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Correspondences to abelian varieties II

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ABSTRACT. When S is an algebraic scheme, and $X \to S$ and $Y \to S$ proper schemes over S, we define the notion of correspondences from X to Y over S. And when $Y \to S$ is a relative abelian scheme and X is a normal variety, we give a characterization for a correspondence from X to Y over S to be a graph of some morphism $X \to Y$ over S, which is a generalization of the result for classical correspondences in [4].

1. Introduction

Let $\alpha : X \vdash Y$ be a correspondence over a point, i.e., an element of the Chow group of $X \times Y$ where X and Y are smooth complete algebraic varieties [1, Chapter 16]. When $\alpha = \Gamma_f$ is a graph of some morphism $f : X \to Y$, then α satisfies the following 3 conditions:

(1) $\dim(\alpha) = \dim(X)$
(2) $\pi_*(\alpha) = [X]$
(3) $\Delta_Y \circ \alpha = (\alpha \times \alpha) \circ \Delta_X$.

Conversely, if the conditions (1), (2) and (3) are satisfied and Y is an Abelian variety, then α is a graph of some morphism [4, Theorem 2.7]. In the paper [5], the notion of correspondences is generalized to the situation where the base scheme can be any algebraic scheme, as far as the structure morphism is proper. In this paper, we will show that the result of [4] is valid in this general case, namely when Y is a relative abelian scheme over an algebraic scheme S, X is scheme over S with the structure morphism proper, and α a correspondence over S.

Convention and Notation. We work in the category of algebraic schemes over a fixed field κ. A variety means a reduced and irreducible scheme. A scheme X is smooth when the structure morphism $X \to \operatorname{Spec} \kappa$ is smooth. A_*X is the Chow group of X and A^*X the Chow cohomogloy group (see [1, Chapter 17]), with rational coefficients. The bivariant groups also have rational coefficients. The notation $X \overset{@}{\to} Y$ means that there exists a morphism $X \to Y$ and an element of the bivariant intersection group $\alpha \in A(X \to Y)$.

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2. Bivariant intersection theory and correspondences

The bivariant intersecction theory is defined and studied in [2] and [1, Chapter 17]. The notion of correspondences over any base scheme is defined and treated in [5]. In this section, we will briefly review them.

For a fixed morphism $f : X \to Y$, an element of the bivariant intersection group $\alpha \in A(X \to Y)$ determines a collection of homomorphisms between the Chow groups; for any morphism $\varphi : \tilde{Y} \to Y$, α determines a homomorphism $\alpha_\varphi : A_*(\tilde{Y}) \to A_*(\tilde{Y} \times_Y X)$. When moreover there is another morphism $\psi : \tilde{\tilde{Y}} \to \tilde{Y}$ which is proper (resp. flat, resp. base extension of a regular imbedding), then the homomorphisms α_φ and $\alpha_{\varphi\circ\psi}$ commute with the proper pushforwards (resp. flat pull-backs, resp. Gysin maps). Conversely, if a collection of homomorphisms of Chow groups

$$\{\alpha_\varphi : A_*(\tilde{Y}) \to A_*(\tilde{Y} \times_Y X) \,|\, \varphi : \tilde{Y} \to Y\}$$

commutes with these three functorialities of Chow groups, then this collection uniquely determines $\alpha \in A(X \to Y)$.

The evaluation map $ev : A(X \to Y) \to A_*X$ is defined by $ev(\alpha) := \alpha_{\mathrm{id}_Y}[Y]$. When Y is smooth, the evaluation map is bijective [1, Proposition 17.4.2], and the functor sending $\tilde{Y} \to Y$ to $A(X \times_Y \tilde{Y} \to \tilde{Y})$ is a sheaf in the proper topology [3, Theorem 2.3]. One can compute the bivariant intersection groups by using these facts (see [3]).

Definition 2.1. *Let X and Y be algebraic schemes over S such that the structure morphisms $X \to S$ and $Y \to S$ are proper. We define* a correspondence from X to Y over S *to be an element of $A(X \times_S Y \to X)$. We write $\alpha : X \vdash Y$ for the correspondence $\alpha \in A(X \times_S Y \to X)$. When $\alpha \in A^{-d}(X \times_S Y \to X)$, we define the dimension of α by* $\dim \alpha = d$.

Remark 2.2. A justification of this notion of correspondence is given in the appendix, from the viewpoint of bivariant sheaves.

Remark 2.3. When X and Y are smooth and the base scheme S is the point $\operatorname{Spec} \kappa$, the group $A(X \times_S Y \to X)$ is isomorphic to $A_*(X \times Y)$ by the evaluation map, and by this identification, we consider this correspondence as a generalization of the classical correspondences [1, Chapter 16].

Remark 2.4. Let $\alpha : X \vdash Y$ be a correspondence from X to Y. For $c \in A_*X$, we define $\alpha_*(c) := \pi_{Y*}(\alpha_{\mathrm{id}_X}(c))$ where $\pi_Y : X \times_S Y \to Y$ is the second projection. Also for $c \in A^*Y$, we define $\alpha^*(c) := \pi_{X*}(\pi_Y^* c \circ \alpha) \in A^*X$ where

$\pi_X : X \times_S Y \to X$ is the first projetion. One can easily check that these definitions agree with the classical correspondences when X and Y are smooth and the base scheme is the point.

Definition 2.5. *When $f : X \to Y$ is a morphism over S, with X and Y proper over S, we define its graph correspondence $\gamma_f : X \vdash Y$ to be $\Gamma_{f*}1 \in A(X \times_S Y \to X)$ where $\Gamma_f : X \to X \times_S Y$ is the graph morphism of f sending $x \in X$ to $(x, f(x)) \in X \times_S Y$ and $1 \in A(X \to X)$ is the bivariant intersection class defined as the identity operation.*

Remark 2.6. For a morphism $f : X \to Y$ over S with X and Y proper over S, it is easy to check that $f_* = \gamma_{f*} : A_*X \to A_*Y$ and $f^* = \gamma_f^* : A^*Y \to A^*X$.

Definition 2.7. *When $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$ are correspondences, we define their* composition *$\beta \circ \alpha : X \vdash Z$ to be $\pi_{XZ*}(p_Y^*\beta \circ \alpha) \in A(X \times_S Z \to X)$, where $\pi_{XZ} : X \times_S Y \times_S Z \to X \times_S Z$ and $p_Y : X \times_S Y \to Y$ are the natural projections, as in the diagram below:*

$$\begin{tikzcd}
X \times_S Z \arrow[rr, bend left] \arrow[ddd, bend right] & & X \\
X \times_S Y \times_S Z \arrow[u, "\pi_{XZ}"] \arrow[r, "p_Y^*\beta"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S Y \arrow[r, "\alpha"] \arrow[d, "p_Y"] & X \\
Y \times_S Z \arrow[r, "\beta"] \arrow[d] & Y & \\
Z & &
\end{tikzcd}$$

Remark 2.8. The composition of correspondences is associative, and the push-forward and pull-backs of Chow groups as in Remarks 2.4 are functorial. When $f : X \to Y$ and $g : Y \to Z$ are morphisms over S, then $\gamma_g \circ \gamma_f = \gamma_{g \circ f}$.

Definition 2.9. *When $\alpha_1 : X_1 \vdash Y_1$ and $\alpha_2 : X_2 \vdash Y_2$ are correspondences, we define $\alpha_1 \times \alpha_2 : X_1 \times_S X_2 \vdash Y_1 \times_S Y_2$ to be*

$$q_2^* p_2^* \alpha_2 \circ p_1^* \alpha_1 = q_1^* p_1^* \alpha_1 \circ p_2^* \alpha_2 \in A(X_1 \times_S X_2 \times_S Y_1 \times_S Y_2),$$

where the equality follows from the fact that there is an alteration for $X_1 \times_S X_2$, as in the fiber diagram below.

$$\begin{tikzcd}
X_1 \times_S X_2 \times_S Y_1 \times_S Y_2 \arrow[r, "\boxed{q_1^* p_1^* \alpha_1}"] \arrow[d, "\boxed{q_2^* p_2^* \alpha_2}"'] & X_1 \times_S X_2 \times_S Y_2 \arrow[r] \arrow[d, "\boxed{p_2^* \alpha_2}"', "q_1"] & X_2 \times_S Y_2 \arrow[d, "\alpha_2"'] \\
X_1 \times_S X_2 \times_S Y_1 \arrow[r, "\boxed{p_1^* \alpha_1}", "q_2"'] \arrow[d] & X_1 \times_S X_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & X_2 \\
X_1 \times_S Y_1 \arrow[r, "\alpha_1"] & X_1 &
\end{tikzcd}$$

We will need the following proposition.

PROPOSITION 2.10. *Let $f : X \to Y$ be a morphism over S, and $\alpha : Y \vdash Z$ a correspondence over S. When $\gamma_f : X \vdash Y$ is the graph correspondence of f, the composition $\alpha \circ \gamma_f$ equals $f^*\alpha$ by the pull-back $f^* : A(Y \times_S Z \to Y) \to A(X \times_S Z \to X)$.*

PROOF. Consider the following diagram:

$$\begin{tikzcd}
X \times_S Z \arrow[r] \arrow[d, "\tilde{\Gamma}_f"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Gamma_f"] \arrow[dr, "\mathrm{id}_X"] & \\
X \times_S Y \times_S Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S Y \arrow[r] \arrow[d, "\pi_Y"] & X \\
Y \times_S Z \arrow[r, "\alpha"'] \arrow[d] & Y & \\
Z & &
\end{tikzcd}$$

where Γ_f is the graph morphism of f, $\tilde{\Gamma}_f$ its base extension, and π_Y the projection. Also define $\pi_{XZ} : X \times_S Y \times_S Z \to X \times_S Z$ to be the projection. Then we have

$$\begin{aligned}
\alpha \circ [\gamma_f] &= \pi_{XZ*}(\pi_Y^*\alpha \circ (\Gamma_{f*}1)) \quad \text{(definition of the composition)} \\
&= \pi_{XZ*}(\tilde{\Gamma}_{f*}(\Gamma_f^*\pi_Y^*\alpha \circ 1) \quad \text{(commute with proper push-foward)} \\
&= \Gamma_f^*\pi_Y^*\alpha \quad (\pi_{XZ} \circ \tilde{\Gamma}_f = \mathrm{id}) \\
&= f^*\alpha \quad (\pi_Y \circ \Gamma_f = f).
\end{aligned}$$

□

3. Cycles on abelian schemes

In this section, we review the results from [4]. Throughout this section, we assume that $\pi : T \to X$ is a relative abelian scheme of relative dimension g

over a d-dimensional smooth scheme X, with $\mu : T \times_X T \to T$ the multiplication morphism.

Definition 3.1. *Let $\Delta_{X/\kappa} : X \to X \times X$ be the diagonal morphism over the point* Spec κ. *For cycles $\alpha, \beta \in A_* T$, we define their* Pontryagin product *$\alpha * \beta \in A_* T$ by $\alpha * \beta = \mu_*(\Delta^!_{X/\kappa}(\alpha \times \beta))$ where the Gysin map $\Delta^!_{X/\kappa}$ acts through the following diagram:*

$$\begin{tikzcd} T & T \times_X T \arrow[l, "\mu"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \times T \arrow[d] \\ & X \arrow[r, "\Delta_{X/\kappa}"] & X \times X \end{tikzcd}$$

Definition 3.2. *When n is an integer, we define $\mathbf{n}_T : T \to T$ to be the multiplication by n. A cycle $\alpha \in A_e T$ is in $A_{e,(s)} T$ if $\mathbf{n}_{T*}\alpha = n^{2e-2d+s}\alpha$ for all $n \in \mathbf{Z}$.*

Proposition 3.3 ([4, Prop. 1.9]). *The Chow group of T decomposes into the direct sum $A_e T = \bigoplus_s A_{e,(s)} T$ where s ranges from* $\max(d-e, 2(d-e))$ *to* $\min(2(g+d-e), 2d+g-e)$. $\square$

Definition 3.4. *Define $A_{d,(+)} T$ by $A_{d,(+)} T = \bigoplus_{s>0} A_{d,(s)} T$.*

Remark 3.5. A cycle $\alpha \in A_e T$ lies in $A_{e,(s)} T$ if and only if $\mathbf{n}_{T*}\alpha = n^{2e-2d+s}\alpha$ for some n with $|n| > 1$. For cycles $\alpha \in A_{*,(s)} T$ and $\beta \in A_{*,(t)} T$, we have $\alpha * \beta \in A_{*,(s+t)} T$ ([4, Prop. 1.10]). In particular, the elements of $A_{d,(+)} T$ are nilpotent for the Pontryagin products.

Proposition 3.6 ([4, Prop. 1.13]). *A d-dimensional cycle $\alpha \in A_d T$ lies in $A_{d,(+)} T$ if and only if $\pi_*\alpha = 0$.* $\square$

Definition 3.7. *When $\alpha \in A_d T$ satisfies $\pi_*\alpha = [X]$, we define* $\log \alpha$ *to be $\sum_{k=1}^{\infty} (-1)^{k-1}(y^{*k}/k) \in A_d T$ where $y = \alpha - [0_T]$ and $y^{*k} := y * y * \cdots * y$ is the k-th power of y in terms of the Pontryagin product.*

Theorem 3.8 ([4, Thm. 2.5]). *If $\alpha \in A_d T$ satisfies $\pi_*\alpha = [X]$ and* $\log(\alpha) \in A_{d,(1)} T$, *then α is a push-foward image of the cycle $[X]$ by some section $\varphi : X \to T$ of the projection π.* $\square$

4. Smooth case

In this section, we prove the main theorem for the case when X is a smooth variety.

Theorem 4.1. *Let $\alpha : X \vdash Y$ be a correspondence from X to Y over S, and assume that X is a smooth variety and $Y \to S$ a relative abelian scheme. The* *correspondence* α *is a graph of some morphism if and only if the following three conditions hold:*

(1) $\dim \alpha = \dim X$

(2) *The following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, mapsto, "\alpha"] \arrow[dr, mapsto, "p_X"'] & Y \arrow[d, mapsto, "p_Y"] \\ & S \end{tikzcd}$$

where p_X *and* p_Y *are the graphs of the structure morphisms.*

(3) *The following diagram commutes*

$$\begin{tikzcd} X \arrow[r, mapsto, "\alpha"] \arrow[d, mapsto, "\Delta_X"'] & Y \arrow[d, mapsto, "\Delta_Y"] \\ X \times_S X \arrow[r, mapsto, "\alpha\times\alpha"] & Y \times_S Y \end{tikzcd}$$

where $\Delta_X : X \vdash X \times_S X$ *and* $\Delta_Y : Y \vdash Y \times_S Y$ *are the graphs of the diagonal morphisms.*

PROOF. The conditions (1), (2) and (3) are satisfied when α is a graph of some morphism, because the construction of the graph correspondence is functorial (Remark 2.8). Conversely assume that $\alpha : X \vdash Y$ satisfies these three conditions. Define $c \in A(X \times_S Y)$ to be $c := \alpha_{\mathrm{id}}[X]$, the image of α by the evaluation map. By the condition (1), we have $c \in A_d(X \times_S Y)$ where $d = \dim X$. The condition (2) says that $\pi_{X*}\alpha = 1$ where $\pi_X : X \times_S Y \to X$ is the first projection and we consider α as an element of the bivariant intersection group $A(X \times_S Y \to X)$. It implies that $\pi_{X*}c = [X]$.

Now let us interpret the condition (3) in terms of c. The group of correspondences from X to $Y \times_S Y$ is $A(X \times_S Y \times_S Y \to X)$, which, by the evaluation map, is identified with $A(X \times_S Y \times_S Y)$. Let us calculate $\mathrm{ev}((\alpha \times \alpha) \circ \Delta_X)$ and $\mathrm{ev}(\Delta_Y \circ \alpha)$ in terms of c.

To calculate $\mathrm{ev}(\Delta_Y \circ \alpha)$, consider the following diagram:

$$\begin{tikzcd} X \times_S Y \arrow[r, "\tilde{\Delta}_Y^{(3)}"'] \arrow[d] \arrow[rr, bend left, "\alpha"] & X \times_S Y \times_S Y \times_S Y \arrow[r] \arrow[d] & X \times_S Y \arrow[r, "\alpha"'] \arrow[d] & X \\ Y \arrow[r, "\Delta_Y^{(3)}"'] & Y \times_S Y \times_S Y \arrow[r] \arrow[d] & Y & \\ & Y \times_S Y & & \end{tikzcd}$$

where $\Delta_Y^{(3)} : Y \to Y \times_S Y \times_S Y$ is the triple diagonal sending $y \in Y$ to $(y, y, y) \in Y \times_S Y \times_S Y$, and $\tilde{\Delta}_Y^{(3)} : X \times_S Y \to X \times_S Y \times_S Y \times_S Y$ is its base extension. By definition of the composition of correspondences, $\Delta_Y \circ \alpha$ is the push-forward of $\tilde{\Delta}_{Y*}^{(3)}\alpha \in A(X \times_S Y \times_S Y \times_S Y \to X)$ to $A(X \times_S Y \times_S Y \to X)$, which is $(\mathrm{id}_X \times \Delta_Y)_* \alpha$. Hence its evaluation is $(\mathrm{id}_X \times \Delta_Y)_* c \in A_*(X \times_S Y \times_S Y)$.

To calculate $\mathrm{ev}((\alpha \times \alpha) \circ \Delta_X)$, consider the following diagram:

$$\begin{tikzcd}
X \times_S Y \times_S Y \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X^{(3)}"'] \arrow[dr, "\mathrm{id}_X"] \arrow[ddd, bend left, "\Delta_{X/\kappa}"] & \\
X \times_S X \times_S X \times_S Y \times_S Y \arrow[r] \arrow[d] & X \times_S X \times_S X \arrow[r] \arrow[d] & X \\
X \times_S X \times_S Y \times_S Y \arrow[r, "\boxed{\alpha\times\alpha}"] \arrow[d] & X \times_S X \arrow[d] & \\
(X \times_S Y) \times (X \times_S Y) \arrow[r, "\boxed{\alpha\times_\kappa\alpha}"] & X \times X &
\end{tikzcd}$$

where $\Delta_{X/\kappa} : X \to X \times X$ is the diagonal over $\mathrm{Spec}\,\kappa$, $\Delta_X^{(3)} : X \to X \times_S X \times_S X$ is the triple diagonal and $\alpha \times_\kappa \alpha \in A(X \times_S Y) \times (X \times_S Y) \to A(X \times X)$ is the exterior product over $\mathrm{Spec}\,\kappa$. As the graph of Δ_X sends $[X]$ to $\Delta_{X*}^{(3)}[X] \in A(X \times_S X \times_S X)$ and $\alpha \times \alpha$ is the pull-back of $\alpha \times_\kappa \alpha$ by definition, the image of the evalutaion map of $(\alpha \times \alpha) \circ \Delta_X$ is the push-forward image of $\Delta_{X/\kappa}^!((\alpha \times_\kappa \alpha)[X \times X])$, which is $\Delta_{X/\kappa}^!(c \times_\kappa c)$. Hence the condition (3) is equivalent to saying that

$$\Delta_{X/\kappa}^!(c \times_\kappa c) = (\mathrm{id}_X \times \Delta_Y)_* c \in A_*(X \times_S Y \times_S Y) \cdots\cdots (*)$$

Now we apply the push-forward $(\mathrm{id}_X \times \mu)_*$ to the equality $(*)$ above, where $\mu : Y \times_S Y \to Y$ is the multiplication morphism. When we consider $X \times_S Y \to X$ as a relative abelian scheme over X, the morphism $\mathrm{id}_X \times \mu : X \times_S Y \times_S Y \to X \times_X Y$ is the multiplication morphism for this abelian scheme, and the identity $(\mathrm{id}_X \times \mu)_*(*)$ becomes $c * c = \mathbf{2}_{X \times_S Y *} c$. The logarithm (Definition 3.7) of the both sides, we obtain $2\log(c) = \mathbf{2}_{X \times_S Y *} \log(c)$. We can apply Theorem 3.8 to conclude that c is a push-forward image of some section $\varphi : X \to X \times_S Y$ of the projection $X \times_S Y \to X$. On the other hand, let $\psi : X \to Y$ be the composition $\pi_Y \circ \varphi : X \to Y$ where $\pi_Y : X \times_S Y \to Y$ is the projection. Then the graph correspondence of ψ sends $[X]$ to $\varphi_*[X] = c$. Because X is smooth, the evaluation morphism is bijective, hence α is the graph correspondence of $\psi : X \to Y$. We are done. $\square$

5. General case

In this section, we prove the main theorem in general.

Let $X \to S$ be any proper morphism. Let $p : \tilde{X} \to X$ be an alteration, a proper surjective morphism from a smooth scheme. Take the fiber product $\tilde{X} \times_S \tilde{X}$, and again take an alteration $\tilde{\tilde{X}} \to \tilde{X} \times_S \tilde{X}$. There are two morphisms $\pi_i : \tilde{\tilde{X}} \to \tilde{X}$, $(i = 1, 2)$, which passes through the first projection (resp. second projection) $\tilde{X} \times_X \tilde{X} \to \tilde{X}$.

Lemma 5.1. *There exists a scheme $\overline{X}$ which satisfies the following conditions.*

(1) *The morphism $\tilde{X} \to X$ factors through $\overline{X}$.*

(2) *The morphism $\overline{X} \to X$ is a universal homeomorphism. If X is normal, then the morphism $\overline{X} \to X$ is an isomorphism.*

(3) *For any scheme Y over S, a morphism $f : \tilde{X} \to Y$ factors through $\overline{X}$ if and only if $f \circ \pi_1 = f \circ \pi_2$. In other words,* $\mathrm{Hom}_S(\overline{X}, Y)$ *is the equalizer of* $\mathrm{Hom}_S(\tilde{X}, Y) \underset{\_\circ\pi_2}{\overset{\_\circ\pi_1}{\rightrightarrows}} \mathrm{Hom}_S(\tilde{\tilde{X}}, Y)$.

Proof. Define $\overline{X}$ to be $\mathrm{Spec}\,\mathrm{Ker}(p_*\mathcal{O}_{\tilde{X}} \to (p \circ \pi_i)_*\mathcal{O}_{\tilde{\tilde{X}}})$. Notice that $p \circ \pi_1 = p \circ \pi_2$, and we have two canonical morphisms $p_*\mathcal{O}_{\tilde{X}} \to (p \circ \pi_i)_*\mathcal{O}_{\tilde{\tilde{X}}}$. The kernel means the difference kernel of these two morphisms. Because these two morphisms are ring homomorphisms, the difference kernel is a subring sheaf, which contains $\mathcal{O}_X$, and coherent because p and $p \circ \pi_i$ are proper. The equalizer property follows from the construction. The universal homeomorphismness follows from [3, Prop. 3.6]. In particular, when X is normal, such a birational morphism $\overline{X} \to X$ must be an isomorphism. □

Remark 5.2. From the viewpoint of the intersection theory, universal homemorphisms behave like isomorphisms. For example, in the situation above, for any proper morphism $Y \to S$, the group of correspondences from X to Y is canonically isomorphic to the group of correspondences from $\overline{X}$ to Y by the composition with the graph of $\overline{X} \to X$.

Lemma 5.3. *Let X be a reduced scheme over S and $Y \to S$ an abelian scheme. Then the graph morphism*

$$\mathit{graph} : \mathrm{Hom}_S(X, Y) \to A(X \times_S Y \to X)$$

is injective.

Proof. Let $f : X \to Y$ be a morphism over S, and $x \in X$ a closed point on X, namely $x = \mathrm{Spec}\,K$ with K/κ a finite field extension. Because X is reduced, it is enough to show that the image $f(x) \in \mathrm{Hom}_S(\mathrm{Spec}\,K, Y)$ can be recovered from the graph $\gamma_f \in A(X \times_S Y \to X)$. Because $Y \to S$ is a relative abelian scheme, the base extension $Y \times_S \{x\}$ is an abelian variety over $\{x\} = \operatorname{Spec} K$. Let $\iota : x \to X$ be the closed immersion, and take the bivariant action $\gamma_{f,\iota}([x]) \in A_0(Y \times_S \operatorname{Spec} K)$. Sending the image by the albanese map $\mathrm{Alb} : A_0(Y \times_S \operatorname{Spec} K) \to Y \times_S \operatorname{Spec} K$, we recover the image $f(x)$. $\square$

THEOREM 5.4. *Let $X \to S$ be any proper morphism from an equidimensional scheme, $Y \to S$ a relative abelian scheme, and $\overline{X} \to X$ the universally homeomorphism as in Lemma* 5.1. *Let $\alpha : X \to Y$ be a correspondence. Then the composition of α with the graph of the morphism $\overline{X} \to X$ is a graph of some morphism $\overline{X} \to Y$, if and only if α satisfies the following three conditions:*

(1) $\dim(\alpha) = \dim(X)$

(2) $\pi_*(\alpha) = [X]$ *where $\pi : X \times_S Y \to X$ is the projection*

(3) $\Delta_Y \circ \alpha = (\alpha \times \alpha) \circ \Delta_X$.

In particular, when X is normal, then α itself is a graph of some morphism $X \to Y$.

PROOF. When α is a graph of some morphism, then obviously the conditions (1), (2) and (3) hold. Conversely, assume the conditions (1), (2) and (3). Take $\tilde{X}$ and $\tilde{\tilde{X}}$ as above, and consider the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}_S(\overline{X}, Y) & \xrightarrow{\ _\circ p\ } & \operatorname{Hom}_S(\tilde{X}, Y) & \underset{_\circ\pi_2}{\overset{_\circ\pi_1}{\rightrightarrows}} & \operatorname{Hom}_S(\tilde{\tilde{X}}, Y) \\
\downarrow & & \downarrow & & \downarrow \\
A(\overline{X} \times_S Y \to \overline{X}) & \longrightarrow & A(\tilde{X} \times_S Y \to \tilde{X}) & \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} & A(\tilde{\tilde{X}} \times_S Y \to \tilde{\tilde{X}})
\end{array}$$

For the upper horizontal sequence, the equalizer property of Lemma 5.1 (3) says that the upper left corner $\operatorname{Hom}_S(\overline{X}, Y)$ is the equalizer of $\circ\pi_1$ and $\circ\pi_2$. On the other hand, for the bottom horizontal sequence, the compositions with the graph correspondences are just the pull-backs of bivariant intersection groups by Proposition 2.10, and hence $A(\overline{X} \times_S Y \to \overline{X})$ is the kernel of the sequence by the sheaf property of the bivariant intersection groups [3, Thm. 2.3]. If $\alpha \in A(\overline{X} \times_S X \to \overline{X})$, an element in the left bottom corner, satisfies the conditions (1), (2) and (3), then we need to show that it is in the image from $\operatorname{Hom}_S(\overline{X}, Y)$, the top left corner.

The bivariant intersection class $\alpha \in A(\overline{X} \times_S Y \to \overline{X})$ is sent to $\alpha \circ \gamma_p : \tilde{X} \vdash Y$ where $\gamma_p : \overline{X} \to X$ is the graph correspondence, hence the image also satisfies the conditions (1), (2) and (3). Because $\tilde{X}$ is smooth, it is in the image of some morphism $\tilde{\varphi} \in \operatorname{Hom}_S(\tilde{X}, Y)$ by Theorem 4.1. Its composition with π_1 and π_2 is equal, because their images in $A(\tilde{\tilde{X}} \times_S Y \to \tilde{\tilde{X}})$ are the same. Therefore $\tilde{\varphi}$ can be written as $\tilde{\varphi} = \varphi \circ p$ by some morphism $\varphi : \overline{X} \to Y$.

The graph correspondence of φ agrees with α, because their images in $A(\tilde{X} \times_S Y \to \tilde{X})$ agree. We are done. $\square$

6. Appendix: bivariant sheaves

In this section, we give a brief explanation of bivariant sheaves, which is behind the notion of correspondences, which is defined and studied in [5]. More details will be published elsewhere.

Let S be a base scheme. For a morphism $T \to S$, we define the contravariant functor $\mathscr{A}_T$ from the category of schemes over S to abelian groups by $\mathscr{A}_T(X) := A(X \times_S T \to X)$, sending morphisms to the pull-backs. This functor $\mathscr{A}_T$ enjoys the following two properties.

(1) ([3, Thm. 3.1]) $\mathscr{A}_T$ is a sheaf on the proper cite on S. Namely, when $X \to Y$ is a proper morphisms over S, then we have an exact sequence

$$0 \to \mathscr{A}(Y) \to \mathscr{A}(X) \rightrightarrows \mathscr{A}(X \times_Y X).$$

(2) When $\alpha : X \vdash Y$ is a correspondence (as in Definition 2.1), then we have a pull-back $\alpha^* : \mathscr{A}(Y) \to \mathscr{A}(X)$. This pull-back is functorial, namely when $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$ are correspondences, then $(\beta \circ \alpha)^* = \alpha^* \beta^* : \mathscr{A}(Z) \to \mathscr{A}(X)$. When $\alpha = \gamma_f : X \to Y$ is a graph correspondence of some morphism $f : X \to Y$, then $f^* = \alpha^* : \mathscr{A}(Y) \to \mathscr{A}(X)$.

We define bivariant sheaves to be the sheaves on the proper cite (over S), which satisfies the condition (2) above. Morphisms between bivariant sheaves are the morphisms of sheaves, which is compatible with the correspondence pull-backs. The category of bivariant sheaves is an abelian category ([5, Thm. 6.4]).

The bivariant sheaf $\mathscr{A}_T$ behaves like "the motif of T over S". For example, if S is smooth, its global section $\mathscr{A}_T(S)$ is canonically isomorphic to the Chow group $A_* T$. Moreover, the morphisms between the bivariant sheaves agree with the correspondences between the schemes:

PROPOSITION 6.1 (Yoneda Lemma). *Let $X \to S$ be a scheme over S with the proper structure morphism, and $\mathscr{F}$ a bivariant sheaf on S. Then the group of morphisms from $\mathscr{A}_X$ to $\mathscr{F}$ as bivariant sheaves is canonically isomorphic to $\mathscr{F}(X)$.*

PROOF. For each $X \to S$, we have the graph of the identity morphism $\gamma_{\Delta X} = \Delta_{X*} 1 \in A(X \times_S X \to X) = \mathscr{A}_X(X)$. Send each morphism of bivariant sheaves $\varphi : \mathscr{A}_X \to \mathscr{F}$ to $\varphi(X)(\gamma_{\Delta X}) \in \mathscr{F}(X)$.

Conversely, when $\alpha \in \mathscr{F}(X)$, define $\varphi_\alpha : \mathscr{A}_X \to \mathscr{F}$ by sending $\beta \in \mathscr{A}_X(T) = A(X \times_S T \to T)$ to $\beta^*\alpha$, where we consider $\beta : T \vdash X$ as a correspondence. One easily checks that φ is a morphism of bivariant sheaves, and by these maps, we have the bijection $\mathrm{Hom}_{Biv}(\mathscr{A}_X, \mathscr{F}) \simeq \mathscr{F}(X)$. □

In particular, when $\mathscr{F} = \mathscr{A}_Y$, then the morphisms of bivariant sheaves from $\mathscr{A}_X$ (the motif of X) to $\mathscr{A}_Y$ (the motif of Y) is identified with $\mathscr{A}_Y(X) = A(X \times_S Y \to X)$, which is the group of correspondences from X to Y. In this way, the category of pure motives is a faithful subcategory of the category of bivariant sheaves.

The category of bivariant sheaves has more properties. For example, when $\mathscr{F}$ and $\mathscr{G}$ are bivariant sheaves, then for each $T \to S$, define $\mathscr{H}om(\mathscr{F}, \mathscr{G})(T) = \mathrm{Hom}_{Biv}(\mathscr{F}_{|T}, \mathscr{G}_{|T})$, then the functor $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ has a natural structure of bivariant sheaf. By defining $\mathscr{F} \otimes \mathscr{G}$ to be the unique bivariant sheaf which satisfies $\mathrm{Hom}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \simeq \mathrm{Hom}(\mathscr{F}, \mathscr{H}om(\mathscr{G}, \mathscr{H}))$, we have a Künneth formula $\mathscr{A}_X \otimes \mathscr{A}_Y \simeq \mathscr{A}_{X \times_S Y}$.

It seems that the bivariant sheaves and correspondences can be a powerful tool to study pure motives.

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