Maximal functions for Lebesgue spaces with variable exponent approaching 1

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his seventieth birthday

Toshihide Futamura and Yoshihiro Mizuta
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ABSTRACT. Our aim in this paper is to deal with maximal functions for Lebesgue spaces with variable exponent approaching 1.

1. Introduction

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. We denote by $B(x, r)$ the open ball centered at $x$ of radius $r$. For a locally integrable function $f$ on $\mathbb{R}^n$, we consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$ 

Following Orlicz [4] and Kovačič and Rákosník [2], we consider a positive continuous function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and a measurable function $f$ satisfying

$$\int |f(y)|^{p(y)} dy < \infty.$$ 

In this paper we are concerned with $p(\cdot)$ satisfying the following log-Hölder condition

$$p(r) = 1 + \frac{a \log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)},$$

for $0 < r \leq r_0 < 1/e$, where $a > 0$ and $b$ is a real number; set $p(0) = 1$ and $p(r) = p(r_0)$ for $r > r_0$. For a bounded open set $G$ in $\mathbb{R}^n$, consider

$$p(x) = p(\delta(x)),$$

where $\delta(x)$ denotes the distance of $x$ from the boundary $\partial G$ of $G$. 

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Cruz-Uribe, Fiorenza and Neugebauer [1] proved the maximal operator $M$ is not bounded on $L^p(G)$ if $\inf_{x \in G} p(x) = 1$. Recently, Hästö [3] has proved that the maximal operator $M$ is bounded from $L^p(G)$ to $L^1(G)$ when $a > 1$ and $G$ satisfies a certain regular condition. Our aim in this note is to show that the same conclusion is still valid for $a = 1$.

2. Maximal functions

Throughout this paper, let $C$ denote various constants independent of the variables in question.

Let $G$ be a bounded open set in $\mathbb{R}^n$, and consider a positive continuous function $p(\cdot)$ on $G$ such that

\[
p(x) = 1 + \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))} + \frac{b}{\log(1/\delta(x))}\]

when $0 < \delta(x) \leq r_0 < 1/e$, where $b$ is a real number; assume always that $p(x) > 1$ when $\delta(x) > 0$.

For simplicity, we denote the Lebesgue measure of $E$ by $|E|$.

Let us begin with the following elementary lemmas.

**Lemma 2.1.** Let $G$ be a bounded open set in $\mathbb{R}^n$. For $0 < k \leq n$ and $r > 0$, set $G_r = \{x \in G : \delta(x) < r\}$ and assume that

\[
|G_r| \leq C r^k,
\]

or the Minkowski $(n-k)$-content of $\partial G$ is finite. Then

\[
\int_G \delta(x)^{-k}(\log(1+\delta(x)^{-1}))^{-\alpha} dx < \infty
\]

for every $\alpha > 1$.

**Lemma 2.2.** Set

\[
\varphi(r) = \frac{\log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}
\]

for a real number $b$. Then there exists $r_0 > 0$ such that

(i) $\varphi'(r) > 0$ when $0 < r < 2r_0$;

(ii) $\varphi''(r) < 0$ when $0 < r < 2r_0$;

(iii) $\varphi(s+t) \leq \varphi(s) + \varphi(t)$ when $0 < s, t < r_0$.

For a locally integrable function $f$ on $G$, we consider the maximal function $Mf$ defined by

\[
Mf(x) = \sup_B \frac{1}{|B|} \int_{G \cap B} |f(y)| dy,
\]
where the supremum is taken over all balls $B = B(x, r)$. Define the $L^{p(\cdot)}(G)$ norm by
\[
\|f\|_{p(\cdot), G} = \inf \left\{ \lambda > 0 : \frac{1}{\lambda} \int_{G} |f(y)|^{p(y)} \, dy \leq 1 \right\}
\]
and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} < \infty$.

**Lemma 2.3.** Suppose the Minkowski $(n-1)$-content of $\partial G$ is finite. If $f$ is a measurable function on $G$ with $\|f\|_{p(\cdot)} \leq 1$, then
\[
\int_{G} |f(x)| \log(1 + |f(x)|) \, dx \leq C.
\]

**Proof.** Consider the set
\[
G' = \{ x \in G : |f(x)| < \delta(x)^{-1} (\log(1/\delta(x)))^{-\alpha} \}
\]
for $\alpha > 2$. If $x \in G'$ and $\delta(x) < r_0$ ($< 1/e$), then
\[
|f(x)| \log(1 + |f(x)|) \leq C\delta(x)^{-1} (\log(1/\delta(x)))^{-\alpha+1}.
\]
Hence we have by Lemma 2.1
\[
\int_{G_0 \cap G'} |f(x)| \log(1 + |f(x)|) \, dx \leq C.
\]
If $x \notin G'$ and $\delta(x) < r_0$ ($< 1/e$), then
\[
\delta(x) \geq (C|f(x)| (\log|f(x)|)^{\alpha})^{-1},
\]
so that Lemma 2.2 yields
\[
|f(x)|^{p(x)} \geq |f(x)| \exp\left( \frac{\log \log(C|f(x)| (\log|f(x)|)^{\alpha})}{\log(C|f(x)| (\log|f(x)|)^{\alpha})} b \right) \left( \frac{\log|f(x)|}{\log(C|f(x)| (\log|f(x)|)^{\alpha})} \log|f(x)| \right)
\]
\[
\geq C|f(x)| \exp\left( \frac{\log|f(x)| + \log(C|f(x)| (\log|f(x)|)^{\alpha})}{\log|f(x)|} \log|f(x)| \right)
\]
\[
\geq C|f(x)| \exp\left( \frac{\log|f(x)|}{\log|f(x)|} \log|f(x)| \right)
\]
\[
= C|f(x)| \log|f(x)|.
\]
Here note that
\[
\left| \frac{\log(t + s)}{t + s} - \frac{\log t}{t} \right| \leq C \left( \frac{\log t}{t} \right)^2 \leq \frac{C}{t} \quad \text{when} \quad 1 < s \leq C \log t.
\]
Hence it follows that
\[
\int_{G_\alpha \setminus G'} |f(x)| \log(1 + |f(x)|) \, dx \leq C \int_{G} |f(x)|^{p(x)} \, dx \leq C.
\]
Finally, since \( p(x) \geq p_0 > 1 \) when \( \delta(x) \geq r_0 \), we find
\[
\int_{G \setminus G_\alpha} |f(x)| \log(1 + |f(x)|) \, dx \leq C \int_{G} |f(x)|^{p(x)} \, dx + C \leq C.
\]
Consequently, the required assertion is proved.

Now we are ready to show our main result, which gives an improvement of Hästö [3].

**Theorem 2.4.** Suppose the Minkowski \((n-1)\)-content of \( \delta G \) is finite. Then
\[
\|\text{Mf}\|_1 \leq C\|f\|_{p(\cdot)} \quad \text{for all } f \in L^{p(\cdot)}(G).
\]
This is a consequence of Lemma 2.3 and the well-known fact of maximal functions (see Stein [5]).

**Remark 2.5.** Theorem 2.4 is sharp in the following sense: for instance, if
\[
p(x) = 1 + \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))} - \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))}
\]
when \( 0 < \delta(x) \leq r_0 < 1/e \) and \( \inf_{\{x: \delta(x) > r_0\}} p(x) > 1 \), then we can find \( f \in L^{p(\cdot)}(B) \) for which
\[
\int_B |f(x)| \log(1 + |f(x)|) \, dx = \infty,
\]
where \( B = B(0,1) \) and \( \delta(x) = 1 - |x| \) for \( x \in B \).

For this purpose, letting \( \log_{(1)} t = \log t \) and \( \log_{(m+1)} t = \log(\log_{(m)} t) \) for \( m = 1, 2, \ldots \), we consider the function
\[
f(x) = \delta(x)^{-1}(\log(1/\delta(x)))^{-2}(\log_{(2)}(1/\delta(x)))^{-1}
\]
for \( x \in B \) with \( \delta(x) < r_0 \); set \( f(x) = 0 \) when \( \delta(x) \geq r_0 \). Then
\[
\int_B f(x) \log(1 + f(x)) \, dx \geq C \int_0^{r_0} t^{-1}(\log_{(1)}(1/t))^{-1}(\log_{(2)}(1/t))^{-1} \, dt = \infty.
\]
Further, we have for \( t = \delta(x) < r_0 \)
\[
f(x)^{p(x)-1} \leq \exp((\log(1/t))(\log_{(2)}(1/t))/\log(1/t) - (\log_{(3)}(1/t))/\log(1/t)))
\]
\[
= (\log_{(1)}(1/t))(\log_{(2)}(1/t))^{-1},
\]
so that
\[
\int_B f(x)^{p(x)} \, dx \leq C \int_0^{r_0} t^{-1} (\log(1/t))^{-1} (\log_2(1/t))^{-2} \, dt < \infty.
\]

3. Variable exponent approaching 1 at a point

Suppose \( p(\cdot) \) satisfies \( \inf_{|x| > r_0} p(x) > 1 \) and
\[
p(x) = 1 + \frac{1}{n} \frac{\log(\log(1/|x|))}{\log(1/|x|)} + \frac{b}{\log(1/|x|)}
\]
for \( 0 < |x| \leq r_0 < 1/e \), where \( b \) is a real number. Of course, \( p(0) = 1 \) as before.

**Theorem 3.1.** If \( \|f\|_{p(\cdot)} \leq 1 \), then
\[
\int_B |f(x)| \log(1 + |f(x)|) \, dx \leq C;
\]
and hence
\[
\|Mf\|_1 \leq C \|f\|_{p(\cdot)} \quad \text{for all } f \in L^{p(\cdot)}(B).
\]

**Proof.** As in the proof of Lemma 2.3, we consider the set
\[
B' = \{ x \in B : |f(x)| < |x|^{-n}(\log(1 + |x|^{-1}))^{-2} \}
\]
for \( \alpha > 2 \). Then we have
\[
\int_{B'} |f(x)| \log(1 + |f(x)|) \, dx \leq C \int_B |x|^{-n}(\log(1 + |x|^{-1}))^{-\alpha + 1} \, dx
\]
\[
\leq C \int_0^1 t^{-1} (\log(1 + t^{-1}))^{-\alpha + 1} \, dt < \infty
\]
with the aid of Lemma 2.1. If \( x \in B \setminus B' \), then we see that
\[
|x| \geq (C|f(x)|)^{1/n}(\log|f(x)|)^{2/n},
\]
which yields
\[
|f(x)|^{p(x)} \geq C|f(x)| \log|f(x)|.
\]
Hence we obtain
\[
\int_{B \setminus B'} |f(x)| \log(1 + |f(x)|) \, dx \leq C \int_{B \setminus B'} |f(x)|^{p(x)} \, dx \leq C,
\]
as required. \( \square \)

**Remark 3.2.** As stated in Hästö [3] we have a general result:
Consider a compact subset \( F \) of a bounded open set \( G \), and denote by
\(\delta(x) = \text{dist}(x, F)\) the distance of \(x\) from \(F\). For \(0 \leq m < n\) and \(0 < r < r_0\) with \(r_0\) small enough let \(G_r = \{x \in G : \delta(x) < r\}\) and assume

\begin{equation}
|G_r| \leq Cr^{n-m}.
\end{equation}

Further \(p(\cdot)\) is a continuous function on \(G\) such that

\begin{equation}
p(x) = 1 + \frac{1}{n-m} \log(\log(1/\delta(x))) + \frac{b}{\log(1/\delta(x))}
\end{equation}

when \(0 < \delta(x) \leq r_0 < 1/e\) for some real number \(b\) and \(\inf_{\{x: \delta(x) > r_0\}} p(x) > 1\). Then we claim that if \(f\) is a locally integrable function on \(G\) satisfying \(\|f\|_{p(\cdot)} \leq 1\), then

\[
\int_G |f(x)| \log(1 + |f(x)|) dx \leq C,
\]

so that \(M : L^{p(\cdot)}(G) \to L^1(G)\) is bounded.

To prove this, as in proofs of Theorems 2.4 and 3.1, it suffices to consider the set

\[G' = \{x \in G : |f(x)| < \delta(x)^{m-n}(\log(1/\delta(x))^{-2}\}
\]

for \(\alpha > 2\).

References


Toshihide Futamura
Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530, Japan
E-mail address: futamura@daido-it.ac.jp

Yoshihiro Mizuta
The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail address: mizuta@mis.hiroshima-u.ac.jp