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Unitary convolution for arithmetical functions in several variables

Emre Alkan, Alexandru Zaharescu and Mohammad Zaki

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Abstract. In this paper we investigate the ring $A_r(R)$ of arithmetical functions in r variables over an integral domain R with respect to the unitary convolution. We study a class of norms, and a class of derivations on $A_r(R)$. We also show that the resulting metric structure is complete.

1. Introduction

The ring A of complex valued arithmetical functions has a natural structure of commutative **C**-algebra with addition and multiplication by scalars, and with the Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

In [1], Cashwell and Everett proved that $(A, +, \cdot)$ is a unique factorization domain. Yokom [5] investigated the prime factorization of arithmetical functions in a certain subring of the regular convolution ring. He determined a discrete valuation subring of the unitary ring of arithmetical functions. Schwab and Silberberg [3] constructed an extension of $(A, +, \cdot)$ which is a discrete valuation ring. In [4], they showed that A is a quasi-noetherian ring. In the present paper we study the ring of arithmetical functions in several variables with respect to the unitary convolution over an arbitrary integral domain. Let R be an integral domain with identity 1_R. Let $r \geq 1$ be an integer number, and denote $A_r(R) = \{f : \mathbf{N}^r \to R\}$. Given $f, g \in A_r(R)$, let us define the unitary convolution $f \oplus g$ of f and g by

$$(f \oplus g)(n_1, \dots, n_r) = \sum_{\substack{d_1 e_1 = n_1 \\ (d_1, e_1) = 1}} \cdots \sum_{\substack{d_r e_r = n_r \\ (d_r, e_r) = 1}} f(d_1, \dots, d_r) g(e_1, \dots, e_r).$$

Note that R has a natural embedding in the ring $A_r(R)$, and $A_r(R)$ with addition and unitary convolution defined above becomes an R-algebra. We define and study a family of norms on $A_r(R)$. Then we show that $A_r(R)$ endowed with any of the above norms is complete. A class of derivations on $A_r(R)$ is then constructed and examined. We also study the logarithmic derivatives of multiplicative arithmetical functions with respect to these derivations.

2. Norms

Let $U(R)$ denote the group of units of R. Let $U(A_r(R))$ be the group of units of $A_r(R)$. Thus, $U(A_r(R)) = \{f \in A_r(R) : f(1,\dots,1) \in U(R)\}$. In this section R will denote an integral domain. We start by defining a norm on $A_r(R)$. Fix $\underline{t} = (t_1,\dots,t_r) \in \mathbb{R}^r$ with $t_1,\dots,t_r$ linearly independent over $\mathbb{Q}$, and $t_i > 0$, $(i = 1,2,\dots,r)$. Given $n \in \mathbf{N}$, we define $\Omega(n)$ to be the total number of prime factors of n counting multiplicities, i.e., if $n = p_1^{\alpha_1}\dots p_k^{\alpha_k}$, then $\Omega(n) = \alpha_1 + \dots + \alpha_k$. We now define $\Omega_r : \mathbf{N}^r \to \mathbf{N}^r$ by

$$\Omega_r(n_1,\dots,n_r) = (\Omega(n_1),\dots,\Omega(n_r)).$$

Given $\underline{n} = (n_1,\dots,n_r)$ and $\underline{m} = (m_1,\dots,m_r)$ in $\mathbf{N}^r$, we denote $\underline{n}\cdot\underline{m} = n_1m_1 + \dots + n_rm_r$. For $f \in A_r(R)$, we define the support of f, $supp(f) = \{\underline{n} \in \mathbf{N}^r \mid f(\underline{n}) \neq 0\}$. We also define for $f \in A_r(R)$,

$$V_{\underline{t}}(f) = \begin{cases} \infty & \text{if } f = 0; \\ \min\limits_{\underline{n} \in supp(f)} \underline{t}\cdot\Omega_r(\underline{n}) & \text{if } f \neq 0. \end{cases}$$

Note that if $f \neq 0$ then $V_{\underline{t}}(f) = \underline{t}\cdot\Omega_r(\underline{n})$ for some $\underline{n} \in supp(f)$.

PROPOSITION 1. (i) *For any* $f,g \in A_r(R)$, *we have*

$$V_{\underline{t}}(f+g) \geq \min\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}.$$

(ii) *For any* $f,g \in A_r(R)$, *we have*

$$V_{\underline{t}}(f \oplus g) \geq V_{\underline{t}}(f) + V_{\underline{t}}(g).$$

PROOF. (i) Let $f,g \in A_r(R)$. If $f+g = 0$, then clearly $V_{\underline{t}}(f+g) \geq \min\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}$. Suppose $f+g \neq 0$. Let $\underline{n} \in supp(f+g)$. Then either $\underline{n} \in supp(f)$, or $\underline{n} \in supp(g)$. If $\underline{n} \in supp(f)$, then $\underline{t}\cdot\Omega_r(\underline{n}) \geq V_{\underline{t}}(f)$, and if $\underline{n} \in supp(g)$, then $\underline{t}\cdot\Omega_r(\underline{n}) \geq V_{\underline{t}}(g)$. It follows that for all $\underline{n} \in supp(f+g)$, $\underline{t}\cdot\Omega_r(\underline{n}) \geq \min\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}$. Hence,

$$V_{\underline{t}}(f+g) \geq \min\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}.$$

(ii) Again let $f,g \in A_r(R)$. If $f \oplus g = 0$, then the inequality holds trivially. So assume that $f \oplus g \neq 0$, and let $a_1,\dots,a_r$ be positive integers such that $(a_1,\dots,a_r) \in supp(f \oplus g)$ and $V_{\underline{t}}(f \oplus g) = t_1\Omega(a_1) + \dots + t_r\Omega(a_r)$. Then

$$0 \neq (f \oplus g)(a_1,\dots,a_r) = \sum_{\substack{d_1e_1=a_1 \\ (d_1,e_1)=1}} \dots \sum_{\substack{d_re_r=a_r \\ (d_r,e_r)=1}} f(d_1,\dots,d_r)g(e_1,\dots,e_r).$$

Therefore $f(d_1,\dots,d_r) \neq 0$ and $g(e_1,\dots,e_r) \neq 0$ for some d_i, e_i with $d_ie_i = a_i$, $(d_i,e_i) = 1$, $(i = 1,\dots,r)$. It follows that

$$\begin{aligned} V_{\underline{t}}(f) + V_{\underline{t}}(g) &\le t_1\Omega(d_1) + \cdots + t_r\Omega(d_r) + t_1\Omega(e_1) + \cdots + t_r\Omega(e_r) \\ &= t_1\Omega(a_1) + \cdots + t_r\Omega(a_r) \\ &= V_{\underline{t}}(f \oplus g). \end{aligned}$$

This completes the proof of the proposition.

Next, we define a family of norms on $A_r(R)$. Fix a $\underline{t}$ as above and a number $\rho \in (0,1)$. Then define a norm $\|.\| = \|.\|_{\underline{t}} : A_r(R) \to \mathbb{R}$ by

$$\|x\|_{\underline{t}} = \rho^{V_{\underline{t}}(x)} \quad \text{if } x \neq 0, \qquad \text{and} \qquad \|x\|_{\underline{t}} = 0 \quad \text{if } x = 0.$$

By the above proposition it follows that $\|x + y\| \le \max\{\|x\|, \|y\|\}$, and $\|x \oplus y\| \le \|x\| \, \|y\|$ for all $x, y \in A_r(R)$. Associated with the norm $\|.\|$ we have a distance d on $A_r(R)$ defined by $d(x, y) = \|x - y\|_{\underline{t}}$, for all $x, y \in A_r(R)$.

Theorem 1. *Let R be an integral domain, and let r be a positive integer. Then $A_r(R)$ is complete with respect to each of the norms $\|.\|_{\underline{t}}$.*

Proof. Let $(f_n)_{n\ge0}$ be a Cauchy sequence in $A_r(R)$. Then for each $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ depending on ε such that $\|f_m - f_n\| < \varepsilon$ for all $m, n \ge N$. For each $k \in \mathbf{N}$, taking $\varepsilon = \rho^k$, there exists $N_k \in \mathbf{N}$ such that $\|f_m - f_n\| < \rho^k$ for all $m, n \ge N_k$. Equivalently, $V_{\underline{t}}(f_m - f_n) > k$ for all $m, n \ge N_k$, i.e., we have that for all $m, n \ge N_k$,

$$f_m(l_1, \ldots, l_r) = f_n(l_1, \ldots, l_r)$$

whenever $t_1\Omega(l_1) + \cdots + t_r\Omega(l_r) \le k$, $l_1, \ldots, l_r \in \mathbf{N}$. We choose inductively for each $k \in \mathbf{N}$, the smallest natural number N_k with the above property such that

$$N_1 < N_2 < \cdots < N_k < N_{k+1} < \cdots.$$

Let us define $f : \mathbf{N}^r \to R$ as follows. Given $\underline{l} = (l_1, \ldots, l_r) \in \mathbf{N}^r$, let k be the smallest positive integer such that $k < t_1\Omega(l_1) + \cdots + t_r\Omega(l_r) \le k + 1$. We set $f(\underline{l}) = f_{N_{k+1}}(\underline{l})$. Then we will have $f(\underline{l}) = f_n(\underline{l})$, for all $n \ge N_{k+1}$. Since this will hold for all $\underline{l}$ and k as above, it follows that the sequence $(f_n)_{n\ge0}$ converges to f. This completes the proof of Theorem 1.

3. Derivations

We use the same notation as in the previous section.

Definition 1. We call an arithmetical function $f \in A_r(R)$ multiplicative provided that f is not identically zero and

$$f(n_1m_1, \ldots, n_rm_r) = f(n_1, \ldots, n_r)f(m_1, \ldots, m_r)$$

for any $n_1, \dots, n_r, m_1, \dots, m_r \in \mathbf{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$. We say that an $f \in A_r(R)$ is additive provided that

$$f(n_1m_1, \dots, n_rm_r) = f(n_1, \dots, n_r) + f(m_1, \dots, m_r)$$

for any $n_1, \dots, n_r, m_1, \dots, m_r \in \mathbf{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$.

Note that if f is multiplicative then $f(1, \dots, 1) = 1$, while if f is additive then $f(1, \dots, 1) = 0$. We now proceed to define a derivation on $A_r(R)$. For any additive function $\psi \in A_r(R)$, define $D_\psi : A_r(R) \to A_r(R)$ by

$$D_\psi(f)(\underline{n}) = f(\underline{n})\psi(\underline{n}),$$

for all $f \in A_r(R)$ and $\underline{n} \in \mathbf{N}^r$. For $\underline{n} = (n_1, \dots, n_r)$, and $\underline{m} = (m_1, \dots, m_r)$ in $\mathbf{N}^r$, we write $\underline{nm} = (n_1m_1, \dots, n_rm_r)$. We state some basic properties of the map D_ψ in the next proposition.

Proposition 2. *Let R be an integral domain, and let r be a positive integer. Let $\psi \in A_r(R)$ be additive. Then for all $f, g \in A_r(R)$ and $c \in R$,*
(a) $D_\psi(f + g) = D_\psi(f) + D_\psi(g)$,
(b) $D_\psi(f \oplus g) = f \oplus D_\psi(g) + g \oplus D_\psi(f)$,
(c) $D_\psi(cf) = cD_\psi(f)$.

Consequently, we see that D_ψ is a derivation on $A_r(R)$ over R.

Proof. Let $\underline{n} = (n_1, \dots, n_r) \in \mathbf{N}^r$. First, from the definition of D_ψ we see that

$$\begin{aligned} D_\psi(f + g)(\underline{n}) &= (f + g)(\underline{n})\psi(\underline{n}) \\ &= f(\underline{n})\psi(\underline{n}) + g(\underline{n})\psi(\underline{n}) \\ &= D_\psi(f) + D_\psi(g). \end{aligned}$$

Thus, (a) holds. Also from the definition of D_ψ we have that

$$D_\psi(f \oplus g)(\underline{n}) = (f \oplus g)(\underline{n})\psi(\underline{n}).$$

So,

$$\begin{aligned} D_\psi(f \oplus g)(\underline{n}) &= \psi(\underline{n}) \sum_{\substack{d_1e_1=n_1 \\ (d_1,e_1)=1}} \cdots \sum_{\substack{d_re_r=n_r \\ (d_r,e_r)=1}} f(d_1, \dots, d_r)g(e_1, \dots, e_r) \\ &= \sum_{\substack{d_1e_1=n_1 \\ (d_1,e_1)=1}} \cdots \sum_{\substack{d_re_r=n_r \\ (d_r,e_r)=1}} f(d_1, \dots, d_r)g(e_1, \dots, e_r)\psi(\underline{n}) \\ &= \sum_{\substack{d_1e_1=n_1 \\ (d_1,e_1)=1}} \cdots \sum_{\substack{d_re_r=n_r \\ (d_r,e_r)=1}} f(d_1, \dots, d_r)g(e_1, \dots, e_r) \end{aligned}$$

$$\times (\psi(d_1, \ldots, d_r) + \psi(e_1, \ldots, e_r))$$

$$= \sum_{\substack{d_1 e_1 = n_1 \\ (d_1, e_1) = 1}} \cdots \sum_{\substack{d_r e_r = n_r \\ (d_r, e_r) = 1}} f(d_1, \ldots, d_r)\psi(d_1, \ldots, d_r) g(e_1, \ldots, e_r)$$

$$+ \sum_{\substack{d_1 e_1 = n_1 \\ (d_1, e_1) = 1}} \cdots \sum_{\substack{d_r e_r = n_r \\ (d_r, e_r) = 1}} f(d_1, \ldots, d_r)\psi(e_1, \ldots, e_r) g(e_1, \ldots, e_r)$$

$$= f \oplus D_\psi(g) + g \oplus D_\psi(f).$$

Therefore (b) holds. Also, it is clear that (c) holds, and this proves the proposition.

Lemma 1. *Let $f, g \in A_1(R)$. Let p be a prime, and let M_p be the monoid $\{1, p, p^2, \ldots\}$ under multiplication. Suppose that supp(f), supp$(g) \subseteq M_p$. If $f(1) = 0$, and $g(1) = 1$, then $f = f \oplus g$.*

Proof. We have that *supp*$(f \oplus g) \subseteq M_p$ since *supp*(f), *supp*$(g) \subseteq M_p$. Thus both f and $f \oplus g$ vanish outside the monoid M_p. Let now n be a positive integer. Then

$$(f \oplus g)(p^n) = \sum_{\substack{de = p^n \\ (d, e) = 1}} f(d) g(e)$$

$$= f(1) g(p^n) + f(p^n) g(1)$$

$$= f(p^n).$$

Thus, $f = f \oplus g$.

Lemma 2. *Let $f \in A_1(R)$ be multiplicative and $\psi \in A_1(R)$ be additive. Let p be a prime, and let M_p be the monoid $\{1, p, p^2, \ldots\}$ under multiplication as in Lemma 1. Suppose that supp$(f) \subseteq M_p$. Then $\frac{D_\psi(f)}{f} = D_\psi(f)$, where the division on the left side is taken with respect to the unitary convolution.*

Proof. Note first that since f is supported on M_p, both $D_\psi(f)$ and f^{-1} will be supported on M_p. We have moreover that $f^{-1}(1) = 1$ because f is multiplicative. Also since ψ is additive, $\psi(1) = 0$. Applying Lemma 1, we conclude that $\frac{D_\psi(f)}{f} = D_\psi(f)$.

Theorem 2. *Let $f \in A_1(R)$ be multiplicative and $\psi \in A_1(R)$ be additive. Let n be a positive integer, and let $\mathscr{P}_n$ be the set of all prime divisors of n. For each prime p, let M_p be as in Lemma 1, and let $f_p = f|_{M_p}$, i.e.,*

$$f_p(m) = \begin{cases} f(m) & \textit{if } m = p^k,\ k \geq 1 \\ 0 & \textit{else}. \end{cases}$$

Then

$$\frac{D_\psi(f)}{f}(n) = \sum_{p \in \mathscr{P}_n} D_\psi(f_p)(n) = \begin{cases} \psi(n) f(n) & \textit{if } n = p^k \textit{ for some } p \textit{ prime} \\ & \textit{and } k \geq 1, \\ 0 & \textit{else}. \end{cases}$$

Proof. Fix an n and let $n = p_1^{s_1} \dots p_t^{s_t}$ be the prime factorization of n. Let $\mathfrak{M}$ be the set of all $m \in \mathbf{N}$ such that whenever p is a prime and p divides m, p also divides n. Note that $\mathfrak{M}$ is a monoid under multiplication, generated by the primes $p_1, \dots, p_t$. Let $g = f|_{\mathfrak{M}}$, i.e., for any $m \in \mathbf{N}$,

$$g(m) = \begin{cases} f(m) & \text{if } m \in \mathfrak{M} \\ 0 & \text{else}. \end{cases}$$

Suppose that m, k are in $\mathbf{N}$, and $(m,k) = 1$. If $m \notin \mathfrak{M}$, or $k \notin \mathfrak{M}$, then $f|_{\mathfrak{M}}(m) = 0$, or $f|_{\mathfrak{M}}(k) = 0$, and so, $g(m)g(k) = f|_{\mathfrak{M}}(m) f|_{\mathfrak{M}}(k) = 0 = f|_{\mathfrak{M}}(mn) = g(mn)$ since $mn \notin \mathfrak{M}$ whenever one of m, or n does not belong to $\mathfrak{M}$. If m, n are relatively prime and $m, n \in \mathfrak{M}$ then $mn \in \mathfrak{M}$ and $g(mn) = f|_{\mathfrak{M}}(mn) = f(mn) = f(m)f(n) = f|_{\mathfrak{M}}(m) f|_{\mathfrak{M}}(n) = g(m)g(n)$. Thus g is multiplicative. We claim that

$$g = \prod_{p \in \mathscr{P}_n} f_p.$$

Indeed, let us first observe that if $h_1, h_2 \in A_1(R)$ are such that $supp(h_1)$, $supp(h_2)$ are contained in $\mathfrak{M}$, then $supp(h_1 \oplus h_2) \subseteq \mathfrak{M}$. To see this, let $m \notin \mathfrak{M}$. Then there exists a prime p such that $p|m$, but p does not divide n. Now

$$(h_1 \oplus h_2)(m) = \sum_{\substack{de=m \\ (d,e)=1}} h_1(d) h_2(e).$$

Since either $p|d$, or $p|e$ whenever $m = de$, every term in this sum is 0 because $supp(h_i) \subseteq \mathfrak{M}$ $(i = 1, 2)$. Thus, $(h_1 \oplus h_2)(m) = 0$ for any $m \notin \mathfrak{M}$. Hence $supp(h_1 \oplus h_2) \subseteq \mathfrak{M}$. Using the above observation and induction, it follows that $supp(\prod_{p \in \mathscr{P}_n} f_p) \subseteq \mathfrak{M}$. Since g is also supported on $\mathfrak{M}$, it follows that in order to prove the above claim it is enough to show that g equals $\prod_{p \in \mathscr{P}_n} f_p$ on $\mathfrak{M}$. Let $m \in \mathfrak{M}$ with

$$m = p_1^{a_1} \dots p_t^{a_t},$$

where all a_i are nonnegative integers for $i = 1, \dots, t$. We have that

$$\prod_{p\in\mathscr{P}_n} f_p(m) = \sum_{\substack{d_1\ldots d_t=m\\(d_i,d_j)=1,\ (i\neq j)}} f_{p_1}(d_1)\ldots f_{p_t}(d_t)$$

$$= \sum_{\substack{b_1,\ldots,b_t\\p_1^{b_1}\ldots p_t^{b_t}=m}} f_{p_1}(p_1^{b_1})\ldots f_{p_t}(p_t^{b_t})$$

$$= f_{p_1}(p_1^{a_1})\ldots f_{p_t}(p_t^{a_t})$$

$$= f(m)$$

$$= g(m),$$

where in the above computation $b_1,\ldots,b_t$ are forced to have unique values equal to $a_1,\ldots,a_t$ respectively. Hence $g = \prod_{p\in\mathscr{P}_n} f_p$, as claimed. Next, we claim that

$$\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = D_\psi(g)\oplus g^{-1}.$$

In order to prove this, we first show that

$$f^{-1}|_{\mathfrak{M}} = g^{-1}.$$

Note that by the previous claim we know that

$$g^{-1} = \left(\prod_{p\in\mathscr{P}_n} f_p\right)^{-1} = \prod_{p\in\mathscr{P}_n} f_p^{-1},$$

and as a consequence g^{-1} is supported on $\mathfrak{M}$. We now proceed by induction. First, since $f(1) = g(1) = 1$, it follows immediately that $f^{-1}|_{\mathfrak{M}}(1) = g^{-1}(1) = 1$. Next, let $m > 1$, and assume that for all $k < m$, $g^{-1}(k) = f^{-1}|_{\mathfrak{M}}(k)$. If $m \notin \mathfrak{M}$, then $f^{-1}|_{\mathfrak{M}}(m) = 0 = g^{-1}(m)$. Now suppose that $m \in \mathfrak{M}$. Then, using the equalities $(f \oplus f^{-1})(m) = 0 = (g \oplus g^{-1})(m)$ in combination with the induction hypothesis we derive

$$f^{-1}|_{\mathfrak{M}}(m) = f^{-1}(m)$$

$$= \frac{-1}{f(1)} \sum_{\substack{de=m\\(d,e)=1\\e<m}} f(d)f^{-1}(e)$$

$$= \frac{-1}{g(1)} \sum_{\substack{de=m\\(d,e)=1\\e<m}} g(d)g^{-1}(e)$$

$$= g^{-1}(m).$$

Thus,

$$f^{-1}|_{\mathfrak{M}} = g^{-1}.$$

Further, it is clear that

$$D_\psi(f)|_{\mathfrak{M}} = D_\psi(f|_{\mathfrak{M}}) = D_\psi(g).$$

By the above two relations we conclude that

$$D_\psi(g) \oplus g^{-1} = D_\psi(f)|_{\mathfrak{M}} \oplus f^{-1}|_{\mathfrak{M}}.$$

Therefore in order to prove the claim it remains to show that

$$\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = D_\psi(f)|_{\mathfrak{M}} \oplus f^{-1}|_{\mathfrak{M}}.$$

Here the left side is supported on $\mathfrak{M}$, while the right side is the unitary convolution of two arithmetical functions supported on $\mathfrak{M}$, so it is also supported on $\mathfrak{M}$. So we only need to check the desired equality at an arbitrary point $m \in \mathfrak{M}$. For such an m, any representation of m as a product $m = de$ forces both d, e to belong to $\mathfrak{M}$. Thus

$$\begin{aligned}\left(\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}}\right)(m) = \frac{D_\psi(f)}{f}(m) &= \sum_{\substack{de=m\\(d,e)=1}} D_\psi(f)(d) f^{-1}(e)\\ &= \sum_{\substack{de=m\\(d,e)=1}} D_\psi(f)|_{\mathfrak{M}}(d) f^{-1}|_{\mathfrak{M}}(e) = (D_\psi(f)|_{\mathfrak{M}} \oplus f^{-1}|_{\mathfrak{M}})(m).\end{aligned}$$

We conclude that $\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = D_\psi(f)|_{\mathfrak{M}} \oplus f^{-1}|_{\mathfrak{M}}$, and hence

$$\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = D_\psi(g) \oplus g^{-1},$$

as claimed. On the other hand, by applying Proposition 2 (b) repeatedly, we obtain

$$D_\psi(g) \oplus g^{-1} = \frac{D_\psi(\prod_{p \in \mathscr{P}_n} f_p)}{\prod_{p \in \mathscr{P}_n} f_p} = \sum_{p \in \mathscr{P}_n} \frac{D_\psi(f_p)}{f_p}.$$

By the above two relations we deduce that

$$\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = \sum_{p \in \mathscr{P}_n} \frac{D_\psi(f_p)}{f_p}.$$

But by Lemma 2, $\sum_{p\in\mathscr{P}_n}\frac{D_\psi(f_p)}{f_p}$ equals $\sum_{p\in\mathscr{P}_n}D_\psi(f_p)$. Therefore, we have that

$$\left.\frac{D_\psi(f)}{f}\right|_{\mathfrak{M}} = \sum_{p\in\mathscr{P}_n} D_\psi(f_p).$$

Since n is in $\mathfrak{M}$, it follows in particular that

$$\frac{D_\psi(f)}{f}(n) = \sum_{p\in\mathscr{P}_n} D_\psi(f_p)(n).$$

This completes the proof of the theorem.

We now proceed to generalize this theorem to the case of arithmetical functions of several variables.

Lemma 3. *Let $f \in A_r(R)$ be multiplicative and consider the monoids*

$$M_1 = \{(k,1,\dots,1)\in\mathbf{N}^r : k\in\mathbf{N}\},\dots,M_r = \{(1,\dots,1,k)\in\mathbf{N}^r : k\in\mathbf{N}\}.$$

Let $f_1 = f|_{M_1},\dots,f_r = f|_{M_r}$. Then

$$f = \prod_{i=1}^r f_i = f_1\oplus\cdots\oplus f_r.$$

Proof. Let $\underline{m} = (m_1,m_2,\dots,m_r)\in\mathbf{N}^r$. We have that

$$\begin{aligned}\left(\prod_{i=1}^r f_i\right)(\underline{m}) &= \sum_{\substack{d_{11}\dots d_{1r}=m_1\\(d_{1i},d_{1j})=1,\ (i\neq j)}}\cdots\sum_{\substack{d_{11}\dots d_{1r}=m_1\\(d_{1i},d_{1j})=1,\ (i\neq j)}}\prod_{i=1}^r f_i(d_{1i},\dots,d_{ri})\\ &= \prod_{i=1}^r f_i(1,\dots,1,m_i,1,\dots,1)\\ &= \prod_{i=1}^r f(1,\dots,1,m_i,1,\dots,1)\\ &= f(m_1,\dots,m_r)\end{aligned}$$

Hence, $f = \prod_{i=1}^r f_i$, and the lemma is proved.

Theorem 3. *Let R be an integral domain, and let r be a positive integer. Then, for any multiplicative function $f\in A_r(R)$, any additive function $\psi\in A_r(R)$, and any $\underline{n} = (n_1,\dots,n_r)\in\mathbf{N}^r$, we have*

$$\frac{D_\psi(f)}{f}(\underline{n}) = \begin{cases} \psi(\underline{n})f(\underline{n}) & \textit{if } n_1 = \cdots = n_{i-1} = n_{i+1} = \cdots = n_r = 1 \textit{ and } n_i = p^k \\ & \textit{for some } p \textit{ prime}, \; k \geq 1, \textit{ and } 1 \leq i \leq r, \\ 0 & \textit{else}, \end{cases}$$

where the division on the left side is taken with respect to the unitary convolution.

PROOF. Let f be multiplicative and consider the monoids

$$M_1 = \{(k,1,\dots,1) \in \mathbf{N}^r : k \in \mathbf{N}\}, \dots, M_r = \{(1,\dots,1,k) \in \mathbf{N}^r : k \in \mathbf{N}\}$$

as in Lemma 3. Let $f_1 = f|_{M_1}, \dots, f_r = f|_{M_r}$. Then by Lemma 3, $f = \prod_{i=1}^{r} f_i$. Applying Proposition 2 (b) repeatedly, we get

$$\frac{D_\psi(f)}{f} = \frac{D_\psi\left(\prod_{i=1}^{r} f_i\right)}{\prod_{i=1}^{r} f_i} = \sum_{i=1}^{r} \frac{D_\psi(f_i)}{f_i}.$$

Therefore the desired equality from the statement of Theorem 3 will hold for f provided it holds for each function f_i. On the other hand, each of the functions f_i is supported on a one dimensional monoid isomorphic to $\mathbf{N}$, so the desired equality for each function f_i follows directly from Theorem 2. This completes the proof of Theorem 3.

We remark that if f and ψ are known, then Theorem 2 and Theorem 3 can be used to compute the logarithmic derivative $\frac{D_\psi(f)}{f}$. We end this paper with a few very explicit examples. Take R to be the field of complex numbers and $r = 1$. An additive arithmetical function is for instance $\psi(n) = \log n$.

1. With R, r and ψ as above, let f be the Möbius function μ, which is a multiplicative function. By its definition, $\mu(1) = 1$, and if $n > 1$, $n = p_1^{a_1} \dots p_k^{a_k}$, then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = \cdots = a_k = 1, \\ 0 & \text{else.} \end{cases}$$

By Theorem 2 we then have

$$\frac{D_\psi(\mu)}{\mu}(n) = \begin{cases} -\log p & \text{if } n = p \text{ for some prime } p, \\ 0 & \text{else.} \end{cases}$$

2. Take R, r and ψ as above and choose f to be the Euler totient function $\phi(n)$ which is multiplicative. By Theorem 2 we see that

$$\frac{D_\psi(\phi)}{\phi}(n) = \begin{cases} k(p^k - p^{k-1}) \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0 & \text{else.} \end{cases}$$

3. With the same R, r and ψ as before, let f be the sum of divisors function σ, given by $\sigma(n) = \sum_{d|n} d$, which is also a multiplicative arithmetical function. By Theorem 2 we find that

$$\frac{D_\psi(\sigma)}{\sigma}(n) = \begin{cases} \frac{k(p^{k+1}-1)\log p}{p-1} & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0 & \text{else.} \end{cases}$$

One can of course consider many other interesting arithmetical functions. For instance one can take f to be the number of divisors function, or the sum of k-th powers of divisors function for some fixed k, which are multiplicative functions, or one can let f be a Dirichlet character, which is completely multiplicative. One may also take f to be the Ramanujan tau function $\tau(n)$ defined in terms of the Delta function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \qquad q = e^{2\pi i z},$$

which is the unique normalized cusp form of weight 12 on $SL_2(\mathbf{Z})$. Ramanujan first studied many of the beautiful properties of this arithmetical function (see his collected works [2]). In particular he conjectured that $\tau(n)$ is multiplicative, a fact that was later proved by Mordell. One can also replace ψ by other additive functions, for instance the logarithm of any multiplicative arithmetical function is additive. Clearly applying Theorems 2 and 3 to various combinations of such examples is equivalent in some sense to providing identities for such arithmetical functions with respect to the unitary convolution.

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Emre Alkan
Department of Mathematics
University of Illinois at Urbana-Champaign
273 *Altgeld Hall, MC*-382, 1409 *W. Green Street*
Urbana, Illinois 61801-2975, *USA*
e-mail: *alkan@math.uiuc.edu*

Alexandru Zaharescu
Department of Mathematics
University of Illinois at Urbana-Champaign
273 *Altgeld Hall, MC*-382, 1409 *W. Green Street*
Urbana, Illinois 61801-2975, *USA*
e-mail: *zaharesc@math.uiuc.edu*

Mohammad Zaki
Department of Mathematics
University of Illinois at Urbana-Champaign
273 *Altgeld Hall, MC*-382, 1409 *W. Green Street*
Urbana, Illinois 61801-2975, *USA*
e-mail: *mzaki@math.uiuc.edu*