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Varieties with nonconstant Gauss fibers

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ABSTRACT. We construct a 4-dimensional projective variety whose general fibers of the Gauss map γ are one-parameter hyperplane sections of the given surface in $\mathbf{P}^3$ when the characteristic is positive. As an application, we have a projective variety whose general fibers of the Gauss map are not constant. In particular, this is a new example of a variety with non-linear Gauss fibers.

1. Introduction

The *Gauss map* γ on a projective variety $X \subset \mathbf{P}^N$ is the rational map from X to the Grassmannian $\mathbf{G}(\dim X, N)$ which assigns to a smooth point $p \in X$ the projective embedded tangent space $\mathbf{T}_p X$.

It is classically known that, if the characteristic of the base field K is 0, the general fiber of the Gauss map is a linear space (see, for example, [8]). In positive characteristic case, this is no longer true. There exist a curve which has infinitely many multiple tangent lines, hence the fibers of the Gauss map of this curve contain two distinct points ([7]). (A multiple tangent line is a line which has two or more distinct tangent points.) H. Kaji ([3], [4]), J. Rathmann ([6]) and A. Noma ([5]) found smooth varieties whose general fiber of the Gauss map has finitely many distinct points. By a result of F. L. Zak, the Gauss map on a smooth variety is a finite map onto its image ([8, I. 2.8]). Recently, the author found (singular) varieties whose general Gauss fiber is *not* a finite union of linear subspaces ([1]). More strongly, he proved that any given projective variety Y is (the reduced structure of) the general fiber of the Gauss map on some variety X ([2]). Note that in this construction, all the general fibers (with reduced structures) are isomorphic to each other.

In this paper we will construct first examples of "nonconstant" Gauss fiber structures. More concretely, when $Y \subset \mathbf{P}^3$ is a generic surface, we construct a 4-dimensional variety X such that the Gauss fibers are one-parameter hyperplane sections of Y.

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Notation

Unless otherwise stated, the base field K is an algebraically closed field of characteristic $p > 0$. $\mathbf{G}(k, N)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbf{P}^N$. Varieties are integral algebraic schemes. Points mean closed points. $[v] \in \mathbf{P}^N$ denotes the point of $\mathbf{P}^N$ corresponding to the equivalence class of $v \in \mathbf{A}^{N+1} \backslash 0$. Given a linear subspace $V \subset \mathbf{A}^{N+1}$, $\mathbf{P}(V) \subset \mathbf{P}^N$ means the linear subspace of $\mathbf{P}^N$ corresponding to V.

2. Construction

Let $p_i \in \mathbf{A}^7$ be the point such that i-th coordinate is 1 and the other coordinates are 0 for $i = 0, \dots, 6$. Let $Y \subset \mathbf{P}^3$ be a hypersurface and let ρ_0, ρ_1, ρ_2, ρ_3 be the morphisms from $\mathbf{A}^2$ to $\mathbf{A}^7$ defined as follows,

$$\rho_0 = (1 \quad 0 \quad 0 \quad u \quad 0 \quad v \quad v^p)$$

$$\rho_1 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) = p_1$$

$$\rho_2 = (0 \quad 0 \quad 1 \quad 0 \quad u \quad 0 \quad 0)$$

$$\rho_3 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0) = p_5.$$

Define a morphism $\eta : \mathbf{A}^2 \times \mathbf{P}^3 \to \mathbf{P}^6$;

$$(s) \times (Y_0 : Y_1 : Y_2 : Y_3) \mapsto [Y_0\rho_0(s) + Y_1\rho_1(s) + Y_2\rho_2(s) + Y_3\rho_3(s)],$$

and let X be the closure of $\eta(\mathbf{A}^2 \times Y)$. Then, X is the closed subvariety in $\mathbf{P}^6$ of dimension ≤ 4. Let $\tau := \eta|_{(\mathbf{A}^2 \times Y)} : \mathbf{A}^2 \times Y \to X$. Let $\hat{Y} \subset \mathbf{A}^4$ be the affine cone of $Y \subset \mathbf{P}^3$. By changing the coordinate system if necessary, we may assume that $Y_0 - y_0$, $Y_1 - y_1$, $Y_2 - y_2$ are a local parameter system of $\hat{Y}$ at a smooth point $(y_0, y_1, y_2, y_3) \in \hat{Y}$.

PROPOSITION 2.1. *The morphism τ is generically étale. And the fiber $(\gamma \circ \tau)^{-1}(\gamma \circ \tau((u, v) \times (1 : y_1 : y_2 : y_3)))$ is the union of all irreducible components, except the line given by $Y_0 = Y_2 = 0$, of the hyperplane section $Y \cap \{Y_2 - y_2 Y_0 = 0\}$ for a general point $(u, v) \times (1 : y_1 : y_2 : y_3)$.*

PROOF. Let $\hat{\tau} : \mathbf{A}^2 \times \hat{Y} \to \mathbf{A}^7$ be the affine lifting of τ. We have

$$\frac{\partial \hat{\tau}}{\partial u} = Y_0 p_3 + Y_2 p_4$$

$$\frac{\partial \hat{\tau}}{\partial v} = Y_0 p_5$$

$$\frac{\partial \hat{\tau}}{\partial Y_0} = \rho_0 + \frac{\partial Y_3}{\partial Y_0} p_5$$

$$\frac{\partial \hat{\tau}}{\partial Y_1} = \rho_1 + \frac{\partial Y_3}{\partial Y_1} p_5$$

$$\frac{\partial \hat{\tau}}{\partial Y_2} = \rho_2 + \frac{\partial Y_3}{\partial Y_2} p_5.$$

This implies that τ is generically étale. Furthermore, we have

$$\mathbf{T}_{\tau((u,v)\times(1:y_1:y_2:y_3))}X = \mathbf{P}(\langle p_3 + y_2 p_4, \rho_0(u,v), \rho_1(u,v), \rho_2(u,v), \rho_3(u,v)\rangle)$$

for a general point $(u,v) \times (1 : y_1 : y_2 : y_3)$. This implies our assertion. $\square$

By Proposition 2.1 we have the following:

PROPOSITION 2.2. *Let E_s be $\eta(s \times \mathbf{P}^3)$, let Y_λ be the hyperplane section $Y \cap \{Y_2 - \lambda Y_0 = 0\}$, and let Y_λ^s be $\eta(s \times Y_\lambda)$. Assume that Y does not contain the line $Y_0 = Y_2 = 0$. Then X has the following property: the one-parameter family $\{Y_\lambda^s\}_{\lambda \in \mathbf{A}^1}$ and the Gauss fibers $\{\gamma^{-1}(\gamma(p))\}_{p \in E_s}$ on E_s almost coincide for a general point $s \in \mathbf{A}^2$.*

A general surface Y satisfies the assumption. In particular, the following examples have hyperplane sections whose isomorphism classes vary.

EXAMPLE 2.3. *Let* char $K > 2$. *Let $Y \subset \mathbf{P}^3$ be the surface given by $Y_1^2 Y_0 - Y_3(Y_3 - Y_0)(Y_3 - Y_2)$, and let Y_λ be the hyperplane section $Y \cap \{Y_2 - \lambda Y_0 = 0\}$. Then all isomorhism classes of elliptic curves defined over K appear as Y_λ.*

When char $K = 2$, *one can use $Y_1^2 Y_0 + Y_0 Y_1 Y_3 + Y_3^3 + Y_2 Y_0^2$ for the similar result.*

Our construction can be generalized on the dimensions of Y and linear sections; we can construct the Gauss fiber structures which are l-parameter and l-codimensional linear sections of r-codimensional subvariety $Y \subset \mathbf{P}^k$ with $l + r \leq k$.

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References

[1] S. Fukasawa, Developable varieties in positive characteristic, Hiroshima Math. J., **35** (2005), 167–182.

[2] S. Fukasawa, Varieties with non-linear Gauss fibers, Math. Ann., **334** (2006), 235–239.
[3] H. Kaji, On the tangentially degenerate curves, J. London Math. Soc. (2), **33** (1986), 430–440.
[4] H. Kaji, On the Gauss maps of space curves in characteristic p, Compositio Math., **70** (1989), 177–197.
[5] A. Noma, Gauss maps with nontrivial separable degree in positive characteristic, J. Pure Appl. Algebra, **156** (2001), 81–93.
[6] J. Rathmann, The uniform position principle for curves in characteristic p, Math. Ann., **276** (1987), 565–579.
[7] A. H. Wallace, Tangency and duality over arbitrary fields, Proc. London Math. Soc. (3), **6** (1956), 321–342.
[8] F. L. Zak, Tangents and secants of algebraic varieties, Transl. Math. Monographs, **127**, Amer. Math. Soc., Providence, RI, (1993).

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