HIROSHIMA MATH. J.
35 (2005), 471–483

Stable extendibility of $m\tau_n$ over real projective spaces

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(Received December 21, 2004)
(Revised April 1, 2005)

ABSTRACT. The purpose of this paper is to study the stable extendibility of the m-times Whitney sum $m\tau_n$ of the tangent bundle $\tau_n = \tau(RP^n)$ of the n-dimensional real projective space RP^n. We determine the dimension N for which $m\tau_n$ is stably extendible to RP^N but is not stably extendible to RP^{N+1} for $m \leq 10$.

1. Introduction

Let X be a space and A its subspace. A t-dimensional real vector bundle ζ over A is said to be *extendible* (respectively *stably extendible*) to X, if there is a t-dimensional real vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ, that is, ζ is equivalent (respectively stably equivalent) to the induced bundle $i^*\eta$ of a t-dimensional real vector bundle η over X under the inclusion map $i : A \to X$ (cf. [10, p. 20] and [3, p. 273]). Let RP^n denote the n-dimensional real projective space. For a real vector bundle ζ over RP^n, define an integer $s(\zeta)$ by

$$s(\zeta) = \max\{m \mid m \geq n \text{ and } \zeta \text{ is stably extendible to } RP^m\},$$

where we put $s(\zeta) = \infty$ if ζ is stably extendible to RP^m for every $m \geq n$.

The following theorem is known.

THEOREM 1 ([7, Theorem 4.2]). *For the tangent bundle $\tau_n = \tau(RP^n)$ of RP^n,*

$$s(\tau_n) = \infty \quad \text{if } n = 1, 3 \text{ or } 7; \qquad \text{and} \qquad s(\tau_n) = n \quad \text{if } n \neq 1, 3, 7.$$

The purpose of this paper is to study $s(m\tau_n)$ for $m \geq 2$. Our main results are as follows.

We write simply $s(m, n)$ instead of $s(m\tau_n)$.

THEOREM 2. (1) *If $1 \leq n \leq 8$, then $s(2, n) = \infty$.*
(2) *If $n \geq 9$, then $s(2, n) = 2n + 1$.*

2000 *Mathematics Subject Classification*. Primary 55R50; Secondary 55N15.
Keywords and Phrases. vector bundle, tangent bundle, Whitney sum, real projective space, stably extendible, *KO*-theory.

THEOREM 3. (1) *If* $1 \leq n \leq 8$, *then* $s(3,n) = \infty$.
(2) *If* $n \geq 9$, *then*
  (a) $s(3,n) = 3n$ *for* $n \equiv 0,1 \mod 4$,
  (b) $s(3,n) = 3n+1$ *for* $n \equiv 2 \mod 4$,
  (c) $s(3,n) = 3n+2$ *for* $n \equiv 3 \mod 4$.

THEOREM 4. (1) *If* $1 \leq n \leq 9$, *then* $s(4,n) = \infty$.
(2) *If* $n \geq 10$, *then* $s(4,n) = 4n+3$.

THEOREM 5. (1) *If* $1 \leq n \leq 9$, *then* $s(5,n) = \infty$.
(2) *If* $n \geq 10$, *then*
  (a) $s(5,n) = 5n$ *for* $n \equiv 0,2,3,5 \mod 8$,
  (b) $s(5,n) = 5n+1$ *for* $n \equiv 6 \mod 8$,
  (c) $s(5,n) = 5n+2$ *for* $n \equiv 1 \mod 8$,
  (d) $s(5,n) = 5n+3$ *for* $n \equiv 4 \mod 8$,
  (e) $s(5,n) = 5n+4$ *for* $n \equiv 7 \mod 8$.

THEOREM 6. (1) *If* $1 \leq n \leq 11$, *then* $s(6,n) = \infty$.
(2) *If* $n \geq 12$, *then*
  (a) $s(6,n) = 6n+1$ *for* $n \equiv 0,1 \mod 4$,
  (b) $s(6,n) = 6n+3$ *for* $n \equiv 2 \mod 4$,
  (c) $s(6,n) = 6n+5$ *for* $n \equiv 3 \mod 4$.

THEOREM 7. (1) *If* $1 \leq n \leq 11$, *then* $s(7,n) = \infty$.
(2) *If* $n \geq 12$, *then*
  (a) $s(7,n) = 7n$ *for* $n \equiv 0,1 \mod 8$,
  (b) $s(7,n) = 7n+i-1$ *for* $n \equiv i \mod 8$ *with* $2 \leq i \leq 7$.

THEOREM 8. (1) *If* $1 \leq n \leq 11$ *or* $n = 15$, *then* $s(8,n) = \infty$.
(2) *If* $n = 12, 13, 14$ *or* $n \geq 16$, *then* $s(8,n) = 8n+7$.

THEOREM 9. (1) *If* $1 \leq n \leq 11$, $n = 14$ *or* 15, *then* $s(9,n) = \infty$.
(2) *If* $n = 12, 13$ *or* $n \geq 16$, *then*
  (a) $s(9,n) = 9n$ *for* $n \equiv 0,2,4,6,7,9,11,13 \mod 16$,
  (b) $s(9,n) = 9n+1$ *for* $n \equiv 14 \mod 16$,
  (c) $s(9,n) = 9n+2$ *for* $n \equiv 5 \mod 16$,
  (d) $s(9,n) = 9n+3$ *for* $n \equiv 12 \mod 16$,
  (e) $s(9,n) = 9n+4$ *for* $n \equiv 3 \mod 16$,
  (f) $s(9,n) = 9n+5$ *for* $n \equiv 10 \mod 16$,
  (g) $s(9,n) = 9n+6$ *for* $n \equiv 1 \mod 16$,
  (h) $s(9,n) = 9n+7$ *for* $n \equiv 8 \mod 16$,
  (i) $s(9,n) = 9n+8$ *for* $n \equiv 15 \mod 16$.

THEOREM 10. (1) *If* $1 \leq n \leq 15$, *then* $s(10,n) = \infty$.
(2) *If* $n \geq 16$, *then*

(a) $s(10,n) = 10n+1$ *for* $n \equiv 0,2,3,5 \mod 8$,
(b) $s(10,n) = 10n+3$ *for* $n \equiv 4,6,7,14 \mod 16$,
(c) $s(10,n) = 10n+4$ *for* $n \equiv 1 \mod 16$,
(d) $s(10,n) = 10n+5$ *for* $n \equiv 9 \mod 16$,
(e) $s(10,n) = 10n+7$ *for* $n \equiv 12 \mod 16$,
(f) $s(10,n) = 10n+9$ *for* $n \equiv 15 \mod 16$.

This paper is arranged as follows. In §2 we state some known theorems that are used to prove Theorems 2–10. In §3 we state some applications. In §4 we study on $m\tau_n$. In §5 we prove Theorem 10. In §6 and §7 we prove Theorems 2–9.

2. Some known theorems

Let ξ_n denote the canonical real line bundle over RP^n.

Theorem 2.1 ([1, Theorem 7.4]). (1) *The reduced KO-group* $\widetilde{KO}(RP^n)$ *is isomorphic to the cyclic group* $Z/2^{\phi(n)}$, *generated by* $\xi_n - 1$, *where* $\phi(n)$ *is the number of integers s such that* $0 < s \leq n$ *and* $s \equiv 0,1,2$ *or* 4 mod 8.

(2) $(\xi_n)^2 (= \xi_n \otimes \xi_n) = 1$, *where* $\otimes$ *denotes the tensor product.*

For a real vector bundle ζ, we denote by Span ζ the maximum number of linearly independent cross-sections of ζ.

Theorem 2.2 ([5, Theorem 1]). *Let l, n and t be integers with* $t \geq 0$ *and* $0 \leq t+l < 2^{\phi(n)}$, *and let* ζ *be a t-dimensional real vector bundle over* RP^n *which is stably equivalent to* $(t+l)\xi_n$. *Then the following hold.*

(1) $s(\zeta) = \infty$ *if and only if* $l \leq 0$.

(2) *Let* $l \geq 1$ *and* $m \geq n$. *Then,* $s(\zeta) \geq m$ *if and only if* $\mathrm{Span}(a2^{\phi(n)} + t + l)\xi_m \geq a2^{\phi(n)} + l$ *for some integer* $a \geq 0$.

For a non-negative integer t and a positive integer l, define an integer $\varepsilon(t,l)$ as follows.

$$\varepsilon(t,l) = \min\left\{ j \,\middle|\, t < j \text{ and } \binom{t+l}{j} \equiv 1 \mod 2 \right\},$$

where $\binom{n}{r}$ denotes the binomial coefficient $n!/(r!(n-r)!)$. Clearly $t < \varepsilon(t,l) \leq t+l$.

Theorem 2.3 ([6, Theorem 2]). *Let* ζ *be a t-dimensional real vector bundle over* RP^n *and assume that there is a positive integer l satisfying the following properties:*

(1) ζ *is stably equivalent to* $(t+l)\xi_n$,

(2) $t+l < 2^{\phi(n)}$.

Then $n < \varepsilon(t,l)$ *and* $s(\zeta) < \varepsilon(t,l)$.

The following theorems are useful.

THEOREM 2.4 ([12, Theorem 2.4], [4, (1.1) and Section 4]). *Let* $n+1=(2b+1)2^{c+4d}$, *where* b, c, d *are non-negative integers and* $0 \le c \le 3$. *Then*

$$\mathrm{Span}(n+1)\xi_n = 2^c + 8d.$$

THEOREM 2.5 ([9, Theorem 1.1]). *Let* $k = 8l + p$, $n = 8m + q$, *where* $0 \le m \le l$ *and* $0 \le p, q \le 7$. *If the binomial coefficient* $\binom{l}{m}$ *is odd, then* Span $k\xi_n = (k-n) + j$, *with* j *given by* Table I *below:*

TABLE I

q \ p	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	1	0	1	0	1	0	1	0
2	2	1	0	0	2	1	0	0
3	3	2	1	0	3	2	1	0
4	4	3	2	1	0	0	0	0
5	5	4	3	2	1	0	1	0
6	6	5	4	3	2	1	0	0
7	7	6	5	4	3	2	1	0

THEOREM 2.6 ([9, Theorem 3.1]). (A) *Let* $k = 8l + p$, $n = 8m + q$, *where* $1 < m < l$ *and* $\binom{l}{m}$ *is even. Then for* $0 \le p \le 6$ *and* $1 \le q \le 7$, Span $k\xi_n \ge (k-n) + j$, *with* j *given by* Table II *below:*

TABLE II

q \ p	0	1	2	3	4	5	6
1	3	4	3	3	4	3	3
2	3	5	4	4	5	4	3
3	4	6	5	5	6	5	4
4	5	4	3	5	5	4	3
5	6	5	4	3	5	4	3
6	7	6	5	4	6	5	4
7	8	7	6	5	4	3	3

(B) *Furthermore, if* $\binom{l}{m} \equiv 2 \bmod 4$, *then* $\mathrm{Span}\, k\xi_n = (k-n)+j$ *except when* $(p,q)=(3,4)$.

Theorem 2.7 ([11, Lemma 2.6, p. 5]). *Let* $a = a_0 2^0 + a_1 2^1 + \cdots + a_l 2^l$ *and* $b = b_0 2^0 + b_1 2^1 + \cdots + b_l 2^l$ $(0 \le a_i, b_i < 2)$. *Then*

$$\binom{a}{b} \equiv 0 \bmod 2 \quad \text{if and only if} \quad a_i = 0 \text{ and } b_i = 1 \text{ for some } i.$$

The following theorem may be well-known. For completeness, we give a proof.

Theorem 2.8. *Let* $a = \sum_{i=0}^{l} a_i 2^i$ *and* $b = \sum_{i=0}^{l} b_i 2^i$ $(0 \le a_i, b_i < 2)$. *Then* $\binom{a}{b} \equiv 2 \bmod 4$ *if* $\binom{a}{b}$ *can be described as follows:*

$$\binom{a}{b} = \binom{m + 2^{p+1} + 0 + r}{n + 0 + 2^p + s},$$

where $m, n \equiv 0 \bmod 2^{p+2}$; $2^p > r, s \ge 0$; $\binom{m}{n}, \binom{r}{s} \equiv 1 \bmod 2$.

Proof. For a positive integer N, let $v(N)$ denote the non-negative integer such that $N = (2q+1)2^{v(N)}$, where q is a non-negative integer. Clearly, $N \equiv 1 \bmod 2$ if and only if $v(N) = 0$, and $N \equiv 2 \bmod 4$ if and only if $v(N) = 1$. It is known that (cf. [2, Lemma 4.8]), for a positive integer M, $v(M!) = M - \alpha(M)$, where $\alpha(M)$ is the number of the non-zero terms in the 2-adic expansion of M. Hence we have

$$v\binom{a}{b} = \alpha(a-b) + \alpha(b) - \alpha(a).$$

Thus $\binom{a}{b} \equiv 1 \bmod 2$ if and only if $\alpha(a-b) + \alpha(b) - \alpha(a) = 0$, and $\binom{a}{b} \equiv 2 \bmod 4$ if and only if $\alpha(a-b) + \alpha(b) - \alpha(a) = 1$. Therefore, if $\binom{a}{b}$ is described as in the theorem, we see that $v\binom{a}{b} = 1$. □

3. Some applications

Proposition 3.1. *Let* ζ *be a t-dimensional real vector bundle over* RP^n. *Then*

$$s(\zeta) = s(\zeta \otimes \xi_n).$$

Proof. If ζ is stably extendible to RP^m $(m \ge n)$, then there is a t-dimensional real vector bundle α over RP^m such that ζ is stably equivalent to $i^*\alpha$, where $i : RP^n \to RP^m$ is the standard inclusion. Since $\zeta \otimes \xi_n$ is stably equivalent to $i^*\alpha \otimes \xi_n = i^*(\alpha \otimes \xi_m)$, $\zeta \otimes \xi_n$ is stably extendible to RP^m

$(m \geq n)$. On the other hand, by using Theorem 2.1(2) and the above result, we see that $\zeta = (\zeta \otimes \xi_n) \otimes \xi_n$ is stably extendible to RP^m $(m \geq n)$ if $\zeta \otimes \xi_n$ is stably extendible to RP^m. □

In the same way as the above proof, we have the following.

REMARK. Let ζ be a t-dimensional real vector bundle over RP^n. Then, ζ is extendible to RP^m $(m \geq n)$ if and only if $\zeta \otimes \xi_n$ is extendible to RP^m.

For a non-negative integer t and a positive integer l, we define an integer $\mu(t, l; n)$ as the maximum of integers m satisfying

$$t + l \geq m + 1 = (2b+1)2^{c+4d} > n \qquad \text{and} \qquad 2^c + 8d \geq l,$$

where b, c and d are non-negative integers with $0 \leq c \leq 3$. We remark that the above $\mu(t, l; n)$ does not necessarily exist, even if $t + l > n$.

PROPOSITION 3.2. *Let ζ be a t-dimensional real vector bundle over RP^n and assume that there is a positive integer l satisfying the following properties:*
(1) *ζ is stably equivalent to $(t+l)\xi_n$,*
(2) *$t + l < 2^{\phi(n)}$.*
If moreover $\mu(t, l; n)$ exists, then $\mu(t, l; n) \leq s(\zeta)$.

PROOF. Put $\mu(t, l; n) = m$. Then, $\mathrm{Span}(t+l)\xi_m \geq \mathrm{Span}(m+1)\xi_m = 2^c + 8d \geq l$ by the definition and Theorem 2.4. Therefore, by using Theorem 2.2(2), we have $s(\zeta) \geq m$. □

PROPOSITION 3.3. *Let $t \geq 0$, $l \geq 1$ and $t + l < 2^{\phi(n)}$. Then*

$$\varepsilon(t, l) = \varepsilon(t, 2^{\phi(n)} - t - l).$$

PROOF. We put $2^{\phi(n)} - l = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_m}$ $(q_1 > q_2 > \cdots > q_m \geq 0)$, $\varepsilon = \varepsilon(t, l)$ and $\tilde{\varepsilon} = \varepsilon(t, 2^{\phi(n)} - t - l)$.

The proof is given by induction on t. When $t = 0$, by Theorem 2.7 we see that $\varepsilon = \tilde{\varepsilon} = 2^{q_m}$ and hence the proposition holds. Now we may assume that $\varepsilon = \tilde{\varepsilon}$ for all t' with $0 \leq t' < t$, and consider the case for t by putting $t = 2^p + s$, where $p \geq 0$ and $0 \leq s < 2^p$.

Now in order to apply Theorem 2.7 more easily, we put $\varepsilon_0(h, k) = \varepsilon(k, h - k)$, that is

$$\varepsilon_0(h, k) = \min\left\{ j \;\middle|\; \binom{h}{j} \equiv 1 \bmod 2 \text{ and } j > k \right\}.$$

(1) Let $q_m > p$. Then $\varepsilon_0(2^{q_1} + \cdots + 2^{q_m}, 2^p + s) = 2^{q_m}$ by Theorem 2.7. And $\varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_m} + 2^p + s, 2^p + s) = 2^{q_m}$ by Theorem 2.7. Hence $\varepsilon = \tilde{\varepsilon}$.

(2) Let $q_m = p$. Then $m \geq 2$ and $\varepsilon_0(2^{q_1} + \cdots + 2^{q_{m-1}} + 2^{q_m}, 2^{q_m} + s) = 2^{q_{m-1}}$. And $\varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_{m-1}} + s, 2^{q_m} + s) = 2^{q_{m-1}}$. Hence $\tilde{\varepsilon} = \varepsilon$.

(3) Let $q_i > p > q_{i+1}$ $(1 \leq i \leq m-1)$. Then $\varepsilon_0(2^{q_1} + \cdots + 2^{q_m}, 2^p + s) = 2^{q_i}$. And $\varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_i} + 2^p - 2^{q_{i+1}} - \cdots - 2^{q_m} + s, 2^p + s) = 2^{q_i}$. Hence $\varepsilon = \tilde{\varepsilon}$.

(4) Let $p = q_i$ $(1 \leq i \leq m-1)$. If $2^{q_{i+1}} + \cdots + 2^{q_m} \leq s$ then $i \geq 2$ and $\varepsilon_0(2^{q_1} + \cdots + 2^{q_m}, 2^{q_i} + s) = 2^{q_{i-1}}$. And $\varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_{i-1}} - 2^{q_{i+1}} - \cdots - 2^{q_m} + s, 2^{q_i} + s) = 2^{q_{i-1}}$. Hence $\varepsilon = \tilde{\varepsilon}$.

If $2^{q_{i+1}} + \cdots + 2^{q_m} > s$ then $\varepsilon_0(2^{q_1} + \cdots + 2^{q_m}, 2^{q_i} + s) = 2^{q_i} + \varepsilon_0(2^{q_1} + \cdots + 2^{q_m}, s)$. And $\varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_{i-1}} - 2^{q_{i+1}} - \cdots - 2^{q_m} + s, 2^{q_i} + s) = 2^{q_i} + \varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_{i-1}} - 2^{q_{i+1}} - \cdots - 2^{q_m} + s, s) = 2^{q_i} + \varepsilon_0(2^{\phi(n)} - 2^{q_1} - \cdots - 2^{q_m} + s, s)$. Hence we see that $\varepsilon = \tilde{\varepsilon}$ by using the assumption of induction. □

Theorem 3.4. *Let ζ be a t-dimensional real vector bundle over RP^n and assume that there is a positive integer l satisfying the following properties:*

(1) *ζ is stably equivalent to $(t+l)\xi_n$,*

(2) *$t + l < 2^{\phi(n)}$.*

If moreover $\mu(t, l; n)$ exists, then $\mu(t, l; n) \leq s(\zeta) < \varepsilon(t, l) = \varepsilon(t, 2^{\phi(n)} - t - l)$.

Proof. The proof follows from Theorem 2.3 and Propositions 3.2 and 3.3. □

4. Study on $m\tau_n$

Lemma 4.1. *Let $m\tau_n$ be the m-times Whitney sum of the tangent bundle $\tau_n = \tau(RP^n)$ of RP^n. Then the equality*

$$m\tau_n = m(n+1)\xi_n - m$$

holds in $KO(RP^n)$.

Proof. Since $\tau_n \oplus 1$ is equivalent to $(n+1)\xi_n$, the equality holds. □

Proposition 4.2. *If $m \geq 2^{\phi(n)}$, $s(m\tau_n) = \infty$.*

Proof. Let $m = \beta 2^{\phi(n)} + \gamma$, where $\beta \geq 1$ and $0 < \gamma < 2^{\phi(n)}$. Then Lemma 4.1 implies $m\tau_n = (\beta 2^{\phi(n)} + \gamma)(n+1)\xi_n - (\beta 2^{\phi(n)} + \gamma)$ in $KO(RP^n)$. So, we see that $m\tau_n$ is stably equivalent to $\gamma(n+1)\xi_n$ by using Theorem 2.1. Then, since $\gamma(n+1) < mn$, we see that $s(m\tau_n) = \infty$ by using Theorem 2.2(1). □

Proposition 4.3. *Let $0 < m < 2^{\phi(n)}$. Then $s(m\tau_n) = \infty$ if and only if $2^{\phi(n)} \leq m(n+1)$.*

Proof. Since $m\tau_n = m(n+1)\xi_n - m$ holds in $KO(RP^n)$, we see that $m\tau_n \otimes \xi_n$ is stably equivalent to $(2^{\phi(n)} - m)\xi_n$ by using Theorem 2.1. Now

$0 < 2^{\phi(n)} - m < 2^{\phi(n)}$ by the assumption. Hence, by Theorem 2.2(1), $2^{\phi(n)} - m - mn \leq 0$ if and only if $s(m\tau_n) = s(m\tau_n \otimes \xi_n) = \infty$. □

Proposition 4.4.

$$s(m\tau_n) \geq mn$$

Proof. Propositions 4.2 and 4.3 imply that $s(m\tau_n) = \infty$ if $m(n+1) \geq 2^{\phi(n)}$.

Let $m(n+1) < 2^{\phi(n)}$. Then, since $\mathrm{Span}(mn+m)\xi_{mn} \geq (mn+m) - mn = m$, we see that $s(m\tau_n) \geq mn$ by using Theorem 2.2(2). □

5. Proof of Theorem 10

In this section we give a proof of Theorem 10. Before proving the theorem, we prepare two lemmas.

Lemma 5.1. *$\varepsilon(10n, 10) = 10n + 2$ for $n \equiv 0, 2, 3, 5 \bmod 8$, $= 10n + 4$ for $n \equiv 6 \bmod 8$, $= 10n + 6$ for $n \equiv 1 \bmod 8$, $= 10n + 8$ for $n \equiv 4 \bmod 8$, $= 10n + 10$ for $n \equiv 7 \bmod 8$.*

Proof. The results follow from the definition of $\varepsilon(10n, 10)$ and Theorem 2.7. □

Lemma 5.2. *$\mu(10n, 10; n) \geq 10n + 5$ for $n \equiv 9 \bmod 16$, $\geq 10n + 7$ for $n \equiv 12 \bmod 16$, $\geq 10n + 9$ for $n \equiv 15 \bmod 16$.*

Proof. Let $n = 16k + 9$ $(k \geq 1)$ and put $10n + 6 (= 2^5(5k+3)) = (2b+1)2^{c+4d}$ $(0 \leq c \leq 3)$. Then we see $2^c + 8d \geq 10$. So we get $\mu(10n, 10; n) \geq 10n + 5$ by the definition of $\mu(10n, 10; n)$.

For $n \equiv 12, 15 \bmod 16$, by putting $10n + 8$, $10n + 10 = (2b+1)2^{c+4d}$ respectively, we obtain the results similarly. □

Proof of Theorem 10. We recall the following

$$10\tau_n = (10n + 10)\xi_n - 10 \in KO(RP^n).$$

(1) Since $10 \geq 2^{\phi(n)}$ for $1 \leq n \leq 7$, $s(10\tau_n) = \infty$ for $1 \leq n \leq 7$ by Proposition 4.2. And since $10 < 2^{\phi(n)}$ and $2^{\phi(n)} - 10 - 10n \leq 0$ for $8 \leq n \leq 15$, $s(10\tau_n) = \infty$ for $8 \leq n \leq 15$ by Proposition 4.3.

(2) Let $n \geq 16$. Then $0 \leq 10n + 10 < 2^{\phi(n)}$ and $10\tau_n$ is stably equivalent to $(10n+10)\xi_n$.

(a) Let $n \equiv 0, 2, 3, 5 \bmod 8$. Since $10n + 10$ is even and $10n + 1$ is odd, we have $\mathrm{Span}(10n+10)\xi_{10n+1} \geq (10n+10) - (10n+1) + 1 = 10$ by Theorems 2.5 and 2.6(A). Hence, by Theorem 2.2(2), $s(10\tau_n) \geq 10n + 1$. On the other hand, we see $\varepsilon(10n, 10) = 10n + 2$ by Lemma 5.1. Hence $s(10\tau_n) < 10n + 2$ by Theorem 2.3. Therefore $s(10\tau_n) = 10n + 1$.

(b) Let $n = 16k+4$ $(k \geq 1)$. We see $10n+10 = 8(20k+6)+2$ and $10n+3 = 8(20k+5)+3$. Here $\binom{20k+6}{20k+5} \equiv 0 \bmod 2$. So, by Theorem 2.6(A), we see $\mathrm{Span}(10n+10)\xi_{10n+3} \geq 10n+10-10n-3+5 = 12 \geq 10$. Hence, by using Theorem 2.2(2), we obtain $s(10\tau_n) \geq 10n+3$. On the other hand, we see $a2^{\phi(n)}+10n+10 = 8(a2^{\phi(16k+4)-3}+20k+6)+2$ and $10n+4 = 8(20k+5)+4$. Here, by Theorem 2.8, $\binom{a2^{\phi(16k+4)-3}+20k+6}{20k+5} \equiv 2 \bmod 4$ for any $a \geq 0$. Hence, by Theorem 2.6(B), $\mathrm{Span}(a2^{\phi(n)}+10n+10)\xi_{10n+4} = a2^{\phi(n)}+10n+10-10n-4+3 < a2^{\phi(n)}+10$ for any $a \geq 0$. So we obtain $s(10\tau_n) < 10n+4$ by using Theorem 2.2(2). Therefore we have $s(10\tau_n) = 10n+3$.

Let $n = 8k+6$ $(k \geq 2)$. We see $10n+10 = 8(10k+8)+6$ and $10n+3 = 8(10k+7)+7$. Here $\binom{10k+8}{10k+7} \equiv 0 \bmod 2$. So, by Theorem 2.6(A), we see $\mathrm{Span}(10n+10)\xi_{10n+3} \geq 10n+10-10n-3+3 = 10$. Hence we obtain $s(10\tau_n) \geq 10n+3$. On the other hand, we see $\varepsilon(10n,10) = 10n+4$ by Lemma 5.1. Hence $s(10\tau_n) < 10n+4$. Therefore we have $s(10\tau_n) = 10n+3$.

Let $n = 16k+7$ $(k \geq 1)$. We see $10n+10 = 8(20k+10)$ and $10n+3 = 8(20k+9)+1$. Here $\binom{20k+10}{20k+9} \equiv 0 \bmod 2$. So, by Theorem 2.6(A), we see $\mathrm{Span}(10n+10)\xi_{10n+3} \geq 10n+10-10n-3+3 = 10$. Hence we obtain $s(10\tau_n) \geq 10n+3$. On the other hand, we see $a2^{\phi(n)}+10n+10 = 8(a2^{\phi(16k+7)-3}+20k+10)$ and $10n+4 = 8(20k+9)+2$. Here, by Theorem 2.8, $\binom{a2^{\phi(16k+7)-3}+20k+10}{20k+9} \equiv 2 \bmod 4$ for any $a \geq 0$. Hence, by using Theorem 2.6(B), $\mathrm{Span}(a2^{\phi(n)}+10n+10)\xi_{10n+4} = a2^{\phi(n)}+10n+10-10n-4+3 < a2^{\phi(n)}+10$ for any $a \geq 0$. So we obtain $s(10\tau_n) < 10n+4$. Therefore we have $s(10\tau_n) = 10n+3$.

(c) Let $n = 16k+1$ $(k \geq 1)$. We see $10n+10 = 8(20k+2)+4$ and $10n+4 = 8(20k+1)+6$. Here $\binom{20k+2}{20k+1} \equiv 0 \bmod 2$. So, by Theorem 2.6(A), we see $\mathrm{Span}(10n+10)\xi_{10n+4} \geq 10n+10-10n-4+6 = 12 \geq 10$. Hence we obtain $s(10\tau_n) \geq 10n+4$. On the other hand, we see $a2^{\phi(n)}+10n+10 = 8(a2^{\phi(16k+1)-3}+20k+2)+4$ and $10n+5 = 8(20k+1)+7$. Here, by Theorem 2.8, $\binom{a2^{\phi(16k+1)-3}+20k+2}{20k+1} \equiv 2 \bmod 4$ for any $a \geq 0$. Hence, by Theorem 2.6(B), $\mathrm{Span}(a2^{\phi(n)}+10n+10)\xi_{10n+5} = a2^{\phi(n)}+10n+10-10n-5+4 < a2^{\phi(n)}+10$ for any $a \geq 0$. So we obtain $s(10\tau_n) < 10n+5$. Therefore we have $s(10\tau_n) = 10n+4$.

(d) Let $n = 16k+9$ $(k \geq 1)$. Then $\varepsilon(10n,10) = 10n+6$ by Lemma 5.1. Hence $s(10\tau_n) < 10n+6$ by Theorem 2.3. And we get $\mu(10n,10;n) \geq 10n+5$ by Lemma 5.2. Therefore, by Proposition 3.2, $s(10\tau_n) \geq 10n+5$.

(e) Let $n = 16k+12$ $(k \geq 1)$. Then $\varepsilon(10n,10) = 10n+8$ by Lemma 5.1. Hence $s(10\tau_n) < 10n+8$. And we get $\mu(10n,10;n) \geq 10n+7$ by Lemma 5.2. Therefore $s(10\tau_n) \geq 10n+7$.

(f) Let $n = 16k + 15$ $(k \geq 1)$. Then $\varepsilon(10n, 10) = 10n + 10$ by Lemma 5.1. Hence $s(10\tau_n) < 10n + 10$. And we get $\mu(10n, 10; n) \geq 10n + 9$ by Lemma 5.2. Therefore $s(10\tau_n) \geq 10n + 9$. □

6. Proof of Theorem 6

In this section we prove Theorem 6, since the method of the proofs of Theorems 2–5, 7–9 is simpler than and similar to that of Theorem 6.

PROOF OF THEOREM 6. We recall the following

$$6\tau_n = (6n + 6)\xi_n - 6 \in KO(RP^n).$$

(1) Since $6 \geq 2^{\phi(n)}$ for $1 \leq n \leq 3$, $s(6\tau_n) = \infty$ for $1 \leq n \leq 3$ by Proposition 4.2. And since $6 < 2^{\phi(n)}$ and $2^{\phi(n)} - 6 - 6n \leq 0$ for $4 \leq n \leq 11$, $s(6\tau_n) = \infty$ for $4 \leq n \leq 11$ by Proposition 4.3.

(2) Let $n \geq 12$. Then $0 \leq 6n + 6 < 2^{\phi(n)}$ and $6\tau_n$ is stably equivalent to $(6n + 6)\xi_n$.

(a) Let $n \equiv 0, 1 \bmod 4$. We use the method similar to the proof of Theorem 10(a). Since $6n + 6$ is even and $6n + 1$ is odd, we have $\mathrm{Span}(6n + 6)\xi_{6n+1} \geq (6n + 6) - (6n + 1) + 1 = 6$ by Theorems 2.5 and 2.6(A). Hence, by Theorem 2.2(2), $s(6\tau_n) \geq 6n + 1$. On the other hand, $\varepsilon(6n, 6) = 6n + 2$ by Theorem 2.7. Hence $s(6\tau_n) < 6n + 2$ by Theorem 2.3. Therefore $s(6\tau_n) = 6n + 1$.

(b) We use the method similar to the proof of Theorem 10(2)(d). Let $n = 4k + 2$ $(k \geq 3)$. Then $\varepsilon(6n, 6) = 6n + 4$. Hence $s(6\tau_n) < 6n + 4$ by Theorem 2.3. Also putting $6n + 4 = 2^3(3k + 2) = (2b + 1)2^{c+4d}$ $(0 \leq c \leq 3)$, we see $2^c + 8d \geq 6$. So we get $\mu(6n, 6; n) \geq 6n + 3$. Therefore, by Proposition 3.2, $s(6\tau_n) \geq 6n + 3$.

(c) Let $n = 4k + 3$ $(k \geq 3)$. Then $\varepsilon(6n, 6) = 6n + 6$. Hence $s(6\tau_n) < 6n + 6$. Putting $6n + 6 = 2^3(3k + 3) = (2b + 1)2^{c+4d}$ $(0 \leq c \leq 3)$, we see $2^c + 8d \geq 6$. So we get $\mu(6n, 6; n) \geq 6n + 5$. Therefore $s(6\tau_n) \geq 6n + 5$. □

7. Proofs of Theorems 2–5, 7–9

In this section we give an outline of the proofs of Theorems 2–5, 7–9 in the way similar to the proofs of Theorem 6(1) and Theorem 6(2)(b) by using Theorem 2.3, Propositions 3.2, 4.2–4.4.

PROOF OF THEOREM 2. We recall the following

$$2\tau_n = (2n + 2)\xi_n - 2 \in KO(RP^n).$$

(1) Since $2 \geq 2^{\phi(n)}$ for $n = 1$, $s(2\tau_n) = \infty$ for $n = 1$ by Proposition 4.2. And since $2 < 2^{\phi(n)}$ and $2^{\phi(n)} - 2 - 2n \leq 0$ for $2 \leq n \leq 8$, $s(2\tau_n) = \infty$ for $2 \leq n \leq 8$ by Proposition 4.3.

(2) Let $n \geq 9$. Then $0 \leq 2n + 2 < 2^{\phi(n)}$ and $2\tau_n$ is stably equivalent to $(2n + 2)\xi_n$.

We use the method similar to the proof of Theorem 6(2)(b). Now $\varepsilon(2n, 2) = 2n + 2$. Hence $s(2\tau_n) < 2n + 2$ by Theorem 2.3. Also putting $2n + 2(= 2(n + 1)) = (2b + 1)2^{c+4d}$ $(0 \leq c \leq 3)$, we see $2^c + 8d \geq 2$. So we get $\mu(2n, 2; n) \geq 2n + 1$. Therefore, by Proposition 3.2, $s(2\tau_n) \geq 2n + 1$. □

Proof of Theorem 3. We recall the following

$$3\tau_n = (3n + 3)\xi_n - 3 \in KO(RP^n).$$

(1) Since $3 \geq 2^{\phi(n)}$ for $n = 1$, $s(3\tau_n) = \infty$ for $n = 1$. And since $3 < 2^{\phi(n)}$ and $2^{\phi(n)} - 3 - 3n \leq 0$ for $2 \leq n \leq 8$, $s(3\tau_n) = \infty$ for $2 \leq n \leq 8$.

(2) Let $n \geq 9$. Then $0 \leq 3n + 3 < 2^{\phi(n)}$ and $3\tau_n$ is stably equivalent to $(3n + 3)\xi_n$.

(a) Let $n \equiv 0, 1 \mod 4$. Then $\varepsilon(3n, 3) = 3n + 1$. Hence $s(3\tau_n) < 3n + 1$ by Theorem 2.3. And we get $s(3\tau_n) \geq 3n$ by Proposition 4.4.

(b), (c) Now $\varepsilon(3n, 3) = 3n + 2$ for $n \equiv 2 \mod 4$, $= 3n + 3$ for $n \equiv 3 \mod 4$. Hence $s(3\tau_n) < 3n + 2$ for $n \equiv 2 \mod 4$, $< 3n + 3$ for $n \equiv 3 \mod 4$. Also let $n = 4k + 2$ $(k \geq 2)$ and put $3n + 2(= 2^2(3k + 2)) = (2b + 1)2^{c+4d}$ $(0 \leq c \leq 3)$. Then we see $2^c + 8d \geq 3$. So $\mu(3n, 3; n) \geq 3n + 1$. For $n \equiv 3 \mod 4$, we see similarly $\mu(3n, 3; n) \geq 3n + 2$. Therefore $s(3\tau_n) \geq 3n + 1$ for $n \equiv 2 \mod 4$, $\geq 3n + 2$ for $n \equiv 3 \mod 4$. □

Proof of Theorem 4. We recall the following

$$4\tau_n = (4n + 4)\xi_n - 4 \in KO(RP^n).$$

(1) Since $4 \geq 2^{\phi(n)}$ for $1 \leq n \leq 3$, $s(4\tau_n) = \infty$ for $1 \leq n \leq 3$. And since $4 < 2^{\phi(n)}$ and $2^{\phi(n)} - 4 - 4n \leq 0$ for $4 \leq n \leq 9$, $s(4\tau_n) = \infty$ for $4 \leq n \leq 9$.

(2) Let $n \geq 10$. Then $0 \leq 4n + 4 < 2^{\phi(n)}$ and $4\tau_n$ is stably equivalent to $(4n + 4)\xi_n$.

Now $\varepsilon(4n, 4) = 4n + 4$. Also we get $\mu(4n, 4; n) \geq 4n + 3$. □

Proof of Theorem 5. We recall the following

$$5\tau_n = (5n + 5)\xi_n - 5 \in KO(RP^n).$$

(1) Since $5 \geq 2^{\phi(n)}$ for $1 \leq n \leq 3$, $s(5\tau_n) = \infty$ for $1 \leq n \leq 3$. And since $5 < 2^{\phi(n)}$ and $2^{\phi(n)} - 5 - 5n \leq 0$ for $4 \leq n \leq 9$, $s(5\tau_n) = \infty$ for $4 \leq n \leq 9$.

(2) Let $n \geq 10$. Then $0 \leq 5n + 5 < 2^{\phi(n)}$ and $5\tau_n$ is stably equivalent to $(5n + 5)\xi_n$.

(a) Let $n \equiv 0, 2, 3, 5 \mod 8$. Then $\varepsilon(5n, 5) = 5n + 1$. And we get $s(5\tau_n) \geq 5n$ by Proposition 4.4.

(b), (c), (d), (e) Now $\varepsilon(5n, 5) = 5n + 2$ for $n \equiv 6 \mod 8$, $= 5n + 3$ for $n \equiv 1 \mod 8$, $= 5n + 4$ for $n \equiv 4 \mod 8$, $= 5n + 5$ for $n \equiv 7 \mod 8$. Also we get $\mu(5n, 5; n) \geq 5n + 1$ for $n \equiv 6 \mod 8$, $\geq 5n + 2$ for $n \equiv 1 \mod 8$, $\geq 5n + 3$ for $n \equiv 4 \mod 8$, $\geq 5n + 4$ for $n \equiv 7 \mod 8$. □

Proof of Theorem 7. We recall the following

$$7\tau_n = (7n + 7)\xi_n - 7 \in KO(RP^n).$$

(1) Since $7 \geq 2^{\phi(n)}$ for $1 \leq n \leq 3$, $s(7\tau_n) = \infty$ for $1 \leq n \leq 3$. And since $7 < 2^{\phi(n)}$ and $2^{\phi(n)} - 7 - 7n \leq 0$ for $4 \leq n \leq 11$, $s(7\tau_n) = \infty$ for $4 \leq n \leq 11$.

(2) Let $n \geq 12$. Then $0 \leq 7n + 7 < 2^{\phi(n)}$ and $7\tau_n$ is stably equivalent to $(7n + 7)\xi_n$.

(a) Let $n \equiv 0, 1 \mod 8$. Then $\varepsilon(7n, 7) = 7n + 1$. Also we get $s(7\tau_n) \geq 7n$ by Proposition 4.4.

(b) Now $\varepsilon(7n, 7) = 7n + i$ for $n \equiv i \mod 8$ with $2 \leq i \leq 7$. Also we get $\mu(7n, 7; n) \geq 7n + i - 1$ for $n \equiv i \mod 8$ with $2 \leq i \leq 7$. □

Proof of Theorem 8. We recall the following

$$8\tau_n = (8n + 8)\xi_n - 8 \in KO(RP^n).$$

(1) Since $8 \geq 2^{\phi(n)}$ for $1 \leq n \leq 7$, $s(8\tau_n) = \infty$ for $1 \leq n \leq 7$. And since $8 < 2^{\phi(n)}$ and $2^{\phi(n)} - 8 - 8n \leq 0$ for $8 \leq n \leq 11$ or $n = 15$, $s(8\tau_n) = \infty$ for $8 \leq n \leq 11$ or $n = 15$.

(2) Let $n = 12, 13, 14$ or $n \geq 16$. Then $0 \leq 8n + 8 < 2^{\phi(n)}$ and $8\tau_n$ is stably equivalent to $(8n + 8)\xi_n$.

Now $\varepsilon(8n, 8) = 8n + 8$. Also we get $\mu(8n, 8; n) \geq 8n + 7$. □

Proof of Theorem 9. We recall the following

$$9\tau_n = (9n + 9)\xi_n - 9 \in KO(RP^n).$$

(1) Since $9 \geq 2^{\phi(n)}$ for $1 \leq n \leq 7$, $s(9\tau_n) = \infty$ for $1 \leq n \leq 7$. And since $9 < 2^{\phi(n)}$ and $2^{\phi(n)} - 9 - 9n \leq 0$ for $8 \leq n \leq 11$, $n = 14$ or 15, $s(9\tau_n) = \infty$ for $8 \leq n \leq 11$, $n = 14$ or 15.

(2) Let $n = 12, 13$ or $n \geq 16$. Then $0 \leq 9n + 9 < 2^{\phi(n)}$ and $9\tau_n$ is stably equivalent to $(9n + 9)\xi_n$.

(a) Let $n \equiv 0, 2, 4, 6, 7, 9, 11, 13 \mod 16$. Then $\varepsilon(9n, 9) = 9n + 1$. Also we get $s(9\tau_n) \geq 9n$ by Proposition 4.4.

(b), (c), (d), (e), (f), (g), (h), (i) Now $\varepsilon(9n, 9) = 9n + 2$ for $n \equiv 14 \mod 16$, $= 9n + 3$ for $n \equiv 5 \mod 16$, $= 9n + 4$ for $n \equiv 12 \mod 16$, $= 9n + 5$ for $n \equiv 3 \mod 16$, $= 9n + 6$ for $n \equiv 10 \mod 16$, $= 9n + 7$ for $n \equiv 1 \mod 16$, $= 9n + 8$ for

$n \equiv 8 \mod 16$, $= 9n+9$ for $n \equiv 15 \mod 16$. Also we get $\mu(9n, 9; n) \geq 9n+1$ for $n \equiv 14 \mod 16$, $\geq 9n+2$ for $n \equiv 5 \mod 16$, $\geq 9n+3$ for $n \equiv 12 \mod 16$, $\geq 9n+4$ for $n \equiv 3 \mod 16$, $\geq 9n+5$ for $n \equiv 10 \mod 16$, $\geq 9n+6$ for $n \equiv 1 \mod 16$, $\geq 9n+7$ for $n \equiv 8 \mod 16$, $\geq 9n+8$ for $n \equiv 15 \mod 16$. □

Remark. According to Theorem 2.2 of [8], $m\tau_n$ is extendible to RP^N if and only if $m\tau_n$ is stably extendible to RP^N, provided $n \geq 1$ and $m > 1$.

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