15-NODAL QUARTIC SURFACES. PART II: THE AUTOMORPHISM GROUP: COMPUTATIONAL DATA

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This note explains the contents of the computational data about the results of the paper [1] (joint work with Igor Dolgachev). The data is available at

http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html

in the text file `15nodalcompdata.txt`. In this data, we use the `Record`-format of `GAP` [2].

1. Lattices

We use L_{26} to denote the even unimodular hyperbolic lattice $II_{1,25}$ of rank 26. We fix bases of the lattices L_{26}, S_{16} and S_{15}. The following data are with respect to these bases.

- `GramL26`. The Gram matrix of L_{26}.
- `GramS16`. The Gram matrix of S_{16}.
- `GramS15`. The Gram matrix of S_{15}.
- `embS16L26`. The matrix M such that $v \mapsto vM$ is the primitive embedding $\epsilon_{16} \colon S_{16} \hookrightarrow L_{26}$.
- `embS15L26`. The matrix M such that $v \mapsto vM$ is the primitive embedding $\epsilon_{15} \colon S_{15} \hookrightarrow L_{26}$.
- `embS15S16`. The matrix M such that $v \mapsto vM$ is the primitive embedding $\epsilon_{15,16} \colon S_{15} \hookrightarrow S_{16}$.
- `projL26S16`. The matrix M such that $v \mapsto vM$ is the orthogonal projection $L_{26} \otimes \mathbb{Q} \to S_{16} \otimes \mathbb{Q}$.
- `projL26S15`. The matrix M such that $v \mapsto vM$ is the orthogonal projection $L_{26} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$.
- `projS16S15`. The matrix M such that $v \mapsto vM$ is the orthogonal projection $S_{16} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$.
- `weyl0`. The Weyl vector $\mathbf{w}_0 \in L_{26}$.
- `ample16`. The class $\alpha_{16} \in S_{16}$, that is, the image of $\mathbf{w}_0$ by the orthogonal projection $L_{26} \otimes \mathbb{Q} \to S_{16} \otimes \mathbb{Q}$. This vector is the class of a hyperplane section of the $(2,2,2)$-complete intersection model of Y_{16}.
- `ample15`. The class $\alpha_{15} \in S_{15} \otimes \mathbb{Q}$, that is, the image of $\mathbf{w}_0$ by the orthogonal projection $L_{26} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$. Note that $\alpha_{15} \notin S_{15}$ but $2\alpha_{15} \in S_{15}$.
- `h4X16`. The class $h_4 \in S_{16}$ of the hyperplane section of the quartic surface X_{16}.
- `h4X16dual`. The class of the hyperplane section of the quartic surface X'_{16} (the dual of $X_{16} \subset \mathbb{P}^3$).
- `h4X15`. The class $h_4 \in S_{15}$ of the hyperplane section of the quartic surface X_{15}.

- `h6Y15`. The class $h_6 \in S_{15}$ of the hyperplane section of the $(2,3)$-complete intersection model $X_{15}^{(6)}$ of Y_{15} (see (5.5) of [1]).

2. Y_{16} and the induced chamber D_{16} of S_{16}

2.1. Groups.

- `GeneratorsOS16D16` is a generating set of the group $\mathrm{O}(S_{16}, D_{16})$ of order 23040.
- `GeneratorsAutY16D16` is a generating set of the group $\mathrm{Aut}(Y_{16}, \alpha_{16}) = \mathrm{O}(S_{16}, D_{16}) \cap \mathrm{O}(S_{16})^{\omega} \cong (\mathbb{Z}/2\mathbb{Z})^5$.

2.2. Rational curves. The following are the data related with Remark 4.4 of [1].

- `RatCurvesOnY16deg5` is the list of classes of smooth rational curves on Y_{16} with degree 5 with respect to α_{16}.
- `RatCurvesOnY16deg7` is the list of classes of smooth rational curves on Y_{16} with degree 7 with respect to α_{16}.

2.3. Walls. The data `D16WallRecs` is the list of 316 records `wallrec`, each of which describes a wall $w = D_{16} \cap (v)^{\perp}$ of D_{16} and consists of the following items.

- `no`. The number k such that the record `wallrec` is at the kth position of `D16WallRecs`.
- `orbit`. The number i of the orbit containing w (see Table 4.2 of [1]).
- `innout`. `"inner"` or `"outer"`.
- `vector`. The primitive defining vector v of w.
- `n`. $n = \langle v, v \rangle$.
- `a`. $a = \langle v, \alpha_{16} \rangle$.
- `adjacentweyl`. The Weyl vector $\mathbf{w}' \in L_{26}$ that induces the chamber D' adjacent to D_{16} across w.
- `d`. $d = \langle \alpha_{16}, \mathbf{w}'_S \rangle$, where $\mathbf{w}'_S$ is the image of $\mathbf{w}'$ by the orthogonal projection $L_{26} \to S_{16} \otimes \mathbb{Q}$.
- `isomL26`. An isometry $\tilde{g} \in \mathrm{O}(L_{26})$ that preserves $S_{16} \subset L_{26}$ and maps the Conway chamber $\mathcal{D}(\mathbf{w}_0)$ to the Conway chamber $\mathcal{D}'$ such that $\epsilon_{16}^{-1}(\mathcal{D}')$ is the induced chamber D' adjacent to D_{16} across w.
- `extraaut`. An isometry $g_w \in \mathrm{O}(S_{16})$ that maps D_{16} to the induced chamber D' adjacent to D_{16} across w. When w is inner, this isometry g_w is chosen from $\mathrm{Aut}(Y_{16})$.
- `index`. The combinatorial data of w. The 32 lines

$$N_0, N_{ij}, T_i, T_{ij}$$

  in the $(2,2,2)$-complete intersection model of Y_{16} are expressed by

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["nodal", [0]],  ["nodal", [i, j]],  ["trope", [i]],  ["trope", [i, j]],
```

  respectively.
  - When `orbit` is 1, `index` indicates the curve whose class defines the wall w.
  - When `orbit` is 2, `index` indicates the Göpel-tetrad.
  - When `orbit` is 3, `index` indicates the curve that is the exceptional curve over the center of the projection $X_{16} \dashrightarrow \mathbb{P}^2$ or $X'_{16} \dashrightarrow \mathbb{P}^2$ that induces the involution g_w.
  - When `orbit` is 4, `index` indicates the Weber-hexad.

When `orbit` is 1, the record `wallrec` has the following items:

- `octad`. The corresponding octad (see Table 4.1 of [1]).
- `Leechroot`. The Leech root $\epsilon_{16}(v) \in L_{26}$.

3. Y_{15} and the induced chamber D_{15} of S_{15}

3.1. Groups.

- `OS15D15` is the list of elements of the group $\mathrm{O}(S_{15}, D_{15})$.
- `OS15D15permutation` describes the natural isomorphism $\mathrm{O}(S_{15}, D_{15}) \cong \mathfrak{S}_6$. The ith isometry g of `OS15D15` is mapped to the ith permutation $\sigma = [i_1, \dots, i_6]$ in `OS15D15permutation` such that $\nu^\sigma = i_\nu$ for $\nu = 1, \dots, 6$.

3.2. Walls.

The data `D15WallRecs` is the list of 314 records `wallrec`, each of which describes a wall $w = D_{15} \cap (v)^\perp$ of D_{15} and consists of the following items.

- `no`. The number k such that the record `wallrec` is at the kth position of `D15WallRecs`.
- `orbit`. The number i of the orbit O_i containing w (see Table 5.1 of [1]).
- `innout`. `"inner"` or `"outer"`.
- `vector`. The primitive defining vector v of w.
- `n`. $n = \langle v, v\rangle$.
- `a`. $a = \langle v, \alpha_{15}\rangle$.
- `adjacentweyl`. The Weyl vector $\mathbf{w}' \in L_{26}$ that induces the chamber D' adjacent to D_{15} across w.
- `d`. $d = \langle \alpha_{15}, \mathbf{w}'_S\rangle$, where $\mathbf{w}'_S$ is the image of $\mathbf{w}'$ by the orthogonal projection $L_{26} \to S_{15} \otimes \mathbb{Q}$.
- `isomL26`. An isometry $\tilde{g} \in \mathrm{O}(L_{26})$ that preserves $S_{15} \subset L_{26}$ and maps the Conway chamber $\mathcal{D}(\mathbf{w}_0)$ to the Conway chamber $\mathcal{D}'$ such that $\epsilon_{15}^{-1}(\mathcal{D}')$ is the induced chamber D' adjacent to D_{15} across w.
- `extraaut`. An isometry $g_w \in \mathrm{O}(S_{15})$ that maps D_{15} to the induced chamber D' adjacent to D_{15} across w. When w is inner, this isometry g_w is the unique extra-automorphism $g_w \in \mathrm{Aut}(X_{15})$.
- `index`. The combinatorial data of w.
  - When `orbit` is 1, then `index` is a double trio $(ijk)(lmn) = [[\mathtt{i},\mathtt{j},\mathtt{k}],[\mathtt{l},\mathtt{m},\mathtt{n}]]$.
  - When `orbit` is 2, then `index` is a duad $(ij) = [\mathtt{i},\mathtt{j}]$.
  - When `orbit` is 3, then `index` is a syntheme $(ij)(kl)(mn) = [[\mathtt{i},\mathtt{j}],[\mathtt{k},\mathtt{l}],[\mathtt{m},\mathtt{n}]]$.
  - When `orbit` is 4, then `index` is a double trio $(ijk)(lmn) = [[\mathtt{i},\mathtt{j},\mathtt{k}],[\mathtt{l},\mathtt{m},\mathtt{n}]]$.
  - When `orbit` is 5, then `index` is a number $\nu \in \{1, \dots, 6\}$.
  - When `orbit` is 6, then `index` is a pair of double trios

$$\begin{aligned} &\{(i_1j_1k_1)(l_1m_1n_1), (i_2j_2k_2)(l_2m_2n_2),\} \\ = \quad &[\ [[\mathtt{i}_1,\mathtt{j}_1,\mathtt{k}_1],[\mathtt{l}_1,\mathtt{m}_1,\mathtt{n}_1]], [[\mathtt{i}_2,\mathtt{j}_2,\mathtt{k}_2],[\mathtt{l}_2,\mathtt{m}_2,\mathtt{n}_2]]\ ] \end{aligned}$$

  - When `orbit` is 7, then `index` is a number $\nu \in \{1, \dots, 6\}$.
  - When `orbit` is 8, then `index` is a duad $(ij) = [\mathtt{i},\mathtt{j}]$.
  - When `orbit` is 9, then `index` is an index $[t(a), t(b), \dots, t(f)]$ of $\mathit{\Gamma}_{\mathrm{tripod}}$ (see Figure 5.3 of [1]).
  - When `orbit` is 10, then `index` is an index $[p(a), p(b), \dots, p(e)]$ of $\mathit{\Gamma}_{\mathrm{penta}}$ (see Figure 5.4 of [1]).

Remark 3.1. When there exist several choices of representatives of a combinatorial data, we choose the minimal one. For example, a double trio (123)(456) can be written in 72 ways

$$[[1,2,3],[4,5,6]],[[1,3,2],[4,5,6]],\dots,[[6,5,4],[3,2,1]],$$

and we choose $[[1,2,3],[4,5,6]]$ as a representative.

3.3. **Involutions.** Let $\iota \in \mathrm{Aut}(Y_{15})$ be an involution that is obtained from a rational double covering $Y_{15} \to \mathbb{P}^2$ (in several ways). Then ι is described by a record `involrec` with the following items:

- `invol` is the matrix representation of the action of ι in S_{15}.
- `degree` is the α_{15}-degree $\langle \alpha_{15}, \alpha_{15}^{\iota} \rangle$.
- `index` indicates a combinatorial data that specifies ι. The content of `index` depends on the type of ι.
- `h2recs` is a list of records `h2rec` that describe polarizations h_2 of degree 2 such that $|h_2|$ gives a rational double covering $Y_{15} \to \mathbb{P}^2$ inducing ι. Each `h2rec` has the following items:
  - `h2` is the vector $h_2 \in S_{15}$.
  - `sing` describes the singular points P of the branch curve B of the covering $Y_{15} \to \mathbb{P}^2$. Each singular point P of B is given by a pair of an ADE-type such as `"A1", "A2", "D4", ...` and the list of classes of smooth rational curves contracted to P by $Y_{15} \to \mathbb{P}^2$.

We have the following lists of involutions of Y_{15}.

3.3.1. `Sigmas`. The list of six involutions $\sigma^{(\nu)}$ that make Y_{15} the focal surface of a congruence of bi-degree $(2,3)$. See Section 5.2.1 of [1]. The index `involrec.index` is the number $\nu \in \{1,\dots,6\}$. Each `involrec.h2recs` consists of five records.

3.3.2. `X15Projections`. The list of 15 involutions obtained by the projection of X_{15} with the center being a node p_δ of X_{15}. See Example 5.6 of [1]. The index `involrec.index` is the duad δ corresponding to the center p_δ. Each `involrec.h2recs` consists of a single record.

3.3.3. `SevenNodals`. The list of 360 involutions obtained by the linear system on Y_{15} cut out by quadric surfaces passing through a set of 7 nodes of X_{15} obtained from the 7 edges of the graph in Figure 5.2 of [1]. See Example 5.7 of [1]. The index `involrec.index` is the list of 7 duads corresponding to the 7 nodes of X_{15}. Each `involrec.h2recs` consists of a single record.

3.3.4. `Pentagons`. The list of 72 involutions obtained by the linear system on Y_{15} cut out by cubic surfaces passing through a certain set of 5 nodes of X_{15} with given multiplicities. See Example 5.8 of [1]. The index `involrec.index` is the list of 5 duads corresponding to the 5 nodes of X_{15}. Each `involrec.h2recs` consists of five records.

3.3.5. `X6ModelProjections`. The list of 45 involutions obtained from the projections of the $(2,3)$-complete intersection $X_{15}^{(6)}$ with the center being the line passing through two ordinary nodes of $X_{15}^{(6)} \subset \mathbb{P}^4$. See Section 5.2.3 of [1]. The index `involrec.index` is the pair of double trios corresponding the two nodes of $X_{15}^{(6)}$. Each `involrec.h2recs` consists of a single record.

3.4. **Faces.** The data `D15InnFaceRecs` is the list of 5235 records `facerec`, each of which describes a face $f = w_1 \cap w_2$ of D_{15} with codimension 2 and consists of the following items.

- `no`. The number k such that the record `facerec` is at the kth position of `D15InnFaceRecs`.
- `orbit`. The number i of the orbit containing f (see Table in Theorem 5.11 of [1]).
- `walls`. The pair of `wallrec1.no` and `wallrec2.no`, where `wallrec1` and `wallrec2` are the records that describe the walls w_1 and w_2 containing f, respectively.
- `relation`. Let $(D_0, \dots, D_m)$ be one of the two simple chamber loops around f from D_0 to $D_m = D_0$, and let $g_1, \dots, g_m$ be the extra-automorphisms such that
$$D_i = {D_0}^{g_i \cdots g_1}$$
for $i = 1, \dots m$. Then we have a relation
$$g_m \cdots g_1 = 1.$$
The item `facerec.relation` is the list $[\nu_m, \dots, \nu_1]$ of numbers, where ν_i is the number `wallrec.no` of the record `wallrec` such that `wallrec.extraaut` is equal to g_i.

References

[1] Igor Dolgachev and Ichiro Shimada. 15-nodal quartic surfaces. Part II: The automorphism group, 2019.

[2] The GAP Group. *GAP - Groups, Algorithms, and Programming*. Version 4.8.6; 2016 (http://www.gap-system.org).

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