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A 2-variable polynomial invariant for a virtual link derived from magnetic graphs

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Abstract. We introduce a 2-variable polynomial invariant for a virtual link derived from virtual magnetic graph diagrams, and using this invariant we prove the splitting of the Jones-Kauffman polynomial with respect to the powers module four.

1. Introduction

A virtual link diagram is a link diagram in $\mathbf{R}^2$ possibly with some encircled crossings without over/under information, called *virtual crossings*. A virtual link ([6]) is the equivalence class of such a link diagram by generalized Reidemeister moves (Reidemeister moves of type I, of type II, of type III and virtual Reidemeister moves of type I, of type II, of type III, and of type IV illustrated in Figure 1).

In [6], Kauffman defined a polynomial invariant $f_L(A) \in \mathbf{Z}[A^2, A^{-2}]$ for a virtual link L, which we call the Jones-Kauffman polynomial. For a classical link L, it is the Jones polynomial $V_L(t)$ after substituting $\sqrt{t}$ for A^2. In particular, for a classical link L, we have $f_L(A) \in \mathbf{Z}[A^4, A^{-4}]$ or $f_L(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^2$ according as the number of components of L is odd or even, respectively. However, for a virtual link L, $f_L(A)$ is decomposed nontrivially into two parts belonging to $\mathbf{Z}[A^4, A^{-4}]$ and to $\mathbf{Z}[A^4, A^{-4}] \cdot A^2$. For example, the Jones-Kauffman polynomial of L in Figure 2 is $A^8 - A^4 - A^2 + 1 + A^{-2}$, which is $(A^8 - A^4 + 1) + (-A^2 + A^{-2})$.

The purpose of this paper is to define a 2-variable polynomial invariant for a virtual link by using magnetic graphs and to show how the splitting of the Jones-Kauffman polynomial in the sense above is related to this invariant.

Our invariant was introduced by the second author at the regional conference on knot theory held in Yamagata, Japan in January of 2004.

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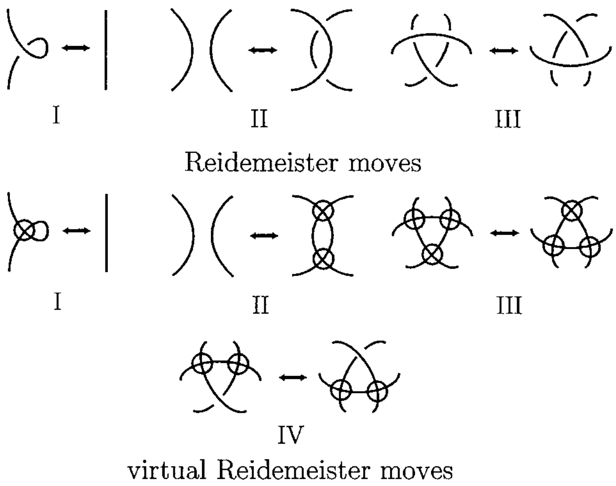

Fig. 1

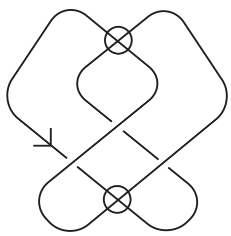

Fig. 2

In Section 2, the definitions and results are given. Sections 3 and 4 are devoted to the proofs. In Section 5, we observe the invariants for checkerboard colorable virtual link diagrams.

2. Preliminaries and results

A *magnetic graph* in the 3-sphere S^3 is a 2-valent graph G in S^3 such that the edges of G are oriented alternately as in Figure 3. We allow G to have components consisting of closed edges without vertices.

Fig. 3

A *magnetic graph diagram* is a projection image on a plane equipped with over/under information on each crossing. See Figure 4, for example.

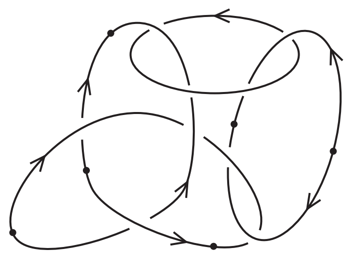

Fig. 4

A *virtual magnetic graph diagram*, which is written as VMG diagram for short, is a magnetic graph diagram possibly with some encircled crossings without over/under information, called virtual crossings. See Figure 5. Crossings that are not encircled are called *real crossings*. If two VMG diagrams are related by a finite sequence of generalized Reidemeister moves, they are said to be *equivalent*.

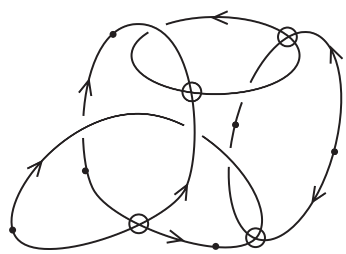

Fig. 5

Let D be a VMG diagram whose crossings are all virtual and $e(D)$ the set of edges of D. A *weight map* of D is a map $\sigma : e(D) \to \{+1, -1\}$ such that $\sigma(e) \neq \sigma(e')$ for magnetically adjacent edges e and e' of D. The image of an edge e of D by a weight map σ is called the *weight* of e with respect to σ. See Figure 6, for example, where each edge e is labeled its weight $\sigma(e)$.

When a weight map σ is given, for a virtual crossing v of D where edges e and e' intersect, the *parity* of v with respect to σ is defined to be the product of two weights $\sigma(e)$ and $\sigma(e')$ and denoted by $\mathrm{i}_\sigma(v)$. We call v a *regular crossing* or an *irregular crossing* with respect to σ according as $\mathrm{i}_\sigma(v) = +1$ or $\mathrm{i}_\sigma(v) = -1$, respectively. The *parity* of D is defined to be the product of parities over all virtual crossings. By Lemma 1 below, we denote it by $\mathrm{i}(D)$ regardless of σ. In other words, $\mathrm{i}(D) = +1$ if the number of irregular crossings of D is even; otherwise $\mathrm{i}(D) = -1$.

For example, there are 5 irregular crossings in Figure 6 and $\mathrm{i}(D) = -1$.

LEMMA 1. *Let D be a VMG diagram whose crossings are all virtual. Then, the parity of D does not depend on the choice of the weight map.*

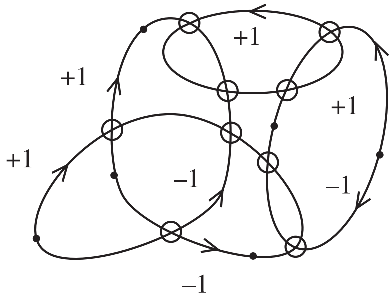

Fig. 6

PROOF. Let D be a VMG diagram whose crossings are all virtual and $e(D)$ the set of edges of D. Let σ and σ' be two weight maps of D. $e(D)$ can be split into two subsets $e_1(D)$ and $e_2(D)$ so that $\sigma(e) = \sigma'(e)$ if e is an element of $e_1(D)$ and $\sigma(e) = -\sigma'(e)$ if e is an element of $e_2(D)$. Note that all the edges of a component of D belong to the same subset $e_1(D)$ or $e_2(D)$, where a component of D means a closed curve in D as a component of a link. Let D_i, $i = 1, 2$, be the set of components of D whose edges belong to $e_i(D)$. Let v be a crossing of D. The parity of v with respect to σ is different from the parity of v with respect to σ' if and only if v is a crossing between an edge from D_1 and an edge from D_2. However the number of such crossings is even because two different components of D have even crossings. Thus, the parity of D is well defined. $\square$

Let D be a VMG diagram, and let p be a real crossing. By *doing* 0-*splice* (resp. ∞-*splice*) at p, we mean the local replacement nearby p illustrated in Figure 7.

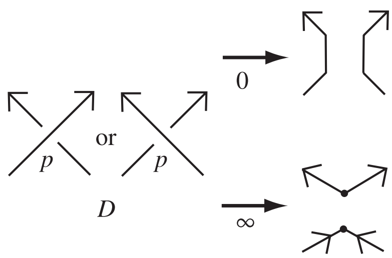

Fig. 7

A *state* of D is a VMG diagram obtained from D by doing 0-splice or ∞-splice at each real crossing. (Our states correspond to *oriented states* in [5].) The set of states of D is denoted by $s(D)$. We denote by $C_0(D;S)$ (resp. $C_\infty(D;S)$) the set of real crossings of D where 0-splices (resp. ∞-splices) are applied to obtain a state S from D. We also denote the sign of a real crossing p by $\operatorname{sign}(p)$.

For a VMG diagram D, we define

$$H_D(A,h) = \sum_{S\in s(D)} A^{\natural S}(-A^2 - A^{-2})^{\#S-1} h^{(1-\mathrm{i}(S))/2} \in \mathbf{Z}[A, A^{-1}, h],$$

where $\natural S$ is $\sum_{p\in C_0(D;S)} \operatorname{sign}(p) - \sum_{p\in C_\infty(D;S)} \operatorname{sign}(p)$ and $\#S$ is the number of components of S.

We define $R_D(A,h)$ to be $(-A^3)^{-\omega(D)} H_D(A,h)$, where $\omega(D)$, called the *writhe*, is the number of positive crossings minus that of negative crossings of D.

Theorem 2. *Let D and D' be VMG diagrams.*
(1) *If D' is related to D by a finite sequence of generalized Reidemeister moves but Reidemeister move of type* I, *then $H_D(A,h) = H_{D'}(A,h)$.*
(2) *If D' is equivalent to D, then $R_D(A,h) = R_{D'}(A,h)$.*

For a VMG diagram D, we decompose $R_D(A,h)$ to $\Phi_D(A) + \Psi_D(A)h$, where $\Phi_D(A)$ and $\Psi_D(A)$ are elements of $\mathbf{Z}[A, A^{-1}]$. Then $\Phi_D(A)$ and $\Psi_D(A)$ are invariants of the equivalence class of D.

Remark. Let D be a virtual link diagram. By the definition of the bracket polynomial $\langle D\rangle$ and the Jones-Kauffman polynomial $f_D(A)$ in [6], we have $\langle D\rangle = H_D(A,1)$ and $f_D(A) = R_D(A,1)$. In particular, $f_D(A) = \Phi_D(A) + \Psi_D(A)$.

Theorem 3. *Let D be a μ-component virtual link diagram. Then $\Phi_D(A) \in \mathbf{Z}[A^4, A^{-4}]\cdot A^{2(\mu-1)}$ and $\Psi_D(A) \in \mathbf{Z}[A^4, A^{-4}]\cdot A^{2\mu}$.*

Theorem 3 gives the splitting of the Jones-Kauffman polynomial $f_L(A)$ as stated in Section 1.

3. Proof of Theorem 2

Let D be a VMG diagram and p a real crossing on D. We denote by Z_pD and I_pD the diagrams obtained from D by doing 0-splice and ∞-splice at p, respectively. For example, for different real crossings p_1 and p_2 on D, $I_{p_2}Z_{p_1}D$ means the diagram obtained from D by doing ∞-splice at p_2 after applying 0-splice at p_1. Note that two diagrams $I_{p_2}Z_{p_1}D$ and $Z_{p_1}I_{p_2}D$ are identical.

LEMMA 4. *Let D be a VMG diagram and p a real crossing on D. Then, we have*

$$H_D(A,h) = A^{\mathrm{sign}(p)}H_{Z_pD}(A,h) + A^{-\mathrm{sign}(p)}H_{I_pD}(A,h).$$

PROOF. Let $s(D)$, $s(Z_pD)$ and $s(I_pD)$ be the sets of states of D, Z_pD and I_pD, respectively. Then, $s(D)$ is the direct sum of $s(Z_pD)$ and $s(I_pD)$. Let S be a state of Z_pD. It can be also regarded as a state of D. Then, it is clear that

$$\begin{aligned}&\sum_{q\in C_0(D;S)}\mathrm{sign}(q) - \sum_{q\in C_\infty(D;S)}\mathrm{sign}(q)\\ &\quad = \mathrm{sign}(p) + \sum_{q\in C_0(Z_pD;S)}\mathrm{sign}(q) - \sum_{q\in C_\infty(Z_pD;S)}\mathrm{sign}(q),\end{aligned}$$

since $C_0(D;S) = C_0(Z_pD;S)\cup\{p\}$ and $C_\infty(D;S) = C_\infty(Z_pD;S)$. If S is a state of I_pD, then we see that

$$\begin{aligned}&\sum_{q\in C_0(D;S)}\mathrm{sign}(q) - \sum_{q\in C_\infty(D;S)}\mathrm{sign}(q)\\ &\quad = -\mathrm{sign}(p) + \sum_{q\in C_0(I_pD;S)}\mathrm{sign}(q) - \sum_{q\in C_\infty(I_pD;S)}\mathrm{sign}(q).\end{aligned}$$

Thus, we have

$$\begin{aligned}H_D(A,h) &= \sum_{S\in s(D)} A^{\natural S}(-A^2-A^{-2})^{\#S-1}h^{(1-\mathrm{i}(S))/2}\\ &= \sum_{S\in s(Z_pD)} A^{\natural S+\mathrm{sign}(p)}(-A^2-A^{-2})^{\#S-1}h^{(1-\mathrm{i}(S))/2}\\ &\quad + \sum_{S\in s(I_pD)} A^{\natural S-\mathrm{sign}(p)}(-A^2-A^{-2})^{\#S-1}h^{(1-\mathrm{i}(S))/2}\\ &= A^{\mathrm{sign}(p)}H_{Z_pD}(A,h) + A^{-\mathrm{sign}(p)}H_{I_pD}(A,h).\end{aligned}$$

□

LEMMA 5. *Let D be a VMG diagram and D' a disjoint union of D and a trivial circle. Then,*

$$H_{D'}(A,h) = (-A^2-A^{-2})H_D(A,h).$$

PROOF. Let S be any state of D. Then, there exists a unique state S' of D' such that S' is a disjoint union of S and a trivial circle U. Since there is no crossing on U, we have $\natural S' = \natural S$ and $\mathrm{i}(S') = \mathrm{i}(S)$. Since $\# S' = \# S + 1$, we obtain

$$\begin{aligned} H_{D'}(A, h) &= \sum_{S' \in s(D')} A^{\natural S'}(-A^2 - A^{-2})^{\# S' - 1} h^{(1 - \mathrm{i}(S'))/2} \\ &= \sum_{S \in s(D)} A^{\natural S}(-A^2 - A^{-2})^{\# S} h^{(1 - \mathrm{i}(S))/2} \\ &= (-A^2 - A^{-2}) H_D(A, h), \end{aligned}$$

where $s(D)$ and $s(D')$ mean the sets of states of D and D', respectively. $\square$

Let D be a VMG diagram and B a local disk in $\mathbf{R}^2$. We denote the restriction of D to B by $T_B(D)$. Let $T_B(D)$ be an arc which has neither self crossing nor vertex. If we divide the edge of D including $T_B(D)$ into two or three edges by putting two vertices v_1 and v_2 on $T_B(D)$ and reverse the orientation of the edge on $T_B(D)$, whose endpoints are v_1 and v_2, we obtain a new VMG diagram D'. We say that D' is obtained from D by the *division* of an edge.

Lemma 6. *Let D and D' be VMG diagrams. If D' is obtained from D by the division of an edge, then*

$$H_D(A, h) = H_{D'}(A, h).$$

Proof. Let S be any state of D. Then, there exists a unique state S' of D' satisfying $\natural S' = \natural S$, $\# S' = \# S$ and $\mathrm{i}(S') = \mathrm{i}(S)$. This completes the proof. $\square$

A local move for a diagram such as a Reidemeister move is applied in a local disk in $\mathbf{R}^2$. We call such a disk a *stage* for the move.

Lemma 7. *Let D and D' be VMG diagrams. If D' is obtained from D by applying a Reidemeister move of type* I*, as shown in Figure* 1*, which eliminates a crossing p of D, then*

$$H_{D'}(A, h) = (-A^3)^{\mathrm{sign}(p)} H_D(A, h).$$

Proof. By Lemma 4, we have

$$H_D(A, h) = A^{\mathrm{sign}(p)} H_{Z_p D}(A, h) + A^{-\mathrm{sign}(p)} H_{I_p D}(A, h).$$

Since $Z_p D$ is a disjoint union of D' and a trivial circle, by Lemma 5, we obtain

$$H_{Z_p D}(A, h) = (-A^2 - A^{-2}) H_{D'}(A, h).$$

Lemma 6 gives the coincidence of the two polynomials $H_{I_pD}(A,h)$ and $H_{D'}(A,h)$. Since $A^{\mathrm{sign}(p)}(-A^2-A^{-2})+A^{-\mathrm{sign}(p)}=(-A^3)^{\mathrm{sign}(p)}$, we have the desired formula. $\square$

Lemma 8. *Let D and D' be VMG diagrams. If D' is obtained from D by applying a Reidemeister move of type* II *as shown in Figure* 1*, then*

$$H_D(A,h)=H_{D'}(A,h).$$

Proof. We may assume that the number of crossings of D' is less than that of crossings of D. Let B be a stage for a Reidemeister move of type II. Since D is oriented, we need to consider two cases according to orientations of the arcs of D in B. Suppose that the two arcs of D in B have parallel orientations. Let p_1 and p_2 be the crossings of D in B. The sign of p_1 is different from that of p_2. We may assume that p_1 is a positive crossing and p_2 is a negative crossing. By Lemma 4, we have

$$\begin{aligned}
H_D(A,h) &= AH_{Z_{p_1}D}(A,h)+A^{-1}H_{I_{p_1}D}(A,h)\\
&= A\{A^{-1}H_{Z_{p_2}Z_{p_1}D}(A,h)+AH_{I_{p_2}Z_{p_1}D}(A,h)\}\\
&\quad +A^{-1}\{A^{-1}H_{Z_{p_2}I_{p_1}D}(A,h)+AH_{I_{p_2}I_{p_1}D}(A,h)\}\\
&= H_{Z_{p_2}Z_{p_1}D}(A,h)+A^2H_{I_{p_2}Z_{p_1}D}(A,h)\\
&\quad +A^{-2}H_{Z_{p_2}I_{p_1}D}(A,h)+H_{I_{p_2}I_{p_1}D}(A,h).
\end{aligned}$$

It is clear that $H_{Z_{p_2}Z_{p_1}D}(A,h)=H_{D'}(A,h)$ and $H_{I_{p_2}Z_{p_1}D}(A,h)=H_{Z_{p_2}I_{p_1}D}(A,h)$. Since $I_{p_2}I_{p_1}D$ is a disjoint union of $Z_{p_2}I_{p_1}D$ and a trivial circle with two vertices, by Lemmas 5 and 6, we obtain

$$H_{I_{p_2}I_{p_1}D}(A,h)=(-A^2-A^{-2})H_{Z_{p_2}I_{p_1}D}(A,h).$$

It follows that $H_D(A,h)=H_{D'}(A,h)$. The proof of the other case is similar to the above. $\square$

If a diagram is oriented, there exist some kinds of Reidemeister moves of type III in a stage B according to signs of the three crossings in B. A Reidemeister move of type III is called *basic* if all the signs of the three crossings in B coincide.

A Reidemeister move of type III in B can be regarded as a passage of one of the three arcs over the crossing p between the others. The arc which pass over p is called the *top* arc and the remaining arcs are called the *middle* and the *bottom* arcs, where the middle arc and the bottom arc correspond to the overpath and the underpath at p, respectively. When a diagram D is related to another one by a Reidemeister move of type III in B, the three arcs of D can be named according to the above.

Lemma 9. *Let D and D' be VMG diagrams. If D' is obtained from D by applying a basic Reidemeister move of type* III, *then*

$$H_D(A,h) = H_{D'}(A,h).$$

Proof. Let B be a stage for a Reidemeister move of type III. Let p_1, p_2 and p_3 be crossings between the top and the middle arcs, the top and the bottom arcs and the middle and the bottom arcs of D in B, respectively. We denote the three crossings of D' in B by p_1', p_2' and p_3' similarly. Suppose that all the signs of the three crossings p_1, p_2 and p_3 are $+1$. Then, all the signs of p_1', p_2' and p_3' are also $+1$. By Lemma 4, we have

$$H_D(A,h) = AH_{Z_{p_3}D}(A,h) + A^{-1}H_{I_{p_3}D}(A,h)$$

and

$$H_{D'}(A,h) = AH_{Z_{p_3'}D'}(A,h) + A^{-1}H_{I_{p_3'}D'}(A,h).$$

Since the diagrams $Z_{p_3}D$ and $Z_{p_3'}D'$ are the same, The polynomial $H_{Z_{p_3}D}(A,h)$ coincides with the polynomial $H_{Z_{p_3'}D'}(A,h)$. By Lemmas 4, 5 and 6, we obtain

$$\begin{aligned} H_{I_{p_3}D}(A,h) &= A^2H_{Z_{p_2}Z_{p_1}I_{p_3}D}(A,h) + H_{Z_{p_2}I_{p_1}I_{p_3}D}(A,h) \\ &\quad + H_{I_{p_2}Z_{p_1}I_{p_3}D}(A,h) + A^{-2}H_{I_{p_2}I_{p_1}I_{p_3}D}(A,h) \\ &= A^2H_{Z_{p_2}Z_{p_1}I_{p_3}D}(A,h) + (-A^2 - A^{-2})H_{Z_{p_2}Z_{p_1}I_{p_3}D}(A,h) \\ &\quad + H_{I_{p_2}Z_{p_1}I_{p_3}D}(A,h) + A^{-2}H_{I_{p_2}I_{p_1}I_{p_3}D}(A,h) \\ &= -A^{-2}H_{Z_{p_2}Z_{p_1}I_{p_3}D}(A,h) + H_{I_{p_2}Z_{p_1}I_{p_3}D}(A,h) \\ &\quad + A^{-2}H_{I_{p_2}I_{p_1}I_{p_3}D}(A,h). \end{aligned}$$

Since $I_{p_2}I_{p_1}I_{p_3}D$ is obtained from $Z_{p_2}Z_{p_1}I_{p_3}D$ by applying the division of an edge twice, by Lemma 6, we have

$$H_{Z_{p_2}Z_{p_1}I_{p_3}D}(A,h) = H_{I_{p_2}I_{p_1}I_{p_3}D}(A,h).$$

It follows that $H_{I_{p_3}D}(A,h) = H_{I_{p_2}Z_{p_1}I_{p_3}D}(A,h)$. It is shown that $H_{I_{p_3'}D'}(A,h) = H_{I_{p_2'}Z_{p_1'}I_{p_3'}D'}(A,h)$ similarly. Since two diagrams $I_{p_2}Z_{p_1}I_{p_3}D$ and $I_{p_2'}Z_{p_1'}I_{p_3'}D'$ are the same, the two facts above give the coincidence of two polynomials $H_{I_{p_3}D}(A,h)$ and $H_{I_{p_3'}D'}(A,h)$. It follows that $H_D(A,h) = H_{D'}(A,h)$. The other case is easily verified by a similar argument. $\square$

Any Reidemeister move of type III can be realized by a sequence of one basic Reidemeister move of type III and some Reidemeister moves of type II. Using this, by Lemmas 8 and 9, we have the following.

LEMMA 10. *For a VMG diagram D, $H_D(A,h)$ is invariant under the Reidemeister move of type* III.

The following two lemmas are about virtual Reidemeister moves.

LEMMA 11. *Let D and D' be VMG diagrams. If D' is obtained from D by applying any virtual Reidemeister move of type* I, *of type* II *or of type* III *as shown in Figure* 1, *then*

$$H_D(A,h) = H_{D'}(A,h).$$

PROOF. We may assume that the number of virtual crossings of D is greater than or equal to that of virtual crossings of D' in either case. Let B be a stage for a virtual Reidemeister move. Since D and D' have no real crossing in B and the restrictions of D and D' to $\mathbf{R}^2 \backslash B$ are identical, for any state S of D, there exists a unique state S' of D' such that the restrictions of S and S' to $\mathbf{R}^2 \backslash B$ are identical. Then, it is easy to see that $\# S = \# S'$ and $\natural S = \natural S'$. We also see that $\mathrm{i}(S) = \mathrm{i}(S')$ by the following. As for the move of type I, the virtual crossing of S in B is always regular. So, the parity of the crossing is equal to 1. As for the move of type II, the product of parities of the two virtual crossings of S in B is equal to 1 for any weight map for S. As for the move of type III, the product of parities of the three crossings of S in B is equal to that of S' in B regardless of the weight map. Hence, we have $H_D(A,h) = H_{D'}(A,h)$. □

LEMMA 12. *Let D and D' be VMG diagrams. If D' is obtained from D by applying a virtual Reidemeister move of type* IV *as shown in Figure* 1, *then*

$$H_D(A,h) = H_{D'}(A,h).$$

PROOF. Let B be a stage for a virtual Reidemeister move of type IV. D (resp. D') has exactly three crossings in B, one of which is a real crossing p_1 (resp. p_1') and the others are virtual crossings. We have two cases according to the sign of p_1. Suppose that the sign of p_1 is equal to -1. Then, the sign of p_1' is equal to -1. By Lemma 4, we have

$$H_D(A,h) = A^{-1} H_{Z_{p_1} D}(A,h) + A H_{I_{p_1} D}(A,h)$$

and

$$H_{D'}(A,h) = A^{-1} H_{Z_{p_1'} D'}(A,h) + A H_{I_{p_1'} D'}(A,h).$$

Since the diagrams $Z_{p_1}D$ and $Z_{p_1'}D'$ are the same, their polynomials coincide. It is clear that $I_{p_1}D$ and $I_{p_1'}D'$ have no real crossing in B and the restrictions of $I_{p_1}D$ and $I_{p_1'}D'$ to $\mathbf{R}^2\backslash B$ are identical. Hence, for any state S of $I_{p_1}D$, there exists a unique state S' of $I_{p_1'}D'$ such that the restrictions of S and S' to $\mathbf{R}^2\backslash B$ are identical. For any weight map of S, the product of parities of the two virtual crossings of S in B is -1 because adjacent edges have different weights. The product of parities of the two crossings of S' in B is also -1 by the same reason. Since S and S' are identical on the outside of B, we have $\mathrm{i}(S) = \mathrm{i}(S')$. It follows that $H_{I_{p_1}D}(A,h) = H_{I_{p_1'}D'}(A,h)$, completing the proof of this case. The other case is proved in the same manner. $\square$

Proof of Theorem 2. By Lemmas 8, 10, 11 and 12, the first assertion of Theorem 2 is easily verified. The writhe of a diagram does not change by generalized Reidemeister moves except Reidemeister move of type I. By a Reidemeister move of type I which eliminate crossing p from a diagram, the writhe changes by $\mathrm{sign}(p)$. Lemma 7 completes the proof of the second assertion of Theorem 2. $\square$

4. Proof of Theorem 3

For a VMG diagram D, we define the chord diagram as follows. Forgetting vertices and the orientations of edges, we regard D as an "unoriented" virtual link diagram, say $\bar{D}$. Let $\eta : \coprod_{\mu} S^1 \to \mathbf{R}^2$ be an immersion of the union $\coprod_{\mu} S^1$ of μ circles which covers $\bar{D}$, where μ is the number of components of D. For each real crossing point p of $\bar{D}$, attach a chord to $\coprod_{\mu} S^1$ spanning the pair of points of $\eta^{-1}(p)$. The union of $\coprod_{\mu} S^1$ and chords corresponding to the real crossings is called the *chord diagram* of D. We say that D is *connected* if the chord diagram of D is connected.

Let D be a VMG diagram and S a state of D. By $H_D|_S$, we mean $A^{\natural S}(-A^{-2} - A^2)^{\#S-1}h^{(1-\mathrm{i}(S))/2}$ and by $R_D|_S$ we mean $(-A^3)^{-\omega(D)}H_D|_S$. We denote by $\mathrm{Mx}_A(R_D|_S)$ the maximal degree on the variable A of $R_D|_S$, namely, $\mathrm{Mx}_A(R_D|_S) = -3\omega(D) + \natural S + 2(\#S - 1)$. Note that any power of A in $R_D|_S$ is congruent to $\mathrm{Mx}_A(R_D|_S)$ modulo 4.

We prove Theorem 3 by using Lemmas 13, 14 and 15.

Lemma 13. *Let D be a connected μ-component virtual link diagram and S_0 the state of D obtained from D by doing 0-splice at each real crossing. Then* $R_D|_{S_0} \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$.

PROOF. Note that S_0 is a VMG diagram without vertices. Considering the weight map of S_0 sending all edges to 1, we see that $\mathrm{i}(S_0) = 1$ and $R_D|_{S_0} \in \mathbf{Z}[A^2, A^{-2}]$. Since D is connected, there is a set of $\mu - 1$ real crossings, say $\{p_1, \ldots, p_{\mu-1}\}$, of D such that the VMG diagram obtained from D by doing 0-splices at these crossings is a 1-component VMG diagram, say D^1.

If there is a pair of chords in the chord diagram of D^1 such that their attaching points appear alternately on the circle, let p_μ and $p_{\mu+1}$ be the corresponding real crossings of D^1 (and of D), do 0-splices at them. The result is a 1-component VMG diagram, say D^2. See Figure 8.

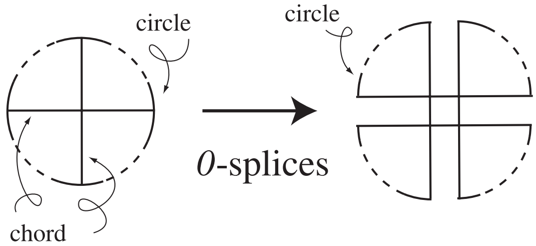

Fig. 8

If there is a pair of chords in the chord diagram of D^2 such that their attaching points appear alternately on the circle, let $p_{\mu+2}$ and $p_{\mu+3}$ be the corresponding real crossings of D^2 (and of D), and do 0-splices at them. The result is a 1-component VMG diagram, say D^3.

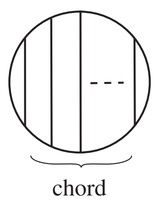

Fig. 9

Repeat this procedure until we have a 1-component VMG diagram, say D', whose chord diagram is as in Figure 9 where all chords are parallel. Let ν be the number of real crossings of D'. Then the number of real crossings of D is congruent to $\mu - 1 + \nu$ modulo 2. Since the writhe $\omega(D)$ is congruent to the number of real crossings of D modulo 2, we have

$$\omega(D) \equiv \mu + \nu - 1 \mod 2.$$

Note that we obtain S_0 from D' by doing 0-splices at the v real crossings of D'. Thus, we have

$$\#S_0 = v + 1.$$

Since S_0 is the state obtained from D by doing 0-splices at all real crossings of D, $\natural S_0 = \sum_p \operatorname{sign}(p) = \omega(D)$.

Therefore, we have

$$\begin{aligned}
\mathrm{Mx}_A(R_D|_{S_0}) &= -3\omega(D) + \natural S_0 + 2(\#S_0 - 1) \\
&= -3\omega(D) + \omega(D) + 2v \\
&= -2(\omega(D) - v) \\
&\equiv 2(\mu - 1) \mod 4.
\end{aligned}$$

Since all power of A in $R_D|_{S_0}$ are congruent to $\mathrm{Mx}_A(R_D|_{S_0})$, we have the conclusion. □

Let D be a virtual link diagram and S a state of D. A real crossing p of D is said to be *normal* with respect to S if $\#S \neq \#S'$, where S' is the state obtained from S by changing the splice at p.

Lemma 14. *Let D be a virtual link diagram and p a real crossing of D. Let S be a state of D and let S' be the state obtained from S by changing the splice at p.*

(1) *If p is normal with respect to S, then* $\mathrm{i}(S) = \mathrm{i}(S')$.
(2) *If p is not normal with respect to S, then* $\mathrm{i}(S) = -\mathrm{i}(S')$.
(3) *If p is normal with respect to S, then* $\mathrm{Mx}_A(R_D|_S) \equiv \mathrm{Mx}_A(R_D|_{S'}) \mod 4$.
(4) *If p is not normal with respect to S, then* $\mathrm{Mx}_A(R_D|_S) \equiv \mathrm{Mx}_A(R_D|_{S'}) + 2 \mod 4$.

Proof. We suppose that S is a state of D done 0-splice at p and that S' is the state obtained from S by switching 0-splice to ∞-splice at p. Let B be the stage where we switch the splice. There are the following three cases.

(i) $\#S = \#S' - 1$,
(ii) $\#S = \#S' + 1$,
(iii) $\#S = \#S'$.

The crossing p is normal with respect to S in the cases (i) and (ii), and it is not in the case (iii). Let e_1 and e_2 be the two edges of $S \cap B$. The edge e_j splits to two edges e_j' and e_j'' of $S' \cap B$ for $j = 1, 2$ as in Figure 10. Put $G = S \backslash B = S' \backslash B$, which is a union of two arc components, say C_1 and C_2, and some loop components. A weight map σ of S (or σ' of S') induces a weight map of G, which we denote by the same symbol σ (or σ', respectively).

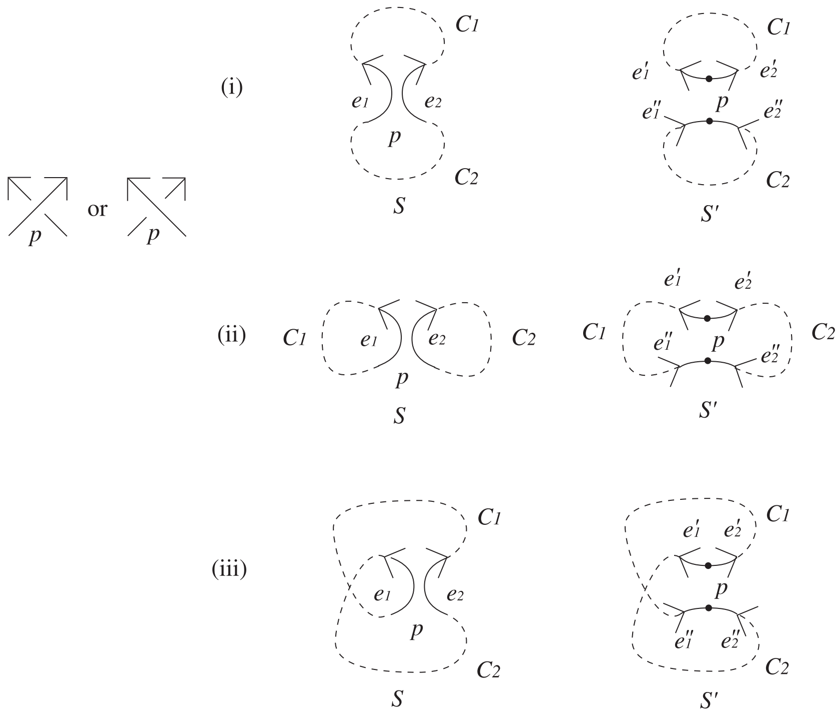

Fig. 10

Consider the case (i). Observing the orientations of e_1 and e_2, we see that there are odd numbers of vertices on C_1 and C_2, respectively. Fix a weight map σ of S such that $\sigma(e_1) = 1$ and $\sigma(e_2) = -1$. There is a weight map σ' of S' such that $\sigma'(e_1') = \sigma'(e_1'') = 1$, $\sigma'(e_2') = \sigma'(e_2'') = -1$ and $\sigma(e) = \sigma'(e)$ for $e \in G$. For a virtual crossing v of G, the parity $\mathrm{i}_\sigma(v)$ with respect to S is equal to the parity $\mathrm{i}_{\sigma'}(v)$ with respect to S'. Thus $\mathrm{i}(S) = \mathrm{i}(S')$.

Consider the case (ii). Let σ be a weight map of S such that $\sigma(e_1) = 1$ and $\sigma(e_2) = -1$. There is a weight map σ' of S' such that $\sigma'(e_1') = \sigma'(e_1'') = 1$, $\sigma'(e_2') = \sigma'(e_2'') = -1$ and $\sigma(e) = \sigma'(e)$ for $e \in G$. Thus $\mathrm{i}(S) = \mathrm{i}(S')$.

Consider the case (iii). There are even numbers of vertices on C_1 and C_2, respectively. Let σ be a weight map of S with $\sigma(e_1) = \sigma(e_2) = 1$. There is a weight map σ' of S' such that $\sigma'(e_1') = \sigma'(e_2'') = 1$, $\sigma'(e_1'') = \sigma'(e_2') = -1$, $\sigma(e) = \sigma'(e)$ for $e \in G \backslash C_1$ and $\sigma(e) = -\sigma'(e)$ for $e \in C_1$. For a virtual crossing v of G, the parity $\mathrm{i}_\sigma(v)$ with respect to S is not equal to the parity $\mathrm{i}_{\sigma'}(v)$ with respect to S' if and only if v is a virtual crossing where C_1 and $G \backslash C_1$ intersect. Since the number of intersections of C_1 and $G \backslash C_1$ is odd, we have $\mathrm{i}(S) = -\mathrm{i}(S')$. Therefore, we obtain the first two assertions of the Lemma.

Note that $\mathrm{Mx}_A(R_D|_S) = -\omega(D) + \natural S + 2(\#S - 1)$ and $\mathrm{Mx}_A(R_D|_{S'}) = -\omega(D) + \natural S' + 2(\#S' - 1)$. Since $\natural S - \natural S' = 2\,\mathrm{sign}(p)$,

$$\mathrm{Mx}_A(R_D|_S) - \mathrm{Mx}_A(R_D|_{S'}) = 2(\mathrm{sign}(p) + \#S - \#S').$$

If p is normal, then $\#S - \#S' = \pm 1$ and $\mathrm{Mx}_A(R_D|_S) - \mathrm{Mx}_A(R_D|_{S'}) \equiv 0 \bmod 4$. If p is not normal, then $\#S = \#S'$ and $\mathrm{Mx}_A(R_D|_S) - \mathrm{Mx}_A(R_D|_{S'}) \equiv 2 \bmod 4$. $\square$

Lemma 15. *Let D be a virtual link diagram which is a union of two virtual link diagrams D_1 and D_2 such that $D_1 \cap D_2$ is empty or consists of virtual crossings. Let $R_D(A,h) = \Phi + \Psi h$ and $R_{D_j}(A,h) = \Phi_j + \Psi_j h$ for $j = 1,2$, where $\Phi, \Psi, \Phi_j, \Psi_j \in \mathbf{Z}[A, A^{-1}]$. Then $\Phi = (-A^2 - A^{-2})(\Phi_1\Phi_2 + \Psi_1\Psi_2)$ and $\Psi = (-A^2 - A^{-2})(\Phi_1\Psi_2 + \Psi_1\Phi_2)$.*

Proof. If $D_1 \cap D_2$ consists of virtual crossings, then we can change D so that $D_1 \cap D_2 = \varnothing$ by virtual Reidemeister moves. Since $R_D(A,h)$ is a virtual link invariant (Theorem 2), we may assume that $D_1 \cap D_2 = \varnothing$. By definition we have

$$\Phi = (-A^3)^{-\omega(D)} \sum_{S \in s(D);\, \mathrm{i}(S)=1} A^{\natural S}(-A^2 - A^{-2})^{\#S-1} \qquad \text{and}$$

$$\Psi = (-A^3)^{-\omega(D)} \sum_{S \in s(D);\, \mathrm{i}(S)=-1} A^{\natural S}(-A^2 - A^{-2})^{\#S-1}.$$

For $j = 1,2$, we have

$$\Phi_j = (-A^3)^{-\omega(D_j)} \sum_{S_j \in s(D_j);\, \mathrm{i}(S_j)=1} A^{\natural S_j}(-A^2 - A^{-2})^{\#S_j-1} \qquad \text{and}$$

$$\Psi_j = (-A^3)^{-\omega(D_j)} \sum_{S_j \in s(D_j);\, \mathrm{i}(S_j)=-1} A^{\natural S_j}(-A^2 - A^{-2})^{\#S_j-1}.$$

Let S be a state of D, which is the union of a state S_1 of D_1 and a state S_2 of D_2. Then we have $\natural S = \natural S_1 + \natural S_2$ and $\#S = \#S_1 + \#S_2$. Since $D_1 \cap D_2 = \varnothing$, we have $\mathrm{i}(S) = \mathrm{i}(S_1)\mathrm{i}(S_2)$ by the definition of parity. Hence, $\mathrm{i}(S) = 1$ if $\mathrm{i}(S_1) = \mathrm{i}(S_2)$, and $\mathrm{i}(S) = -1$ if $\mathrm{i}(S_1) = -\mathrm{i}(S_2)$. Note that $\omega(D) = \omega(D_1) + \omega(D_2)$. Therefore we have

$$\begin{aligned}\Phi = (-A^3)^{-\omega(D)} &\sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2);\\ \mathrm{i}(S_1)=\mathrm{i}(S_2)=1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\#S_1 + \#S_2 - 1} \\ + (-A^3)^{-\omega(D)} &\sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2);\\ \mathrm{i}(S_1)=\mathrm{i}(S_2)=-1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\#S_1 + \#S_2 - 1}.\end{aligned}$$

Since

$$\begin{aligned}\Phi_1\Phi_2 &= \left\{(-A^3)^{-\omega(D_1)} \sum_{S_1 \in s(D_1);\, \mathrm{i}(S_1)=1} A^{\natural S_1}(-A^2 - A^{-2})^{\# S_1 - 1}\right\} \\ &\quad \times \left\{(-A^3)^{-\omega(D_2)} \sum_{S_2 \in s(D_2);\, \mathrm{i}(S_2)=1} A^{\natural S_2}(-A^2 - A^{-2})^{\# S_2 - 1}\right\} \\ &= (-A^3)^{-(\omega(D_1)+\omega(D_2))} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2); \\ \mathrm{i}(S_1)=\mathrm{i}(S_2)=1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 2} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\Psi_1\Psi_2 &= \left\{(-A^3)^{-\omega(D_1)} \sum_{S_1 \in s(D_1);\, \mathrm{i}(S_1)=-1} A^{\natural S_1}(-A^2 - A^{-2})^{\# S_1 - 1}\right\} \\ &\quad \times \left\{(-A^3)^{-\omega(D_2)} \sum_{S_2 \in s(D_2);\, \mathrm{i}(S_2)=-1} A^{\natural S_2}(-A^2 - A^{-2})^{\# S_2 - 1}\right\} \\ &= (-A^3)^{-(\omega(D_1)+\omega(D_2))} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2); \\ \mathrm{i}(S_1)=\mathrm{i}(S_2)=-1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 2},\end{aligned}$$

we conclude $\Phi = (-A^2 - A^{-2})(\Phi_1\Phi_2 + \Psi_1\Psi_2)$. Similarly we have

$$\begin{aligned}\Psi &= (-A^3)^{-\omega(D)} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2); \\ \mathrm{i}(S_1)=1,\, \mathrm{i}(S_2)=-1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 1} \\ &\quad + (-A^3)^{-\omega(D)} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2); \\ \mathrm{i}(S_1)=-1,\, \mathrm{i}(S_2)=1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 1}.\end{aligned}$$

Since

$$\begin{aligned}\Phi_1\Psi_2 &= \left\{(-A^3)^{-\omega(D_1)} \sum_{S_1 \in s(D_1);\, \mathrm{i}(S_1)=1} A^{\natural S_1}(-A^2 - A^{-2})^{\# S_1 - 1}\right\} \\ &\quad \times \left\{(-A^3)^{-\omega(D_2)} \sum_{S_2 \in s(D_2);\, \mathrm{i}(S_2)=-1} A^{\natural S_2}(-A^2 - A^{-2})^{\# S_2 - 1}\right\} \\ &= (-A^3)^{-(\omega(D_1)+\omega(D_2))} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2); \\ \mathrm{i}(S_1)=1,\, \mathrm{i}(S_2)=-1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 2} \quad \text{and}\end{aligned}$$

$$\Psi_1\Phi_2 = \left\{ (-A^3)^{-\omega(D_1)} \sum_{S_1 \in s(D_1);\, \mathrm{i}(S_1)=-1} A^{\natural S_1}(-A^2 - A^{-2})^{\# S_1 - 1} \right\}$$
$$\times \left\{ (-A^3)^{-\omega(D_2)} \sum_{S_2 \in s(D_2);\, \mathrm{i}(S_2)=1} A^{\natural S_2}(-A^2 - A^{-2})^{\# S_2 - 1} \right\}$$
$$= (-A^3)^{-(\omega(D_1)+\omega(D_2))} \sum_{\substack{S_1 \in s(D_1),\, S_2 \in s(D_2);\\ \mathrm{i}(S_1)=-1,\, \mathrm{i}(S_2)=1}} A^{\natural S_1 + \natural S_2}(-A^2 - A^{-2})^{\# S_1 + \# S_2 - 2},$$

we conclude $\Phi = (-A^2 - A^{-2})(\Phi_1\Psi_2 + \Psi_1\Phi_2)$. □

Proof of Theorem 3. Let $D = D_1' \cup D_2' \cup \cdots \cup D_m'$, where D_j' $(j = 1, \ldots, m)$ is a connected virtual link diagram. We prove the theorem by induction on m.

If $m = 1$, then D is connected. By Lemma 13, we have $R_D|_{S_0} \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$. Let S be a state of D obtained from S_0 by changing 0-splice to ∞-splice at a real crossing p of D. By Lemma 14, we see that $R_D|_S \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$ if p is normal with respect to S_0 and $R_D|_S \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu} \cdot h$ if p is not normal. Since any state of D is obtained from S_0 by changing splices at some real crossings in D, applying Lemma 14 inductively, we have the conclusion.

If $m > 1$, then put $D_1 = D_1' \cup D_2' \cup \cdots \cup D_{m-1}'$ and $D_2 = D_m'$. Then, as the induction hypothesis, we assume that

$$\Phi_{D_j}(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu_j - 1)} \quad \text{and}$$
$$\Psi_{D_j}(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu_j},$$

where μ_j is the number of components of D_j for $j = 1, 2$. Since $\mu = \mu_1 + \mu_2$, by Lemma 15, we have

$$\Phi_D(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)} \quad \text{and}$$
$$\Psi_D(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu}.$$ □

5. An observation on checkerboard colorable diagrams

In [3] the notion of checkerboard coloring for a virtual link diagram was defined by using the corresponding abstract link diagram [1].

Theorem 16. *Let D be a μ-component virtual knot diagram. If D admits a checkerboard coloring, then* $R_D(A, h) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$.

As a consequence of Theorem 16, we see that if a μ-component virtual link daigram D admits a checkerboard coloring, then $f_D(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$. This result was first proved in [3]. In [2] checkerboard colorable virtual link diagrams are investigated in terms of link diagrams on closed oriented surfaces. We need the following fact from [2] to prove Theorem 16.

LEMMA 17 ([2]). *Let D be a virtual link diagram. Any real crossing is normal with respect to all states of D, if and only if D admits a checkerboard coloring.*

PROOF OF THEOREM 16. Let S_0 be the state of D obtained from D by doing 0-splice at each real crossing of D. By Lemma 13, $R_D|_{S_0} \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$. Any state of D is obtained from S_0 by switching splices at some crossings. Since D admits a checkerboard coloring, any real crossing is normal with respect to all states of D by Lemma 17. Hence we have the conclusion by applying Lemma 14 inductively. □

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