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Analogues of sampling theorems for some homogeneous spaces

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ABSTRACT. Sampling theorems are one of the basic tools in information theory. The signal function f whose band–region is contained in a certain interval can be reconstructed from their values $f(x_k)$ at the sampling points $\{x_k\}$. We obtain analogues of this theorem for the cases of the Fourier–Jacobi series, the complex sphere S_c^{n-1} and the complex semisimple Lie groups. And as an application of these formulae, we show a version of the sampling theorem for the Radon transform on the complex hyperbolic space.

1. Introduction

Sampling theorems are one of the basic tools in information theory and various types of sampling theorems are obtained in many papers. The Shannon sampling theorem is well known as a fundamental tool. A signal function is called to be band–limited if its band–region is contained in a certain interval. In the terminology of Fourier analysis, the band–limitedness condition is equivalent to the condition that the support of the Fourier transform $\tilde{f}$ of $f \in L^2(\mathbf{R})$ is contained in a certain interval. The Shannon sampling theorem yields that if a function $f \in L^2(\mathbf{R})$ is band–limited, then f can be reconstructed by samples taken at the equidistant sampling points. We are interested in generalizing sampling theorems to the cases of homogeneous spaces. In this paper, we obtain analogues of this theorem in the cases of the Fourier–Jacobi series, the complex sphere $S_c^{n-1} = U(n)/U(n-1)$ and the complex semisimple Lie groups.

On the other hand, the problem how to recover the values of the functions from the samples of their Radon transforms is studied in the theory of the

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computerized tomography. There are also many papers related to these tomographic inversion problems. We can find a number of algorithms in the book of Natterer [9]. An irregular version of this problem is studied in the paper [4]. In [2] we study the Fourier reconstruction algorithm and extend this algorithm to the case of Riemannian symmetric spaces. In [2] we fix a K-type δ and give the reconstruction formula for the function of type δ on the Riemannian symmetric space G/K. By using this, the reconstruction formula for the band–limited function can be formally constructed. In this paper, by taking sampling points suitably, we concretely construct the sampling function for the complex sphere. And using this, we obtain a version of the sampling theorem for the Radon transform on the complex hyperbolic space.

We shall describe here the context of this paper. Section 2 is devoted to the overview of the Shannon sampling theorem on $\mathbf{R}^d$ and the regular or irregular sampling theorems on the torus $\mathbf{T}^d$. These are directly proved by using the Lagrange interpolation theorem. In Section 3, applying the sampling theorem on $\mathbf{T}^d$, we show a sampling theorem for the Fourier–Jacobi series. In Section 4, with the help of the Shannon sampling theorem on $\mathbf{R}^d$, we give a sampling theorem for the complex semisimple Lie group. Section 5 is devoted to showing a sampling theorem for the complex sphere $S_c^{n-1} = U(n)/U(n-1)$. In this case, the spherical functions of the $U(n-1)$-invariant irreducible representations of $U(n)$ are written in terms of the Jacobi polynomials. So by using the sampling formula for the Fourier–Jacobi series given in Section 3 and the sampling formula for the torus given in Section 2, we can obtain a sampling theorem for S_c^{n-1}. In Section 6 we consider the Fourier reconstruction algorithm for the case of the complex hyperbolic space. Applying the Shannon sampling theorem on $\mathbf{R}^d$ and the one on S_c^{n-1} to this theorem, we can get a version of the sampling formula for the Radon transform on the complex hyperbolic space.

2. Sampling theorems on the Euclidean space and the torus

We shall first survey the sampling theorems on the Euclidean space and the torus. These theorems are used to derive sampling theorems for the complex semisimple Lie groups and the complex sphere S_c^{n-1}. For details, see [2, 7]. The Fourier transform $\tilde{f}$ of $f \in L^1(\mathbf{R}^d)$ is defined by

$$\tilde{f}(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-i\lambda \cdot x} f(x)dx, \qquad (\lambda \in \mathbf{R}^d).$$

Here $\lambda \cdot x$ denotes the natural inner product of λ and x. $f \in L^2(\mathbf{R}^d)$ is called to be band–limited on $[-L, L]^d$ if $\operatorname{supp} \tilde{f} \subseteq [-L, L]^d$. Then the following proposition is called the Shannon sampling theorem.

PROPOSITION 2.1 ([7, Theorem 14.1]). *Let $f \in L^2(\mathbf{R}^d)$ be band–limited on $[-L, L]^d$. Then f is reconstructed by*

$$f(x) = \sum_{k \in \mathbf{Z}^d} f\left(\frac{\pi}{L}k\right) \prod_{j=1}^{d} \operatorname{sinc}\left(\frac{L}{\pi}x_j - k_j\right),$$

where $k = (k_1, \dots, k_d) \in \mathbf{Z}^d$ and $x = (x_1, \dots, x_d) \in \mathbf{R}^d$.

The Shannon sampling theorem is called a regular sampling theorem because, in this theorem, the samples are taken at the equidistant sampling points.

On the other hand, in the case of the torus $\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z})^d$, various types of sampling theorems are given in [2, 7]. For $p \in L^1(\mathbf{T}^d)$, its Fourier transform $\tilde{p}$ is defined by

$$\tilde{p}(m) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} p(\theta) e^{-im\cdot\theta}\, d\theta, \qquad (m \in \mathbf{Z}^d).$$

And the inversion formula is the following:

$$p(\theta) = \sum_{m \in \mathbf{Z}^d} \tilde{p}(m) e^{im\cdot\theta}. \tag{2.1}$$

In this case, the band–limitedness condition is interpreted as the condition that the support of $\tilde{p}$ is a finite set. We call $p \in L^2(\mathbf{T}^d)$ is band–limited on $\{-N_1, \dots, N_1\} \times \cdots \times \{-N_d, \dots, N_d\}$ if $\operatorname{supp} \tilde{p} \subseteq \{-N_1, \dots, N_1\} \times \cdots \times \{-N_d, \dots, N_d\}$. In this case we can directly obtain an irregular sampling theorem on $\mathbf{T}^d$ by way of the Lagrange interpolation formula.

PROPOSITION 2.2 ([2, Lemma 3]). *Let $\theta_{\ell,-N_\ell}, \dots, \theta_{\ell,N_\ell}$ be $2N_\ell + 1$ distinct numbers in $[-\pi, \pi)$ for each $\ell = 1, \dots, d$. If a function $p \in L^2(\mathbf{T}^d)$ is band–limited on $\{-N_1, \dots, N_1\} \times \cdots \times \{-N_d, \dots, N_d\}$, then we have*

$$p(\theta_1, \dots, \theta_d) = \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_d=-N_d}^{N_d} p(\theta_{1,k_1}, \dots, \theta_{d,k_d}) S_{k_1,\dots,k_d}^{N_1,\dots,N_d}(\theta_1, \dots, \theta_d),$$

where

$$S_{k_1,\dots,k_d}^{N_1,\dots,N_d}(\theta_1, \dots, \theta_d) = e^{-i\sum_{\ell=1}^{d} N_\ell(\theta_\ell - \theta_{\ell,k_\ell})} \prod_{\ell=1}^{d} \left\{ \prod_{j_\ell \neq k_\ell} \frac{e^{i\theta_\ell} - e^{i\theta_{\ell,j_\ell}}}{e^{i\theta_{\ell,k_\ell}} - e^{i\theta_{\ell,j_\ell}}} \right\}. \tag{2.2}$$

REMARK. If we take $\theta_{\ell,k_\ell} = k_\ell\pi/(2N_\ell + 1)$, $(k_\ell = -N_\ell, \dots, N_\ell, \ell = 1, \dots, d)$ in (2.2), then we have

$$S_{k_1,\dots,k_d}^{N_1,\dots,N_d}(\theta_1,\dots,\theta_d)=\prod_{\ell=1}^{d}\left\{\prod_{j_\ell\neq k_\ell}\frac{\sin\frac{1}{2}\left(\theta_\ell-\frac{j_\ell\pi}{2N_\ell+1}\right)}{\sin\frac{(k_\ell-j_\ell)\pi}{2N_\ell+1}}\right\},\tag{2.3}$$

and Proposition 2.2 gives a version of the regular sampling theorem on $\mathbf{T}^d$.

In [7], there is another version of the sampling theorem on $\mathbf{T}$ which is deduced from Cauchy's formula.

Proposition 2.3 ([7, Example 4.1]). *If $p\in L^2(\mathbf{T})$ is band–limited on $\{-N,\dots,N\}$, then we have*

$$p(\theta)=\sum_{k=-N}^{N}p\left(\frac{2k\pi}{2N+1}\right)\frac{1}{2N+1}\sum_{\ell=-N}^{N}e^{-i\ell(\theta-2k\pi/(2N+1))}.$$

We shall next describe a sampling theorem for the Radon transform on $\mathbf{R}^2$ that is called the Fourier reconstruction algorithm (see [9, Chapter 5]). Let $\mathscr{C}(\mathbf{R}^2)$ denote the set of rapidly decreasing functions on $\mathbf{R}^2$. For $f\in\mathscr{C}(\mathbf{R}^2)$, its Radon transform Rf is defined by

$$(Rf)(\omega_\phi,r)=\int_{-\infty}^{\infty}f(r\cos\phi-t\sin\phi,r\sin\phi+t\cos\phi)dt,$$
$$(r\in\mathbf{R},\omega_\phi=(\cos\phi,\sin\phi)\in\mathbf{T}).$$

It is known that the Fourier transform of f is the composition of the Radon transform Rf and the 1-dimensional Fourier transform $\mathscr{F}_2$ with respect to the second variable r:

$$\tilde{f}(\tau\omega_\phi)=(2\pi)^{-1/2}(\mathscr{F}_2(Rf))(\omega_\phi,\tau).\tag{2.4}$$

This formula is called the Fourier slice formula. Let $L,N>0$. We call that $f\in\mathscr{C}(\mathbf{R}^2)$ is band–limited if

(1) $\operatorname{supp}\tilde{f}\subseteq\{\xi\in\mathbf{R}^2;|\xi|\le L\}$;

(2) $\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\tau\omega_\phi)e^{-im\phi}\,d\phi=0$ for $|m|>N$.

The Fourier reconstruction algorithm is performed by the following process. Since

$$(\mathscr{F}_2(Rf))(\omega_\phi,\tau)=(2\pi)^{1/2}\tilde{f}(\tau\omega_\phi)=0\qquad\text{for }|\tau|>L,$$

it follows from the Shannon sampling theorem that

$$(Rf)(\omega_\phi,r)=\sum_{n\in\mathbf{Z}}(Rf)\left(\omega_\phi,\frac{\pi}{L}n\right)\operatorname{sinc}\left(\frac{L}{\pi}r-n\right).$$

And hence from the Fourier slice formula (2.4) we have

$$\begin{aligned}\tilde{f}(\tau\omega_\phi) &= \sum_{n\in\mathbf{Z}}(Rf)\left(\omega_\phi, \frac{\pi}{L}n\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{sinc}\left(\frac{L}{\pi}r-n\right)e^{-i\tau r}\,dr\\ &= \sum_{n\in\mathbf{Z}}(Rf)\left(\omega_\phi, \frac{\pi}{L}n\right)\frac{1}{2L}e^{-in\pi\tau/L}\chi_{(-L,L)}(\tau),\end{aligned}$$

where $\chi_{(-L,L)}(\tau)$ denotes the characteristic function of the open interval $(-L, L)$. Applying Proposition 2.3 to the last equation, we have

$$\tilde{f}(\tau\omega_\phi) = \sum_{k=-N}^{N}\sum_{n\in\mathbf{Z}}(Rf)\left(\omega_{\phi_k}, \frac{\pi}{L}n\right)\frac{1}{2L(2N+1)}e^{-in\pi\tau/L}\chi_{(-L,L)}(\tau)\sum_{\ell=-N}^{N}e^{-i\ell(\phi-\phi_k)},$$

where $\phi_k = 2k\pi/(2N+1)$. Noting

$$\begin{aligned}\tilde{f}_m(\tau) &= \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\tau\omega_\phi)e^{-im\phi}\,d\phi\\ &= \sum_{k=-N}^{N}\sum_{n\in\mathbf{Z}}(Rf)\left(\omega_{\phi_k}, \frac{\pi}{L}n\right)\frac{1}{2L(2N+1)}e^{-in\pi\tau/L}\chi_{(-L,L)}(\tau)e^{-im\phi_k},\end{aligned}$$

we have

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{\mathbf{R}^2}\tilde{f}(\xi)e^{ix\cdot\xi}\,d\xi\\ &= \sum_{m=-\infty}^{\infty}i^m e^{im\theta}\int_0^{\infty}\tilde{f}_m(\tau)J_m(r\tau)\tau\,d\tau \qquad (x = r\omega_\theta)\\ &= \sum_{k=-N}^{N}\sum_{n\in\mathbf{Z}}(Rf)\left(\omega_{\phi_k}, \frac{\pi}{L}n\right)\frac{1}{2L(2N+1)}\\ &\quad\times\sum_{m=-N}^{N}i^m e^{im(\theta-\phi_k)}\int_0^L e^{-in\pi\tau/L}J_m(r\tau)\tau\,d\tau.\end{aligned}$$

Here J_m denotes the Bessel function of the first kind. We use the sampling theorem on the Fourier–Bessel transform (cf. [8]) to compute the integral appeared in the last equation. We set

$$\varphi(r) = \int_0^L e^{-in\pi\tau/L}J_m(r\tau)\tau\,d\tau.$$

Because φ is band–limited on $[0, L]$ with respect to the Fourier–Bessel transform

$$\tilde{\varphi}(\tau) = \int_0^{\infty}\varphi(r)J_m(r\tau)r\,dr,$$

we have from the sampling theorem on the Fourier–Bessel transform that

$$\varphi(r) = \sum_{\ell=1}^{\infty} \frac{2r_\ell J_m(Lr)}{(r_\ell^2 - L^2 r^2) J_{m+1}(r_\ell)} \int_0^L e^{-in\pi\tau/L} J_m\left(\frac{r_\ell \tau}{L}\right) \tau \, d\tau,$$

where $\{r_\ell\}$ denotes the set of positive zeroes of J_m. Summarizing these, we have the following proposition.

Proposition 2.4. *Let $L, N > 0$ and assume that $f \in \mathscr{C}(\mathbf{R}^2)$ is band–limited in the above sence. Then f is reconstructed as follows:*

$$f(r\omega_\theta) = \sum_{k=-N}^{N} \sum_{n \in \mathbf{Z}} (Rf)\left(\omega_{\phi_k}, \frac{\pi}{L} n\right) \frac{L}{2N+1}$$

$$\times \sum_{m=-N}^{N} i^m e^{i(\theta - \phi_k)} \sum_{\ell=1}^{\infty} \frac{c_{mn} r_\ell J_m(Lr)}{(r_\ell^2 - L^2 r^2) J_{m+1}(r_\ell)},$$

where $\phi_k = 2k\pi/(2N+1)$ and $c_{mn} = \int_0^1 e^{-in\pi\tau} J_m(r_\ell \tau)\tau \, d\tau$.

The concepts of the Radon transforms are generalized by various homogeneous spaces (cf. e.g. [5, 6]). In [1], Berenstein explaines how the Radon transform on the hyperbolic plane is utilized to solve the problem of Electrical Impedance Tomography. So we think that it is meaningful to study the generalization of the above proposition to the cases of homogeneous spaces. In Section 6, we shall treat the case of the complex hyperbolic space.

3. A sampling theorem on the Fourier–Jacobi series

We shall first summarize the notation of the Jacobi polynomials and the Fourier–Jacobi series. For the detail of the Fourier–Jacobi series, see [10]. Let $\alpha, \beta > -1$ and put $\rho = \alpha + \beta + 1$. Let $n \in \mathbf{Z}_{\geq 0}$. The polynomial

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1\left(-n, n+\rho; \alpha+1; \frac{1-x}{2}\right), \qquad (-1 \leq x \leq 1)$$

is called the Jacobi polynomial. Here ${}_2F_1$ denotes the Gauss hypergeometric function. It is known that the system $\{R_n^{(\alpha,\beta)}(x); n \in \mathbf{Z}_{\geq 0}\}$ is an orthogonal system with respect to the measure

$$d\mu^{(\alpha,\beta)}(x) = \frac{\Gamma(\rho+1)}{2^\rho \Gamma(\alpha+1)\Gamma(\beta+1)} (1-x)^\alpha (1+x)^\beta dx.$$

Moreover, it is satisfied that

$$\int_{-1}^{1} R_n^{(\alpha,\beta)}(x)^2 d\mu^{(\alpha,\beta)}(x) = \frac{n!\Gamma(\rho+1)\Gamma(\alpha+1)\Gamma(\beta+n+1)}{(\rho+2n)\Gamma(\rho+n)\Gamma(\alpha+n+1)\Gamma(\beta+1)}$$
$$(= (d_n^{(\alpha,\beta)})^{-1}, \text{ say}). \tag{3.1}$$

For $f \in L^2([-1,1], d\mu^{(\alpha,\beta)}(x))$, its Fourier–Jacobi transform $\tilde{f}$ is defined by

$$\tilde{f}(n) = \int_{-1}^{1} f(x) R_n^{(\alpha,\beta)}(x) d\mu^{(\alpha,\beta)}(x).$$

And the inversion formula is the following:

$$f(x) = \sum_{n=0}^{\infty} d_n^{(\alpha,\beta)} \tilde{f}(n) R_n^{(\alpha,\beta)}(x). \tag{3.2}$$

The above series is called the Fourier–Jacobi series of f.

Take $x = \cos 2\theta$, $(0 \le \theta \le \pi/2)$. Then it follows from (3.2) that

$$f(\cos 2\theta) = \sum_{n=0}^{N} d_n^{(\alpha,\beta)} \tilde{f}(n) R_n^{(\alpha,\beta)}(\cos 2\theta).$$

By definition, $R_n^{(\alpha,\beta)}(\cos 2\theta)$ is a polynomial of $\sin\theta$ with degree $2n$ and hence f is a polynomial of $e^{\pm i\theta}$ with degree at most $2N$. Then the following theorem is easily follows from Proposition 2.2 with $d = 1$.

Theorem 3.1. *Let $x_{-2N}, \dots, x_{2N}$ be $4N+1$ distinct points on $[-1,1)$. If $f \in L^2([-1,1], d\mu^{(\alpha,\beta)}(x))$ is band–limited to $\{0, \dots, N\}$, then we have*

$$f(x) = \sum_{k=-2N}^{2N} f(x_k) S_k^{2N}\left(\frac{1}{2}\cos^{-1} x\right),$$

where S_k^{2N} is given in (2.2).

Remark. As is well-known (cf. [6, 10]), the spherical functions on the compact isotropic Riemannian spaces are written in terms of the Jacobi polynomial. We thus have a sampling theorem for the spherical transform on the compact isotropic Riemannian spaces as special cases of Theorem 3.1.

4. A sampling theorem on the complex Lie group

In the case of the complex semisimple Lie groups, since the zonal spherical functions and the Harish-Chandra c-functions are explicitly written, we can get a sampling theorem on such groups by way of the Shannon sampling theorem on the Euclidean space.

Let G be a complex semisimple Lie group of rank r and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and Σ the corresponding set of restricted roots. The Killing form $\langle \cdot , \cdot \rangle$ induces an inner product on $\mathfrak{a}$ and on its dual space $\mathfrak{a}^*$. Let M' and M be the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively, and denote by $W = M'/M$, the Weyl group of G/K, and let w be its order. Fix a positive Weyl chamber $\mathfrak{a}^+$ and put $A^+ = \exp \mathfrak{a}^+$. We then obtain the Cartan decomposition $G = K\overline{A^+}K$. Let Σ^+ be the corresponding set of positive restricted roots and $|\Sigma^+|$ be its order. For $\alpha \in \Sigma^+$, $\mathfrak{g}_\alpha$ denotes the root subspace and $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of α. Let $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. Let N denote the analytic subgroup of $\mathfrak{n}$. Then we have $G = KAN$. For $x \in G$, $H(x) \in \mathfrak{a}$ denotes the element uniquely determined by $x \in K\exp(H(x))N$. For $a \in A$, we sometimes write $\log a$ instead of $H(a)$. We choose an orthonormal basis $\{H_1, \dots, H_r\}$ of $\mathfrak{a}$ and its dual basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ of $\mathfrak{a}^*$ and identify both $\mathfrak{a}$ and $\mathfrak{a}^*$ to $\mathbf{R}^r$. We normalize them by multiplying $(2\pi)^{-r/2}$ and denote them by dH and $d\lambda$, respectively. According to the Cartan decomposition, we have a Haar measure $dg = \Delta(H)dk_1 dH dk_2$ on G. Here $\Delta(H) = \prod_{\alpha \in \Sigma^+} \sinh^2 \alpha(H)$. The zonal spherical function φ_λ on G is given by

$$\varphi_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))}\, dk, \qquad (g \in G, \lambda \in \mathfrak{a}^*).$$

We set

$$I_L = \{\lambda \in \mathfrak{a}^*; -L \le \langle \lambda, \varepsilon_i \rangle \le L \text{ for } i = 1, 2, \dots, r\}.$$

For $f \in \mathscr{C}(K\backslash G/K)$, the space of rapidly decreasing functions on $K\backslash G/K$, its spherical Fourier transform $\tilde{f}(\lambda)$ is given by

$$\tilde{f}(\lambda) = \int_{\mathfrak{a}^+} f(\exp H)\varphi_{-\lambda}(\exp H)\Delta(H)dH.$$

And the inversion formula is the following:

$$f(\exp H) = \int_{\mathfrak{a}^{*+}} \tilde{f}(\lambda)\varphi_\lambda(\exp H)|c(\lambda)|^{-2} d\lambda.$$

Let $f \in \mathscr{C}(K\backslash G/K)$ be such that $\operatorname{supp}(\tilde{f}) \subseteq I_L$. Similarly to the case of Euclidean sapce, f is called to be band–limited on I_L. From [3, p. 251], we have

$$\pi(\rho)\Delta(H)^{1/2} f(\exp H) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\pi(-i\lambda)e^{i\lambda(H)}\, d\lambda,$$

where $\pi(\lambda) = \prod_{\alpha \in \Sigma^+} \langle \lambda, \alpha \rangle$. By definition, $\pi(-i\lambda)$ is a polynomial on $\mathfrak{a}^*$ and hence the support of $\tilde{f}(\lambda)\pi(-i\lambda)$ is also contained in I_L. And the function $H \to \Delta(H)^{1/2} f(\exp H)$ on $\mathfrak{a}$ is square–integrable with respect to the usual Euclidean measure dH. Consequently, applying Proposition 2.1 to the function $H \to (H)^{1/2} f(\exp H)$, we have the following theorem.

THEOREM 4.1. *Retain the above notation. Suppose that $f \in \mathscr{C}(K\backslash G/K)$ is band–limited on I_L. Then f is reconstructed as follows:*

$$f(\exp H) = \sum_{(n_1,\dots,n_r) \in \mathbf{Z}^r} \Delta(H)^{-1/2} \Delta\left(\sum_{i=1}^{r} \frac{\pi}{L} n_i H_i\right)^{1/2} f\left(\exp \sum_{i=1}^{r} \frac{\pi}{L} n_i H_i\right)$$
$$\times \prod_{i=1}^{r} \operatorname{sinc}\left(\frac{L}{\pi} \varepsilon_i(H) - n_i\right).$$

5. A sampling theorem on the complex sphere

Let S_c^{n-1} be the unit sphere $|\xi_1|^2 + \cdots + |\xi_n|^2 = 1$ in $\mathbf{C}^n$. The unitary group $U = U(n)$ acts naturally on S_c^{n-1} and the stabilizer of the element $\boldsymbol{e}_n = {}^t(0, \dots, 0, 1) \in S_c^{n-1}$ is isomorphic to $K = U(n-1)$. So $S_c^{n-1} \cong U/K$. We define the elements $d_j(\varphi)$ and $g_j(\theta)$ in U by

$$d_j(\varphi) = \operatorname{diag}(1, \dots, 1, \overset{(j)}{e^{i\varphi}}, 1, \dots, 1), \qquad g_j(\theta) = \begin{pmatrix} I_{j-1} & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & I_{n-j-1} \end{pmatrix},$$

where $0 \le \varphi_j < 2\pi$ and $0 \le \theta_j \le \pi/2$. For any $g \in U$, we define $\varphi_1, \dots, \varphi_n$, $\theta_1, \dots, \theta_{n-1}$ as follows:

Let $g\boldsymbol{e}_n = {}^t(\xi_1, \dots, \xi_n)$. We set $\varphi_j = \arg \xi_j$ and $r_j = \sqrt{\sum_{s=1}^{j} |\xi_s|^2}$. If $r_k = 0$ and $r_{k+1} \neq 0$ for some k, we set $\theta_1 = \cdots = \theta_k = 0$. And for $j > k$, we give θ_j by $\cos\theta_j = |\xi_{j+1}|/r_{j+1}$ and $\sin\theta_j = r_j/r_{j+1}$. Then an arbitrary $g \in U$ is written as

$$g = d_1(\varphi_1) d_2(\varphi_2) g_1(\theta_1) \dots d_n(\varphi_n) g_{n-1}(\theta_{n-1}) k, \qquad (k \in K). \tag{5.1}$$

By using this polar coordinate system, we write $g \in U/K$ as

$$\begin{aligned} g(\varphi, \theta) &= g(\varphi_1, \dots, \varphi_n, \theta_1, \dots, \theta_{n-1}) \\ &= d_1(\varphi_1) d_2(\varphi_2) g_1(\theta_1) \dots d_n(\varphi_n) g_{n-1}(\theta_{n-1}). \end{aligned} \tag{5.2}$$

We set $A = \{g_{n-1}(\theta_{n-1})\}$ and $U_n(1) = \{d_n(\varphi_n)\}$. We consequently have the Cartan decomposition $U = KU_n(1)AK$. Let dg_K denote the Haar measure on

U/K normalized so that the total measure is 1. Then under the above Cartan decomposition, we have

$$dg_K = \frac{(n-1)!}{2\pi^n} \prod_{j=1}^{n} d\varphi_j \prod_{k=1}^{n-1} \sin^{2k-1}\theta_k \cos\theta_k \, d\theta_k.$$

Let $V_{p,q}$ be the set of harmonic polynomials in $z \in \mathbf{C}^n$ of bidegree (p,q). We define the action $\tau_{p,q}$ of $U(n)$ on $V_{p,q}$ by

$$(\tau_{p,q}(g)\varphi)(z,\bar{z}) = \varphi(g^{-1}z, g^{-1}\bar{z}), \qquad (\varphi \in V_{p,q}).$$

Then $(\tau_{p,q}, V_{p,q})$ is a class 1 representation with respect to K. We use the terminology of the Gel'fand–Tsetline basis for $U(n)$ to denote the elements in $V_{p,q}$. We set

$$\left(\frac{1}{n+1}\mathbf{Z}\right)^{\ell}_{\geq} = \left\{(m_1,\dots,m_\ell) \in \mathbf{R}^{\ell}; m_j \in \frac{1}{n+1}\mathbf{Z} \text{ and } m_j - m_{j+1} \in \mathbf{Z}_{\geq 0} \text{ for all } j\right\}.$$

A sequence $\boldsymbol{M} = (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \dots, \boldsymbol{m}_1)$ is called a Gel'fand–Tsetline data if

(1) $\boldsymbol{m}_j = (m_{1,j},\dots,m_{j,j}) \in \left(\frac{1}{n+1}\mathbf{Z}\right)^{j}_{\geq}$;

(2) $m_{j,k} - m_{j,k-1} \in \mathbf{Z}_{\geq 0}$ and $m_{j,k-1} - m_{j+1,k} \in \mathbf{Z}_{\geq 0}$.

Let $(\tau_\lambda, V_\lambda)$ be a finite dimensional irreducible representation of $U(n)$ with highest weight λ. It is known that for any Gel'fand–Tsetline data $\boldsymbol{M} = (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \dots, \boldsymbol{m}_1)$ with $\boldsymbol{m}_n = \lambda$, there exists an element $v(\boldsymbol{M}) \in V_\lambda$ such that the set $\{v(\boldsymbol{M})\}$ forms a basis for V_λ. This basis is called a Gef'fand–Tsetline basis. For detail, see [11, Vol. 3, p. 363].

We restrict our attention to the case of $(\tau_{p,q}, V_{p,q})$. Let $p_2,\dots,p_n \in \mathbf{Z}_{\geq 0}$, $q_2,\dots,q_n \in \mathbf{Z}_{\geq 0}$ and $r \in \mathbf{Z}$ be such that $p = p_n \geq p_{n-1} \geq \cdots \geq p_2$, $q = q_n \geq q_{n-1} \geq \cdots \geq q_2$ and $-q_2 \leq r \leq p_2$, respectively. For a previous r, we define $p_1, q_1 \in \mathbf{Z}_{\geq 0}$ by $p_1 = \max\{r, 0\}$ and $q_1 = -\min\{r, 0\}$. In the following we interprets p_0, q_0 as 0 when they appear in a calculation. We define $\boldsymbol{m}_j$ by $\boldsymbol{m}_j = (p_j, \overbrace{0,\dots,0}^{j-2}, -q_j)$, $(j \geq 2)$ and $\boldsymbol{m}_1 = (r)$. We define the Gel'fand–Tsetline datas $\boldsymbol{M}$ and $\boldsymbol{M}_0$ by $\boldsymbol{M} = (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \dots, \boldsymbol{m}_1)$ and $\boldsymbol{M}_0 = (\boldsymbol{m}_n, \mathbf{0}_{n-1}, \dots, \mathbf{0}_1)$, respectively. Here $\mathbf{0}_i = \overbrace{(0,\dots,0)}^{i}$. Putting $\boldsymbol{p} = (p_n,\dots,p_1)$ and $\boldsymbol{q} = (q_n,\dots,q_1)$, we frequently write $(\boldsymbol{p},\boldsymbol{q})$ instead of $\boldsymbol{M}$. Keeping these notation, we define the spherical function $\Phi_{(\boldsymbol{p},\boldsymbol{q})}(g)$ on U/K by

$$\Phi_{(\boldsymbol{p},\boldsymbol{q})}(gK) = \langle \tau_{p,q}(gK)v(\boldsymbol{M}_0), v(\boldsymbol{p},\boldsymbol{q})\rangle_{V_{p,q}}.$$

The explicit expression of $\Phi_{(\boldsymbol{p},\boldsymbol{q})}(g)$ is given as follows (see [11, Vol. 2, p. 313]):

For the above Gel'fand–Tsetline data $\boldsymbol{M} = (\boldsymbol{p},\boldsymbol{q})$, we put $\alpha_j = p_j + q_j + j - 1$, $b_j = p_{j+1} - p_j - q_{j+1} + q_j$, $\beta_j = |b_j|$ and $\gamma_j = \min\{p_{j+1} - p_j, q_{j+1} - q_j\}$. Moreover we put

$$b_{(\boldsymbol{p},\boldsymbol{q})} = \left\{\frac{1}{(n-1)!2^{n-1}}\prod_{j=1}^{n-1} d_{\gamma_j}^{(\alpha_j,\beta_j)}\right\}^{1/2},$$

where $d_\gamma^{(\alpha,\beta)}$ is the constant given in (3.1). Using these notation and the Cartan decomposition (5.1), we have

$$\Phi_{(\boldsymbol{p},\boldsymbol{q})}(g(\varphi,\theta)K) = b_{(\boldsymbol{p},\boldsymbol{q})}\prod_{j=1}^{n} e^{-ib_{j-1}\varphi_j}\prod_{k=1}^{n-1}\sin^{p_k+q_k}\theta_k\cos^{\beta_k}\theta_k R_{\gamma_k}^{(\alpha_k,\beta_k)}(\cos 2\theta_k). \quad (5.3)$$

For $f \in L^2(U/K)$, its Fourier transform $\tilde{f}(\boldsymbol{p},\boldsymbol{q})$ is defined by

$$\tilde{f}(\boldsymbol{p},\boldsymbol{q}) = \int_{U/K} f(g(\varphi,\theta)K)\overline{\Phi_{(\boldsymbol{p},\boldsymbol{q})}(g(\varphi,\theta)K)}dg_K.$$

And the Plancherel inversion formula is the following:

$$f(g(\varphi,\theta)K) = \sum_{p_n=p_{n-1}}^{\infty}\sum_{q_n=q_{n-1}}^{\infty}\sum_{p_{n-1}=p_{n-2}}^{p_n}\sum_{q_{n-1}=q_{n-2}}^{q_n}\cdots$$
$$\sum_{p_2=0}^{p_3}\sum_{q_2=0}^{q_3}\sum_{r=-q_2}^{p_2}\tilde{f}(\boldsymbol{p},\boldsymbol{q})\Phi_{(\boldsymbol{p},\boldsymbol{q})}(g(\varphi,\theta)K).$$

After these preparations, we can deduce a sampling theorem for the complex sphere. By the explicit expression of $\Phi_{(\boldsymbol{p},\boldsymbol{q})}$, the Fourier transform $\tilde{f}(\boldsymbol{p},\boldsymbol{q})$ can be regarded as the composition of the Fourier transform on $\mathbf{T}^n$ related to the variable $(\varphi_1,\dots,\varphi_n) \in \mathbf{T}^n$ and the Fourier–Jacobi transforms related to the variables $\theta_1,\dots,\theta_{n-1}$. Therefore we obtain a sampling formula for the complex sphere by combining the sampling theorem for $\mathbf{T}^n$ (Proposition 2.2) and the one for the Fourier–Jacobi series (Theorem 3.1). This theorem is used to show a sampling theorem for the Radon transform of the complex hyperbolic space in Section 6.

In Section 2, to get the expression of the reconstruction formula on $\mathbf{R}^2$ (Proposition 2.4), we use the regular sampling theorem on $\mathbf{T}$. For constructing a similar sampling function to the case of $\mathbf{R}^2$, we shall here take the samples at the equidistant sampling points on $\mathbf{T}^n$ and use the sampling function given in

(2.3). Let $N_1, \dots, N_n \in \mathbf{Z}_{\geq 0}$ be such that $N_n \geq N_{n-1} \geq \cdots \geq N_1$. We call that $f \in L^2(U/K)$ is band–limited on the n-tuple $(N_1, \dots, N_n)$ if $f(\boldsymbol{p}, \boldsymbol{q}) = 0$ unless $0 \leq \max\{p_j, q_j\} \leq N_j$ for $j = 1, \dots, n$. We suppose that $f \in L^2(U/K)$ is band–limited on the n-tuple $(N_1, \dots, N_n)$. Then it is easy to check that $-N_j \leq b_{j-1} \leq N_j$ for $j = 1, \dots, n$. And we have from the Plancherel inversion formula that

$$f(g(\varphi, \theta)K) = \sum_{r=-N_1}^{N_1} \left\{ \sum_{p_2=p_1}^{N_2} \sum_{q_2=q_1}^{N_2} \sum_{p_3=p_2}^{N_3} \sum_{q_3=q_2}^{N_3} \cdots \right.$$
$$\left. \sum_{p_n=p_{n-1}}^{N_n} \sum_{q_n=q_{n-1}}^{N_n} \tilde{f}(\boldsymbol{p}, \boldsymbol{q}) \Phi_{(\boldsymbol{p}, \boldsymbol{q})}(g(\varphi, \theta)K) \right\}. \tag{5.4}$$

We first note that $\sin^{p_k+q_k} \theta_k \cos^{\beta_k} \theta_k R_{\gamma_k}^{(\alpha_k, \beta_k)}(\cos 2\theta_k)$ is a polynomial of $e^{\pm i\theta_k}$ with degree $p_k + q_k + \beta_k + 2\gamma_k = p_{k+1} + q_{k+1}$. Therefore the right-hand side of (5.4) is a polynomial of $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_{n-1}}$ with degree at most $2N_2, \dots, 2N_n$, respectively. Therefore by using Proposition 2.2 and remark, we have the following theorem.

Theorem 5.1. *Let $N_1, \dots, N_n \in \mathbf{Z}_{\geq 0}$ be such that $N_n \geq N_{n-1} \geq \cdots \geq N_1$ and suppose that $f \in L^2(U/K)$ is band–limited on the n-tuple $(N_1, \dots, N_n)$. Let θ_{j,k_j}, $(k_j = -2N_{j+1}, \dots, 2N_{j+1})$ be distinct points in $[0, \pi/2]$ for each $j = 1, \dots, n-1$. And we put $\varphi_{j,k_j} = k_j\pi/(2N_j + 1)$ for $j = 1, \dots, n$ and $k_j = -N_j, \dots, N_j$. Then f is reconstructed by*

$$\begin{aligned} &f(g(\varphi, \theta)K) \\ &\quad = \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_n=-N_n}^{N_n} \sum_{\ell_1=-2N_2}^{2N_2} \cdots \sum_{\ell_{n-1}=-2N_n}^{2N_n} \\ &\qquad \times f(g(\varphi_{1,k_1}, \dots, \varphi_{n,k_n}, \theta_{1,\ell_1}, \dots, \theta_{n-1,\ell_{n-1}})K) \\ &\qquad \times S_{k_1,\dots,k_n}^{N_1,\dots,N_n}(\varphi_1, \dots, \varphi_n) S_{\ell_1}^{2N_2}(\theta_1) \dots S_{\ell_{n-1}}^{2N_n}(\theta_{n-1}), \end{aligned}$$

where $S_{k_1,\dots,k_n}^{N_1,\dots,N_n}(\varphi_1, \dots, \varphi_n)$ are given in (2.3) *and $S_{\ell_k}^{2N_k}(\theta_k)$ are given in* (2.2) *with $d = 1$.*

6. A sampling theorem on the complex hyperbolic space

In this section, as an application of Theorem 5.1, we give a sampling theorem for the Radon transform on the complex hyperbolic space. Let $G = SU(n, 1)$, $(n \geq 2)$ and define subgroups K, A and N of G by

$$K = \left\{ \begin{pmatrix} X & \\ & u \end{pmatrix} ; X \in U(n), u \in U(1), u \det X = 1 \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} I_{n-1} & & \\ & \cosh t & \sinh t \\ & \sinh t & \cosh t \end{pmatrix} ; t \in \mathbf{R} \right\},$$

$$N = \left\{ n(z,u) = \begin{pmatrix} I_{n-1} & z & -z \\ -z^* & 1-\omega/2 & \omega/2 \\ -z^* & -\omega/2 & 1+\omega/2 \end{pmatrix} ; z \in \mathbf{C}^{n-1}, u \in i\mathbf{R}, \omega = |z|^2 - 2u \right\}.$$

Then we have an Iwasawa decomposition $G = KAN$. For $g \in G$, we define $t(g) \in \mathbf{R}$ by $g \in Ka_{t(g)}N$. By a simple calculation we have $t(g) = \log|g(\boldsymbol{e}_n + \boldsymbol{e}_{n+1})/\sqrt{2}|$. The Lie algebra $\mathfrak{a}$ of A and the centralizer M of $\mathfrak{a}$ in K are given by

$$\mathfrak{a} = \left\{ H_t = \begin{pmatrix} O_{n-1} & & \\ & 0 & t \\ & t & 0 \end{pmatrix} ; t \in \mathbf{R} \right\},$$

$$M = \left\{ \begin{pmatrix} X & & \\ & u & \\ & & u \end{pmatrix} ; X \in U(n-1), u \in U(1), u^2 \det X = 1 \right\}.$$

In our case K/M can be identified with the complex sphere S_c^{n-1}. We thus give the coordinate system on K/M induced from the polar coordinate system on S_c^{n-1} described in the previous section. Let dk denote the invariant measure on K normalized so that $\int_K dk = 1$. We identify $\mathbf{R}$ with A via the mapping $t \mapsto a_t$ and da_t denotes the measure on A indeuced from the measure $(2\pi)^{-1/2}dt$ on $\mathbf{R}$. We write dn for the invariant measure on N given by

$$dn = dn(z,u) = \frac{2^n(n-1)!}{\pi^n}\, dz d\bar{z} du.$$

Denoting $\mathfrak{a}^*$ by the real dual of $\mathfrak{a}$, we identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbf{R}$ via the correspondence $H_t \mapsto t$ and $\lambda \mapsto \lambda(H_1)$, respectively.

We define the invariant measure dg on G by $dg = e^{2nt}\, dk da_t dn$. Let dg_K be the measure on G/K such that

$$\int_G f(g) dg = \int_{G/K} \int_K f(gk) dk dg_K.$$

Let $\mathscr{C}(G/K)$ denote the space of rapidly decreasing functions on G/K. For $f \in \mathscr{C}(G/K)$, its Radon transform Rf and Helgason–Fourier transform $\tilde{f}$ are defined by

$$(Rf)(kM, a_t) = \int_N f(ka_t n)dn,$$

$$\tilde{f}(kM, \lambda) = \int_{G/K} f(g)e^{(i\lambda - n)(t(g^{-1}k))}\, dg_K.$$

And the Fourier inversion formula is the following:

$$f(g) = \frac{1}{2}\int_{K/M}\int_{\mathfrak{a}^*} e^{-(i\lambda+n)(t(g^{-1}k))}\tilde{f}(kM, \lambda)|c(\lambda)|^{-2}dk_m d\lambda. \tag{6.1}$$

Here $c(\lambda)$ is the Harish-Chandra c-function.

In the case of the Helgason–Fourier transform on the Riemannian symmetric space, the Fourier slice formula is given by the following form:

$$\tilde{f}(kM, \lambda) = \int_A e^{(-i\lambda+n)t}(Rf)(kM, a_t)da_t. \tag{6.2}$$

We shall here prove a sampling theorem for the Radon transform by using the Fourier reconstruction algorithm that is similar to the case of $\mathbf{R}^2$. Let $L > 0$ and $N_1, \dots, N_n \in \mathbf{Z}_{\geq 0}$ be such that $N_n \geq \cdots \geq N_1$. We call that $f \in \mathscr{C}(G/K)$ is band–limited if

(1) $\operatorname{supp}(\tilde{f}) \subseteq K/M \times \{\lambda \in \mathbf{R}; |\lambda| \leq L\}$;

(2) $\int_{K/M} \tilde{f}(kM, \lambda)\overline{\Phi_{(\boldsymbol{p},\boldsymbol{q})}(kM)}dk_M = 0$ unless $0 \leq \max\{p_j, q_j\} < N_j$.

The Fourier slice theorem (6.2) yields that the Fourier transform of $\tilde{f}(kM, \lambda)$ on A is equal to $e^{nt}(Rf)(kM, a_t)$. Therefore using the Shannon sampling theorem, we have

$$e^{nt}(Rf)(kM, a_t) = \sum_{p \in \mathbf{Z}} e^{np\pi/L}(Rf)(kM, a_{p\pi/L})\operatorname{sinc}\left(\frac{L}{\pi}t - p\right), \tag{6.3}$$

(see [2, Lemma 3.1]). Substituting (6.3) into (6.2), we have

$$\begin{aligned}\tilde{f}(kM, \lambda) &= \sum_{p \in \mathbf{Z}}(Rf)(kM, a_{p\pi/L})\int_A e^{np\pi/L}\operatorname{sinc}\left(\frac{L}{\pi}t - p\right)e^{-i\lambda t}\, da_t \\ &= \frac{\sqrt{\pi}}{\sqrt{2}L}\sum_{p \in \mathbf{Z}}(Rf)(kM, a_{p\pi/L})e^{(-i\lambda+n)(p\pi/L)}\chi_{(-L,L)}(\lambda). \end{aligned} \tag{6.4}$$

In this case, by using Theorem 5.1, we can explicitly construct the reconstruction formula for $(Rf)(kM, a_{p\pi/L})$ (cf. [2, Corollary 3.4]). From the assumption (2), $(Rf)(kM, a_{p\pi/L})$ is band–limited to the n-tuple $(N_1, \dots, N_n)$ as a function of S_c^{n-1} and hence

$$
\begin{aligned}
&(Rf)(kM, a_{p\pi/L}) \\
&\quad = \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_n=-N_n}^{N_n} \sum_{\ell_1=-2N_2}^{2N_2} \cdots \sum_{\ell_{n-1}=-2N_n}^{2N_n} \\
&\qquad \times (Rf)(k(\varphi_{1,k_1}, \dots \varphi_{n,k_n}, \theta_{1,\ell_1}, \dots \theta_{n-1,\ell_{n-1}})M, a_{p\pi/L}) \\
&\qquad \times S_{k_1,\dots,k_n}^{N_1,\dots,N_n}(\varphi_1, \dots, \varphi_n) S_{\ell_1}^{2N_2}(\theta_1) \dots S_{\ell_{n-1}}^{2N_n}(\theta_{n-1}), \qquad (6.5)
\end{aligned}
$$

where θ_{j,k_j} are chosen as arbitrary distinct points in $[0, \pi/2]$ and $\varphi_{j,k_j} = k_j\pi/(2N_j+1)$. We set

$$
S_{k_1,\dots,k_n,\ell_1,\dots,\ell_{n-1}}^{N_1,\dots,N_n}(k_M) = S_{k_1,\dots,k_n}^{N_1,\dots,N_n}(\varphi_1, \dots, \varphi_n) S_{\ell_1}^{2N_2}(\theta_1) \dots S_{\ell_{n-1}}^{2N_n}(\theta_{n-1}).
$$

Substituting (6.4) and (6.5) into the Fourier inversion formula (6.1), we finally obtain the following theorem.

THEOREM 6.1. *Let $L > 0$ and $N_1, \dots, N_n \in \mathbf{Z}_{\geq 0}$ be such that $N_n \geq \cdots \geq N_1$. Assume that $f \in \mathscr{C}(G/K)$ is band–limited. Then f is reconstructed as follows:*

$$
\begin{aligned}
f(g) = &\sum_{p \in \mathbf{Z}} \sum_{k_1=-N_1}^{N+1} \cdots \sum_{k_n=-N_n}^{N_n} \\
&\sum_{\ell_1=-2N_2}^{2N_2} \cdots \sum_{\ell_{n-1}=-2N_n}^{2N_n} (Rf)(k(\varphi_{1,k_1}, \dots \varphi_{n,k_n}, \theta_{1,\ell_1}, \dots \theta_{n-1,\ell_{n-1}})M, a_{p\pi/L}) \\
&\frac{\sqrt{\pi}}{2\sqrt{2}L} \int_{-L}^{L} \int_{K/M} S_{k_1,\dots,k_n,\ell_1,\dots,\ell_{n-1}}^{N_1,\dots,N_n}(k_M) e^{(-i\lambda+n)(p\pi/L)} \\
&\times e^{-(i\lambda+n)(t(g^{-1}k))} |c(\lambda)|^{-2} dk_M d\lambda.
\end{aligned}
$$

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