LECTURES ON ZARISKI VAN-KAMPEN THEOREM

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1. Introduction

Zariski van-Kampen Theorem is a tool for computing fundamental groups of complements to curves (germs of curve singularities, affine plane curves and projective plane curves). It gives you the fundamental groups in terms of generators and relations.

2. The fundamental group

2.1. **Homotopy between continuous maps.** We denote by I the closed interval $[0,1]$ of $\mathbb{R}$. Let X and Y be two topological spaces, and let $f_0 : X \to Y$ and $f_1 : X \to Y$ be two continuous maps. A continuous map $F : X \times I \to Y$ is called a *homotopy* from f_0 to f_1 if it satisfies

$$F(x,0) = f_0(x), \quad F(x,1) = f_1(x) \quad \text{for all} \quad x \in X.$$

We say that f_0 and f_1 are *homotopic* and write $f_0 \sim f_1$ if there exists a homotopy from f_0 to f_1. The relation $\sim$ is an equivalence relation as is seen below.

• Reflexive law. For any continuous map $f : X \to Y$, define $F : X \times I \to Y$ by

$$F(x,s) := f(x) \quad \text{for all} \quad s \in I.$$

Thus $f \sim f$ follows.

• Symmetric law. Let $F : X \times I \to Y$ be a homotopy from f_0 to f_1. Define $\overline{F} : X \times I \to Y$ by

$$\overline{F}(x,s) := F(x, 1-s),$$

which is a homotopy from f_1 to f_0. Hence $f_0 \sim f_1$ implies $f_1 \sim f_0$.

• Transitive law. Let $F : X \times I \to Y$ be a homotopy from f_0 to f_1, and $F' : X \times I \to Y$ a homotopy from f_1 to f_2. Define $G : X \times I \to Y$ by

$$G(x,s) = \begin{cases} F(x,2s) & \text{if } 0 \le s \le 1/2 \\ F'(x, 2s-1) & \text{if } 1/2 \le s \le 1, \end{cases}$$

which is a homotopy from f_0 to f_2. Hence $f_0 \sim f_1$ and $f_1 \sim f_2$ imply $f_0 \sim f_2$.

We call the equivalence class under the relation $\sim$ the *homotopy class.*

If there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity of X, and $f \circ g$ is homotopic to the identity of Y, then X and Y are said to be *homotopically equivalent.*

Let A be a subspace of X. A homotopy $F : X \times I \to Y$ from f_0 to f_1 is said to be *stationary on A* if

$$F(a,s) = f_0(a) \quad \text{for all} \quad (a,s) \in A \times I.$$

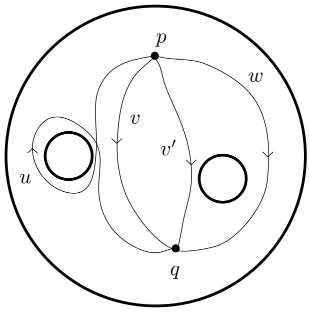


FIGURE 2.1. $[u] \neq [v] = [v'] \neq [w]$

If there exists a homotopy stationary on A from f_0 to f_1, we say that f_0 and f_1 are *homotopic relative to* A, and write $f_0 \sim_A f_1$. It is easy to see that $\sim_A$ is an equivalence relation.

2.2. **Definition of the fundamental group.** Let p and q be points of a topological space X. A continuous map $u : I \to X$ satisfying

$$u(0) = p, \quad u(1) = q$$

is called a *path* from p to q. We denote by $[u]$ the homotopy class relative to $\partial I = \{0, 1\}$ containing u. We define a path $\bar{u} : I \to X$ from q to p by

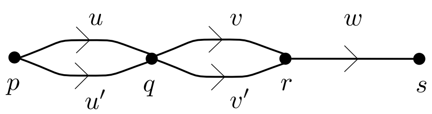
$$\bar{u}(t) := u(1 - t),$$

and call $\bar{u}$ the *inverse path* of u. A constant map to the point p is a path with both of the initial point and the terminal point being p. This path is denoted by 0_p. Let p, q, r be three points of X. Let u be a path from p to q, and v a path from q to r. We define a path $uv : I \to X$ from p to r by

$$uv(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

and call it the *concatenation* of u and v.

Lemma 2.2.1. *Let p, q, r, s be points of X. Let u and u' be paths from p to q, v and v' paths from q to r, and w a path from r to s.*



(1) If $[u] = [u']$ and $[v] = [v']$, then $[uv] = [u'v']$.
(2) $[0_p u] = [u 0_q] = [u]$.
(3) $[\bar{u}u] = [0_q]$, $[u\bar{u}] = [0_p]$.
(4) $[u(vw)] = [(uv)w]$.

Proof. (1) Let $F : I \times I \to X$ be a homotopy stationary on ∂I from u to u', and let $G : I \times I \to X$ be a homotopy stationary on ∂I from v to v'. We can construct

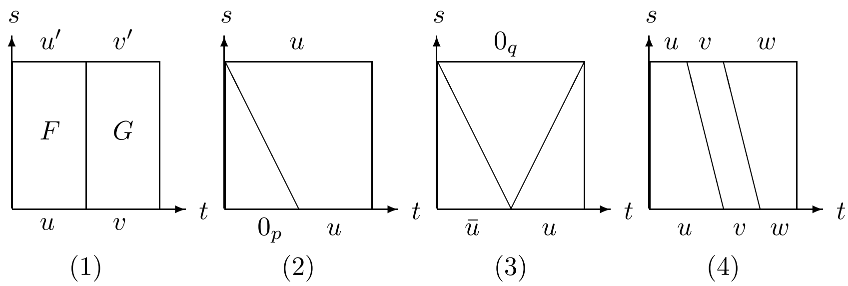


FIGURE 2.2. Proof of Lemma 2.2.1.

a homotopy $H : I \times I \to X$ stationary on ∂I from uv to $u'v'$ by

$$H(t,s) := \begin{cases} F(2t,s) & \text{if } 0 \le t \le 1/2 \\ G(2t-1,s) & \text{if } 1/2 \le t \le 1. \end{cases}$$

(2) We can construct a homotopy $F : I \times I \to X$ stationary on ∂I from $0_p u$ to u by

$$F(t,s) := \begin{cases} p & \text{if } 0 \le t \le (1-s)/2 \\ u(1-2(1-t)/(s+1)) & \text{if } (1-s)/2 \le t \le 1. \end{cases}$$

A homotopy stationary on ∂I from $u0_p$ to u can be constructed in a similar way.

(3) We can construct a homotopy $F : I \times I \to X$ stationary on ∂I from $\bar{u}u$ to 0_q by

$$F(t,s) := \begin{cases} \bar{u}(2t) & \text{if } 0 \le t \le (1-s)/2 \\ \bar{u}(1-s) = u(s) & \text{if } (1-s)/2 \le t \le (1+s)/2 \\ u(2t-1) & \text{if } (1+s)/2 \le t \le 1. \end{cases}$$

A homotopy stationary on ∂I from $u\bar{u}$ to 0_p can be constructed in a similar way.

(4) We can construct a homotopy $F : I \times I \to X$ stationary on ∂I from $u(vw)$ to $u(vw)$ by

$$F(t,s) := \begin{cases} u(4t/(2-s)) & \text{if } 0 \le t \le (2-s)/4 \\ v(4t+s-2) & \text{if } (2-s)/4 \le t \le (3-s)/4 \\ w((4t+s-3)/(s+1)) & \text{if } (3-s)/4 \le t \le 1. \end{cases}$$

□

The following is obvious from the definition:

Lemma 2.2.2. *Let u and v be paths on X with $u(1) = v(0)$, and $\phi : X \to Y$ a continuous map. Then $\phi \circ u$ and $\phi \circ v$ are paths on Y with $(\phi \circ u)(1) = (\phi \circ v)(0)$ and they satisfy $\phi \circ (uv) = (\phi \circ u)(\phi \circ v)$.* □

We fix a point b of X, and call it a *base point* of X. A path from b to b is called a *loop* with the base point b. Let $\pi_1(X, b)$ denote the set of homotopy classes (relative to ∂I) of loops with the base point b. We define a structure of the group on $\pi_1(X, b)$ by

$$[u] \cdot [v] := [uv].$$

From Lemma 2.2.1 (1), this product is well-defined; that is, $[uv]$ does not depend on the choice of the representatives u of $[u]$ and v of $[v]$. By Lemma 2.2.1 (4), this product satisfies the associative law. By Lemma 2.2.1 (3), $[0_b]$ yields the neutral element 1. By Lemma 2.2.1 (2), $[\bar{u}]$ gives the inverse $[u]^{-1}$ of $[u]$. Therefore $\pi_1(X, b)$ is a group.

Definition 2.2.3. The group $\pi_1(X, b)$ is called the *fundamental group* of X with the base point b.

Definition 2.2.4. On the set of points of X, we can introduce an equivalence relation by

$$p \sim q \iff \text{there exists a path from } p \text{ to } q.$$

An equivalence class of this relation is called a *path-connected component* of X. When X has only one path-connected component, we say that X is *path-connected.*

Lemma 2.2.5. *Let b_1 and b_2 be two base points of X. Suppose that $w : I \to X$ is a path from b_1 to b_2. The map $[w]_\sharp([u]) := [wu\bar{w}]$ defines an isomorphism*

$$[w]_\sharp : \pi_1(X, b_2) \xrightarrow{\sim} \pi_1(X, b_1)$$

of groups. The inverse is given by $[\bar{w}]_\sharp$. This isomorphism depends only on the homotopy class $[w]$ of w; that is, if w and w' are homotopic with respect to ∂I, then $[w]_\sharp = [w']_\sharp$ holds. □

Corollary 2.2.6. *If X is path-connected, then, for any two points b_1 and b_2, the fundamental group $\pi_1(X, b_1)$ is isomorphic to $\pi_1(X, b_2)$* □

The following theorem is well-known.

Theorem 2.2.7. *If X is path-connected, then the abelianization $\pi_1/[\pi_1, \pi_1]$ of $\pi_1 := \pi_1(X, b)$ is isomorphic to $H_1(X; \mathbb{Z})$.* □

Definition 2.2.8. A topological space X is said to be *simply connected* if X is path-connected and $\pi_1(X, b)$ is trivial for any base point b.

Example 2.2.9. If $n \geq 2$, then S^n is simply connected. The circle S^1 is path-connected, but $\pi_1(S^1) \cong \mathbb{Z}$.

2.3. **Homotopy invariance.** Let $\phi : X \to Y$ be a continuous map. We choose a base point b of X, and let $b' := \phi(b)$ be a base point of Y. Suppose that an element $[u] \in \pi_1(X, b)$ is represented by a loop $u : I \to X$ with the base point b. Then $\phi \circ u : I \to Y$ is a loop on Y with the base point b'. If $[u] = [u']$, then $[\phi \circ u] = [\phi \circ u']$. Indeed, let $F : I \times I \to X$ be a homotopy (stationary on ∂I) from u to u'. Then $\phi \circ F : I \times I \to Y$ is a homotopy (stationary on ∂I) from $\phi \circ u$ to $\phi \circ u'$. Accordingly, we can define a well-defined map $\phi_* : \pi_1(X, b) \to \pi_1(Y, b')$ by

$$\phi_*([u]) := [\phi \circ u].$$

For $[u], [v] \in \pi_1(X, b)$, we have

$$\phi \circ (uv) = (\phi \circ u)(\phi \circ v).$$

Hence ϕ_* is a homomorphism, which is called the *the homomorphism induced from $\phi : X \to Y$.*

Let Z be another topological space, and let $\psi : Y \to Z$ be a continuous map. We put $b'' := \psi(b')$. Then we have

$$(\psi \circ \phi)_* = \psi_* \circ \phi_*.$$

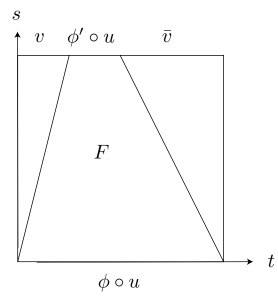


FIGURE 2.3. The homotopy G in the proof of Proposition 2.3.1

Let $w : I \to X$ be a path from $b_1 \in X$ to $b_2 \in X$. The following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X, b_2) & \xrightarrow{\phi_*} & \pi_1(Y, \phi(b_2)) \\ {\scriptstyle [w]_\sharp}\downarrow\wr & & \downarrow\wr\,{\scriptstyle [\phi\circ w]_\sharp} \\ \pi_1(X, b_1) & \xrightarrow[\phi_*]{} & \pi_1(Y, \phi(b_1)). \end{array} \tag{2.1}$$

Indeed, for a loop $u : I \to X$ with the base point b, we have

$$(\phi_* \circ [w]_\sharp)([u]) = \phi \circ (wu\bar{w}) = (\phi \circ w)(\phi \circ u)(\phi \circ \bar{w}) = ([\phi \circ w]_\sharp \circ \phi_*)([u]).$$

Proposition 2.3.1. *Let $F : X \times I \to Y$ be a homotopy from a continuous map $\phi : X \to Y$ to a continuous map $\phi' : X \to Y$. Let $v : I \to Y$ be a path on Y from $\phi(b)$ to $\phi'(b)$ defined by $t \mapsto F(b, t)$. Then the composite of $\phi'_* : \pi_1(X, b) \to \pi_1(Y, \phi'(b))$ and $[v]_\sharp : \pi_1(Y, \phi'(b)) \to \pi_1(Y, \phi(b))$ coincides with $\phi_* : \pi_1(X, b) \to \pi_1(Y, \phi(b))$.*

Proof. Let $u : I \to X$ be a loop with the base point b. We can construct a homotopy $G : I \times I \to Y$ stationary on ∂I from $\phi \circ u$ to $v(\phi' \circ u)\bar{v}$ by

$$G(t, s) := \begin{cases} v(4t) & \text{if } 0 \le t \le s/4 \\ F(u((4t - s)/(4 - 3s)), s) & \text{if } s/4 \le t \le (2 - s)/2 \\ \bar{v}(2t - 1) & \text{if } (2 - s)/2 \le t \le 1 \ . \end{cases}$$

□

Proposition 2.3.2. *Suppose that X and Y are homotopically equivalent, and that X is path-connected. Then Y is also path-connected, and, for any base points $b \in X$ and $b' \in Y$, the fundamental group $\pi_1(X, b)$ and $\pi_1(Y, b')$ are isomorphic.*

Proof. By the assumption, there are continuous maps $\phi : X \to Y$ and $\psi : Y \to X$, and homotopies $F : X \times I \to X$ from the identity map of X to $\phi \circ \psi$, and $G : Y \times I \to X$ from the identity map of Y to $\psi \circ \phi$. For any $y \in Y$, the map $t \mapsto G(y, t)$ defines a path $v_y : I \to Y$ from y to $(\phi \circ \psi)(y)$. Let b'_1 and $b'_2 \in Y$ be chosen arbitrary. Since X is path-connected, there is a path $u : I \to X$ from $\psi(b'_1)$ to $\psi(b'_2)$. Then $v_{b'_1}(\phi \circ u)\bar{v}_{b'_2}$ is a path on Y from b'_1 to b'_2. Therefore Y is path-connected.

Since $\psi \circ \phi$ is homotopic to the identity map of X, $\psi_* \circ \phi_*$ is an isomorphism of $\pi_1(X)$. Hence ϕ_* is surjective. Since $\phi \circ \psi$ is homotopic to the identity map of Y, $\phi_* \circ \psi_*$ is an isomorphism of $\pi_1(Y)$. Hence ϕ_* is injective. Therefore $\phi_* : \pi_1(X, b) \to \pi_1(Y, \phi(b))$ is an isomorphism. □

Corollary 2.3.3. *Suppose that X and Y are homotopically equivalent, and that X is simply connected. Then Y is also simply connected.* □

3. Presentation of groups

We review the theory of presentation of groups briefly. For the details, see [8] or [10].

3.1. **Amalgam.** First we introduce a notion of amalgam. This notion has been used to construct various interesting examples in group theory.

Proposition 3.1.1. *Let A, G_1, G_2 be groups, and let $f_1 : A \to G_1$, $f_2 : A \to G_2$ be homomorphisms. Then there exists a triple (G, g_1, g_2), unique up to isomorphism, where G is a group and $g_\nu : G_\nu \to G$ are homomorphisms, with the following properties:*
(i) $g_1 \circ f_1 = g_2 \circ f_2$.
(ii) *Suppose we are given a group H with homomorphisms $h_\nu : G_\nu \to H$ for $\nu = 1, 2$ satisfying $h_1 \circ f_1 = h_2 \circ f_2$. Then there exists a unique homomorphism $h : G \to H$ such that $h_1 = h \circ g_1$ and $h_2 = h \circ g_1$.*

$$\begin{tikzcd}
 & G_1 \arrow[dr, "g_1"] \arrow[drr, "h_1"] & & \\
A \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & G \arrow[r, "h"'] & H \\
 & G_2 \arrow[ur, "g_2"'] \arrow[urr, "h_2"'] & &
\end{tikzcd}$$

Figure 3.1. Universality of the amalgam

Proof. We call a finite sequence $(a_1, \dots, a_l)$ of elements of G_1 or G_2 *a word*. The empty sequence () is also regarded as a word. We define a product on the set W of words by the conjunction:

$$(a_1, \dots, a_l) \cdot (b_1, \dots, b_m) := (a_1, \dots, a_l, b_1, \dots, b_m).$$

This product satisfies the associative rule. We then introduce a relation $\succ$ on W by the following rule. Let $w, w' \in W$. Then $w \succ w'$ if and only if one of the following holds:

(1) Successive two elements a_i, a_{i+1} of w belong to a same G_ν, and w' is obtained from w by replacing these two with a single element $a_i a_{i+1} \in G_\nu$.
(2) An element of w is the neutral element of G_ν, and w' is obtained from w by deleting this element.
(3) An element a_i of w is an image $f_\nu(a)$ of some $a \in A$, and w' is obtained from w by replacing a_i with $f_\mu(a)$ where $\mu \neq \nu$.

We introduce a relation $\sim$ on W by the following. Let w and w' be two words. Then $w \sim w'$ if and only if there exists a sequence $w_0, \dots, w_N$ of words with $w_0 = w$ and $w_N = w'$ such that $w_j \succ w_{j+1}$ or $w_j \prec w_{j+1}$ or $w_j = w_{j+1}$ holds for each j. It is easy to check that $\sim$ is an equivalence relation, and that

$$w_1 \sim w'_1, \quad w_2 \sim w'_2 \qquad \Longrightarrow \qquad w_1 \cdot w_2 \sim w'_1 \cdot w'_2.$$

The equivalence class containing $w \in W$ is denoted by $[w]$. We can define a product on $G := W/\sim$ by $[w] \cdot [w'] := [w \cdot w']$. This product is well-defined, and G becomes a group under this product. Moreover, the map $a \mapsto [(a)]$ gives a group homomorphism $g_\nu : G_\nu \to G$. This triple (G, g_1, g_2) possesses the properties (i) and (ii) above. Hence the proof of the existence part is completed.

The uniqueness follows from the universal property. Suppose that both of (G, g_1, g_2) and (G', g'_1, g'_2) enjoy the properties (i) and (ii). Then there exist homomorphisms $\psi : G \to G'$ and $\psi' : G' \to G$ such that $\psi \circ g_\nu = g'_\nu$ and $\psi' \circ g'_\nu = g_\nu$ hold for $\nu = 1, 2$. The composite $\psi' \circ \psi : G \to G$ satisfies $(\psi' \circ \psi) \circ g_\nu = g_\nu$ for $\nu = 1, 2$, and the identity map id_G also satisfies $\mathrm{id}_G \circ g_\nu = g_\nu$ for $\nu = 1, 2$. From the uniqueness of h in (ii), it follows $\mathrm{id}_G = \psi' \circ \psi$. By the same way, we can show $\mathrm{id}_{G'} = \psi \circ \psi'$. Hence (G, g_1, g_2) and (G', g'_1, g'_2) are isomorphic. $\square$

Definition 3.1.2. The triple (G, g_1, g_2) is called the amalgam of $f_1 : A \to G_1$ and $f_2 : A \to G_2$, and G is denoted by $G_1 *_A G_2$ (with the homomorphisms being understood).

When A is the trivial group, then $G_1 *_A G_2$ is simply denoted by $G_1 * G_2$, and called *the free product* of G_1 and G_2.

Example 3.1.3. Let G be a group, and let N_1 and N_2 be two normal subgroups of G. Let N be the smallest normal subgroup of G containing N_1 and N_2. Then the amalgam of the natural homomorphisms $G \to G/N_1$ and $G \to G/N_2$ is G/N.

Definition 3.1.4. We define *free groups* F_n generated by n alphabets by induction on n. We put $F_1 := \mathbb{Z}$ (the infinite cyclic group), and $F_{n+1} := F_n * F_1$.

By definition, F_n is constructed as follows. Let a word mean a sequence of n alphabets $a_1, \dots, a_n$ and their inverse $a_1^{-1}, \dots, a_n^{-1}$. If successive two alphabets of a word w is of the form a_i, a_i^{-1} or a_i^{-1}, a_i, and w' is obtained from w by removing these two letters, then we write $w \succ w'$. Let w and w' be two words. We define an equivalence relation $\approx$ on the set of words by the following: $w \approx w'$ if and only if there exists a sequence $w_0, \dots, w_N$ of words with $w_0 = w$ and $w_N = w'$ such that $w_j \succ w_{j+1}$ or $w_j \prec w_{j+1}$ or $w_j = w_{j+1}$ holds for each j. We can define a product on the set of equivalence classes of words by $[w] \cdot [w'] := [w \cdot w']$, where $w \cdot w'$ is the conjunction of words. Then this set becomes a group, which is F_n.

3.2. **Van Kampen Theorem.**

Theorem 3.2.1 (van Kampen)**.** *Let X be a path-connected topological space, and $b \in X$ a base point. Let U_1 and U_2 be two open subsets of X such that the following hold:*

- *$U_1 \cup U_2 = X$, $U_1 \cap U_2 \ni b$.*
- *U_1, U_2 and $U_{12} := U_1 \cap U_2$ are path-connected.*

Let $i_\nu : U_{12} \hookrightarrow U_\nu$ and $j_\nu : U_\nu \hookrightarrow X$ be the inclusions. Then $(\pi_1(X, b), j_{1*}, j_{2*})$ is the amalgam of $i_{1*} : \pi_1(U_{12}, b) \to \pi_1(U_1, b)$ and $i_{2*} : \pi_1(U_{12}, b) \to \pi_1(U_2, b)$. That

$$\begin{tikzcd} & \pi_1(U_1, b) \arrow[dr, "j_{1*}"] & \\ \pi_1(U_{12}, b) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(X, b) \\ & \pi_1(U_2, b) \arrow[ur, "j_{2*}"'] & \end{tikzcd}$$

is, the above diagram is a diagram of the amalgam. $\square$

For the proof, see [2].

Corollary 3.2.2. *Let X_n be the bouquet of n circles: $X_n = S^1 \vee \cdots \vee S^1$. Then $\pi_1(X_n)$ is isomorphic to F_n.* □

Corollary 3.2.3. *(1) Let Z be a set of distinct n points on the complex plane $\mathbb{C}$. Then $\pi_1(\mathbb{C} \setminus Z)$ is isomorphic to F_n.*

(2) Let Z be a set of distinct n points on the complex projective line $\mathbb{P}^1$. Then $\pi_1(\mathbb{P}^1 \setminus Z)$ is isomorphic to F_{n-1}. □

Here is a simple topological proof of the following classical theorem.

Proposition 3.2.4. *Let G be a subgroup of F_n with $[F_n : G] = r < \infty$. Then G is isomorphic to F_{rn-r+1}.*

Proof. The euler number of X_n is $1-n$. Let $Y_G \to X_n$ be the covering corresponding to G; that is, Y_G is the quotient of the universal covering of X_n by G. Then Y_G is an r-fold covering of X_n, and hence its euler number is $r(1-n)$. On the other hand, Y_G is homotopically equivalent to a bouquet of S^1, and the number circles is $1 - r(1-n)$. Hence $G \cong \pi_1(Y_G)$ is isomorphic to F_{rn-r+1}. □

3.3. **Presentation.**

Definition 3.3.1. Let $\mathcal{R} := \{R_\lambda\}_{\lambda \in \Lambda}$ be a subset of F_n, and let $N(\mathcal{R})$ be the smallest normal subgroup of F_n containing $\mathcal{R}$. We denote the group $F_n/N(\mathcal{R})$ by

$$\langle\, a_1, \dots, a_n \mid R_\lambda = e \quad (\lambda \in \Lambda)\,\rangle,$$

and call it the group generated by $a_1, \dots, a_n$ with defining relations R_λ $(\lambda \in \Lambda)$.

Example 3.3.2. The group $\langle\, a \mid a^n = e\,\rangle$ is isomorphic to $\mathbb{Z}/(n)$.

Example 3.3.3. $\langle\, a, b \mid aba^{-1}b^{-1} = e\,\rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. We write this group sometimes as $\langle\, a, b \mid ab = ba\,\rangle$.

Example 3.3.4. Let n be an integer ≥ 2. Then the group generated by $a_1, \dots, a_{n-1}$ with defining relations

$$\begin{aligned}
&a_i^2 = e \quad \text{for } i = 1, \dots, n-1,\\
&a_i a_j = a_j a_i \quad \text{if } |i-j| > 1,\\
&a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } i = 1, \dots, n-1,
\end{aligned}$$

is isomorphic to the full symmetric group $\mathfrak{S}_n$. The isomorphism is given by $a_i \mapsto (i, i+1)$.

Example 3.3.5. Let p, q be positive integers, $f_1 : \mathbb{Z} \to \mathbb{Z}$ and $f_2 : \mathbb{Z} \to \mathbb{Z}$ the multiplications by p and q, respectively. Then the amalgam of these two homomorphisms is isomorphic to $\langle\, a, b \mid a^p = b^q\,\rangle$.

Remark 3.3.6. In general, it is very difficult to see the structure of a group from its presentation. For example, it is proved that there are *no* universal algorithms for determining whether a finitely presented group is finite or not (abelian or not).

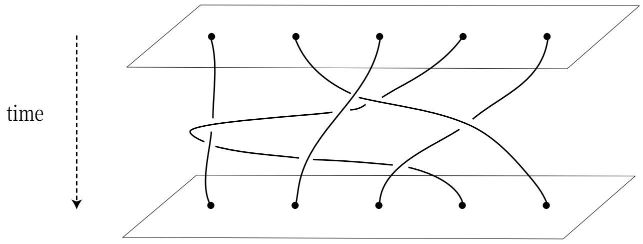


FIGURE 4.1. A braid

4. Braid groups

We put

$$M_n := \mathbb{C}^n \setminus (\text{the big diagonal}) = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j (i \neq j) \}.$$

The symmetric group $\mathfrak{S}_n$ acts on M_n by interchanging the coordinates. We then put $\overline{M}_n := M_n/\mathfrak{S}_n$. This space $\overline{M}_n$ is the space parameterizing non-ordered sets of distinct n points on the complex plane $\mathbb{C}$ (sometimes called the *configuration space* of non-ordered sets of distinct n points on $\mathbb{C}$). By associating to a non-ordered set of distinct n points $\{\alpha_1, \ldots, \alpha_n\}$ the coefficients $a_1, \ldots, a_n$ of

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = (z - \alpha_1) \cdots (z - \alpha_n),$$

we obtain an isomorphism from $\overline{M}_n$ to the complement to the *discriminant hypersurface* of monic polynomials of degree n in $\mathbb{C}^n$. For example, $\overline{M}_4$ is the complement in an affine space $\mathbb{C}^4$ with the affine parameter a_1, a_2, a_3, a_4 to the hypersurface defined by

$$\begin{gathered}
-27\, {a_4}^2 {a_1}^4 + 18\, a_4\, a_3\, a_2\, {a_1}^3 - 4\, a_4\, {a_2}^3 {a_1}^2 - 4\, {a_3}^3 {a_1}^3 + {a_3}^2 {a_2}^2 {a_1}^2 \\
+144\, {a_4}^2 a_2\, {a_1}^2 - 6\, a_4\, {a_3}^2 {a_1}^2 - 80\, a_4\, a_3\, {a_2}^2 a_1 + 16\, a_4\, {a_2}^4 + 18\, {a_3}^3 a_2\, a_1 - 4\, {a_3}^2 {a_2}^3 \\
-192\, {a_4}^2 a_3\, a_1 - 128\, {a_4}^2 {a_2}^2 + 144\, a_4\, {a_3}^2 a_2 - 27\, {a_3}^4 + 256\, {a_4}^3 = 0.
\end{gathered}$$

We put

$$P_n := \pi_1(M_n), \qquad B_n := \pi_1(\overline{M}_n),$$

where the base points are chosen in a suitable way. The group P_n is called the *pure braid group* on n strings, and the group B_n is called the *braid group* on n strings. By definition, we have a short exact sequence

$$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow \mathfrak{S}_n \longrightarrow 1,$$

corresponding to the Galois covering $M_n \to \overline{M}_n$ with Galois group $\mathfrak{S}_n$. The point of the configuration space $\overline{M}_n$ is a set of distinct n points on the complex plane $\mathbb{C}$. Hence a loop in $\overline{M}_n$ is a movement of these distinct points on $\mathbb{C}$, which can be expresses by a braid as in Figure 4.1, whence the name the braid group.

The product in B_n is defined by the conjunction of the braids. In particular, the inverse is represented by the braid upside-down. For $i = 1, \ldots, n-1$, let σ_i be the element of B_n represented by the braid given in Figure 4.4.

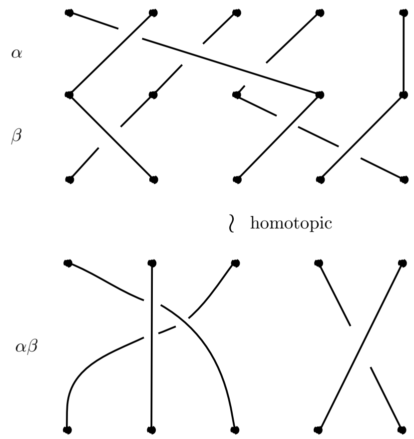


FIGURE 4.2. The product in a braid group

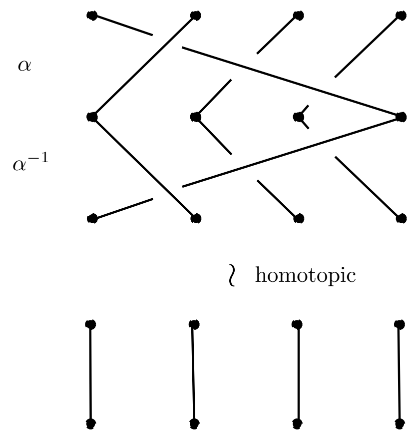


FIGURE 4.3. The inverse in a braid group

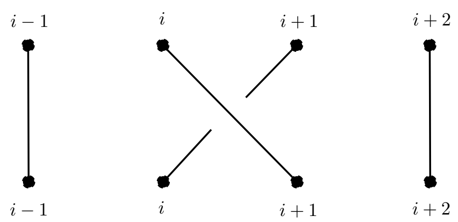


FIGURE 4.4. The element σ_i

Theorem 4.0.7 (Artin). *The braid group B_n is generated by the elements $\sigma_1, \dots,$ σ_{n-1}, and defined by the following relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| > 1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, \dots, n-1.$$

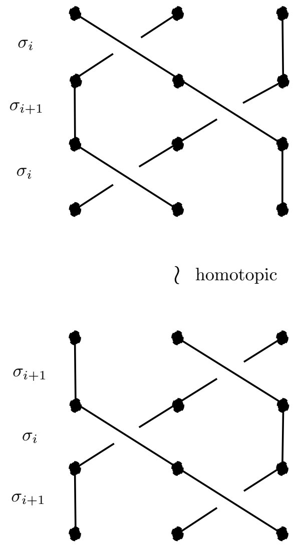


FIGURE 4.5. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

□

The fact that B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ is easy to see. The relations actually hold can be checked easily by drawing braids. The difficult part is that any other relations among the generators can be derived from these relations. See [1] for the proof.

We can define an action from right of the braid group B_n on the free group F_n by the following

$$a_j^{\sigma_i} := \begin{cases} a_j & \text{if } j \neq i, i+1 \\ a_i a_{i+1} a_i^{-1} & \text{if } j = i \\ a_i & \text{if } j = i+1. \end{cases} \tag{4.1}$$

Check that this definition is compatible with the defining relation of the braid group. In the next section, we will explain the geometric meaning of this action.

5. Monodromy on fundamental groups

We denote the conjunction of paths $\alpha : I \to X$ and $\beta : I \to X$ on the topological space X in such a way that $\alpha\beta$ is defined if and only if $\alpha(1) = \beta(0)$.

5.1. **Fundamental groups and locally trivial fiber spaces.** Let $p : E \to B$ be a locally trivial fiber space. Suppose that $p : E \to B$ has a section

$$s : B \to E;$$

that is, s is a continuous map satisfying $p \circ s = \mathrm{id}_B$. We choose a base point $\tilde{b}$ of E and b of B in such a way that $\tilde{b} = s(b)$ holds. We then put

$$F_b := p^{-1}(b).$$

We can regard $\tilde{b}$ as a base point of the fiber F_b. Then $\pi_1(B,b)$ acts on $\pi_1(F_b,\tilde{b})$ from right. This action is called the *monodromy action* on the fundamental group of the fiber.

Indeed, suppose that we are given a loop $u : I \to B$ with the base point b, and a loop $w : I \to F_b$ with the base point $\tilde{b} = s(u(0))$. The pointed fibers

$$(p^{-1}(u(t)), s(u(t))) \quad (t \in I)$$

form a trivial fiber space over I. We can deform the loop w into a loop

$$w_t : I \to p^{-1}(u(t))$$

with the base point $s(u(t))$ continuously. The loop $w_1 : I \to p^{-1}(u(1))$ with the base point $s(u(1)) = \tilde{b}$ represents $[w]^{[u]} \in \pi_1(F_b,\tilde{b})$. We have to check that $[w]^{[u]} = [w_1]$ is independent of the choice of the representing loops $u : I \to B$ and $w : I \to F_b$, and of the choice of the deformation $w_t : I \to p^{-1}(u(t))$. This checking is carried out straightforwardly by means of Serre's lifting property of the locally trivial fiber space.

Example 5.1.1. Suppose that E is $B \times F$ and $p : E \to B$ is the projection. For a point $a \in F$, the map $x \mapsto (x,a)$ defines a section of $p : E \to B$. In this case, $\pi_1(B,b)$ acts on $\pi_1(F,a)$ trivially.

Example 5.1.2. Let $p : E \to B$ be as above. For a continuous map $\alpha : B \to F$, the map $x \mapsto (x,\alpha(x))$ defines a section of $p : E \to B$. In this case, the pointed fibers are $(F,\alpha(u(t)))$. Let $A_t : [0,t] \to F$ be the path defined on F from $a(b)$ to $\alpha(u(t))$ by $A_t(s) := \alpha(u(s))$. Then $w_t := A_t^{-1} w A_t$ is a deformation of w. Hence $\pi_1(B,b)$ acts on $\pi_1(F,\alpha(b))$ by

$$[w]^{[u]} = (\alpha_*[u])^{-1} \cdot [w] \cdot (\alpha_*[u]).$$

5.2. **Monodromy action of the braid group B_n on the free group F_n.** Let R be a sufficiently large positive real number, and let $\Delta_R \subset \mathbb{C}$ be the open unit disc with the radius R. We define a open subset M'_n of the configuration space M_n of ordered distinct n points on $\mathbb{C}$ by

$$M'_n := \{ \ (z_1,\dots,z_n) \in M_n \ \mid \ |z_i| < R \text{ for } i = 1,\dots,n \ \} = \Delta_R^n \setminus (\text{the big diagonal}),$$

and put

$$\overline{M}'_n := M'_n/\mathfrak{S}_n.$$

We choose a base point $\bar{b}$ of $\overline{M}'_n$ to be a point corresponding to a set S_b of distinct n points on the closed interval $[-1,1] \subset \mathbb{C}$. The inclusion $\overline{M}'_n \hookrightarrow \overline{M}_n$ induces an isomorphism

$$\pi_1(\overline{M}'_n, b) \cong \pi_1(\overline{M}_n, b) = B_n.$$

Indeed, there is a homeomorphism $\mathbb{C} \to \Delta_R$ that is a homotopy inverse to the inclusion map $\Delta_R \hookrightarrow \mathbb{C}$. From this homeomorphism, we can construct a homotopy inverse $\overline{M}_n \to \overline{M}'_n$ of the inclusion $\overline{M}'_n \hookrightarrow \overline{M}_n$.

We consider the universal family of the complements on $\overline{M}_n$;

$$\mathcal{C} := \{ \ (S,y) \in \overline{M}_n \times \mathbb{C} \ \mid \ y \notin S \ \},$$

where a point $S \in \overline{M}_n$ is regarded as a subset of $\mathbb{C}$. The projection $\mathcal{C} \to \overline{M}_n$ is a locally trivial fiber space. A fiber over $S \in \overline{M}_n$ is the complement $\mathbb{C} \setminus S$ to S.

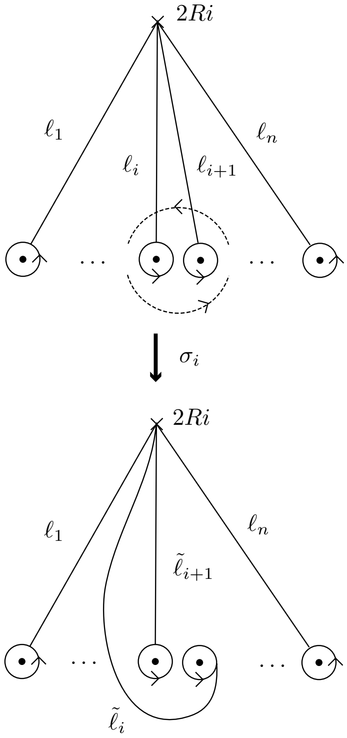


FIGURE 5.1. The braid monodromy

In particular, the fundamental group of a fiber is the free group generated by n elements. We put

$$\mathcal{C}' := p^{-1}(\overline{M}'_n),$$

and let $p' : \mathcal{C}' \to \overline{M}'_n$ be the restriction of p to $\mathcal{C}'$. We can construct a section of $p' : \mathcal{C}' \to \overline{M}'_n$ by

$$S \mapsto (S, 2Ri),$$

because, if $S \in \overline{M}'_n$, then $2Ri \notin S$. Then the monodromy action of the braid group $\pi_1(\overline{M}'_n, S_b) = B_n$ on the free group $\pi_1(\mathbb{C} \setminus S_b, 2Ri) = F_n$ is just the one described in the previous section. Indeed, $\pi_1(\mathbb{C}\setminus S_b, 2Ri)$ is the free group generated by the homotopy classes of the loops $\ell_1, \ldots, \ell_n$ indicated in the upper part of Figure 5.1. By the movement of the points in S_b that represents $\sigma_i \in B_n$, the i-th and $(i+1)$-st points interchange their positions by going around their mid-point counter-clockwise, while the other points remain still. Hence the loops ℓ_i and ℓ_{i+1} are dragged, and deform into the new loops $\tilde{\ell}_i$ and $\tilde{\ell}_{i+1}$ indicated in the lower part of Figure 5.1, while other loops does not change. The homotopy classes of loops $\tilde{\ell}_i$ and $\tilde{\ell}_{i+1}$ are written as a word of the homotopy classes of original loops:

$$[\tilde{\ell}_{i+1}] = [\ell_i], \quad [\tilde{\ell}_i] = [\ell_i][\ell_{i+1}][\ell_i]^{-1}.$$

Therefore we get the action (4.1).

5.3. **Monodromy around a curve singularity.** Let Δ_r denote the open disc $\{z \in \mathbb{C} \mid |z| < r\}$. We consider the curve C on $\Delta_{2\varepsilon} \times \Delta_{2r}$ defined by

$$x^p - y^q = 0,$$

where p, q are integers ≥ 2. Let $\bar{p} : \Delta_{2\varepsilon} \times \Delta_{2r} \to \Delta_{2\varepsilon}$ be the first projection $(x, y) \mapsto x$. We assume that r is large enough compared with ε. We put

$$Y := \bar{p}^{-1}(\Delta_{2\varepsilon}^{\times}) \cap ((\Delta_{2\varepsilon} \times \Delta_{2r}) \setminus C).$$

Then the restriction

$$p : Y \to \Delta_{2\varepsilon}^{\times}$$

of $\bar{p}$ is locally trivial. The fiber over $x \in \Delta_{2\varepsilon}^{\times}$ is Δ_{2r} minus the q-th roots of x^p. We choose the base point of $\Delta_{2\varepsilon}^{\times}$ at

$$b := \varepsilon.$$

Let α be a positive real number such that

$$|2\varepsilon|^{p/q} < \alpha < r.$$

Then the map $x \mapsto (x, \alpha)$ gives us a section of $p : Y \to \Delta_{2\varepsilon}^{\times}$, because α does not overlap any deleted point. We put

$$F_b := p^{-1}(b), \quad \text{and} \quad \tilde{b} := s(b) = (\varepsilon, \alpha).$$

How does $\pi_1(\Delta_{2\varepsilon}^{\times}, b)$ act on $\pi_1(F_b, \tilde{b})$?

The group $\pi_1(\Delta_{2\varepsilon}^{\times}, b)$ is an infinite cyclic group generated by the homotopy class $\gamma = [g]$ of the loop

$$g(t) = \varepsilon \exp(2\pi i t).$$

On the other hand, the fiber F_b is homotopic to the bouquet of q circles, and hence its fundamental group $\pi_1(F_b, \tilde{b})$ is a free group generated by q elements $\ell_0, \dots, \ell_{q-1}$, which are represented by the lassos given in Figure 5.2. (We draw figures for the case $p = 2$ and $q = 5$.) The loop of the type in Figure 5.3 is called a *lasso*.

How does the fiber $p^{-1}(g(t))$ with the base point $s(g(t))$ deform when t goes from 0 to 1? The base point $s(g(t))$ is constantly at α. The deleted points move around the origin with angular speed $2\pi p/q$, because $g(t)^p$ moves around the origin with angular speed $2\pi p$, and hence the angular speed of its q-th roots is $2\pi p/q$. Therefore the lassos around the deleted points are dragged around the origin, and when $g(t)$ comes back to the starting point, the lasso ℓ_i in Figure 5.2 is deformed into the lasso $\tilde{\ell}_i$ in Figure 5.4. Therefore the monodromy action of $\pi_1(\Delta_{2\varepsilon}^{\times}, b) = \langle \gamma \rangle$ on the free group

$$\pi_1(F_b, \tilde{b}) = \langle \ell_0, \dots, \ell_{q-1} \rangle$$

is given by

$$\ell_i^{\gamma} = \tilde{\ell}_i.$$

The homotopy classes $\tilde{\ell}_i \in \pi_1(F_b, \tilde{b})$ should be written as words of $\ell_0, \dots, \ell_{q-1}$. For this purpose, we use the following notation:

$$m := \ell_{q-1}\ell_{q-2} \cdots \ell_1 \ell_0,$$

$$\ell_j = \ell_{aq+r} := m^a \ell_r m^{-a}.$$

for $j < 0$ or $j \geq q$, where r is the remainder of j devided by r

The homotopy class m is represented by the big loop around the origin. Then we have

$$\tilde{\ell}_i = \ell_{i+p}.$$

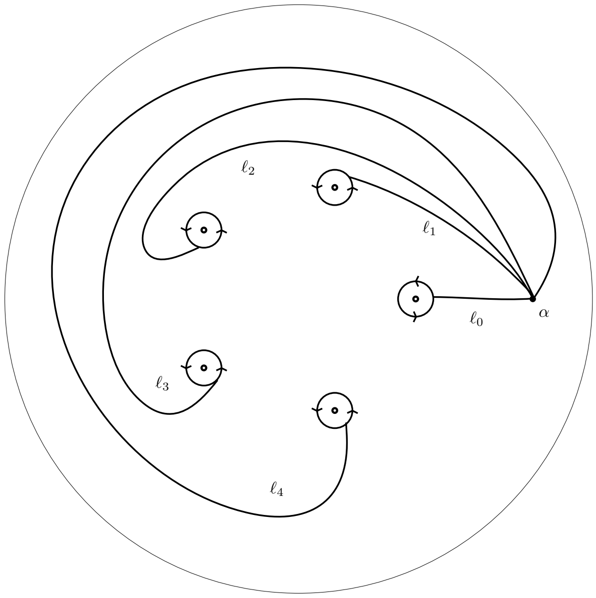


FIGURE 5.2. The generators of $\pi_1(F_b, \tilde{b})$

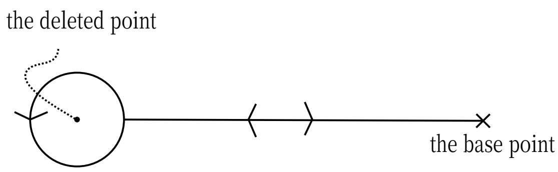


FIGURE 5.3. A lasso around a deleted point

Hence the monodromy action of $\pi_1(\Delta_{2\varepsilon}^{\times}, b)$ on $\pi_1(F_b, \tilde{b})$ is given by

$$\ell_i^{\gamma} = \ell_{i+p}.$$

We will return to this example when we calculate the local fundamental group of the curve singularity C.

5.4. **Semi-direct product.** In order to use the monodromy action in the calculation of the fundamental group of the total space, we need the concept of semi-direct product of groups. Hence let us recall briefly the definition.

Suppose that a group H acts on a group N from right. We denote this action by

$$n \mapsto n^h \qquad (n \in N, h \in H)$$

We can define a product on the set $N \times H$ by

$$(n_1, h_1)(n_2, h_2) = (n_1 n_2^{(h_1^{-1})}, h_1 h_2).$$

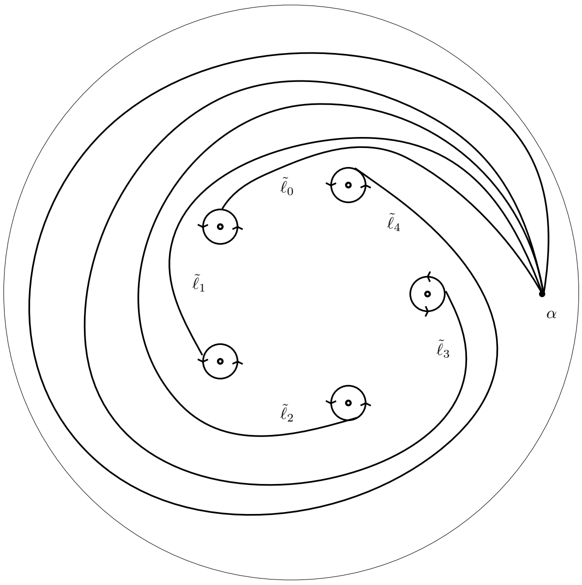


FIGURE 5.4. The dragged generators

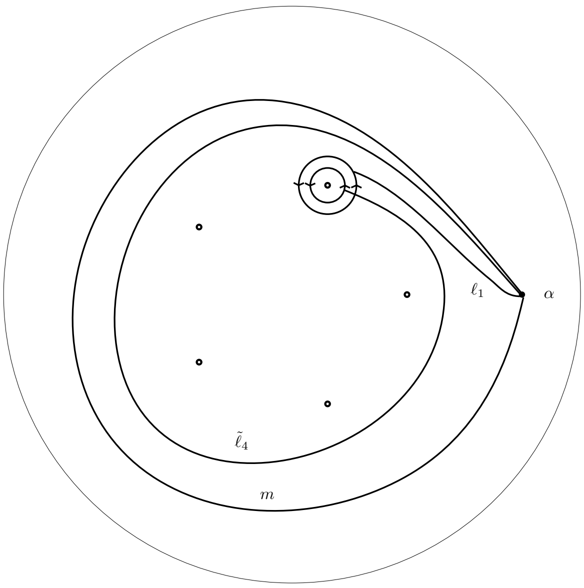


FIGURE 5.5. The loops representing m and $\tilde{\ell}_i$

It is easy to see that, under this product, $N \times H$ becomes a group, which is called the *semi-direct product* of N and H, and denoted by $N \rtimes H$. The map $n \mapsto (n, e_H)$ defines an injective homomorphism $\iota : N \to N \rtimes H$, whose image is a normal subgroup of $N \rtimes H$. By ι, we can regard N as a normal subgroup of $N \rtimes H$. The map $(n,h) \mapsto h$ defines a surjective homomorphism $\rho : N \rtimes H \to H$ whose kernel is N. Hence H can be identified with $(N \rtimes H)/N$. The map $h \mapsto (e_N, h)$ defines an injective homomorphism $\sigma : H \to N \rtimes H$ such that $\rho \circ \sigma = \mathrm{id}_H$. By σ, we can regard H as a subgroup (not normal in general) of $N \rtimes H$. We have thus obtain a splitting short exact sequence

$$1 \longrightarrow N \underset{\iota}{\longrightarrow} N \rtimes H \underset{\rho}{\overset{\sigma}{\longleftarrow\!\!\!\longrightarrow}} H \longrightarrow 1.$$

Example 5.4.1. Suppose that H acts on N trivially. Then $N \rtimes H$ is the direct product $N \times H$.

Example 5.4.2. Suppose that $\mathbb{Z}/(2)$ acts on $\mathbb{Z}/(n)$ by $x \mapsto -x$. Then $N \rtimes H$ is the dihedral group of order $2n$.

Let G be a group, N a normal subgroup of G, and $\iota : N \hookrightarrow G$ the inclusion map. We denote G/N by H, and let $\rho : G \to H$ be the natural surjective homomorphism. An injective homomorphism $\sigma : H \to G$ such that $\rho \circ \sigma = \mathrm{id}_H$ is called a *section* of ρ. Suppose that a section $\sigma : H \to G$ of ρ exists.

$$1 \longrightarrow N \underset{\iota}{\longrightarrow} G \underset{\rho}{\overset{\sigma}{\longleftarrow\!\!\!\longrightarrow}} H \longrightarrow 1.$$

We can define an action of H from right by

$$n \mapsto \sigma(h)^{-1} n \sigma(h) \qquad (n \in N, h \in H).$$

Then the semi-direct product $N \rtimes H$ constructed from this action is isomorphic to G. The isomorphism from G to $N \rtimes H$ is given by

$$g \mapsto (g \cdot \sigma(\rho(g))^{-1}, \rho(g))$$

and its inverse is given by

$$(n,h) \mapsto \iota(n)\sigma(h)$$

5.5. **The fundamental group of the total space.**

Proposition 5.5.1. *Let $p : E \to B$ be a locally trivial fiber space with a section $s : B \to E$. Suppose that E is path-connected. Let b be a base point of B, and put $\tilde{b} := s(b)$, $F_b := p^{-1}(b)$. Then $\pi_1(E, \tilde{b})$ is isomorphic to the semi-direct product $\pi_1(F_b, \tilde{b}) \rtimes \pi_1(B, b)$ constructed from the monodromy action of $\pi_1(B, b)$ on $\pi_1(F_b, \tilde{b})$.*

Proof. First note that F_b and B are path-connected, because E is path-connected and there is a section s. (The union of the path-connected components of fibers that contain the point of the image of s form a path-connected component of E, and hence it coincides with E.) Let $i : F_b \hookrightarrow E$ be the inclusion. We have the homotopy exact sequence

$$\xrightarrow{i_*} \pi_2(E, \tilde{b}) \xrightarrow{p_*} \pi_2(B, b) \longrightarrow \pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{p_*} \pi_1(B, b) \longrightarrow 1.$$

Note that $\pi_0(F_b, \tilde{b}) = 1$. There is a homomorphism $s_* : \pi_2(B, b) \to \pi_2(E, \tilde{b})$ such that

$$\pi_2(B, b) \xrightarrow{s_*} \pi_2(E, \tilde{b}) \xrightarrow{p_*} \pi_2(B, b)$$

is the identity. Therefore $p_* : \pi_2(E, \tilde{b}) \to \pi_2(B, b)$ is surjective, and hence we obtain a short exact sequence

$$1 \longrightarrow \pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{p_*} \pi_1(B, b) \longrightarrow 1.$$

There is a section $s_* : \pi_1(B, b) \to \pi_1(E, \tilde{b})$ of $p_* : \pi_1(E, \tilde{b}) \to \pi_1(B, b)$. We regard $\pi_1(F_b, \tilde{b})$ as a normal subgroup of $\pi_1(E, \tilde{b})$ by i_*. Then $\pi_1(E, \tilde{b})$ is isomorphic to the semi-direct product

$$(\pi_1(F_b, \tilde{b}) \rtimes \pi_1(B, b))'$$

constructed from the action of $\pi_1(B, b)$ on $\pi_1(F_b, \tilde{b})$ given by

$$\gamma \ \mapsto \ s_*(\alpha)^{-1} \cdot \gamma \cdot s_*(\alpha) \qquad (\alpha \in \pi_1(B, b), \gamma \in \pi_1(F_b, \tilde{b}) \subset \pi_1(E, \tilde{b})).$$

Hence it is enough to show that this group-theoretic action of $\pi_1(B, b)$ on $\pi_1(F_b, \tilde{b})$ coincides with the monodromy action of $\pi_1(B, b)$ on $\pi_1(F_b, \tilde{b})$. Let $a : I \to B$ be a loop on B representing $\alpha \in \pi_1(B, b)$, $c_0 : I \to F_b$ a loop in F_b representing $\gamma \in \pi_1(F_b, \tilde{b})$, and $c_1 : I \to F_b$ a loop in F_b representing $\gamma^\alpha \in \pi_1(F_b, \tilde{b})$. Then the conjunction

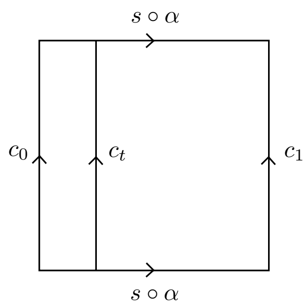
$$s_*(a) \cdot c_1 \cdot s_*(a)^{-1} \cdot c_0^{-1}$$

is null-homotopic in the total space E by the definition of the monodromy. Indeed, let

$$c_t : I \to p^{-1}(a(t)) \qquad (t \in I)$$

be the loops in the fiber $p^{-1}(a(t))$ with the base point $s(a(t))$ that appear in the process of the deformation (dragging) of c_0 into c_1. Then the above conjunction is the boundary of the map $I^2 \to E$ given by

$$(s, t) \ \mapsto \ c_t(s).$$



Hence we have $\gamma^\alpha = s_*(\alpha)^{-1} \cdot \gamma \cdot s_*(\alpha)$. □

As can be seen from the construction above, the isomorphism

$$\pi_1(F_b, \tilde{b}) \rtimes \pi_1(B, b) \ \xrightarrow{\sim} \ \pi_1(E, \tilde{b})$$

is given by

$$(u, v) \mapsto i_*(u) s_*(v),$$

where $i : F_b \hookrightarrow E$ is the inclusion.

5.6. **Fundamental groups of complements to subvarieties.** Let M be a connected complex manifold, and V a proper closed analytic subspace of M. Let $\iota : M \setminus V \hookrightarrow M$ be the inclusion. We choose a base point b of $M \setminus V$. We will investigate the homomorphism $\iota_* : \pi_1(M \setminus V, b) \to \pi_1(M, b)$.

Proposition 5.6.1. (1) *The homomorphism $\iota_* : \pi_1(M \setminus V, b) \to \pi_1(M, b)$ is surjective.* (2) *If the codimension of V in M is at least 2, then ι_* is an isomorphism.*

Proof. (1) Let $f : I \to M$ be an arbitrary loop in M with the base point b. Since V is of real codimension ≥ 2 and I is of real dimension 1, we can perturb f into a new loop f' with the base point fixed so that the image of f' is disjoint from V. Since $[f] = [f']$ in $\pi_1(M, b)$ and f' is a loop of $M \setminus V$, the homotopy class $[f]$ is in the image of ι_*.

(2) Suppose that the homotopy class $[g] \in \pi_1(M \setminus V, b)$ of a loop $g : I \to M \setminus V$ is contained in $\operatorname{Ker} \iota_*$. Then g is null-homotopic in M; that is, there exists a homotopy $G : I \times I \to M$ stationary on the boundary from $G|I \times \{0\} = g$ to the constant loop $G|I \times \{1\} = 0_b$. Since V is of real codimension ≥ 4 by the assumption and $I \times I$ is of real dimension 2, we can perturb G to a new homotopy $G' : I \times I \to M$ from g to 0_b such that the image of G' is disjoint from V. Then G' is a homotopy in $M \setminus V$. Therefore g is actually null-homotopic in $M \setminus V$. Hence ι_* is injective. □

Now let us consider the case when V is a hypersurface D of M; that is, suppose that every irreducible component of $V = D$ is of codimension 1 in M. Suppose also that D has only finitely many irreducible components. Let $D_1, \dots, D_N$ be the irreducible components of D. We put

$$D_i^\circ := D_i \setminus (D_i \cap \operatorname{Sing} D).$$

Note that D_i° is a connected complex manifold. Let p be an arbitrary point of D_i°. We take a sufficiently small open disc Δ in M in such a way that Δ intersects D at only one point p and that the intersection is transverse. Let z be a local coordinate on Δ with the center p. Then, for a small positive real number ε, the map

$$t \mapsto z = \varepsilon \exp(2\pi i t)$$

is a loop in $M \setminus D$. Let $v : I \to M \setminus D$ be a path from the base point b to $u(0) = u(1)$. Then vuv^{-1} is a loop in $M \setminus D$ with the base point b. We call a loop of this type a *lasso around D_i*.

Proposition 5.6.2. *Homotopy classes of lassos around an irreducible component D_i of D constitute a conjugacy class of $\pi_1(M \setminus D, b)$.*

Proof. Let vuv^{-1} be a lasso around D_i. Then any element of $\pi_1(M \setminus D, b)$ conjugate to $[vuv^{-1}]$ is represented by a loop of the type $w(vuv^{-1})w^{-1}$, where $w : I \to M \setminus D$ is a loop with the base point b. Since $w(vuv^{-1})w^{-1} = (wv)u(wv)^{-1}$ is also a lasso around D_i, the conjugacy class of $\pi_1(M \setminus D, b)$ containing $[vuv^{-1}]$ consists of homotopy classes of lassos around D_i.

Next we show that any two lassos $v_0 u_0 v_0^{-1}$ and $v_1 u_1 v_1^{-1}$ around D_i represents homotopy classes conjugate to each other. Since D_i° is connected, there is a homotopy

$$U : I \times I \to M \setminus D$$

from $U|I \times \{0\} = u_0$ to $U|I \times \{1\} = u_1$ such that $U(0, s) = U(1, s)$ holds for any $s \in I$. Let $w : I \to M \setminus D$ be the path from $u_0(0) = u_0(1)$ to $u_1(0) = u_1(1)$ given

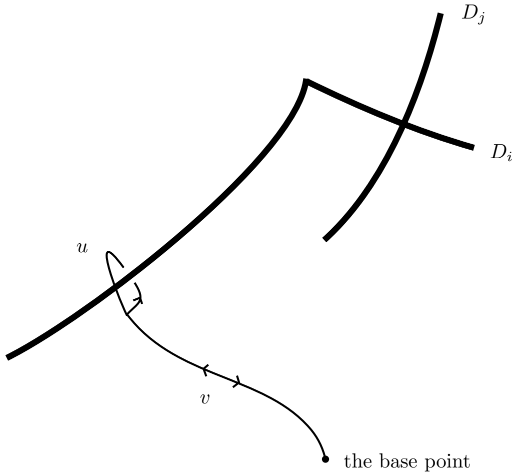


FIGURE 5.6. A lasso around D_i

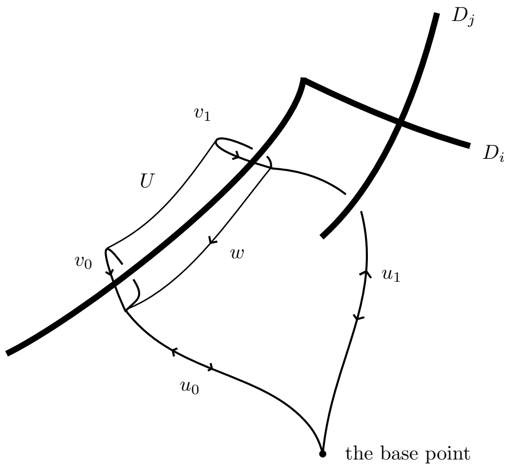


FIGURE 5.7. The path w

by $w(s) := U(s, 0)$. Then we have

$$[v_0 u_0 v_0^{-1}] = [v_0 w u_1 w^{-1} v_0^{-1}] = [a v_1 u_1 v_1^{-1} a^{-1}],$$

where $a = v_0 w v_1^{-1}$ is a loop in $M \setminus D$ with the base point b. Hence $[v_0 u_0 v_0^{-1}]$ and $[v_1 u_1 v_1^{-1}]$ are conjugate to each other in $\pi_1(M \setminus D, b)$. $\square$

Definition 5.6.3. We will denote by $\Sigma(D_i) \subset \pi_1(M \setminus D, b)$ the conjugacy class consisting of homotopy classes of lassos around an irreducible component D_i.

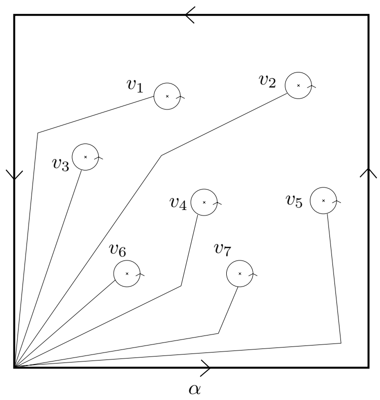
Proposition 5.6.4. *The kernel of $\iota_* : \pi_1(M \setminus D, b) \to \pi_1(M, b)$ is the smallest subgroup of $\pi_1(M \setminus D, b)$ containing $\Sigma(D_1) \cup \cdots \cup \Sigma(D_N)$.*

Proof. Any lasso around an irreducible component of D is null-homotopic in M. Hence $\Sigma(D_i)$ is contained in the kernel of ι_*.

Suppose that a loop $f : I \to M \setminus D$ with the base point b represents an element of $\operatorname{Ker} \iota_*$. Then f is null-homotopic in M, and hence there is a homotopy $F : I \times I \to M$ from $F|I \times \{0\} = f$ to $F|I \times \{1\} = 0_b$ that is stationary on the boundary ∂I. Noting that $\dim \operatorname{Sing} D < \dim M - 1$, we can perturb F into a new homotopy $G : I \times I \to M$ from f to 0_b such that the following hold:

- The image of G is disjoint from $\operatorname{Sing} D$.
- The image $G(\partial(I \times I))$ of $\partial(I \times I)$ by G is disjoint from D.
- The map G intersects D transversely; that is, if $q \in I \times I$ satisfies $G(q) \in D$, then $\operatorname{Im}(dG)_q \oplus T^{\mathbb{R}}_{G(q)} D = T^{\mathbb{R}}_{G(q)} M$ holds, where $T^{\mathbb{R}}$ is the real tangent space.

Let $\{q_1, \dots, q_L\}$ be the inverse image $G^{-1}(D)$. We choose a lasso v_i in $I \times I$ around each point q_i of $G^{-1}(D)$ with the base point $(0, 0)$. Let $\alpha : I \to I \times I$ be the loop with the base point $(0, 0)$ that goes along the boundary of the square in the counter clockwise direction. Then α is homotopically equivalent to a product of lassos v_i in $(I \times I) \setminus G^{-1}(D)$. On the other hand, the image $G \circ v_i$ of this lasso v_i by G is a lasso around an irreducible component of D, or its inverse. Since f is homotopically equivalent in $M \setminus D$ to $G \circ (\alpha^{-1})$, it is also homotopically equivalent in $M \setminus D$ to the product of these loops $G \circ v_i$. Hence $[f]$ is contained in the smallest subgroup containing $\cup\Sigma(D_i) \subset \pi_1(M \setminus D, b)$. ☐



$$[\alpha] = [v_5][v_7][v_4][v_6][v_2][v_3][v_1]$$

5.7. **Zariski van-Kampen theorem in general setting.** Let $f : M \to C$ be a surjective homomorphic map from a connected complex manifold M to a 1-dimensional complex manifold C. Suppose that the following conditions are satisfied.

(a) The curve C is simply connected.

(b) There exists a holomorphic map $s : C \to M$ such that $f \circ s = \mathrm{id}_C$ holds.

(c) There exists a finite set Z of points of C such that the restriction $f_0 : M_0 \to C \setminus Z$ of f to $M_0 := f^{-1}(C \setminus Z)$ is a locally trivial fiber space

Here a locally trivial fiber space means in the category of topological spaces and continuous maps. Let b be a base point of $C \setminus Z$, and let $s(b)$ be a base point of M_0. We denote by $F_b := f^{-1}(b)$ the fiber over b, and by $i : F_b \hookrightarrow M$ the inclusion map. It is easy to see that M_0 is path-connected. Because there is a section, F_b is also path-connected. The fundamental group $\pi_1(C \setminus Z, b)$ acts on $\pi_1(F_b, s(b))$ from right. We denote this action by

$$a \mapsto a^g \quad (\ a \in \pi_1(F_b, s(b)),\ g \in \pi_1(C \setminus Z, b)\ ).$$

The following the theorem of Zariski van-Kampen in this general setting.

Theorem 5.7.1. (1) *Suppose that the conditions (a), (b), (c) are satisfied. Then* $i_* : \pi_1(F_b, s(b)) \to \pi_1(M, s(b))$ *is surjective.*

(2) *Suppose moreover that the following condition is satisfied:*

(d) For each point $p \in Z$*, the fiber* $f^{-1}(p)$ *is irreducible.*

Then the kernel of $i_* : \pi_1(F_b, s(b)) \to \pi_1(M, s(b))$ *is the smallest subgroup of* $\pi_1(F_b, s(b))$ *containing the subset*

$$\{\ a^{-1}a^g \ \mid\ a \in \pi_1(F_b, s(b)),\ g \in \pi_1(C \setminus Z, b)\}$$

of $\pi_1(F_b, s(b))$*.*

Proof. Suppose that the conditions (a), (b), (c) are fulfilled.

We put

$$Z = \{a_1, \dots, a_N\},$$

and let D_i denote the singular fiber $f^{-1}(a_i)$ of $f : M \to C$. The homomorphic section $s : C \to M$ passes through a smooth point of D_i for each $a_i \in Z$. There exist local holomorphic coordinates $(z_1, \dots, z_m)$ of M with the center $s(a_i)$ and a local holomorphic coordinate t of C with the center a_i such that f is given by

$$(z_1, \dots, z_m) \mapsto t = z_1$$

and s is given by

$$t \mapsto (t, 0, \dots, 0).$$

Let D_i^1 be the irreducible component of D_i containing the point $s(a_i)$, and D_i^2 the union of other irreducible components. We put

$$M' := M \setminus \bigcup_i D_i^2,$$

and let $f' : M' \to C$ be the restriction of f. The homomorphic section $s : C \to M$ is also a homomorphic section $s : C \to M'$ of $f' : M' \to C$. The inclusion $M' \hookrightarrow M$ induces a surjective homomorphism $\pi_1(M', \tilde{b}) \twoheadrightarrow \pi_1(M, \tilde{b})$. On the other hand, $f' : M' \to C$ satisfies the condition (d), as well as (a), (b), (c). Hence, if (2) is proved for $f' : M' \to C$, then (1) will be proved for $f : M \to C$. Therefore it is enough to prove (2) under the assumptions (a), (b), (c), (d).

Now we assume that $f : M \to C$ satisfies (a), (b), (c), (d). We put

$$C^0 := C \setminus Z, \quad M^0 := f^{-1}(C^0),$$

and let the restriction of f and s be denoted by

$$f^0 : M^0 \to C^0, \quad s^0 : C^0 \to M^0,$$

respectively. Let $j : M^0 \hookrightarrow M$ and $i_0 : F_b \hookrightarrow M^0$ be the inclusions. Then $\pi_1(M^0, \tilde{b})$ is isomorphic to the semi-direct product

$$\pi_1(F_b, \tilde{b}) \rtimes \pi_1(C^0, b)$$

constructed from the monodromy action of $\pi_1(C^0, b)$ on $\pi_1(F_b, \tilde{b})$. The isomorphism

$$\pi_1(F_b, \tilde{b}) \rtimes \pi_1(C^0, b) \xrightarrow{\sim} \pi_1(M^0, \tilde{b})$$

is given by

$$(u, v) \mapsto i_{0*}(u) s_{0*}(v).$$

Let $g_i : I \to C^0$ be a lasso around $a_i \in Z$ with the base point b. Since C is simply connected, $\pi_1(C^0, b)$ is generated by the homotopy classes $[g_i]$ of these lassos. On the other hand, $s_0 \circ g_i : I \to M^0$ is a lasso around the irreducible hypersurface D_i with the base point $\tilde{b}$. Hence the kernel $\operatorname{Ker} j_*$ of the surjective homomorphism $j_* : \pi_1(M^0, \tilde{b}) \to \pi_1(M, \tilde{b})$ is the smallest normal subgroup containing the homotopy classes $[s_0 \circ g_i] = s_{0*}([g_i])$. In other words, $\operatorname{Ker} j_*$ is the smallest normal subgroup of $\pi_1(M^0, \tilde{b})$ containing $\operatorname{Im} s_{0*}$.

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & \operatorname{Ker} j_{0*} \arrow[d] & & \\
1 \arrow[r] & \pi_1(F_b, \tilde{b}) \arrow[r, "i_{0*}"'] & \pi_1(M^0, \tilde{b}) \arrow[r, shift right, "f_{0*}"'] \arrow[d] & \pi_1(C^0, b) \arrow[l, shift right, "s_{0*}"'] \arrow[ul] \arrow[r] & 1 \, . \\
 & & \pi_1(M, \tilde{b}) \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

Hence the proof of Theorem is reduced to the proof of the following proposition of group-theory. ☐

Proposition 5.7.2. *Suppose that a group H acts on a group N from right, and let $N \rtimes H$ be the semi-direct product. We consider N as a normal subgroup of $N \rtimes H$ by the injective homomorphism $n \mapsto (n, e_H)$, and H as a subgroup of $N \rtimes H$ by the injective homomorphism $h \mapsto (e_N, h)$. Let K be the smallest normal subgroup of $N \rtimes H$ containing H. Then the composite $\gamma : N \to (N \rtimes H)/K$ of $N \hookrightarrow N \rtimes H$ and the natural homomorphism $N \rtimes H \to (N \rtimes H)/K$ is surjective, and its kernel $N \cap K$ is the smallest subgroup of N containing the set*

$$S := \{ \, n^{-1} n^h \ \mid \ n \in N, h \in H \, \}. \tag{5.1}$$

Proof. Recall that the product in $N \rtimes H$ is defined by

$$(n_1, h_1)(n_2, h_2) = (n_1 n_2^{(h_1^{-1})}, h_1 h_2).$$

Since every element $(n, h) \in N \rtimes H$ can be written $(n, h) = (n, e_H)(e_N, h)$, this element (n, h) is equal to (n, e_H) modulo K. Therefore γ is surjective. Because

$$(n^{-1} n^h, e_H) = (n, e_H)^{-1}(e_N, h^{-1})(n, e_H)(e_N, h) \in K,$$

we have $(s, e_H) \in N \cap K$ for any $s \in S$. Hence the smallest subgroup of N containing S is contained in $N \cap K$. For $n \in N$, $g, h \in H$, we have

$$(n^{-1} n^h)^g = (n^g)^{-1} (n^g)^{(g^{-1} h g)}, \quad (n^{-1} n^h)^{-1} = (n^h)^{-1} (n^h)^{(h^{-1})}.$$

Therefore S is invariant under the action of H and the action $s \mapsto s^{-1}$. Because

$$(n,h)^{-1}(e_N, g)(n,h) = ((n^{-1}n^{(g^{-1})})^h, h^{-1}gh),$$

the first component of an arbitrary element of K is contained in the smallest subgroup of N containing S. Therefore $N \cap K$ coincides with the smallest subgroup of N containing S. □

We consider the situation in the previous proposition. Suppose that N is generated by a subset $\Gamma_N \subset N$, and H is generated by a subset $\Gamma_H \subset H$. Then the smallest subgroup of N containing S is equal to the smallest *normal* subgroup of N containing

$$S_\Gamma := \{\ n^{-1}n^h \ \mid \ n \in \Gamma_N, h \in \Gamma_H\ \}.$$

This follows from the following:

$$\begin{aligned}
&(n_1 n_2)^{-1}(n_1 n_2)^h = n_2^{-1}(n_1^{-1}n_1^h)n_2(n_2^{-1}n_2^h),\\
&(n^{-1})^{-1}(n^{-1})^h = (n'^{-1}n'^{(h')})^{-1} \qquad (n' = (n^{-1})^h),\\
&n^{-1}n^{(h_1h_2)} = (n^{-1}n^{h_1})((n^{h_1})^{-1}(n^{h_1})^{h_2}),\\
&n^{-1}n^{(h^{-1})} = (n'^{-1}n'^h)^{-1} \qquad (n' = n^{(h^{-1})}).
\end{aligned}$$

Hence we obtain the following:

Corollary 5.7.3. *Suppose that $f : M \to C$ satisfies the conditions (a), (b), (c), (d) and*

(e) $\pi_1(F_b, \tilde{b})$ is a free group generated by $\alpha_1, \dots, \alpha_d$.

Suppose that $\pi_1(C \setminus Z, b)$ is generated by $\gamma_1, \dots, \gamma_N$. Then $\pi_1(M, \tilde{b})$ is isomorphic to the group defined by the presentation

$$\left\langle\ \alpha_1, \dots, \alpha_d\ \middle|\ \alpha_i^{\gamma_j} = \alpha_i \quad \begin{pmatrix} i = 1, \dots, d \\ j = 1, \dots, N \end{pmatrix}\ \right\rangle.$$

□

6. Local fundamental group of curve singularities

Let us consider the the local fundamental group

$$\pi_1((\Delta_{2\varepsilon} \times \Delta_{2r}) \setminus C)$$

of the curve singularity C defined by $x^p - y^q = 0$, which was considered above. Let X be the complement $(\Delta_{2\varepsilon} \times \Delta_{2r}) \setminus C$, and let $f : X \to \Delta_{2\varepsilon}$ be the projection onto the first factor. As was shown above, there is a holomorphic section $x \mapsto (x, \alpha)$ of $f : X \to \Delta_{2\varepsilon}$. The fundamental group of the fiber is the free group generated by $\ell_0, \dots, \ell_{q-1}$. By introducing the auxiliary elements

$$\begin{aligned}
&m := \ell_{q-1}\ell_{q-2}\cdots\ell_1\ell_0,\\
&\ell_j := m^a \ell_r m^{-a} \qquad \text{when } j = aq + r (0 \le r < q),
\end{aligned}$$

we can write the monodromy action of the generator $\gamma \in \pi_1(\Delta_{2\varepsilon}^\times)$ by

$$\ell_i^\gamma = \ell_{i+p}.$$

Hence, by the corollary, the fundamental group $\pi_1(X)$ is isomorphic to G defined by the presentation below:

$$G := \left\langle \; m,\ \ell_j\ (j \in \mathbb{Z}) \;\middle|\; \begin{array}{l} m = \ell_{q-1}\ell_{q-2}\cdots\ell_1\ell_0, \\ \ell_{j+q} = m\ell_j m^{-1}, \\ \ell_j = \ell_{j+p} \quad \text{(the monodromy relation)} \end{array} \right\rangle.$$

Theorem 6.0.4. *Suppose that p and q are prime to each other. Then the group G has a simpler presentation $G' := \langle\, \alpha, \beta \mid \alpha^p = \beta^q \,\rangle$.*

Proof. For any integer j, let (A_j, B_j) be a pair of integers satisfying $A_j q + B_j p = j$. From the relations $\ell_{j+q} = m\ell_j m^{-1}$ and $\ell_j = \ell_{j+p}$, we have

$$\ell_{j+q-1}\ldots\ell_j = \ell_{A_j q+q-1}\ldots\ell_{A_j q} = m^{A_j}(\ell_{q-1}\ldots\ell_0)m^{-A_j} = m.$$

We define an element $n \in G$ by

$$n = \ell_{p-1}\ell_{p-2}\cdots\ell_1\ell_0.$$

Then we have

$$m^p = \ell_{qp-1}\ldots\ell_1\ell_0 = n^q.$$

Hence we can define a homomorphism $\varphi : G' \to G$ by

$$\alpha \mapsto m, \quad \beta \mapsto n.$$

We have

$$m^{A_j}\ell_0 m^{-A_j} = \ell_{A_j q} = \ell_{A_j q + B_j p} = \ell_j.$$

Let us calculate $m^{A_1}n^{B_1}$. Since $m^{A_1+kp}n^{B_1-kq} = m^{A_1}n^{B_1}$ for any integer k, we can assume $B_1 > 0$ and $A_1 < 0$. Then we have

$$m^{A_1}n^{B_1} = (\ell_{|A_1|q}\ldots\ell_1)^{-1}(\ell_{B_1 p-1}\ldots\ell_0) = \ell_0,$$

because $B_1 p - 1 = |A_1|q$. Hence any ℓ_j can be written as a word of m and n. In particular, φ is surjective. It can be easily checked that

$$m \mapsto \alpha, \quad \text{and} \qquad \ell_j \mapsto \alpha^{A_j}(\alpha^{A_1}\beta^{B_1})\alpha^{-A_j}$$

define an inverse homomorphism $\varphi^{-1} : G \to G'$. Note that, from $\alpha^p = \beta^q$, the right-hand side does not depend on the choice of (A_j, B_j). Thus φ is an isomorphism. $\square$

Corollary 6.0.5. *The local fundamental group of the ordinary cusp $x^2 - y^3 = 0$ is $\langle\, \alpha, \beta \mid \alpha^2 = \beta^3 \,\rangle$.* $\square$

7. Fundamental groups of complements to projective plane curves

7.1. **Zariski-van Kampen theorem for projective plane curves.** Let $C \subset \mathbb{P}^2$ be a complex projective plane curve defined by a homogeneous equation

$$\Phi(X, Y, Z) = 0$$

of degree d. Suppose that C is reduced; that is, Φ does not have any multiple factor. We consider the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$. (Since $\mathbb{P}^2 \setminus C$ is path-connected, $\pi_1(\mathbb{P}^2 \setminus C)$ does not depend on the choice of the base point.) We choose a point $a \in \mathbb{P}^2 \setminus C$. By a linear coordinate transformation, we can assume that

$$a = [0 : 0 : 1].$$

Since $a \notin C$, the coefficient of Z^d in Φ is not zero. Let $L \subset \mathbb{P}^2$ be the line defined by $Z = 0$. For a point $p \in L$, let $\overline{pa} \subset \mathbb{P}^2$ be the line connecting p and a. We put

$$\overline{X} := \{\ (p, q) \in L \times \mathbb{P}^2 \ \mid\ q \in \overline{pa}\ \},$$

and let $\bar{f} : \overline{X} \to L$ and $\rho : \overline{X} \to \mathbb{P}^2$ be the projections onto each factors. If $q \neq a$, then $\rho^{-1}(q)$ consists of a single point, while $E := \rho^{-1}(a)$ is isomorphic to L by $\bar{f}$. The morphism $\rho : \overline{X} \to \mathbb{P}^2$ is called the blowing up of $\mathbb{P}^2$ at a, and E is called the exceptional divisor.

We then put

$$X := \overline{X} \setminus \rho^{-1}(C),$$

and let $f : X \to L$ be the restriction of $\bar{f}$. Since $E \cap \rho^{-1}(C) = \emptyset$, ρ induces an isomorphism from $X \setminus E$ to $\mathbb{P}^2 \setminus (C \cup \{a\})$. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X \setminus E) & \xrightarrow{\sim} & \pi_1(\mathbb{P}^2 \setminus (C \cup \{a\})) \\ \downarrow & & \downarrow \wr \\ \pi_1(X) & \underset{(\rho|X)_*}{\longrightarrow} & \pi_1(\mathbb{P}^2 \setminus C), \end{array}$$

where the vertical arrows are induced from the inclusions. The right vertical arrow is surjective because E is a proper subvariety of X, and the left vertical arrow is an isomorphism because $\{a\}$ is a proper subvariety of $\mathbb{P}^2 \setminus C$ with codimension 2. Hence $\rho|X$ induces an isomorphism from $\pi_1(X)$ to $\pi_1(\mathbb{P}^2 \setminus C)$. Therefore we will calculate $\pi_1(X)$.

For $p \in L$, the intersection points of $\bar{f}^{-1}(p)$ and $\rho^{-1}(C)$ is mapped by ρ to the intersection points of $\overline{pa}$ and C bijectively. Suppose that p is the point $[\xi : \eta : 0]$. Then the line $\overline{pa}$ is given by an affine parameter t as follows:

$$\{ \ [\xi : \eta : t] \ \mid \ t \in \mathbb{C} \cup \{\infty\} \ \},$$

where $t = \infty$ corresponds to a. Hence the intersection points of $\overline{pa}$ and C correspond to the roots of

$$\Phi(\xi, \eta, t) = 0$$

bijectively. Let $D_\Phi(\xi, \eta)$ be the discriminant of $\Phi(\xi, \eta, t)$ regarded as a polynomial of t. We have assumed that Φ has no multiple factors. Therefore $D_\Phi(\xi, \eta)$ is not zero. It is a homogeneous polynomial of degree $d(d-1)$ in ξ and η. We put

$$Z := \{ \ [\xi : \eta : 0] \in L \ \mid \ D_\Phi(\xi, \eta) = 0 \ \}.$$

If $p \in L \setminus Z$, then $f^{-1}(p)$ is the line $\overline{pa}$ minus d distinct points. Hence the restriction of f to $f^{-1}(L \setminus Z)$ is a locally trivial fiber space over $L \setminus Z$.

We choose a base point of X at $\tilde{b} \in E \setminus (E \cap f^{-1}(Z))$, and let $b := f(\tilde{b})$ be the base point of L. Let F be the fiber $f^{-1}(b)$ of f passing through $\tilde{b}$. The map

$$p \mapsto (p, a)$$

is the holomorphic section $s : L \to X$ of $f : X \to L$ that passes through $\tilde{b}$. The image of s is E. Hence $\pi_1(L \setminus Z, b)$ acts on $\pi_1(F, \tilde{b})$ from right. The projective line L is simply connected. Every fiber of f is irreducible because it is a projective line minus some points. Moreover $\pi_1(F, \tilde{b})$ is the free group generated by homotopy classes $\alpha_1, \dots, \alpha_{d-1}$ of $d-1$ lassos around $d-1$ points of $F \cap \rho^{-1}(C)$. Hence we can apply the corollary. Suppose that $Z \subset L$ consists of e points. Then $\pi_1(L \setminus Z, b)$ is the free group generated by homotopy classes $\gamma_1, \dots, \gamma_{e-1}$ of $e-1$ lassos around $e-1$ points of Z.

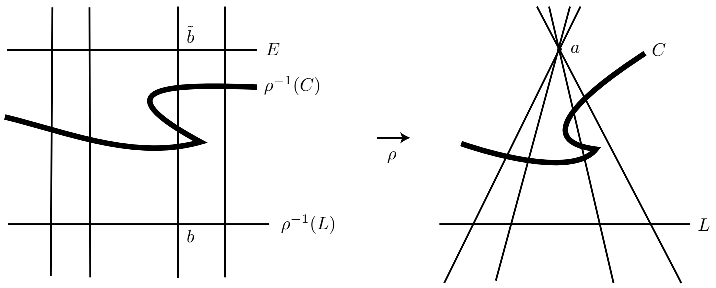


FIGURE 7.1. Blowing up at a

Theorem 7.1.1. *The fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the group defined by the presentation*

$$\left\langle\ \alpha_1, \dots, \alpha_{d-1}\ \middle|\ \alpha_i^{\gamma_j} = \alpha_i \quad \begin{pmatrix} i = 1, \dots, d-1 \\ j = 1, \dots, e-1 \end{pmatrix}\ \right\rangle.$$

□

7.2. **An example.** Let C be a nodal cubic curve defined by

$$F(X, Y, Z) := Y^2 Z - (X + Z)\, X^2 = 0.$$

Its affine part ($Z \neq 0$) is given by

$$y^2 = x^2(x + 1) \qquad (x = X/Z,\ \ y = Y/Z).$$

Let $[U : V : W]$ be the homogeneous coordinates of the dual projective plane; that is, a point $[U : V : W]$ corresponds to the line defined by

$$UX + VY + WZ = 0.$$

The dual curve $C^\vee$ of C is defined by

$$G(U, V, W) := -4\, WU^3 + 36\, UV^2 W - 27\, V^2 W^2 - 8\, U^2 V^2 + 4\, V^4 + 4\, U^4 = 0.$$

The defining polynomial G of $C^\vee$ is obtained by the following method. The incidence variety

$$I := \{(p, \ell) \in C \times C^\vee \mid \ell \text{ is tangent to } C \text{ at } p\}$$

is defined by the equation

$$F = 0, \quad U - \frac{\partial F}{\partial X} = 0, \quad V - \frac{\partial F}{\partial Y} = 0, \quad W - \frac{\partial F}{\partial Z} = 0,$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. We calculate the Gröbner basis of the defining ideal of I with respect to the lexicographic order. Since $C^\vee$ is the image of I by the second projection, you can find G among the Gröbner basis. Solving the equation

$$\frac{\partial G}{\partial U} = \frac{\partial G}{\partial V} = \frac{\partial G}{\partial W} = 0,$$

we see that $C^\vee$ has three singular points

$$[0 : 0 : 1], \quad [\frac{9}{8} : \pm\frac{\sqrt{-27}}{8} : 1].$$

Looking at G locally around these points, we see that these three points are ordinary cusps of $C^\vee$. Conversely, the dual curve of a three-cuspidal quartic curve is a nodal cubic curve. Since any nodal cubic curve is projectively isomorphic to C, any three-cuspidal quartic curve is projectively isomorphic to $C^\vee$.

We blow up $\mathbb{P}^2$ at
$$a = [1 : 0 : 0].$$
Note that $a \notin C^\vee$. We put
$$L := \{U = 0\},$$
and let
$$s := V/W$$
be the affine parameter on L. The line connecting a and a point $p = [0 : s : 1]$ of L is given by
$$\{[t : s : 1] \mid t \in \mathbb{C} \cup \{\infty\}\}$$
in terms of the affine parameter t, where $t = \infty$ corresponds to a. Hence the intesection points of $\overline{pa}$ and $C^\vee$ corresponds to the roots of
$$G(t, s, 1) = 4\,t^4 - 4\,t^3 - 8\,s^2t^2 + 36\,s^2t + 4\,s^4 - 27\,s^2 = 0.$$
The discriminant of this equation is
$$-256\,s^4\left(64\,s^2 + 27\right)^3.$$
You can easily check that the line connecting a and the point $s = \infty$ on L does not intersect $C^\vee$ transversely. It also follows from the fact that $-256\,s^4\left(64\,s^2 + 27\right)^3$ does not have the degree $d(d-1) = 12$ of the homogeneous discriminant. Hence we have
$$Z = \{\ 0,\ \pm\frac{\sqrt{-27}}{8},\ \infty\ \}.$$
The line connecting a and $s = \infty$ is the double tangent to $C^\vee$, which corresponds to the node of $(C^\vee)^\vee = C$. The other three lines connecting a and the points of Z pass throgh the cusps of $C^\vee$. The line connecting a and $s = 0$ is the tangent line at the cusp $[0 : 0 : 1]$; that is, the intersection multiplicity is 3. The other two lines intersect $C^\vee$ at the cusps $[9/8 : \pm\sqrt{-27}/8 : 1]$ with intersection multiplicity 2.

We choose a base point $b \in L \setminus Z$ at $s = 1$. Then the fundamental group $\pi_1(L \setminus Z, b)$ is the free group generated by the homotopy classes α, $\bar{\alpha}$ and β of of the lassos indicated in Figure 7.2. The homotopy class of a lasso around the point $s = \infty$ of Z is equal to $(\alpha\beta\bar{\alpha})^{-1}$.

The line $\overline{ba}$ intersects $C^\vee$ at four points
$$\begin{aligned} A &: t = 0.7886\ldots, \\ B &: t = 1.2200\cdots + 1.3353\ldots\, i. \\ C &: t = -2.2287\ldots, \\ D &: t = 1.2200\cdots - 1.3353\ldots\, i. \end{aligned}$$
The base point $\tilde{b}$ on the fiber is given by
$$t = \infty.$$
The fundamental group $\pi_1(\overline{ba} \setminus (\overline{ba} \cap C^\vee), \tilde{b})$ is generated by the homotopy classes a, b and c of of the lassos indicated in Figure 7.3. The homotopy class of the lasso d in Figure 7.3 is equal to the product $(cba)^{-1}$; that is we have
$$\pi_1(\overline{ba} \setminus (\overline{ba} \cap C^\vee), \tilde{b}) = \langle\ a, b, c, d \mid dcba = 1\ \rangle.$$
When a point p on $L \setminus Z$ moves from the base point $s = 1$ to the point near

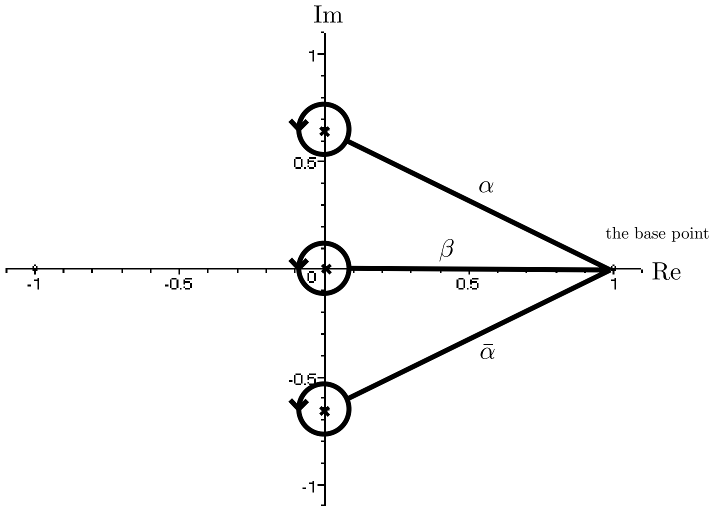


FIGURE 7.2. The generators of $\pi_1(L \setminus Z, b)$

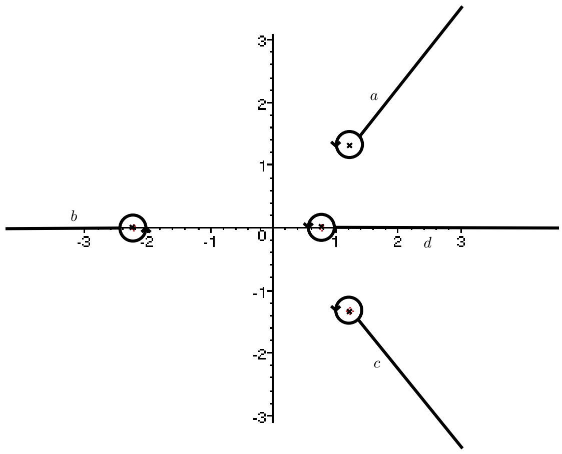


FIGURE 7.3. The generators of $\pi_1(\overline{ba} \setminus (\overline{ba} \cap C^\vee), \tilde{b})$

the deleted point $s = \sqrt{-27}/8$ along the line segment part of the lasso α, the intersection points $\overline{pa} \cap C^\vee$ moves as in Figure 7.5. The two points C and D will collide. This collision corresponds to the cusp $[9/8 : \sqrt{-27}/8 : 1]$ of $C^\vee$. When the point p goes around the deleted point $s = \sqrt{-27}/8$ in a counter-clockwise direction, then the two points go around each other $3/2$ times, and interchange their positions. Actually, they move in the way as indicated in Figure 7.4. This is the case $p = 3, q = 2$ in the previous section. When the point p goes back to the base point, then the four points go back to the original point. The lassos a, b, c, d around the points A, B, C, D are dragged by these movements, and become the lassos $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ indicated in Figure 7.6. Since

$$c^\alpha = \tilde{c} = dcdc^{-1}d^{-1}, \quad d^\alpha = \tilde{d} = dcdcd^{-1}c^{-1}d^{-1},$$

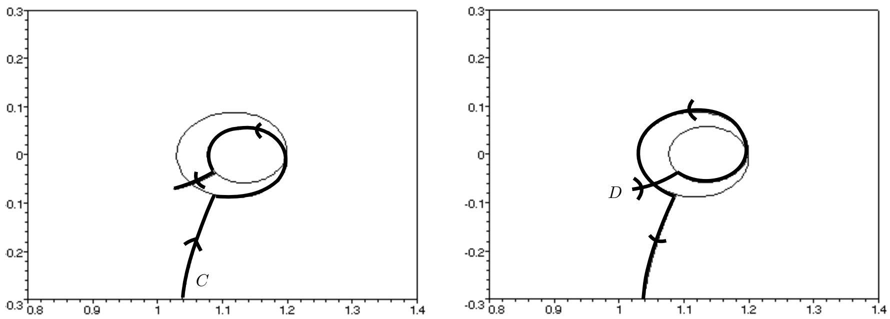


FIGURE 7.4. The movement of C and D around $s = \sqrt{-27}/8$

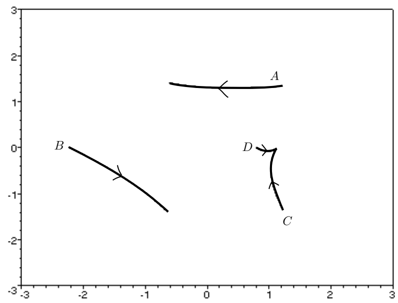


FIGURE 7.5. The movement of points for the monodromy of α

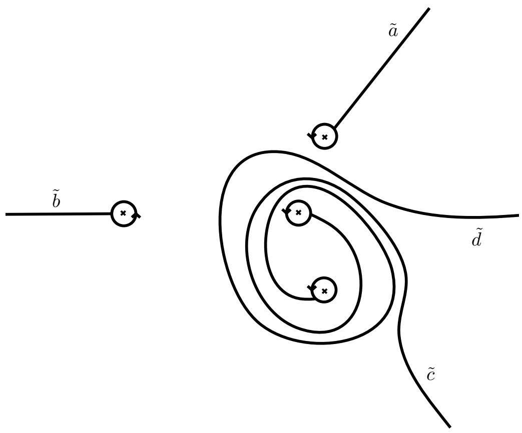


FIGURE 7.6. The new lassos after the monodromy action of α

we have

$$c = dcdc^{-1}d^{-1}, \quad d = dcdcd^{-1}c^{-1}d^{-1},$$

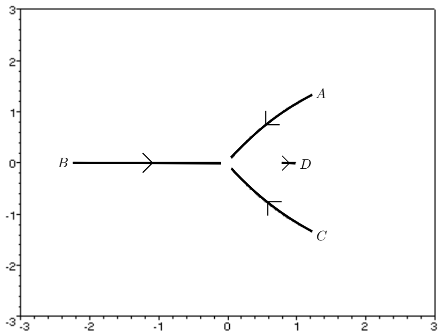


FIGURE 7.7. The movement of points for the monodromy of β

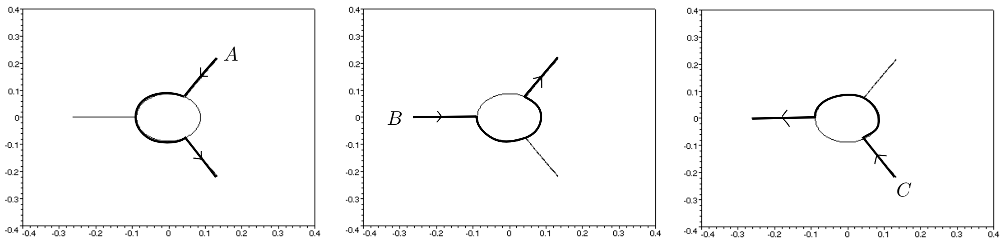


FIGURE 7.8. The movement of A, B, C around $s = 0$

in $\pi_1(\mathbb{P}^2 \setminus C^\vee, a)$. These can be reduced to the simple relation

$$cdc = dcd.$$

By the same method, we see that the monodromy relation in $\pi_1(\mathbb{P}^2 \setminus C^\vee, a)$ corresponding to $\bar{\alpha} \in \pi_1(L \setminus Z, b)$ is

$$ada = dad.$$

When a point p on $L \setminus Z$ moves from the base point $s = 1$ to the point near the deleted point $s = 0$ along the line segment part of the lasso β, the intersection points $\overline{pa} \cap C^\vee$ moves as in Figure 7.7. The three points A, B, C colide. This collision corresponds to the cusp $[0 : 0 : 1]$ of $C^\vee$. When the point p goes around the deleted point $s = 0$ in a counter-clockwise direction, then the three points go around each other 2/3 times, and interchange their positions. See Figure 7.8. This is the case $p = 2, q = 3$ in the previous section. Note that the line connecting $s = 0$ and a intersects $C^\vee$ at $[0 : 0 : 1]$ with multiplicity 3. When the point p goes back to the base point, the lassos a, b, c, d around the points A, B, C, D are dragged and become the lassos $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ indicated in Figure 7.9. Since

$$a^\beta = \tilde{a} = c, \quad b^\beta = \tilde{b} = d^{-1}ad, \quad c^\beta = \tilde{c} = d^{-1}bd,$$

we have

$$a = c, \quad db = ad, \quad dc = bd$$

in $\pi_1(\mathbb{P}^2 \setminus C^\vee, a)$.

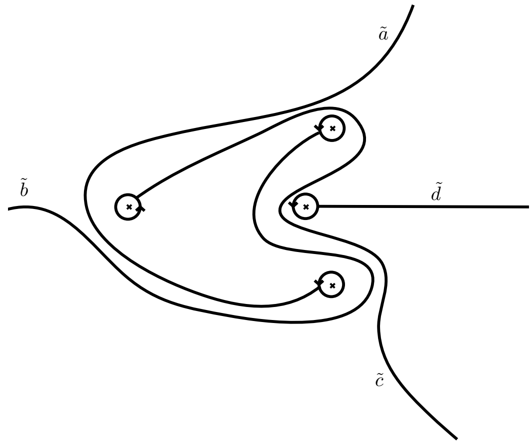


FIGURE 7.9. The new lassos after the monodromy action of β

Combining all of these, we see that

$$\pi_1(\mathbb{P}^2 \setminus C^\vee, a) = \left\langle \; a, b, c, d \;\middle|\; \begin{array}{l} dcba = 1, \\ cdc = dcd, ada = dad, \\ a = c, db = ad, dc = bd \end{array} \right\rangle .$$

We can reduce the relations to

$$\pi_1(\mathbb{P}^2 \setminus C^\vee, a) = \langle\, c, d \mid cdc = dcd, c^2d^2 = 1 \,\rangle.$$

Putting $\alpha = cd$, $\beta = cdc$, we have

$$\pi_1(\mathbb{P}^2 \setminus C^\vee, a) = \langle\, \alpha, \beta \mid \alpha^3 = \beta^2 = (\beta\alpha)^2 \,\rangle.$$

This group is the binary 3-dihedral group. Thus we obtain the following theorem due to Zariski:

Theorem 7.2.1. *The fundamental group of the complement to a three cuspidal quartic curve is isomorphic to the binary* 3*-dihedral group.* $\square$

Remark 7.2.2. Let us see the structure of the group

$$G := \langle\, a, b \mid aba = bab, a^2b^2 = 1 \,\rangle.$$

From $a^2b^2 = 1$, we have $ab^2a = ba^2b = 1$. Hence

$$(aba)^2 = a(ba^2b)a = a^2, \quad (bab)^2 = b(ab^2a)b = b^2.$$

From $aba = bab$, we have $a^2 = b^2$. We put $c := a^2 = b^2$. Then c is of order 2 and in the center of G. Since the group

$$G/\langle c\rangle = \langle\, a, b \mid aba = bab, a^2 = b^2 = 1 \,\rangle$$

is isomorphic t $\mathfrak{S}_3$, we see that G is a central extension of $\mathfrak{S}_3$ by $\mathbb{Z}/(2)$. In particular, G is a non-abelian finite group of order 12.

7.3. **Zariski conjecture and Zariski pairs.** If a reduced plane curve $C \subset \mathbb{P}^2$ consists of irreducible components of degree $d_1, \ldots, d_k$, then $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z})$ is isomorphic to

$$\mathbb{Z}^k/(d_1, \ldots, d_k)\mathbb{Z}.$$

Suppose that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. Then it is isomorphic to $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z})$, and hence it is determined by the degrees of the irreducible components.

When is $\pi_1(\mathbb{P}^2 \setminus C)$ abelian? We have the following theorem, which had been known as Zariski conjecture since the publication of the paper [11], and was proved by Fulton and Deligne around 1970 in [4] and [3].

Theorem 7.3.1. *If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.* □

This theorem was proved, not by Zariski-van kampen's theorem, but by Fulton-Hansen's connectedness theorem [6]. See [7] for the proof.

Several improvements of this theorem are known. One of them is the following theorem, due to Nori [9].

Theorem 7.3.2. *Let C be an irreducible curve of degree d with n nodes and k cusps. If $2n + 6k < d^2$, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

Note that the fundamental group of the complement need not be determined by the number and types of the singularity. The following is the classical example discovered by Zariski.

Example 7.3.3. There exist two curves C_1 and C_2, both of which are of degree 6 and has 6 cusps as their only singularities, such that $\pi_1(\mathbb{P}^2 \setminus C_1)$ is isomorphic to $\mathbb{Z}/(6)$ while $\pi_1(\mathbb{P}^2 \setminus C_2)$ is isomorphic to the free product of $\mathbb{Z}/(2) * \mathbb{Z}/(3)$.

References

[1] Birman J. S., "Braids, links, and mapping class groups" Annals of Mathematics Studies, No. 82, Princeton University Press, Princeton, N.J., 1974.

[2] R. H. Crowell R. H. and Fox, "Introduction to knot theory" Graduate Texts in Mathematics, **57**. Springer-Verlag, New York, 1977.

[3] P. Deligne, Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien (d'après W. Fulton). Bourbaki Seminar, Vol. 1979/80, pp. 1–10, Lecture Notes in Mathenatics **842**, Springer-Verlag, New York, 1981.

[4] W. Fulton, On the fundamental group of the complement of a node curve. Ann. of Math. **111** (1980), 407–409.

[5] W. Fulton, "Algebraic topology." A first course. Graduate Texts in Mathematics **153**. Springer-Verlag, New York, 1995.

[6] W. Fulton and J. Hansen, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings. Ann. of Math. **110** (1979), 159–166.

[7] W. Fulton and R. Lazarsfeld, Connectivity and its applications in algebraic geometry. Algebraic geometry (Chicago, Ill., 1980), pp. 26–92, Lecture Notes in Mathematics **862**, Springer-Verlag, New York, 1981.

[8] W. Magnus, A. Karrass and D. Solitar, "Combinatorial group theory. Presentations of groups in terms of generators and relations. Second revised edition ". Dover Publications, Inc., New York, 1976.

[9] M. V. Nori, Zariski's conjecture and related problems. Ann. Sci. École Norm. Sup. **16** (1983), 305–344.

[10] J.-P. Serre, "Arbres, amalgames, SL_2". Société Mathématique de France, Paris, 1977: "Trees". Translated from the French by John Stillwell. Springer-Verlag, New York, 1980.

[11] O. Zariski, On the problem of existence of algebraic functions of two variables possesing a given branch curve. Amer. J. Math. **51** (1929), 305–328.

[12] O. Zariski, A theorem on the Poincaré group of an algebraic hypersurface. Ann. of Math. **38** (1937), 131-141.

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