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An algorithm for determining the simplicity of (i, j)-curves on a surface

Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

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Abstract. Chillingworth found an algorithm for determining whether a given element of the fundamental group of a surface contains simple closed curves. We extend the theory to 'open' curves on a punctured surface.

1. Introduction

Chillingworth developed in [3], [4] an algorithm for determining whether a given element of the fundamental group of a surface contains simple closed curves. In this paper, we extend the theory to 'open' curves on a punctured surface. Another algorithm in the case of a plane is already known by Kamada and Matsumoto [5].

Let N be a closed surface or the real plane $\mathbf{R}^2$. In this introduction we assume that N is orientable for simplicity. Fix a positive integer n and let $P_n = \{p_i \in N \mid i = 1, \dots, n\}$ be a finite set of points on N. Given a word V in a certain free group related to the fundamental group, we can show that V determines a homotopy class of (i, j)-curves on $(N \backslash P_n) \cup \{p_i, p_j\}$ such that the initial point is p_i and the terminal point is p_j. There, let the homotopies remain fixing end points p_i, p_j of (i, j)-curves. Then we will prove the following algorithm which decides whether the homotopy class determined by V contains a simple curve or not.

The Algorithm. Suppose $V = y_1 \dots y_t$ and let s_i, s_j be letters corresponding to p_i, p_j respectively. Then to determine the above said representability by a simple curve, we have only to check a certain condition $D(u, v) = 0$ on each cyclic divisions $(s_i \circ V \circ s_j \circ V^{-1}) = (uv)$; $u = y_k \dots y_l$ and $v = y_{l+1} \dots y_t \circ s_j \circ V^{-1} \circ s_i \circ y_1 \dots y_{k-1}$, $1 \le k \le l \le t$. Note that there are $\frac{t(t+1)}{2}$ cyclic divisions

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to check. The homotopy class determined by V is represented by a simple (i,j)-curve if and if only $D(u,v)=0$ for each above division. For the details, see §4.

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2. Preliminaries

First of all, we will recall the definition of winding numbers from Chillingworth [3]. Let M be a smooth surface with a finite number (≥ 1) of points deleted from an orientable or non-orientable closed surface N. We may also consider the case where N is a plane. Let X denote a non-vanishing vector field on M, which exists because M is non-compact.

Definition 2.1. A closed curve on M, $f: S^1 \to M$, is *regular* if f is of C^1-class and the tangent map $df: TS^1 \to TM$ is injective on each fiber.

Let v_x be a vector in tangent plane T_xM to M at $x \in M$. A regular curve is *based at* v_x if $f(1)=x$ and $df((1,1))=v_x$ where $(1,1) \in TS^1 = S^1 \times \mathbf{R}$.

Definition 2.2 [3] (Winding numbers). Suppose that M is given a Riemannian structure. Then for each $x \in M$ a norm $\| \ \|_x$ on T_xM is induced. Let T_0M be the bundle of unit tangent vectors *i.e.* $\bigcup_{x\in M}\{v \in T_xM \mid \|v\|_x = 1\}$. Any continuous map $f: S^1 \to M$ pulls back a bundle E^f over S^1 from T_0M over M and the following diagram commutes where p_1 and p^f are bundle projections and F is an isomorphism on each fiber.

$$\begin{array}{ccc} E^f & \xrightarrow{F} & T_0M \\ \Big\downarrow{\scriptstyle p^f} & & \Big\downarrow{\scriptstyle p_1} \\ S^1 & \xrightarrow{f} & M \end{array}$$

Then E^f is a torus if $\gamma = f(S^1)$ is orientation-preserving, or a Klein bottle if γ is orientation-reversing. A non-vanishing vector field X defines a section X_0 of T_0M by $X_0(z) = \frac{X(z)}{\|X(z)\|_z}$, $z \in M$ and the composition $X_0 f: S^1 \to T_0M$ pulls back to a unique section $X^f: S^1 \to E^f$ such that $FX^f = X_0 f$. Let $\gamma = f(S^1)$ be a regular closed curve. The tangent map df defines a normalized map $d_0 f: S^1 \to T_0M$, *i.e.*, $d_0 f(z) = \frac{df((z,1))}{\|df((z,1))\|_{f(z)}}$. Then $d_0 f$ pulls back to a unique

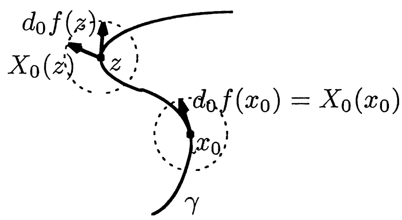

Fig. 1. $d_0f(z)$ and $X_0(z)$

section $Z^f : S^1 \to E^f$ such that $FZ^f = d_0 f$. See Figure 1. Moreover γ based at $X(x_0) \in T_{x_0}M$, $X^f(1) = Z^f(1) = e_0$, so that we can consider that X^f and Z^f represent element $[X^f], [Z^f] \in \pi_1(E^f, e_0)$. On the other hand, since X^f and Z^f are sections over E^f, the element $[X^f] \cdot [Z^f]^{-1}$ belongs to the kernel of the homomorphism $p_*^f : \pi_1(E^f, e_0) \to \pi_1(S^1, 1)$ induced by p^f, *i.e.*, the longitude elements of $\pi_1(E^f, e_0)$ vanish. Therefore $[Z^f] \cdot [X^f]^{-1} = m^t$ for a certain $t \in \mathbf{Z}$ and the normal meridian generator m of $\pi_1(E^f, e_0)$, where we consider m is on the fiber E_0 over $1 \in S^1$. Then, if M is orientable, a choice of an orientation of T_xM induces an orientation of E_0, so that we can define t as the *winding number* of γ based at x_0 with respect to X and we denote it by $\omega_X(\gamma, x_0)$. Finally we define $\omega_X(\gamma, x_0)$ reduced mod 2 in the case when M is non-orientable, since from the above construction the winding number is well-defined only in $\mathbf{Z}_2$ if γ is orientation-reversing.

Definition 2.3. A closed curve γ is *direct* if γ does not contain any non-essential subloops.

For the proofs of following remarks, see [3].

Remark 2.4 [3]. Let γ_1, γ_2 be regular closed curves based at $X(x_0)$, $x_0 \in M$. We assume $[\gamma_1] = [\gamma_2] \neq 1 \in \pi_1(M, x_0)$ and γ_1, γ_2 are direct. Then $\omega_X(\gamma_1, x_0) = \omega_X(\gamma_2, x_0)$ if γ_1 is orientation-preserving, and $\omega_X(\gamma_1, x_0) \equiv \omega_X(\gamma_2, x_0) \bmod 2$ if γ_1 is orientation-reversing, where X is any non-vanishing vector field.

Therefore choosing arbitrary direct regular closed curve γ representing $c \in \pi_1(M, x_0)$, we can define $\omega_X(c) \stackrel{\text{def}}{=} \omega_X(\gamma, x_0)$ if M is orientable or $\omega_X(c) \stackrel{\text{def}}{\equiv} \omega_X(\gamma, x_0) \bmod 2$ if M is non-orientable.

Remark 2.5 [3]. Let γ_1, γ_2 be direct regular closed curves which are freely homotopic but not nullhomotopic. Then $\omega_X(\gamma_1, x_0) = \omega_X(\gamma_2, x_0)$.

Thus we can define winding numbers up to homotopy classes of closed curves. Second, we will define the reading word of a curve.

DEFINITION 2.6 [4] (canonical curve system Σ).

If M is an orientable surface of genus g, then a *canonical curve system* Σ consists of a system of oriented simple closed curves α_i, β_j and σ_k $(1 \le i, j \le g;$ $1 \le k \le r)$ where r is the number of the punctures of M, with the following properties:

i) the curves all meet at a base-point x_0 of M and are disjoint except at x_0;

ii) σ_k bounds a punctured disk R_k surrounding the k-th puncture of M. The interior of R_k is disjoint from all the other curves of Σ;

iii) cutting along the curves of Σ dissects M into the disjoint union of the punctured disks R_k together with a disk whose boundary runs

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\dots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\sigma_1\dots\sigma_r.$$

When M is a punctured plane, we ignore iii).

If M is a non-orientable surface of genus g' then Σ consists of a system of oriented simple closed curves η_i, σ_k $1 \le i \le g'$; $1 \le k \le r$ satisfying i), ii) above together with:

iv) cutting along the curves of Σ dissects M into the disjoint union of the R_k together with a disk whose boundary runs

$$\eta_1\eta_1\eta_2\eta_2\dots\eta_n\eta_n\sigma_1\sigma_2\dots\sigma_r$$

Then, if M is orientable, the fundamental group $\pi_1(M, x_0)$ has the presentation $\{a_i, b_j, s_k (1 \le i, j \le g; 1 \le k \le r) \mid d = 1\}$ where a_i, b_j, s_k denote the homotopy classes of α_i, β_j, σ_k respectively, and

$$d = a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}s_1s_2\dots s_r.$$

Notice d does not exist when M is a punctured plane.

If M is non-orientable, $\pi_1(M, x_0)$ has the representation $\{u_i, s_k (1 \le i \le g';$ $1 \le k \le r) \mid d' = 1\}$ where u_i, s_k denote the homotopy classes of η_i, σ_k respectively and $d' = u_1^2u_2^2\dots u_{g'}^2 s_1\dots s_r$.

DEFINITION 2.7 [4] (Dual dissection).

When M is orientable, the *dual dissection* Σ^*, corresponding to the canonical curve-system Σ, consists of a collection of oriented simple closed curves α_i^*, β_j^* $(1 \le i, j \le g)$ and arcs σ_k^* $(1 \le k \le r)$ with the following properties:

i) all curves and arcs have a common base-point q but are disjoint except at q;

ii) σ_k^* connects q to a sufficiently small circle ϱ_k surrounding the k-th puncture;

iii) each α_i^*, β_j^* or σ_k^* meets α_i, β_j or σ_k transversely exactly once, respectively, and meets none of the other curves of Σ;

iv) cutting along the curves and arcs of Σ^* dissects M into a disk D plus trivial punctured disks inside the small circles ϱ_k whose boundary runs

$$\alpha_1^*\beta_1^{*-1}\alpha_1^{*-1}\beta_1^*\alpha_2^*\beta_2^{*-1}\dots\alpha_g^{*-1}\beta_g^*\sigma_1^*\varrho_1\sigma_1^{*-1}\sigma_2^*\varrho_2\sigma_2^{*-1}\dots\sigma_r^*\varrho_r\sigma_r^{*-1}.$$

If M is a punctured plane, α_i^*, β_j^* do not exist.

When M is non-orientable, Σ^* consists of oriented simple closed curves η_i^* $(1 \le i \le g')$ and arcs σ_k^* $(1 \le k \le r)$ satisfying i), ii) above and

v) each η_i^* or σ_k^* meets η_i, σ_k respectively, transversely exactly once, and meets none of the other curves of Σ;

vi) cutting along the curves and arcs of Σ^* dissects M into a disk D whose boundary runs

$$\eta_1^*\eta_1^*\eta_2^*\eta_2^*\dots\eta_{g'}^*\eta_{g'}^*\sigma_1^*\varrho_1\sigma_1^{*-1}\sigma_2^*\varrho_2\sigma_2^{*-1}\dots\sigma_r^*\varrho_r\sigma_r^{*-1}.$$

Definition 2.8 (some terms).

Let P_n be the set of points $\{p_i \in N \mid i = 1, \dots, n\}$ to be deleted from N. Let M be $N \backslash P_n$. A regular map $f : [0, 1] \to M \cup \{p_i, p_j\}$ is an (i, j)-*curve* if f satisfies $f(0) = p_i$ and $f(1) = p_j$ where $i \neq j$, and $f^{-1}(\{p_i, p_j\}) = \{0, 1\}$.

Let G be a group. For any element $g \in G$, we define *the cyclic word of* g to be the conjugate class of g and denote it by (g).

Let γ be an oriented curve not passing through the base-point q of Σ^*, and in general position with respect to Σ^*. Let χ^* be any curve of Σ^*, and let p be a point of $\gamma \cap \chi^*$. By assigning to a neighborhood of p the orientation induced by the path qp along χ^* in the positive direction from some fixed orientation at q, we can define that γ cuts χ^* from right to left *positively* at p or from left to right *negatively* at p.

Definition 2.9 (the reading word of a curve).

When γ is a closed curve which satisfies the above condition, the *reading word* of γ, $rd(\gamma)$, is a cyclic word in the letters $a_i^{\pm 1}$, $b_j^{\pm 1}$, $s_k^{\pm 1}$, if M is orientable or $u_i^{\pm 1}$, $s_k^{\pm 1}$, if M is non-orientable, obtained from γ as follow. Choose any point y on γ and not on any curve of Σ^* and proceed once round γ in the positive direction. We give each point at which γ crosses a curve or arc of Σ^*

a letter a_i if γ crosses α_i^* positively,

a letter a_i^{-1} if γ crosses α_i^* negatively,

a letter b_j if γ crosses β_j^* positively etc.,

and sum up in order, so that we can get $rd(\gamma)$.

When l is an (i, j)-curve, proceed from p_i to p_j along l and construct a word by the same process as above. Next devide some power of $s_i^{\pm 1}$ from the left of the sequence V' obtained and some power of $s_j^{\pm 1}$ from the right of

V', so that the first letter and the last letter of the sequence are not $s_i^{\pm 1}$, $s_j^{\pm 1}$ respectively. Finally we define the resulting sequence V of letters as the *reading word* $rd(l)$ of an (i, j)-curve l.

For an (i, j)-curve l, since we can remove the intersection points between l and σ_i^* (or σ_j^*) corresponding to the first $s_i^{\pm 1}$ or the last $s_j^{\pm 1}$ of V', if any, by using a homotopy, the reading word $rd(l)$ is realized by the 'natural' reading of an (i, j)-curve which is homotopic (fixing p_i, p_j) to l.

Conversely, given a word A consisting of generators of $\pi_1(M, x_0)$, we mark a point on χ^* such that any two points do not overlap each other and we connect those in order of the letters of A such that each interval between two adjacent marked points is direct without crossing χ^* except marked points, so that we can obtain a closed curve $\gamma(A)$ whose reading word is A. To obtain an (i, j)-curve corresponding to A, we connect p_i, p_j and the end points of the path obtained in the same way respectively and denote it by $l_{ij}(A)$.

DEFINITION 2.10 (reduced words).

Let V be any word consisting of generators of $\pi_1(M, x_0)$, $x_0 \in M$. V is *simply reduced* if it satisfies the conditon that no two adjacent letters in V are mutually inverse.

Let W be any word consisting of generators of $\pi_1(M, x_0)$, $x_0 \in M$. W is *completely reduced* if it satisfies the following conditions;

i) no two cyclically adjacent letters in W are mutually inverse;

ii) neither W nor any cyclic permutation of W contains any subword of the element $d^{\pm 1}$ or $d'^{\pm 1}$, or of any cyclic permutation of $d^{\pm 1}$ or $d'^{\pm 1}$ whose length is more than half of the length of d or d' where d and d' are the words defined just before Definition 2.7;

iii) if W or any cyclic permutation of W contains a subword which consists of exactly half of a cyclic permutation of d^{ε} or d'^{ε} then this half contains the first element of d or d' when $\varepsilon = +1$ or the last element of d^{-1} or d'^{-1} when $\varepsilon = -1$.

We call a word satisfying the condition i) *cyclically reduced*. When M is a punctured plane, we ignore ii) and iii) and a *completely reduced* word is nothing but a *cyclically reduced* word.

Notice that if any word W is given, we can always obtain a completely reduced word $\tilde{W}$ from W by means of canceling so that it satisfies i), ii), iii) in order. Moreover, if we regard W and $\tilde{W}$ as elements of the fundamental group, W and $\tilde{W}$ belong to the same conjugacy class so that those are freely homotopic. Of course, we can apply the same discussion to simply reduced words. Hence in the following, we can suppose that a word V is simply reduced and a word W is completely reduced without loss of generality.

3. Main theorem

Definition 3.1 (the double curve of an (i, j)-curve). Let l be an (i, j)-curve such that $rd(l) = V$.

We call the curve resulting from the following constructions the *double curve* of the (i, j)-curve l.

The case where M is orientable:

1. Draw a curve l' in a sufficiently small tubular neighborhood of the (i, j)-curve l so that l' satisfies the following conditions: l' is parallel to l on the left side of the oriented curve l, l' is contained in M, and the number of the self-intersection points of l' is equal to that of l. We give an orientation to l' so that $rd(l')$ is $(rd(l))^{-1}$. See Figure 2.

Now let the initial point and the terminal point of l' be p_j' and p_i' respectively.

2. Make a curve $l'' : p_i' \to p_j'$ in a sufficietly small tubular neighborhood of l through an isotopy of M in the following way. We move l only in a small neighborhoods of p_i containing p_i' and of p_j containing p_j' so that we make no new self-intersection points and $l'' \cap l' = \{p_i', p_j'\}$. We make l'' cross σ_i^* and σ_j^* so that $rd(l'') = s_i \cdot rd(l) \cdot s_j$. See Figure 3.

3. Define the *double curve* of the (i, j)-curve l to be the smooth loop $l'' \circ l' : p_i' \xrightarrow{l'} p_j' \xrightarrow{l''} p_i'$ where, if nessesary, make $l'' \circ l'$ smooth at p_i' and p_j' by a homotopy deformation. Then $rd(l'' \circ l')$ is $s_i \circ V \circ s_j \circ V^{-1}$.

The case where M is non-orientable: we take l' in the tubular neighborhood of the (i, j)-curve l such that p_i' is on the the left side of l toward

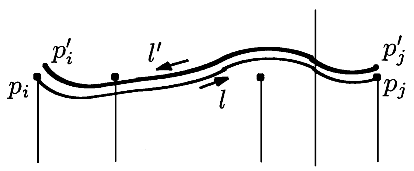

Fig. 2. a curve l' parallel to l

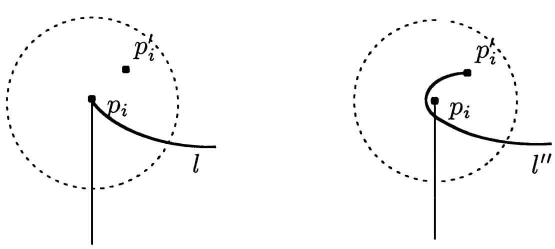

Fig. 3. curves l and l'' in a neighborhood of p_i

positive direction of l. Then p_j' is on the right side of l toward the same direction if $rd(l)$ contains an odd number of u_i, $i = 1, \dots, n$, or p_j' is on the the left side if $rd(l)$ contains an even number of them. We constract the double curve of $l'' \circ l'$ similarly to the orientable case, so that $rd(l'' \circ l')$ is $s_i \circ V \circ s_j^{-1} \circ V^{-1}$ if the number of u_i's is odd, or is $s_i \circ V \circ s_j \circ V^{-1}$ if it is even. ■

If an (i, j)-curve l is simple, we see immediately from the above construction that the double curve of l is also simple.

THEOREM 3.2. *Let V be any given word of the generators of $\pi_1(M, x_0)$.*

i) *The case where M is orientable:*

The homotopy class of (i, j)-curves determined by V is represented by a simple (i, j)-curve on M, if and only if $s_i \circ V \circ s_j \circ V^{-1}$ is represented by a simple closed curve.

ii) *The case where M is non-orientable:*

The homotopy class of (i, j)-curves determined by V is represented by a simple (i, j)-curve on M if and only if $s_i \circ V \circ s_j^{-1} \circ V^{-1}$ is represented by a simple closed curve, if V contains an odd number of u_i, $i = 1, \dots, n$. If V contains an even number of them, the result is the same as in the orientable case.

PROOF. Since the proof of the non-orientable case is almost the same as that of the orientable case, we will prove the theorem in the orientable case only.

Suppose V is represented by a simple (i, j)-curve l. Then the reading word of the double curve of l is $s_i \circ V \circ s_j \circ V^{-1}$ and the double curve is simple by the assumption. Thus $s_i \circ V \circ s_j \circ V^{-1}$ is represented by a simple closed curve.

Conversely, assume that $s_i \circ V \circ s_j \circ V^{-1}$ is represented by a simple closed curve γ and $V = y_1 y_2 \dots y_t$ where y_h is $a_k^{\pm 1}$, $b_j^{\pm 1}$ or $s_k^{\pm 1}$ for $h = 1, \dots, t$. Suppose that p_i' and p_j' are on the arcs σ_i^* and σ_j^* of the dual dissection Σ^* which give the readings s_i and s_j in $s_i \circ V \circ s_j \circ V^{-1}$, respectively. Furthermore let l_1 be the path from p_i' to p_j' along γ^{-1} and l_2 be the path from p_i' to p_j' along γ. Then we notice $rd(l_1) = rd(l_2) = V$ and V cannot be reduced at all. Let q_h, q_h' be the points which give readings y_h^{-1}, y_h for $h = 1, \dots, t$ on l_1, l_2 respectively, $\mathscr{L}_h$ be a subline $q_h q_h'$ on α_i^*, β_j^* or σ_k^* in the right side of l_1 toward the possitive direction of l_1 and $l_1(h)$, $l_2(h)$ be subpaths from q_h to q_{h+1} on l_1, from q_h' to q_{h+1}' on l_2, respectively. Notice distinct $\mathscr{L}_h$ and $\mathscr{L}_{h'}$ are disjoint since they do not contain base-point of Σ^* and γ is a simple closed curve. See Figure 4. When M is a surface of positive genus, we see the fact immediately, considering the disk D obtained by dissecting M along Σ^*.

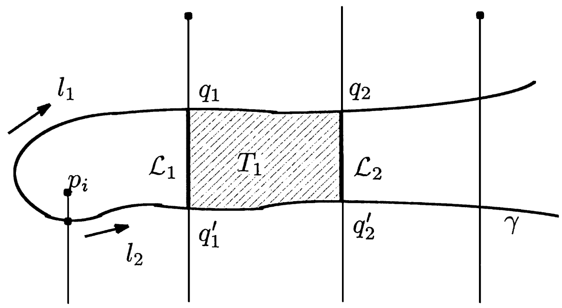

Fig. 4

We define

$$T = \{\textit{the region enclosed by } \mathscr{L}_1 \textit{ and } \gamma \textit{ and containing a small disk centered at } p_i\}$$

$$T^* = \{\textit{the region enclosed by } \mathscr{L}_t \textit{ and } \gamma \textit{ and containing a small disk centered at } p_j\}$$

$$T_h = \{\textit{the region enclosed by } \mathscr{L}_h,\ l_1(h),\ l_2(h) \textit{ and } \mathscr{L}_{h+1}\} \qquad \text{for } 1 \le h \le t-1$$

$$U_s = \bigcup_{k=1}^{s} T_k \qquad \text{for } 1 \le s \le t-1$$

Remember that t is the word length of V. Then we will show that U_s for $1 \le s \le t-1$ is isotopic to an embedding by induction.

First Step: On $U_1 = T_1$.

First notice that the interior of T_1 is an embedding of an open disk. Suppose T_1 is not isotopic to an embedding. Then the boundary ∂T_1 has intersection with itself. Since γ is simple, $l_1(1) \cap l_2(1) = \emptyset$. Thus $\mathscr{L}_1 \cap \mathscr{L}_2 \neq \emptyset$. This means $\mathscr{L}_1$ and $\mathscr{L}_2$ are on the same curve or arc of Σ^*. Then $y_1 = y_2^{\pm 1}$. But in the simply reduced word V, $y_1 = y_2^{-1}$ is prohibitted. Therefore we obtain $y_1 = y_2$. On the other hand, since $l_1(1)$ and $l_2(1)$ are outside of T except at q_1 and q_1', we see that $q_2, q_2' \notin \mathscr{L}_1$. Otherwise y_2 would be y_1^{-1}. Thus $\mathscr{L}_1 \subset \mathscr{L}_2$. However, since $p_i \in T$, it is impossible to realize both $rd(l_1(1)) = rd(l_2(1)) = y_1^2$ and $\mathscr{L}_1 \subset \mathscr{L}_2$. See Figure 5. Hence $T_1 = U_1$ is isotopic to an embedding.

Second Step: On U_{s+1}. Assume that U_s is isotopic to an embedding for $1 \le s \le t-2$. Then U_{s+1} is isotopic to an embedding, if and only if a) T_{s+1} is isotopic to an embedding and b) $U_s \cap T_{s+1} = \mathscr{L}_{s+1}$. We will show a) and b).

a) Suppose T_{s+1} is not isotopic to an embedding. Like the *First Step*, $\mathscr{L}_{s+2} \subset \mathscr{L}_{s+1}$ is impossible. Thus $\mathscr{L}_{s+1} \subset \mathscr{L}_{s+2}$. However this is again impossible due to the fact $p_i \in (T \cup U_s)$.

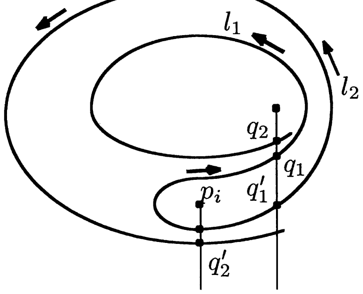

Fig. 5

b) It is obvious that $U_s \cap T_{s+1} \supset \mathscr{L}_{s+1}$. Note that the boundary of $U_s \cup T_{s+1}$ consists of $\mathscr{L}_1$, $\mathscr{L}_{s+2}$ and subarcs of γ. Thus if $U_s \cap T_{s+1}$ contains some more points which are not on $\mathscr{L}_{s+1}$, then these points must be on $\mathscr{L}_1 \cup \mathscr{L}_{s+2}$, because γ is a simple closed curve. However, this is contradiction just like *First Step*.

Therefore U_s is isotopic to an embedding for $1 \le s \le t-1$.

Finally, we also find $T \cup U_{t-1} \cup T^*$ is isotopic to an embedding disk from the above construction. Therefore, we can connect p_i and p_i' on σ_i^* without crossing γ except p_i' because γ doesn't cross the interior of subline $p_i p_i'$ on σ_i^*. Of course, we can also discuss the same about p_j and p_j'. Then connect p_i' and p_j' by a subarc of γ. After this connection, doing a homotopy deformation of M if necessary, we can obtain a simple (i,j)-curve l_0 such that $rd(l_0) = V$. ■

LEMMA 3.3. *Let V be any simply reduced word. Suppose that the first letter (resp. the last letter) of V is not $s_i^{\pm 1}$ (resp. $s_j^{\pm 1}$) and $s_i \neq s_j^{\pm 1}$. Then $s_i \circ V \circ s_j^{\pm 1} \circ V^{-1} \neq U^m$, for any $m \in \mathbf{Z}$ with $m \ge 2$ and any word U.*

PROOF. Now we assume $s_i \circ V \circ s_j^{\pm 1} \circ V^{-1} = U^m$ for some word U and $\exists m \in \mathbf{Z}$ with $m \ge 2$.

If m is even, both s_i and $s_j^{\pm 1}$ are the first letter of U. This is a contradiction because of the assumption that $s_i \neq s_j^{\pm 1}$.

If m is odd, the number of letters of U is $\frac{2k+2}{m}$ where k is the number of letters of V. Thus both $\left(\frac{2k+2}{m} \cdot \frac{m-1}{2} + 1\right)$th letter of $s_i \circ V \circ s_j^{\pm} \circ V^{-1}$ and $\left(\frac{2k+2}{m} \cdot \frac{m+1}{2} + 1\right)$th letter of $s_i \circ V \circ s_j^{\pm} \circ V^{-1}$ are the first letter of U. Denote those letters by y_l, $y_{l'}^{-1}$ respectively.

$$l = \frac{2k+2}{m} \cdot \frac{m-1}{2} + 1 - 1 = \frac{(k+1)(m-1)}{m}$$

$$l' = k - \left(\frac{2k+2}{m} \cdot \frac{m+1}{2} + 1 - 3 - k\right)$$
$$= 2k + 2 - \frac{(k+1)(m+1)}{m} = \frac{(k+1)(m-1)}{m} = l$$

Therefore $y_l = y_l^{-1}$. However $y_l^2 \neq 1$ since M is non-compact. Hence this is a contradiction. ■

Theorem 3.4 (Chillingworth [4]).
Let M be $N \backslash P_n$, not projective plane with two disks removed, i.e., $\pi_1(M, x) \neq \{u_1, s_1, s_2 \mid u_1^2 s_1 s_2 = 1\}$, $x \in M$. Then a given element $c \in \pi_1(M, x)$ is represented by a simple closed curve, if and only if the completely reduced word W read from c satisfies one of the following conditions;

i) *W is empty, i.e., $c = 1$ or consists of one letter,*
ii) *$W = V^2$ where V is represented by an orientation-reversing simple closed curve,*
iii) *when W consists of at least 2 lettters and W is not equal to U^m for any word U and any $m \geq 2$, the equation $\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$ is satisfied for every cyclic division $(W) = (uv)$, where X is any non-vanishing vector field on M and u, v are non-empty subwords. (Note that we can express the cyclic word (W) as (uv) in $\frac{1}{2}p(p-1)$ ways.)*

See [4] for the proof of Theorem 3.4.

Corollary 3.5. *Let V be any given simply reduced word. Assume the first (or the last) letter of V is not $s_i^{\pm 1}$ (or $s_j^{\pm 1}$).*

i) *The case where M is orientable: the homotopy class determined by V is represented by a simple (i, j)-curve if and only if the equation*
$$\omega_X(uv^{-1}) = \omega_X(u) + \omega_X(v^{-1})$$
is satisfied, for every cyclic division $(s_i \circ V \circ s_j \circ V^{-1}) = (uv)$.

ii) *The case where M is non-orientable: the homotopy class determined by V is represented by a simple (i, j)-curve if and only if the equation*
$$\omega_X(uv^{-1}) \equiv \omega_X(u) + \omega_X(v^{-1}) \bmod 2$$
is satisfied, for every cyclic division $(s_i \circ V \circ s_j^{-1} \circ V^{-1}) = (uv)$, if V contains an odd number of u_i, $i = 1, \dots, n$. If V contains an even number of that, the result is the same as orientable case.

Proof. The problem of (i, j)-curves reduces to Theorem 3.4 by virtue of Theorem 3.2. Note that case i) of Theorem 3.4 does not occur in the situation of Corollary 3.5, because the number of letters of $s_i \circ V \circ s_j^{\pm 1} \circ V^{-1}$ is equal to or greater than 2. The case ii) of Theorem 3.4 does not occur either because of Lemma 3.3. ■

REMARK 3.6. We will consider the case excluded by Theorem 3.4. Namely, let $N =$ *projective plane*, $M = N \backslash P_2$. Then it is known that a given non-trivial element $c \in \pi_1(M, x)$ is represented by a simple closed curve if and only if the completely reduced word W read from c is one of those conjugate to following words; u_1, u_1^2, $u_1 s_1$, $(u_1 s_1)^2$, s_1, $u_1^2 s_1$ or their inverses. Hence the homotopy class determined by simply reduced word V is represented by a simple $(1,2)$-curve (or a simple $(2,1)$-curve), if and only if, noticing the number of u_1 which V contains, $s_1 \circ V \circ s_2^{\pm 1} \circ V^{-1}$ (or $s_2 \circ V \circ s_1^{\pm 1} \circ V^{-1}$) is one of those conjugate to above words.

4. The algorithm

Mainly we discuss the orientable case in order to avoid complication. However we can do the same thing in the non-orientable case. First, we recall how to calculate the winding number in the case when M is orientable. See [3] for details. Let γ be a closed curve such that $rd(\gamma) = W = y_1 \dots y_k$ for a cyclically reduced word W. We define;

$$A_i(W) = \{\textit{the number of letter } a_i \textit{ in } W\}$$

$$\widetilde{A_i}(W) = \{\textit{the number of letter } a_i^{-1} \textit{ in } W\}$$

and define $B_j(W)$, $\widetilde{B_j}(W)$, $S_k(W)$, $\widetilde{S_k}(W)$ similarly to $A_i(W)$ and $\widetilde{A_i}(W)$.

$$P(W) = \{\textit{the number of values of } m \textit{ for which } y_m^{-1} \textit{ appears before } y_{m+1} \textit{ in the following order } \Omega \textit{ for } m = 1, \dots, n\}$$

$$N(W) = \{\textit{the number of values of } m \textit{ for which } y_m^{-1} \textit{ appears after } y_{m+1} \textit{ in } \Omega \textit{ for } m = 1, \dots, n\}$$

where the indices are considered cyclically.

The order Ω is defined as follows:

$$\Omega\colon\ a_1, b_1^{-1}, a_1^{-1}, b_1, a_2, b_2^{-1}, \dots, a_g^{-1}, b_g, s_1, s_1^{-1}, s_2, \dots, s_r^{-1}$$

Notice $y_{i+1} \neq y_i$ since W is cyclic reduced. Then we have the following theorem:

THEOREM 4.1 [3].

$$2\pi\omega_X(W) = \sum_{i=1}^{g} (2\pi\omega_X(a_i) + \pi)(A_i - \widetilde{A_i}) + \sum_{j=1}^{g} (2\pi\omega_X(b_j) - \pi)(B_j - \widetilde{B_j})$$
$$+ \sum_{k=1}^{r} (2\pi\omega_X(s_k) + \pi)(S_k - \widetilde{S_k}) + (P(W) - N(W))\pi$$

where X is any non-vanishing vector field.

Hence the *discriminating function* $D(u, v)$ for a division $(W) = (uv)$ is the following,

$$D(u, v) := 2(\omega_X(uv^{-1}) - (\omega_X(u) + \omega_X(v^{-1})))$$
$$= P(uv^{-1}) - N(uv^{-1}) - (P(u) - N(u) + P(v^{-1}) - N(v^{-1})),$$

where we reduce W completely and uv^{-1}, u, v^{-1} cyclically. Therefore $D(u, v)$ is independent of the the vector field but depends only on the order of letters of W.

The homotopy class determined by W is represented by a simple closed curve if $D(u, v) = 0$ for every division, and it is not represented by any simple closed curve if $D(u, v) \neq 0$ for some division.

Remark 4.2. Let W be any cyclic reduced word such that $(W) = (uv)$ for non-empty words u, v. Then $D(u, v) \neq 0$ if and only if the closed curve γ such that $rd(\gamma) = W$ has two non-empty subloops of which the reading words are u, v, respectively.

Proof. When M is not a plane, we consider the problem on the disk obtained by dissecting M by Σ^*. Let l_1, l_2 be subarcs in γ such that $rd(l_1) = u$, $rd(l_2) = v$ and $l_1 \cup l_2 = \gamma$.

First we consider the case where l_1 and l_2 are subloops with the base-point x. Then in $\gamma(uv^{-1})$ the point corresponding to x vanishes by an isotopy. See Figure 6. On the other hand, $\gamma(u)$ and $\gamma(v^{-1})$ are two loops such that the directions are the same. See Figure 7. Therefore the winding number $\omega_X(uv^{-1})$ is not equal to the sum of the winding numbers $\omega_X(u) + \omega_X(v^{-1})$. Thus $D(u, v) \neq 0$.

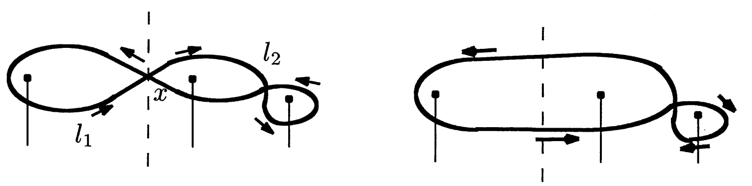

Fig. 6. curves $\gamma(uv)$ and $\gamma(uv^{-1})$

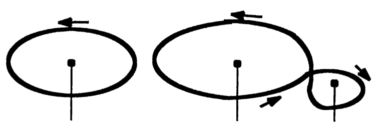

Fig. 7. curves $\gamma(u)$ and $\gamma(v^{-1})$

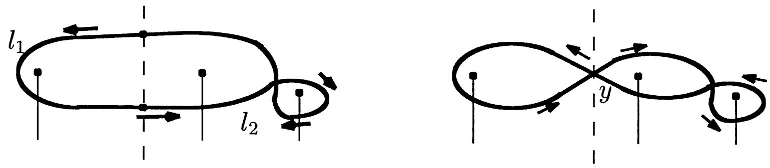

Fig. 8. curves γ and $\tilde{\gamma}$

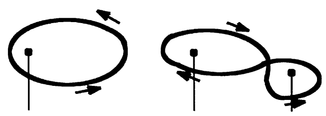

Fig. 9. curves $\gamma(u)$ and $\gamma(v^{-1})$

Next we consider the second case neither l_1 nor l_2 is a loop. (But they might contain subloops.) Then the closed curve $\tilde{\gamma}$ such that $rd(\tilde{\gamma}) = \gamma(uv^{-1})$ have a self-intersection point y and two subloops of which the reading words are u, v^{-1} and which have a common base-point y. See Figure 8. In this case, cutting $\tilde{\gamma}$ off at y to two loops through a homotopy, we get the loops of which the reading words are u, v^{-1}. See Figure 9.

Hence in this case the winding number $\omega_X(uv^{-1})$ is equal to the sum $\omega_X(u) + \omega_X(v^{-1})$. Thus $D(u,v) = 0$. ∎

Let $V = y_1 \dots y_t$ be a word such that V is not represented by any simple (i,j)-curve. Then since a $l_{ij}(V)$ has one subloop at least, the reading word of the subloop is containd in V. Hence it is sufficient to check $D(u,v) = 0$ only in the case $u = y_k \dots y_l$ for $1 \le k \le l \le t$.

THEOREM 4.3. *We can find whether a given word $V = y_1 \dots y_t$ is represented by a simple (i,j)-curve or not, by checking $D(u,v) = 0$ only for the following division $(s_i \circ V \circ s_j \circ V^{-1}) = (uv)$; $u = y_k \dots y_l$ and $v = y_{l+1} \dots y_t \circ s_j \circ V^{-1} \circ s_i \dots y_1 \circ y_{k-1}$, $1 \le k \le l \le t$. (Note that there are $\frac{t(t+1)}{2}$ divisions to check.*

The homotopy class determined by V is represented by a simple (i,j)-curve if $D(u,v) = 0$ for each division above. It is not represented by any simple (i,j)-curve if $D(u,v) \neq 0$ for some division above.

In the case where M is non-orientable, referring to Theorem 3.2, we have only to choose $s_i \circ V \circ s_j \circ V^{-1}$ or $s_i \circ V \circ s_j^{-1} \circ V^{-1}$.

EXAMPLE. The case where M is $\mathbf{R}^2 \backslash P_n$. Frist we construct a dual dissection Σ^* of $\mathbf{R}^2 \backslash P_n$ and how to conclude the reading word of the curve. See Figure 10.

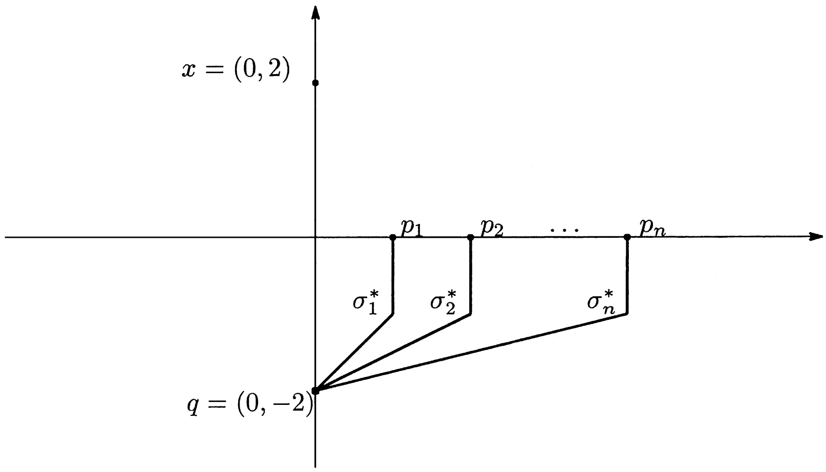

Fig. 10. P_n and the arcs of Σ^* on $\mathbf{R}^2$

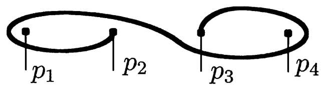

Fig. 11. a simple $(2,3)$-curve whose reading word is $\tilde{V}_1$

Then without loss of generality we can suppose that an (i,j)-curve l is in general position with respect to Σ^* and is contained in $\{(x,t) \in \mathbf{R}^2 \mid t \geq -1\}$. When l crosses an arc σ_k^* possitively with respect to the x-coordinate of $\mathbf{R}$, we give the point the letter x_k and when l cross σ_k^* negatively, we give the letter x_k^{-1}. The reading word of l, $rd(l)$, is obtained by assembling those letters in order as we proceed from p_i to p_j along l. Notice that the equation $d = 1$ does not exist in this case.

i) Suppose $n = 4$. Consider the homotopy class of $(2,3)$-curves given by $V_1 = x_1^{-1}x_3x_2x_2^{-1}x_4$. First we need to reduce V_1 simply. The simply reduced word $\tilde{V}_1$ is $x_1^{-1}x_3x_4$. Then we have only to check $D(u,v) = 0$ for the six cases $u = x_1^{-1}, x_1^{-1}x_3, x_1^{-1}x_3x_4, x_3, x_3x_4$ and x_4. For example, in the case of $u = x_1^{-1}x_3x_4$, $v = x_3x_4^{-1}x_3^{-1}x_1x_2$ from $(x_2 \circ \tilde{V}_1 \circ x_3 \circ \tilde{V}_1^{-1}) = (uv)$, we have

$$\begin{aligned} D(u,v) &= P(uv^{-1}) - N(uv^{-1}) - (P(u) - N(u) + P(v^{-1}) - N(v^{-1})) \\ &= 4 - 4 - (2 - 1 + 2 - 3) = 0. \end{aligned}$$

Calculating each that of the six cases similarly, we find $D(u,v) = 0$ for the six cases. Thus we know the homotopy class determined by V_1 is represented by a simple $(2,3)$-curve. See Figure 11.

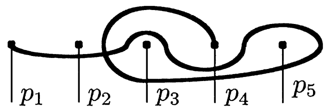

Fig. 12. a $(1,4)$-curve whose reading word is V_2

ii) Suppose $n = 5$. Consider the homotopy class of $(1,4)$-curves given by $V_2 = x_2x_4x_5^{-1}x_4^{-1}x_3^{-1}$. Similarly we inspect the fifteen cases. Then in the case of $u = x_4x_5^{-1}x_4^{-1}x_3^{-1}$, $D(u,v)$ is equal to -2, not 0. Therefore this class has a subloop of which the reading word is $x_4x_5^{-1}x_4^{-1}x_3^{-1}$, thus V_2 does not contain any simple $(1,4)$-curve. See Figure 12.

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