{if } l = 0 \in \mathbf{Z}/3, \\ \mathrm{M}_1 & \text{if } l = 1 \in \mathbf{Z}/3, \\ \mathrm{M}_{-1} & \text{if } l = -1 = 2 \in \mathbf{Z}/3, \end{cases}$$

the assertion follows. $\square$

5. The case $m \equiv 0 \pmod 3$

From now on, we assume that $m \geq 3$ is an odd integer with $m \equiv 0 \pmod 3$, and consider the $\mathscr{E}(X_m)$-action on $\pi_7(X_m)$.

We remark that (by Corollary 4.7) there is a homotopy equivalence $\tilde{\theta} \in \mathscr{E}(X_m)$ such that $\tilde{\theta}|_{L_m} = h_1$ represents the generator of $\mathscr{E}(L_m) \cong \mathbf{Z}_2$. In this case, because $\mathscr{E}(X_m)$ is generated by $\{\theta_k, \theta_l', \tilde{\theta} : k \in \mathbf{Z}/m, l \in \mathbf{Z}/3\}$ and the actions of θ_k's or those of θ_l''s are given in Proposition 4.5, it remains to consider the action of $\tilde{\theta}$ on $\pi_7(X_m)$. For this purpose, we recall self-homotopy equivalences h_1 and $\tilde{\theta}$. First, recall h_1. Because

$$(-\iota_2) \circ (m\eta_2) = m(-\eta_2 + [\iota_2, \iota_2] \circ H(\eta_2)) = m(-\eta_2 + (2\eta_2) \circ \iota_3) = m\eta_2 \tag{10}$$

by [[10]; page 537, (8.12)], there is a map $h_1 : L_m \to L_m$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccccc} S^3 & \xrightarrow{m\eta_2} & S^2 & \xrightarrow{i} & L_m & \xrightarrow{q_m} & S^4 \\ \| & & \downarrow{\scriptstyle -\iota_2} & & \downarrow{\scriptstyle h_1} & & \| \\ S^3 & \xrightarrow{m\eta_2} & S^2 & \xrightarrow{i} & L_m & \xrightarrow{q_m} & S^4. \end{array} \tag{11}$$

Then the following is known:

Lemma 5.1 ([11]). *Let $m \geq 1$ be an odd integer.*
(i) $h_1 \in \mathscr{E}(L_m)$ *and* $\mathscr{E}(L_m) = \{h_1, \mathrm{id}_{L_m}\} = \langle h_1 \mid h_1^2 = \mathrm{id}_{L_m} \rangle \cong \mathbf{Z}_2$.
(ii) *The degree of h_1 on S^2 is -1 and the that of it on e^4 is $+1$.*
(iii) $h_1 \circ b_m = -b_m$.

If $m \equiv 1 \pmod 2$, it follows from Proposition 3.1 that there exists a homotopy equivalence $\tilde{\theta} \in \mathscr{E}(X_m)$ such that

$$\tilde{\theta}|_{L_m} = h_1. \tag{12}$$

We note that $\tilde{\theta}$ also defines a self-homotopy equivalence on (X_m, L_m), and we write it as the same letter $\tilde{\theta}$.

Lemma 5.2. *Let $m \geq 3$ be an odd integer with $m \equiv 0 \pmod 3$.*
(i) $\tilde{\theta} \circ \beta_m = -\beta_m$.
(ii) $\tilde{\theta} \circ \varphi_m \equiv \varphi_m \mod \mathrm{Im}\, j_*$.

Proof. (i) If $x_{2k} \in H^{2k}(X_m, \mathbf{Z}) \cong \mathbf{Z}$ denotes the corresponding generator $(k = 1, 2, 3)$, $x_2 \cdot x_4 = m x_6$. Hence, the degree of $\tilde{\theta}$ on the top cell e^6 on X_m is -1 (by (ii) of Lemma 5.1). So it follows from the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Z}\cdot\beta_m=\pi_6(X_m,L_m) & \xrightarrow[\cong]{h} & H_6(X_m,L_m;\mathbf{Z}) & \xleftarrow[\cong]{j_{1*}} & H_6(X_m;\mathbf{Z})\cong\mathbf{Z} \\
\tilde{\theta}_* \downarrow & & \downarrow & & \downarrow \times(-1) \\
\mathbf{Z}\cdot\beta_m=\pi_6(X_m,L_m) & \xrightarrow[\cong]{h} & H_6(X_m,L_m;\mathbf{Z}) & \xleftarrow[\cong]{j_{1*}} & H_6(X_m;\mathbf{Z})\cong\mathbf{Z}
\end{array}$$

that we have $\tilde{\theta}\circ\beta_m=-\beta_m$.

(ii) Consider the induced homomorphism $j_{1*}(X_m)\to\pi_7(X_m,L_m)$. Then because

$$\begin{aligned}
j_{1*}(\tilde{\theta}\circ\varphi_m) &= \tilde{\theta}_*(j_{1*}(\varphi_m))=\tilde{\theta}_*([\beta_m,i]_r+\beta_m\circ\eta_5') \qquad \text{(by (3))}\\
&= [\tilde{\theta}\circ\beta_m,h_1\circ i]_r+\tilde{\theta}\circ\beta_m\circ\eta_5'\\
&= [-\beta_m,-i]_r+(-\beta_m)\circ E\eta_4' \qquad \text{(by (i), (11))}\\
&= [\beta_m,i]_r+\beta_m\circ\eta_5'=j_{1*}(\varphi_m),
\end{aligned}$$

the assertion (ii) follows form the exact sequence of the pair (X_m,L_m). □

LEMMA 5.3. *There exists a homotoy equivalence $h_P\in\mathscr{E}(\mathrm{P}^4(m))$ such that $h_1\circ f_m=f_m\circ h_P$ with $h_P|_{S^3}=\iota_3$.*

PROOF. Consider the fibration sequence,

$$\mathrm{P}^4(m)\vee S^5\xrightarrow{(f_m,b_m)}L_m\xrightarrow{\ \iota\ }K(\mathbf{Z},2).$$

Since (f_m,b_m) is a 2-connective covering of L_m, we may assume that the map $\iota:L_m\to K(\mathbf{Z},2)$ represents the oriented generator of $[L_m,K(\mathbf{Z},2)]\cong H^2(L_m,\mathbf{Z})\cong\mathbf{Z}$. Now we define the involution $v:\mathbf{Z}\to\mathbf{Z}$ by $v(n)=-n$. It induces a self-homotopy equivalence $\tilde{v}\in\mathscr{E}(K(\mathbf{Z},2))$. Here, because $h_1|_{S^2}=-\iota_2$, $h_1^*:H^2(L_m,\mathbf{Z})\overset{\cong}{\to}H^2(L_m,\mathbf{Z})$ is given by $h_1^*(x)=-x$ for $x\in H^2(L_m,\mathbf{Z})\cong\mathbf{Z}$. Hence, $\tilde{v}\circ\iota=\iota\circ h_1$ (up to homotopy). So there exists a self-homotopy equivalence $\tilde{h}_1\in\mathscr{E}(\mathrm{P}^4(m)\vee S^5)$ such that the diagram

$$\begin{array}{ccccc}
\mathrm{P}^4(m)\vee S^5 & \xrightarrow{(f_m,b_m)} & L_m & \xrightarrow{\ \iota\ } & K(\mathbf{Z},2)\\
\tilde{h}_1\downarrow\simeq & & h_1\downarrow\simeq & & \tilde{v}\downarrow\simeq\\
\mathrm{P}^4(m)\vee S^5 & \xrightarrow{(f_m,b_m)} & L_m & \xrightarrow{\ \iota\ } & K(\mathbf{Z},2)
\end{array}$$

is homotopy commutative, where horizontal sequences are fibration sequences. Now we define the map $h_P:\mathrm{P}^4(m)\to\mathrm{P}^4(m)$ by $h_P=\pi_P\circ\tilde{h}_1\circ i_P$, where $i_P:\mathrm{P}^4(m)\to\mathrm{P}^4(m)\vee S^5$ and $\pi_P:\mathrm{P}^4(m)\vee S^5\to\mathrm{P}^4(m)$ denote the natural inclusion and the natural projection, respectively. By chasing the diagram, we can see $h_p\in\mathscr{E}(\mathrm{P}^4(m))$ and that $f_m\circ h_P=h_1\circ f_m$.

On the other hand, it follows from the diagram (2) that $f_m|_{S^3} = \eta_2$. Hence, using $h_1|_{S^2} = -\iota_2$ and $(-\iota_2) \circ \eta_2 = \eta_2$, we can choose the map h_P such that $h_P|_{S^3} = \iota_3$. □

Lemma 5.4. *If m is an odd integer with $m \equiv 0 \pmod 3$, we may choose the homotopy equivalence $\tilde{\theta} \in \mathscr{E}(X_m)$ such that*

$$\tilde{\theta} \circ \varphi_m = \varphi_m + l_m \cdot j_*(f_m \circ \omega_m) \qquad \text{for some } l_m \in \mathbf{Z}/3. \tag{13}$$

Proof. It follows from Lemma 5.2 that there exists a pair $(k, l_m) \in \mathbf{Z}/m \times \mathbf{Z}/3$ such that, $\tilde{\theta} \circ \varphi_m = \varphi_m + k \cdot j_*([b_m, i_*(\eta_2)]) + l_m \cdot j_*(f_m \circ \omega_m)$.

If we take $\psi = \theta_{-k} \circ \tilde{\theta} \in \mathscr{E}(X_m)$, by Proposition 4.5, we have

$$\begin{aligned}
\psi \circ \varphi_m &= \theta_{-k} \circ \tilde{\theta} \circ \varphi_m \\
&= \theta_{-k} \circ (\varphi_m + k \cdot j_*([b_m, i_*(\eta)]) + l_m \cdot j_*(f_m \circ \omega_m)) \\
&= \theta_{-k} \circ \varphi_m + k \cdot \theta_{-k} \circ j_*([b_m, i_*(\eta_2)]) + l_m \cdot \theta_{-k} \circ j_*(f_m \circ \omega_m) \\
&= \varphi_m - k \cdot j_*([b_m, i_*(\eta_2)]) + k \cdot j_*([b_m, i_*(\eta_2)]) + l_m \cdot j_*(f_m \circ \omega_m) \\
&= \varphi_m + l_m \cdot j_*(f_m \circ \omega_m).
\end{aligned}$$

Then because $\psi|_{L_m} = h_1$, we can change the generator $\psi \mapsto \tilde{\theta}$, and we may assume that $\tilde{\theta} \in \mathscr{E}(X_m)$ satisfies the equality (13). □

Lemma 5.5. *Let $m \geq 3$ be an integer such that $m \equiv 0 \pmod 3$.*

(i) $\tilde{\theta} \circ j_*([b_m, i_*(\eta_2)]) = -j_*([b_m, i_*(\eta_2)])$.

(ii) $\tilde{\theta} \circ j_*(f_m \circ \omega_m) = j_*(f_m \circ \omega_m)$.

Proof. (i) Since $\tilde{\theta} \circ j = j \circ h_1$ (by (12)), we have

$$\begin{aligned}
\tilde{\theta} \circ j_*([b_m, i_*(\eta_2)]) &= j_*(h_1 \circ [b_m, i_*(\eta_2)]) = j_*([h_1 \circ b_m, h_1 \circ i \circ \eta_2]) \\
&= j_*([-b_m, i \circ (-\iota_2) \circ \eta_2]) \qquad \text{(by Lemma 5.1 and (11))} \\
&= -j_*([b_m, i_*(\eta_2)]) \qquad \text{(by (10))}.
\end{aligned}$$

(ii) Since $\tilde{\theta} \circ j = j \circ h_1$, it suffices to prove that $h_1 \circ f_m \circ \omega_m = f_m \circ \omega_m$. Then it follows from Lemma 5.3 that we have $h_1 \circ f_m \circ \omega_m = f_m \circ h_P \circ \omega_m = f_m \circ (\iota_3) \circ \omega_m = f_m \circ \omega_m$. □

Lemma 5.6. *Let $m \geq 3$ be an integer such that $m \equiv 0 \pmod 3$.*

(i) *If $l_m \neq 0 \in \mathbf{Z}/3$, $[\mathrm{M}_0] = [\mathrm{M}_1] = [\mathrm{M}_{-1}]$ in $\mathscr{M}_m^4$.*

(ii) *If $l_m = 0 \in \mathbf{Z}/3$, $[\mathrm{M}_0] \neq [\mathrm{M}_1]$, $[\mathrm{M}_1] \neq [\mathrm{M}_{-1}]$ and $[\mathrm{M}_{-1}] \neq [\mathrm{M}_0]$ in $\mathscr{M}_m^4$.*

Remark. So we can determine the set $\mathscr{M}_m^4$ completely if we know whether $l_m = 0$ or not. In fact, $l_m = 0$ holds and this will be proved in Theorem 5.8.

PROOF. (i) If $l_m \neq 0 \in \mathbf{Z}/3$, $l_m = \pm 1$. Then because

$$\begin{cases} \tilde{\theta} \circ \varphi_m = \varphi_m \pm j_*(f_m \circ \omega_m) & \text{(by (13)), and} \\ \tilde{\theta} \circ (\varphi_m \pm j_*(f_m \circ \omega_m)) = \varphi_m \mp j_*(f_m \circ \omega_m), \end{cases}$$

it follows from Lemma 4.1 that we obtain $[\mathrm{M}_0] = [\mathrm{M}_1] = [\mathrm{M}_{-1}] \in \mathscr{M}_m^4$.

(ii) If $l_m = 0$, by using (13) and Proposition 4.5, we have

$$\begin{cases} \theta \circ \varphi_m \neq \pm(\varphi_m \pm j_*(f_m \circ \omega_m)), \\ \theta \circ ((\varphi_m + j_*(f_m \circ \omega_m)) \neq \pm(\varphi_m - j_*(f_m \circ \omega_m)) \end{cases}$$

for any $\theta \in \mathscr{E}(X_m)$. Hence, by Lemma 4.1, $[\mathrm{M}_0] \neq [\mathrm{M}_1]$, $[\mathrm{M}_1] \neq [\mathrm{M}_{-1}]$ and $[\mathrm{M}_{-1}] \neq [\mathrm{M}_0]$ in $\mathscr{M}_m^4$. □

LEMMA 5.7. *Let $m \geq 3$ be an odd integer with $m \equiv 0 \pmod 3$ and let M be an m-twisted* $\mathbf{CP}^4$.

(i) *There is a homotopy equivalence*

$$M/S^2 \simeq S^4 \vee S^6 \cup_\gamma e^8 = N(n_0) \qquad \text{for some } n_0 \in \mathbf{Z}/12,$$

where $\gamma = i_4 \circ v_4 + i_6 \circ \eta_6 + n_0 \cdot i_4 \circ E\omega \in \pi_7(S^4 \vee S^6)$ and $i_l : S^l \to S^4 \vee S^6$ $(l = 4, 6)$ denotes the corresponding inclusion.

(ii) *In this case, $\mathscr{P}^1 : \mathbf{Z}/3 \cong H^4(M, \mathbf{Z}/3) \to H^8(M, \mathbf{Z}/3) \cong \mathbf{Z}/3$ is isomorphism if $n_0 \not\equiv 0 \pmod 3$ and it is trivial if $n_0 \equiv 0 \pmod 3$.*

PROOF. (i) Since $Sq^2 : H^4(M, \mathbf{Z}/2) \to H^6(M, \mathbf{Z}/2)$ is trivial by [[6], Proposition 4.1], there is a homotopy equivalence $M/S^2 \simeq S^4 \vee S^6 \cup_\gamma e^8 = N_\gamma$ for some $\gamma \in \pi_7(S^4 \vee S^6) = \mathbf{Z} \cdot i_4 \circ v_4 \oplus \mathbf{Z}/12 \cdot i_4 E\omega \oplus \mathbf{Z}/2 \cdot i_6 \circ \eta_6$.

Since $N_{-\gamma} \simeq N_\gamma$, without loss of generalities, we may suppose that

$$\gamma = a \cdot i_4 \circ v_4 + n_0 \cdot i_4 \circ E\omega + \varepsilon \cdot i_6 \circ \eta_6 \qquad (a \geq 0 \in \mathbf{Z}, n_0 \in \mathbf{Z}/12, \varepsilon \in \mathbf{Z}/2).$$

If $x_{2l} \in H^{2l}(M, \mathbf{Z}) \cong \mathbf{Z}$ denotes the corresponding generator $(l = 2, 4)$, since M is an m-twisted $\mathbf{CP}^4$, the equality $x_4 \cdot x_4 = \pm x_8$ holds. Hence, by the solution of Hopf invariant one problem, we have $a = 1$. Moreover, it follows from [[6], Lemma 4.2] that $Sq^2 : \mathbf{Z}/2 = H^6(M, \mathbf{Z}) \xrightarrow{\cong} H^8(M, \mathbf{Z}/2) = \mathbf{Z}/2$ is an isomorphism. Then if $q : M \to M/S^2 \simeq N_\gamma$ denotes the pinch map, because $N_\gamma/S^4 \simeq S^6 \cup_{\varepsilon \cdot \eta_6} e^8$, it follows from the commutative diagram

$$\begin{array}{ccccc} H^6(M, \mathbf{Z}/2) & \xleftarrow[\cong]{q^*} & H^6(N_\gamma, \mathbf{Z}/2) & \xleftarrow[\cong]{} & H^6(S^6 \cup_{\varepsilon \cdot \eta_6} e^8, \mathbf{Z}/2) \cong \mathbf{Z}/2 \\ {\scriptstyle Sq^2} \downarrow \cong & & {\scriptstyle Sq^2} \downarrow & & {\scriptstyle Sq^2} \downarrow \\ H^8(M, \mathbf{Z}/2) & \xleftarrow[\cong]{q^*} & H^8(N_\gamma, \mathbf{Z}/2) & \xleftarrow[\cong]{} & H^8(S^6 \cup_{\varepsilon \cdot \eta_6} e^8, \mathbf{Z}/2) \cong \mathbf{Z}/2 \end{array}$$

that we obtain $\varepsilon = 1$. Therefore, (i) is proved.

(ii) By (i) we may assume that $M/S^2 = N(n_0)$. It follows from the solution of mod 3 Hopf invariant one problem that

$$\mathscr{P}^1 : \mathbf{Z}/3 \cong H^4(N(n_0), \mathbf{Z}/3) \to H^8(N(n_0), \mathbf{Z}/3) \cong \mathbf{Z}/3$$

is an isomorphism if $n_0 \not\equiv 0 \pmod 3$, and it is trivial if $n_0 \equiv 0 \pmod 3$. Then the assertion (ii) follows from the following commutative diagram.

$$\begin{array}{ccc} H^4(M, \mathbf{Z}/3) & \xrightarrow{\mathscr{P}^1} & H^8(M, \mathbf{Z}/3) \\ q^* \uparrow \cong & & q^* \uparrow \cong \\ \mathbf{Z}/3 \cong H^4(N(n_0), \mathbf{Z}/3) & \xrightarrow{\mathscr{P}^1} & H^8(N(n_0), \mathbf{Z}/3) \cong \mathbf{Z}/3. \end{array}$$ □

Theorem 5.8. *Let $m \geq 3$ be an odd integer with $m \equiv 0 \pmod 3$.*
(i) $[\mathrm{M}_0] \neq [\mathrm{M}_1]$, $[\mathrm{M}_0] \neq [\mathrm{M}_{-1}]$ *and* $[\mathrm{M}_1] \neq [\mathrm{M}_{-1}]$ *in* $\mathscr{M}_m^4$.
(ii) $\mathscr{M}_m^4 = \{[\mathrm{M}_0], [\mathrm{M}_1], [\mathrm{M}_{-1}]\}$.
(iii) *If we choose the free generator $\varphi_m \in \pi_7(X_m)$ suitably,*

$$\mathscr{P}^1 : \mathbf{Z}/3 \cong H^4(\mathrm{M}_\varepsilon, \mathbf{Z}/3) \to H^8(\mathrm{M}_\varepsilon, \mathbf{Z}/3) \cong \mathbf{Z}/3$$

is an isomorphism if $\varepsilon = \pm 1$ and it is trivial if $\varepsilon = 0$.

Proof. (i) It follows from Lemma 5.7 that there is a homotopy equivalence

$$\mathrm{M}_0/S^2 \simeq S^4 \vee S^6 \cup_\gamma e^8 = N(n_0) \qquad \text{for some } n_0 \in \mathbf{Z}/12,$$

where $\gamma = i_4 \circ v_4 + n_0 \cdot i_4 \circ E\omega + i_6 \circ \eta_6 \in \pi_7(S^4 \vee S^6)$.

Now we recall the definition of $\{\mathrm{M}_1, \mathrm{M}_{-1}, \mathrm{M}_0\}$; $\mathrm{M}_{\pm 1} = \mathrm{M}(\varphi_m \pm j_*(f_m \circ \omega_m))$ and $\mathrm{M}_0 = M(\varphi_m)$. If we consider the induced homomorphism

$$q'_{m*} : \mathbf{Z}/3 \cdot \omega_m = \pi_7(\mathrm{P}^4(m)) \to \pi_7(S^4) = \mathbf{Z} \cdot v_4 \oplus \mathbf{Z}/12 \cdot E\omega,$$

because $q'_{m*}(\omega_m) = 4E\omega = E\alpha_1(3)$ (by Lemma 2.1), we may assume that there are homotopy equivalences

$$(14) \qquad \mathrm{M}_1/S^2 \simeq N(k_0 + 4) \qquad \text{and} \qquad \mathrm{M}_{-1}/S^2 \simeq N(k_0 - 4).$$

Because $k_0 \pm 4 \equiv k_0 \pm 1 \pmod 3$, one of $\mathscr{N} = \{k_0, k_0 - 4, k_0 + 4\}$ is zero mod 3 and the other two numbers of $\mathscr{N}$ are both non-zero mod 3.

Hence, by Lemma 5.7, there is some $\varepsilon_0 \in \{0, 1, -1\}$ such that $\mathscr{P}^1 : \mathbf{Z}/3 \cong H^4(M_\varepsilon, \mathbf{Z}/3) \to H^8(M_\varepsilon, \mathbf{Z}/3) \cong \mathbf{Z}/3$ is trivial if $\varepsilon = \varepsilon_0$ and an isomorphism if $\varepsilon \in \{0, 1, -1\}$ and $\varepsilon \neq \varepsilon_0$.

So $[M_{\varepsilon_0}] \neq [M_{\varepsilon_1}]$ in $\mathscr{M}_m^4$ if $\varepsilon_0 \neq \varepsilon_1 \in \{0, 1, -1\}$. Then by Lemma 5.6, $[M_0] \neq [M_1]$, $[M_1] \neq [M_{-1}]$, $[M_{-1}] \neq [M_0]$ in $\mathscr{M}_m^4$.

(ii) The assertion (ii) follows from Proposition 4.8 and (i).

(iii) We note that $\pi_7(X_m) = \mathbf{Z} \cdot \varphi_m \oplus \mathbf{Z}/3 \cdot j_*(f_m \circ \omega_m) \oplus \mathbf{Z}/m \cdot [b_m, i_*(\eta_2)]$. Then if we change the free base φ_m by $\varphi_m \mapsto \varphi_m + \varepsilon_0 \cdot j_*(f_m \circ \omega_m)$, the assertion (iii) is also satisfied. □

Now we can complete the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. The assertion (i) follows from Theorem 4.6, and the assertions (ii), (iii) follow from Theorem 5.8. □

Finally we compute the action of $\tilde{\theta}$ on $\pi_7(X_m)$ explicitly.

THEOREM 5.9. *Let $m \geq 3$ be an odd integer with $m \equiv 0 \pmod 3$. Then the left action of $\tilde{\theta}$ on $\pi_7(X_m)$ is determined by the following:*

$$\begin{cases} \tilde{\theta} \circ j_*([b_m, i_*(\eta_2)]) = -j_*([b_m, i_*(\eta_2)]), \\ \tilde{\theta} \circ j_*(f_m \circ \omega_m) = j_*(f_m \circ \omega_m), \\ \tilde{\theta} \circ \varphi_m = \varphi_m. \end{cases}$$

PROOF. It follows from Theorem 5.8 and Lemma 5.6 that $l_m = 0$. Hence, the assertion follows from Lemma 5.5 and (13). □

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Kohhei Yamaguchi
Department of Information Mathematics
University of Electro-Communications
Chofu, Tokyo 182-8585 *Japan*
E-mail: kohhei@im.uec.ac.jp