ZARISKI-VAN KAMPEN METHOD AND TRANSCENDENTAL LATTICES OF CERTAIN SINGULAR $K3$ SURFACES

KEN-ICHIRO ARIMA AND ICHIRO SHIMADA

Abstract. We present a method of Zariski-van Kampen type for the calculation of the transcendental lattice of a complex projective surface. As an application, we calculate the transcendental lattices of complex singular $K3$ surfaces associated with an arithmetic Zariski pair of maximizing sextics of type $A_{10} + A_9$ that are defined over $\mathbb{Q}(\sqrt{5})$ and are conjugate to each other by the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$.

1. Introduction

First we prepare some terminologies about lattices. Let R be $\mathbb{Z}$ or $\mathbb{Z}_p$, where p is a prime integer or ∞, $\mathbb{Z}_p$ is the ring of p-adic integers for $p < \infty$, and $\mathbb{Z}_\infty$ is the field $\mathbb{R}$ of real numbers. An *R-lattice* is a free R-module L of finite rank with a non-degenerate symmetric bilinear form

$$(\ , \ )_L \ : \ L \times L \to R.$$

A $\mathbb{Z}$-lattice is simply called a *lattice*. A lattice L is called *even* if $(v, v)_L \in 2\mathbb{Z}$ holds for any $v \in L$. Two lattices L and L' are said to be *in the same genus* if the $\mathbb{Z}_p$-lattices $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isomorphic for all p (including ∞). Then the set of isomorphism classes of lattices are decomposed into a disjoint union of *genera*. Note that, if L and L' are in the same genus and L is even, then L' is also even, because being even is a 2-adic property. Let L be a lattice. Then L is canonically embedded into $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ as a submodule of finite index, and $(\ , \ )_L$ extends to a symmetric bilinear form

$$(\ , \ )_{L^\vee} \ : \ L^\vee \times L^\vee \to \mathbb{Q}.$$

Suppose that L is even. We put

$$D_L := L^\vee / L,$$

and define a quadratic form $q_L : D_L \to \mathbb{Q}/2\mathbb{Z}$ by

$$q_L(\bar{x}) := (x, x)_{L^\vee} \bmod 2\mathbb{Z}, \quad \text{where} \quad \bar{x} = x + L \in D_L.$$

The pair (D_L, q_L) is called the *discriminant form* of L. By the following result of Nikulin (Corollary 1.9.4 in [8]), each genus of even lattices is characterized by the signature and the discriminant form.

Proposition 1.1. *Two even lattices are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic.*

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For a $K3$ surface X defined over a field k, we denote by $\mathrm{NS}(X)$ the *Néron-Severi lattice* of $X \otimes \bar{k}$; that is, $\mathrm{NS}(X)$ is the lattice of numerical equivalence classes of divisors on $X \otimes \bar{k}$ with the intersection paring $\mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbb{Z}$. Following the terminology of [12, §8] and [22], we say that a $K3$ surface X defined over a field of characteristic 0 is *singular* if the rank of $\mathrm{NS}(X)$ attains the possible maximum 20.

Let S be a *complex* $K3$ surface. Then the second Betti cohomology group $\mathrm{H}^2(S, \mathbb{Z})$ is regarded as a unimodular lattice by the cup-product, which is even of signature $(3, 19)$. The Néron-Severi lattice $\mathrm{NS}(S)$ is embedded into $\mathrm{H}^2(S, \mathbb{Z})$ primitively, because we have $\mathrm{NS}(S) = \mathrm{H}^2(S, \mathbb{Z}) \cap \mathrm{H}^{1,1}(S)$. We denote by $\mathrm{T}(S)$ the orthogonal complement of $\mathrm{NS}(S)$ in $\mathrm{H}^2(S, \mathbb{Z})$, and call $\mathrm{T}(S)$ the *transcendental lattice* of S. Suppose that S is singular in the sense above. Then $\mathrm{T}(S)$ is an even positive-definite lattice of rank 2. The Hodge decomposition $\mathrm{T}(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S)$ induces a canonical orientation on $\mathrm{T}(S)$. We denote by $\widetilde{\mathrm{T}}(S)$ the *oriented transcendental lattice* of S.

We denote by $[2a, b, 2c]$ the symmetric matrix $\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$, and put

$$\mathcal{M} := \{\ [2a, b, 2c]\ \mid\ a, b, c \in \mathbb{Z}, a > 0, c > 0, 4ac - b^2 > 0\ \},$$

on which $GL(2, \mathbb{Z})$ acts by $M \mapsto g^T M g$, where $M \in \mathcal{M}$ and $g \in GL(2, \mathbb{Z})$. The set of isomorphism classes of even positive-definite lattices of rank 2 is equal to

$$\mathcal{L} := \mathcal{M}/GL(2, \mathbb{Z}),$$

while the set of isomorphism classes of even positive-definite *oriented* lattices of rank 2 is equal to

$$\widetilde{\mathcal{L}} := \mathcal{M}/SL(2, \mathbb{Z}).$$

For a matrix $[2a, b, 2c] \in \mathcal{M}$, we denote by $\widetilde{L}[2a, b, 2c] \in \widetilde{\mathcal{L}}$ and $L[2a, b, 2c] \in \mathcal{L}$ the isomorphism classes represented by $[2a, b, 2c]$.

In [22], Shioda and Inose proved the following:

Theorem 1.2. *The map $S \mapsto \widetilde{\mathrm{T}}(S)$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces S to the set $\widetilde{\mathcal{L}}$.*

The injectivity follows from the Torelli theorem by Piatetski-Shapiro and Shafarevich [12]. In the proof of the surjectivity, Shioda and Inose gave an explicit construction of the complex singular $K3$ surface with a given oriented transcendental lattice, and they have proved the following:

Theorem 1.3. *Every complex singular $K3$ surface is defined over a number field.*

Let X be a singular $K3$ surface defined over a number field F. We denote by $\mathrm{Emb}(F)$ the set of embeddings of F into $\mathbb{C}$, and for $\sigma \in \mathrm{Emb}(F)$, we denote by X^σ the *complex* singular $K3$ surface $X \otimes_{F,\sigma} \mathbb{C}$. We define a map

$$\tau_X\ :\ \mathrm{Emb}(F) \to \widetilde{\mathcal{L}}$$

by $\tau_X(\sigma) := \widetilde{\mathrm{T}}(X^\sigma)$. Then we have the following theorem by Schütt [14], which is a generalization of a result that had been obtained in [20].

Theorem 1.4. *Let $\mathcal{G}_X \subset \mathcal{L}$ be the genus of all $L \in \mathcal{L}$ such that (D_L, q_L) is isomorphic to $(D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$, and let $\widetilde{\mathcal{G}}_X \subset \widetilde{\mathcal{L}}$ be the pull-back of $\mathcal{G}_X$ by the natural projection $\widetilde{\mathcal{L}} \to \mathcal{L}$. Then the image of τ_X coincides with $\widetilde{\mathcal{G}}_X$.*

Therefore we obtain a surjective map

$$\tau_X \;:\; \mathrm{Emb}(F) \twoheadrightarrow \widetilde{\mathcal{G}}_X.$$

Remark that, by the classical theory of Gauss [6], we can easily calculate the oriented genus $\widetilde{\mathcal{G}}_X \subset \widetilde{\mathcal{L}}$ from the finite quadratic form $(D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$.

Let Y be a geometrically reduced and irreducible projective surface defined over a number field K, and let $X \to Y \otimes_K F$ be a desingularization of $Y \otimes_K F$ defined over a finite extension F of K. Suppose that X is a singular $K3$ surface. Then we can define a map

$$\tau_Y \;:\; \mathrm{Emb}(K) \twoheadrightarrow \widetilde{\mathcal{G}}_X$$

by the following:

Proposition 1.5. *The map $\tau_X : \mathrm{Emb}(F) \to \widetilde{\mathcal{G}}_X$ factors as*

$$\mathrm{Emb}(F) \;\overset{\rho}{\twoheadrightarrow}\; \mathrm{Emb}(K) \;\xrightarrow{\tau_Y}\; \widetilde{\mathcal{G}}_X,$$

where $\rho : \mathrm{Emb}(F) \twoheadrightarrow \mathrm{Emb}(K)$ is the natural restriction map $\rho(\sigma) := \sigma|K$.

The purpose of this paper is to present a method to calculate the map τ_Y from a defining equation of Y.

More generally, we consider the following problem. Let S be a reduced irreducible complex projective surface. For a desingularization $S^{\sim} \to S$, we put

$$\mathrm{H}^2(S^{\sim}) := \mathrm{H}^2(S^{\sim}, \mathbb{Z})/(\text{the torsion part}),$$

which is regarded as a lattice by the cup-product, and let $\mathrm{NS}(S^{\sim}) \subset \mathrm{H}^2(S^{\sim})$ be the sublattice of the cohomology classes of divisors on $S^{\sim}$. We denote by

$$\mathrm{T}(S^{\sim}) \subset \mathrm{H}^2(S^{\sim})$$

the orthogonal complement of $\mathrm{NS}(S^{\sim})$ in $\mathrm{H}^2(S^{\sim})$. Then we can easily see that the isomorphism class of the lattice $\mathrm{T}(S^{\sim})$ does not depend on the choice of the desingularization $S^{\sim} \to S$, and hence we can define the *transcendental lattice* $\mathrm{T}(S)$ *of S* to be $\mathrm{T}(S^{\sim})$. (See Lemma 3.1 of [23] or Proposition 2.1 of this paper.) We will give a *method of Zariski-van Kampen type* for the calculation of $\mathrm{T}(S)$.

We apply our method to maximizing sextics. Following Persson [10, 11], we say that a reduced projective plane curve $C \subset \mathbb{P}^2$ of degree 6 defined over a field k of characteristic 0 is a *maximizing sextic* if $C \otimes \bar{k}$ has only simple singularities and its total Milnor number attains the possible maximum 19, where $\bar{k}$ is the algebraic closure of k. The *type* of a maximizing sextic C is the ADE-type of the singular points of $C \otimes \bar{k}$.

Let $C \subset \mathbb{P}^2$ be a maximizing sextic defined over a number field K. The double covering $Y_C \to \mathbb{P}^2$ branching exactly along C is defined over K. Let $X_C \to Y_C \otimes_K F$ be the minimal resolution defined over a finite extension F of K. Then X_C is a singular $K3$ surface defined over F. We denote by $\widetilde{\mathcal{G}}_{[C]}$ the oriented genus $\widetilde{\mathcal{G}}_{X_C}$. By Proposition 1.5, we have a surjective map

$$\tau_{[C]} := \tau_{Y_C} \;:\; \mathrm{Emb}(K) \twoheadrightarrow \widetilde{\mathcal{G}}_{[C]}.$$

As an illustration of our Zariski-van Kampen method, we calculate $\tau_{[C_0]}$ for a reducible maximizing sextic C_0 of type $A_{10} + A_9$ defined over $K = \mathbb{Q}(\sqrt{5})$ by the homogeneous equation

$$z \cdot (\ G(x,y,z)\ +\ \alpha \cdot H(x,y,z)\ )\ =\ 0, \tag{1.1}$$

where $\alpha^2 = 5$ and

$$\begin{aligned} G(x,y,z) &= -9\,x^4z - 14\,x^3yz + 58\,x^3z^2 - 48\,x^2y^2z - 64\,x^2yz^2 + \\ &\quad +10\,x^2z^3 + 108\,xy^3z - 20\,xy^2z^2 - 44\,y^5 + 10\,y^4z, \\ H(x,y,z) &= 5\,x^4z + 10\,x^3yz - 30\,x^3z^2 + 30\,x^2y^2z + 20\,x^2yz^2 - \\ &\quad -40\,xy^3z + 20\,y^5. \end{aligned}$$

This equation was discovered by means of Roczen's result [13] (see §5).

We can calculate $\mathrm{NS}(X_{C_0})$ by the method of Yang [24], and obtain

$$\widetilde{\mathcal{G}}_{[C_0]}\ =\ \{\ \widetilde{L}[8,3,8],\ \widetilde{L}[2,1,28]\ \}.$$

Let $\sigma_\pm$ be the embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ given by $\sigma_\pm(\alpha) = \pm\sqrt{5}$. We have two surjective maps from $\mathrm{Emb}(\mathbb{Q}(\alpha)) = \{\sigma_+, \sigma_-\}$ to $\widetilde{\mathcal{G}}_{[C_0]}$. Remark that, since the two complex maximizing sextics $C_0^{\sigma+}$ and $C_0^{\sigma-}$ cannot be distinguished by any algebraic methods, *we have to employ some transcendental method to determine which surjective map is the map* $\tau_{[C_0]}$. By the method described in §3 of this paper, we obtain the following:

Proposition 1.6.

$$\tau_{[C_0]}(\sigma_+) = \widetilde{L}[2,1,28], \quad \tau_{[C_0]}(\sigma_-) = \widetilde{L}[8,3,8].$$

We have shown in [19] and [21] that, for a complex maximizing sextic C, the transcendental lattice $\mathrm{T}_{[C]} := \mathrm{T}(Y_C)$ of the double covering $Y_C \to \mathbb{P}^2$ branching exactly along C is a topological invariant of $(\mathbb{P}^2, C)$. Thus the curves $C_0^{\sigma+}$ and $C_0^{\sigma-}$ form an *arithmetic Zariski pair.* (See [19] for the definition.) The proof of Proposition 1.6 illustrates very explicitly how the action of the Galois group of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ affects the topology of the embedding of C_0 into $\mathbb{P}^2$.

The first example of arithmetic Zariski pairs was discovered by Artal, Carmona and Cogolludo [3] in degree 12 by means of the *braid monodromy.* It will be an interesting problem to investigate the relation between the braid monodromy of a maximizing sextic $C \subset \mathbb{P}^2$ and our lattice invariant $\mathrm{T}_{[C]}$.

In the study of Zariski pairs of complex plane curves, the topological fundamental groups of the complements (or its variations like the Alexander polynomials) have been used to distinguish the topological types. (See, for example, [1], [9] or [16] for the oldest example of Zariski pairs of 6-cuspidal sextics [25, 26].) We can calculate the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C_0^{\sigma+})$ and $\pi_1(\mathbb{P}^2 \setminus C_0^{\sigma-})$ of our example in terms of generators and relations by the classical Zariski-van Kampen theorem. (See, for example, [15, 18].) It will be an interesting problem to determine whether these two groups are isomorphic or not. Note that, by the theory of algebraic fundamental groups, their profinite completions are isomorphic.

The plan of this paper is as follows. In §2, we prove Proposition 1.5. In §3, we present the Zariski-van Kampen method for the calculation of the transcendental lattice in full generality. In §4, we apply this method to the complex maximizing sextics $C_0^{\sigma\pm}$ and prove Proposition 1.6. In §5, we explain how we have obtained the equation (1.1) of C_0.

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2. The map τ_Y

We recall the proof of Theorem 1.4, and prove Proposition 1.5. The main tool is Theorem 1.2 due to Shioda and Inose [22].

Proof of Theorem 1.4. It is easy to see that the image of τ_X is contained in $\widetilde{\mathcal{G}}_X$. (See Theorem 2 in [20] or Proposition 3.5 in [21].) In [20] and [14], using Shioda-Inose construction, we constructed a singular $K3$ surface X^0 defined over a number field F^0 such that $\mathrm{NS}(X) \cong \mathrm{NS}(X^0)$ (and hence $\widetilde{\mathcal{G}}_X = \widetilde{\mathcal{G}}_{X^0}$) holds and that the image of τ_{X^0} coincides with $\widetilde{\mathcal{G}}_X$. (See also §4 of [21].) We choose an arbitrary $\sigma \in \mathrm{Emb}(F)$. Then there exists $\sigma^0 \in \mathrm{Emb}(F^0)$ such that $\tau_{X^0}(\sigma^0) = \tau_X(\sigma)$. Since $(X^0)^{\sigma^0}$ and X^σ are isomorphic over $\mathbb{C}$ by Theorem 1.2, there exists a number field $M \subset \mathbb{C}$ containing both $\sigma^0(F^0)$ and $\sigma(F)$ such that we have an isomorphism

$$X^0 \otimes_{F^0,\sigma^0} M \;\cong\; X \otimes_{F,\sigma} M$$

over M. Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Emb}(F^0) \arrow[dr, "\tau_{X^0}"] & \\ \mathrm{Emb}(M) \arrow[ur, "\rho_{M,\sigma^0(F^0)}"] \arrow[rr, "\tau_{X^0\otimes M} = \tau_{X\otimes M}"] \arrow[dr, "\rho_{M,\sigma(F)}"'] & & \widetilde{\mathcal{G}}_X = \widetilde{\mathcal{G}}_{X^0} \\ & \mathrm{Emb}(F) \arrow[ur, "\tau_X"'] & \end{tikzcd} ,$$

where $\rho_{M,\sigma^0(F^0)}$ and $\rho_{M,\sigma(F)}$ are the natural surjective restriction maps. The surjectivity of τ_X then follows from the surjectivity of τ_{X^0}. □

Proof of Proposition 1.5. Let σ_1 and σ_2 be elements of $\mathrm{Emb}(F)$ such that $\sigma_1|K = \sigma_2|K$. We put

$$\sigma_K := \sigma_1|K = \sigma_2|K \in \mathrm{Emb}(K).$$

Then the complex surfaces X^{σ_1} and X^{σ_2} are desingularizations of the complex surface Y^{σ_K}. Hence Proposition 1.5 follows from Proposition 2.1 below. □

Proposition 2.1. *Let $S_1^\sim$ and $S_2^\sim$ be two desingularizations of a reduced irreducible complex projective surface S. Then $\mathrm{T}(S_1^\sim) \cong \mathrm{T}(S_2^\sim)$. If $S_1^\sim$ and $S_2^\sim$ are singular $K3$ surfaces, then $\widetilde{\mathrm{T}}(S_1^\sim) \cong \widetilde{\mathrm{T}}(S_2^\sim)$.*

Proof. Using a desingularization of $S_1^\sim \times_S S_2^\sim$, we obtain a complex smooth projective surface Σ with birational morphisms $\Sigma \to S_1^\sim$ and $\Sigma \to S_2^\sim$. Since the transcendental lattice of a complex smooth projective surface is invariant under a blowing-up, and any birational morphism between smooth projective surfaces factors into a composite of blowing-ups, we have $\mathrm{T}(\Sigma) \cong \mathrm{T}(S_1^\sim)$ and $\mathrm{T}(\Sigma) \cong \mathrm{T}(S_2^\sim)$. □

3. Zariski-van Kampen method for transcendental lattices

For a $\mathbb{Z}$-module A, we denote by

$$A^{\mathrm{tf}} := A/(\text{the torsion part})$$

the maximal torsion-free quotient of A. If we have a bilinear form $A \times A \to \mathbb{Z}$, then it induces a canonical bilinear form $A^{\mathrm{tf}} \times A^{\mathrm{tf}} \to \mathbb{Z}$.

Let S be a reduced irreducible complex projective surface. Our goal is to calculate $\mathrm{T}(S)$. Let $\delta : S^{\sim} \to S$ be a desingularization. We choose a reduced curve D on S with the following properties:

(D1) the classes of irreducible components of the total transform $D^{\sim} \subset S^{\sim}$ of D span $\mathrm{NS}(S^{\sim}) \otimes \mathbb{Q}$ over $\mathbb{Q}$, and

(D2) the desingularization δ induces an isomorphism $S^{\sim} \setminus D^{\sim} \cong S \setminus D$.

We put

$$S^0 := S \setminus D,$$

and consider the free $\mathbb{Z}$-module

$$\mathrm{H}_2(S^0) := \mathrm{H}_2(S^0, \mathbb{Z})^{\mathrm{tf}}$$

with the intersection paring

$$\iota \;:\; \mathrm{H}_2(S^0) \times \mathrm{H}_2(S^0) \to \mathbb{Z}.$$

We put

$$I(S^0) := \{x \in \mathrm{H}_2(S^0) \mid \iota(x, y) = 0 \text{ for any } y \in \mathrm{H}_2(S^0)\}, \tag{3.1}$$

and set

$$V_2(S^0) := \mathrm{H}_2(S^0)/I(S^0).$$

Then $V_2(S^0)$ is a free $\mathbb{Z}$-module, and the intersection paring ι induces a non-degenerate symmetric bilinear form

$$\bar{\iota} \;:\; V_2(S^0) \times V_2(S^0) \to \mathbb{Z}.$$

Proposition 3.1. *The transcendental lattice* $\mathrm{T}(S) = \mathrm{T}(S^{\sim})$ *is isomorphic to the lattice* $(V_2(S^0), \bar{\iota})$.

Proof. By the condition (D2), we can regard S^0 as a Zariski open subset of $S^{\sim}$. Consider the homomorphism

$$j_* \;:\; \mathrm{H}_2(S^0) \to \mathrm{H}_2(S^{\sim}) := \mathrm{H}_2(S^{\sim}, \mathbb{Z})^{\mathrm{tf}}$$

induced by the inclusion $j : S^0 \hookrightarrow S^{\sim}$. Under the isomorphism of lattices

$$\mathrm{H}_2(S^{\sim}) \cong \mathrm{H}^2(S^{\sim}) := \mathrm{H}^2(S^{\sim}, \mathbb{Z})^{\mathrm{tf}}$$

induced by the Poincaré duality, the image of j_* is contained in $\mathrm{T}(S^{\sim}) \subset \mathrm{H}^2(S^{\sim})$ by the condition (D1) on D. Using the argument in the proof of Theorem 2.6 of [19] or Theorem 2.1 of [21], we see that the homomorphism

$$j_* \;:\; \mathrm{H}_2(S^0) \to \mathrm{T}(S^{\sim})$$

is surjective. Note that we have

$$\iota(x, y) = (j_*(x), j_*(y))_T$$

for any $x, y \in \mathrm{H}_2(S^0)$, where $(\ ,\ )_T$ is the cup-product on $\mathrm{T}(S^{\sim})$. Since $(\ ,\ )_T$ is non-degenerate, we conclude that $\operatorname{Ker} j_* = I(S^0)$. □

Proposition 3.1 shows that, in order to obtain $\mathrm{T}(S)$, it is enough to calculate $\mathrm{H}_2(S^0)$ and ι. Enlarging D if necessary, we have a surjective morphism

$$\phi \;:\; S^0 \to U$$

onto a Zariski open subset U of an affine line $\mathbb{A}^1$ such that its general fiber is a connected Riemann surface. By the condition (D1) on D, the general fiber of ϕ is non-compact. Let $\overline{S^0}$ be a smooth irreducible projective surface containing S^0 as a Zariski open subset such that ϕ extends to a morphism

$$\bar{\phi} \;:\; \overline{S^0} \to \mathbb{P}^1.$$

Let $V_1, \dots, V_M$ and $H_1, \dots, H_N$ be the irreducible components of the boundary $\overline{S^0} \setminus S^0$, where $V_1, \dots, V_N$ are the vertical components (that is, $\bar{\phi}(V_i)$ is a point), and $H_1, \dots, H_M$ are the horizontal components (that is, $\bar{\phi}(H_j) = \mathbb{P}^1$). Since the general fiber of ϕ is non-compact, we have at least one horizontal component. We put

$$\mathbb{A}^1 \setminus U = \{p_1, \dots, p_m\}.$$

Adding to D some fibers of ϕ and making U smaller if necessary, we can assume the following:

(1) the surjective morphism $\phi : S^0 \to U$ has only ordinary critical points,
(2) $\bar{\phi}| \cup_j H_j : \cup_j H_j \to \mathbb{P}^1$ is étale over U, and
(3) $V_1 \cup \dots \cup V_N = \bar{\phi}^{-1}(\infty) \cup \bar{\phi}^{-1}(p_1) \cup \dots \cup \bar{\phi}^{-1}(p_m)$, where $\{\infty\} = \mathbb{P}^1 \setminus \mathbb{A}^1$.

Note that $\bar{\phi}$ has no critical points on $(\cup H_j) \cap \bar{\phi}^{-1}(U)$ by the condition (2). We denote by $c_1, \dots, c_n \in U$ the critical values of ϕ, and put

$$U^\sharp := U \setminus \{c_1, \dots, c_n\}.$$

By the assumptions, ϕ is locally trivial (in the category of topological spaces and continuous maps) over $U^\sharp$ with the fiber being a connected Riemann surface of genus g with r punctured points, where $r > 0$ is the degree of $\bar{\phi}| \cup_j H_j : \cup_j H_j \to \mathbb{P}^1$. We then choose a base point $b \in U^\sharp$, and put

$$F_b := \phi^{-1}(b).$$

For each $p_i \in \mathbb{A}^1 \setminus U$, we choose a loop

$$\lambda_i \;:\; (I, \partial I) \to (U^\sharp, b)$$

that is sufficiently smooth and injective in the sense that $\lambda_i(t) = \lambda_i(t')$ holds only when $t = t'$ or $\{t, t'\} = \partial I$, and that defines the same element in $\pi_1(U^\sharp, b)$ as a simple loop (a lasso) around p_i in $U^\sharp$. For each critical value $c_j \in U \setminus U^\sharp$, we choose a sufficiently smooth and injective path

$$\gamma_j \;:\; I \to U$$

such that $\gamma_j(0) = b$, $\gamma_j(1) = c_j$ and $\gamma_j(t) \in U^\sharp$ for $t < 1$. We choose these loops λ_i and paths γ_j in such a way that any two of them intersect only at b. Then, by a suitable self-homeomorphism of $\mathbb{A}^1$, the objects b, p_i, c_j, λ_i and γ_j on $\mathbb{A}^1$ are mapped as in Figure 3.1. In particular, the union B of λ_i and γ_j is a strong deformation retract of U. Note that ϕ is locally trivial over $U \setminus B$.

Let $\mathbb{S}^1$ be an oriented one-dimensional sphere. We fix a system of oriented simple closed curves

$$a_\nu \;:\; \mathbb{S}^1 \hookrightarrow F_b \quad (\nu = 1, \dots, 2g + r - 1)$$

on F_b in such a way that their union $\bigcup a_\nu(\mathbb{S}^1)$ is a strong deformation retract of F_b. In particular, we have

$$\mathrm{H}_1(F_b, \mathbb{Z}) = \bigoplus \mathbb{Z}[a_\nu],$$

where $[a_\nu]$ is the homology class of a_ν. For each $p_i \in \mathbb{A}^1 \setminus U$ and a_ν, let

$$\Lambda_{i,\nu} \;:\; \mathbb{S}^1 \times I \hookrightarrow S^0$$

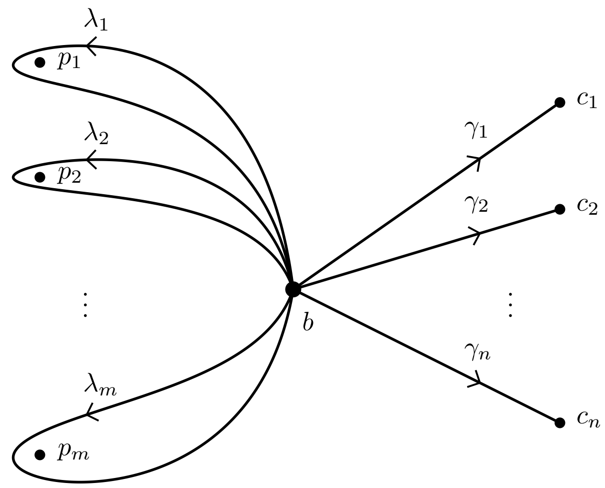

FIGURE 3.1

be an embedding such that the diagram

$$\begin{array}{ccc} \mathbb{S}^1 \times I & \xrightarrow{\Lambda_{i,\nu}} & S^0 \\ \mathrm{pr} \downarrow & & \downarrow \phi \\ I & \xrightarrow[\lambda_i]{} & U \end{array}$$

commutes and that

$$\Lambda_{i,\nu}|\mathbb{S}^1 \times \{0\} \ : \ \mathbb{S}^1 \hookrightarrow F_b$$

is equal to a_ν. We put

$$M_i(a_\nu) := \Lambda_{i,\nu}|\mathbb{S}^1 \times \{1\} \ : \ \mathbb{S}^1 \hookrightarrow F_b,$$

where M_i stands for the *monodromy* along λ_i, and denote the homology class of $M_i(a_\nu)$ by

$$M_i([a_\nu]) \in \mathrm{H}_1(F_b, \mathbb{Z}).$$

Let Θ be the topological space obtained from $\mathbb{S}^1 \times I$ by contracting $\mathbb{S}^1 \times \{1\}$ to a point $v \in \Theta$; that is, Θ is a cone over $\mathbb{S}^1$ with the vertex v. Let $\mathrm{pr} : \Theta \to I$ be the natural projection. Let $c_j \in U \setminus U^\sharp$ be a critical value of ϕ, and let $\tilde{c}_j^1, \dots, \tilde{c}_j^m$ be the critical points of ϕ over c_j. For each critical point $\tilde{c}_j^k \in \phi^{-1}(c_j)$, we choose a *thimble*

$$\Gamma_j^k \ : \ \Theta \hookrightarrow S^0$$

along the path γ_j corresponding to the ordinary node $\tilde{c}_j^k$ of $\phi^{-1}(c_j)$. Namely, the thimble Γ_j^k is an embedding such that

$$\begin{array}{ccc} \Theta & \xrightarrow{\Gamma_j^k} & S^0 \\ \mathrm{pr} \downarrow & & \downarrow \phi \\ I & \xrightarrow[\gamma_j]{} & U \end{array}$$

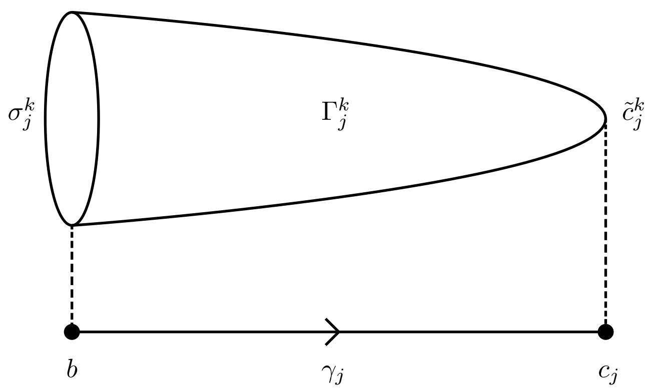

FIGURE 3.2. Thimble

commutes, and that $\Gamma_j^k(v) = \tilde{c}_j^k$. (See [7] for thimbles and vanishing cycles.) Then the simple closed curve

$$\sigma_j^k := \Gamma_j^k|\mathrm{pr}^{-1}(0) = -\partial\Gamma_j^k \ : \ \mathbb{S}^1 \hookrightarrow F_b$$

on F_b represents the *vanishing cycle* for the critical point $\tilde{c}_j^k$ along γ_j. We denote its homology class by

$$[\sigma_j^k] \in \mathrm{H}_1(F_b, \mathbb{Z}).$$

We can assume that Γ_j^k and $\Gamma_j^{k'}$ are disjoint if $k \neq k'$.

Remark 3.2. There are two choices of the orientation of the thimble Γ_j^k (and hence of the vanishing cycle $\sigma_j^k = -\partial\Gamma_j^k$).

Then the union

$$F_b \ \cup \ \bigcup \Lambda_{i,\nu}(\mathbb{S}^1 \times I) \ \cup \ \bigcup \Gamma_j^k(\Theta)$$

is homotopically equivalent to S^0. Since the 1-dimensional CW-complex $\bigcup a_\nu(\mathbb{S}^1)$ is a strong deformation retract of F_b, the homology group $\mathrm{H}_2(S^0, \mathbb{Z})$ is equal to the kernel of the homomorphism

$$\partial \ : \ \bigoplus \mathbb{Z}[\Lambda_{i,\nu}] \oplus \bigoplus \mathbb{Z}[\Gamma_j^k] \ \to \ \mathrm{H}_1(F_b, \mathbb{Z}) = \bigoplus \mathbb{Z}[a_\nu]$$

given by

$$\partial[\Lambda_{i,\nu}] = M_i([a_\nu]) - [a_\nu] \quad \text{and} \quad \partial[\Gamma_j^k] = -[\sigma_j^k].$$

The intersection pairing on $\mathrm{H}_2(S^0, \mathbb{Z})$ is calculated by perturbing the system (λ_i, γ_j) of loops and paths with the base point b to a system (λ_i', γ_j') with the base point $b' \neq b$. We make the perturbation in such a way that the following hold.

- There exists a small open disk $\Delta \subset U^\sharp$ containing both b and b' such that

$$\lambda_i^{-1}(\Delta) = [0, s_i) \cup (1 - r_i, 1], \quad \lambda_i'^{-1}(\Delta) = [0, s_i') \cup (1 - r_i', 1],$$
$$\gamma_j^{-1}(\Delta) = [0, u_j), \qquad \gamma_j'^{-1}(\Delta) = [0, u_j'),$$

  where $s_i, r_i, s_i', r_i', u_j, u_j'$ are small positive real numbers.
- If λ_i intersects $\lambda_{i'}'$ or $\gamma_{j'}'$, then their intersection points are contained in $\Delta \setminus \{b, b'\}$, and the intersections are transverse.

- If γ_j intersects $\lambda'_{i'}$ or $\gamma'_{j'}$ with $j' \neq j$, then their intersection points are contained in $\Delta \setminus \{b, b'\}$, and the intersections are transverse.
- Any intersection point of γ_j and γ'_j is either the common end-point c_j, or a transversal intersection point contained in $\Delta \setminus \{b, b'\}$.

We then perturb the topological 2-chains $\Lambda_{i,\nu}$ over λ_i and Γ^k_j over γ_j to topological 2-chains $\Lambda'_{i,\nu}$over λ'_i and Γ'^k_j over γ'_j, respectively. Let T be one of $\Lambda_{i,\nu}$ or Γ^k_j, and let t be the loop or the path over which T locates. Let T' be one of $\Lambda'_{i,\nu}$ or Γ'^k_j over the loop or the path t'. We can make the perturbation in such a way that T and T' intersect transversely at each intersection points. Suppose that

$$t(I) \cap t'(I) \cap \Delta = \{q_1, \ldots, q_l\}.$$

Then $T \cap T'$ are contained in the union $\bigcup_{\mu=1}^{l} \phi^{-1}(q_\mu)$ of fibers except for the case where $T = \Gamma^k_j$ and $T' = \Gamma'^k_j$ for some j and k. If $T = \Gamma^k_j$ and $T' = \Gamma'^k_j$, then T and T' also intersect at the critical point $\tilde{c}^k_j$ transversely with the local intersection number -1. (See Lemma 4.1 of [17].) For each q_μ, let θ_μ and θ'_μ be the 1-cycles on the open Riemann surface $\phi^{-1}(q_\mu)$ given by

$$\begin{aligned} \theta_\mu &:= T\,|\mathbb{S}^1 \times \{w_\mu\} \;:\; \mathbb{S}^1 \to \phi^{-1}(q_\mu), \quad \text{where } t(w_\mu) = q_\mu, \\ \theta'_\mu &:= T'|\mathbb{S}^1 \times \{w'_\mu\} \;:\; \mathbb{S}^1 \to \phi^{-1}(q_\mu), \quad \text{where } t'(w'_\mu) = q_\mu. \end{aligned}$$

We denote by $(t, t')_\mu$ the local intersection number of the 1-chains t and t' on U at q_μ, which is 1 or -1 by the assumption on the perturbation. We also denote by $(\theta_\mu, \theta'_\mu)_\mu$ the intersection number of θ_μ and θ'_μ on the Riemann surface $\phi^{-1}(q_\mu)$. Then the intersection number (T, T') of T and T' is equal to

$$(T, T') = -\sum_{\mu=1}^{l} (t, t')_\mu (\theta_\mu, \theta'_\mu)_\mu + \delta, \tag{3.2}$$

where

$$\delta := \begin{cases} -1 & \text{if } T = \Gamma^k_j \text{ and } T' = \Gamma'^k_j \text{ for some } j \text{ and } k, \\ 0 & \text{otherwise.} \end{cases}$$

The number $(\theta_\mu, \theta'_\mu)_\mu$ is calculated as follows. Let

$$(\;,\;)_F \;:\; \mathrm{H}_1(F_b, \mathbb{Z}) \times \mathrm{H}_1(F_b, \mathbb{Z}) \to \mathbb{Z}$$

be the intersection pairing on $\mathrm{H}_1(F_b, \mathbb{Z})$, which is anti-symmetric. If $T = \Gamma^k_j$, then the 1-cycle θ_μ on $\phi^{-1}(q_\mu)$ can be deformed to the vanishing cycle $\sigma^k_j = -\partial\Gamma^k_j$ on F_b along the path $t|[0, w_\mu]$ in Δ. We put

$$[\tilde{\theta}_\mu] := [\sigma^k_j] \in \mathrm{H}_1(F_b, \mathbb{Z}).$$

Suppose that $T = \Lambda_{i,\nu}$. Then we have $t^{-1}(\Delta) = \lambda_i^{-1}(\Delta) = [0, s) \cup (1 - r, 1]$, where s and r are small positive real numbers. If the number w_μ such that $t(w_\mu) = q_\mu$ is contained in $[0, s)$, then θ_μ can be deformed to the 1-cycle a_ν on F_b along the path $t|[0, w_\mu]$ in Δ. If $w_\mu \in (1 - r, 1]$, then θ_μ can be deformed to the 1-cycle $M_i(a_\nu)$ on F_b along the path $t|[w_\mu, 1]$ in Δ. We define $[\tilde{\theta}_\mu] \in \mathrm{H}_1(F_b, \mathbb{Z})$ by

$$[\tilde{\theta}_\mu] := \begin{cases} [a_\nu] & \text{if } w_\mu \in [0, s), \\ M_i([a_\nu]) & \text{if } w_\mu \in (1 - r, 1]. \end{cases}$$

We define $[\tilde{\theta}'_\mu] \in \mathrm{H}_1(F_b, \mathbb{Z})$ from T' in the same way. Since ϕ is topologically trivial over Δ, we have

$$(3.3) \qquad (\theta_\mu, \theta'_\mu)_\mu = ([\tilde{\theta}_\mu], [\tilde{\theta}'_\mu])_F.$$

The formulae (3.2) and (3.3) give the intersection number (T, T') of topological 2-chains T and T'. Even though the number (T, T') depends on the choice of the perturbation, it gives the symmetric intersection paring on $\operatorname{Ker}\partial = \mathrm{H}_2(S^0, \mathbb{Z})$. Thus we obtain $\mathrm{H}_2(S^0)$ and ι.

4. Maximizing sextics of type $A_{10} + A_9$

Recall from Introduction that $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is the set of isomorphism classes of even positive-definite lattices (resp. oriented lattices) of rank 2.

Definition 4.1. Let $\varphi : \widetilde{\mathcal{L}} \to \mathcal{L}$ be the map of forgetting orientation. We say that $T \in \mathcal{L}$ is *real* if $\varphi^{-1}(T)$ consists of a single element, and that $\widetilde{T} \in \widetilde{\mathcal{L}}$ is *real* if $\varphi(\widetilde{T}) \in \mathcal{L}$ is real.

Let S be a complex singular $K3$ surface, and let $\overline{S}$ denote $S \otimes_{\mathbb{C},^-} \mathbb{C}$, where $^- : \mathbb{C} \to \mathbb{C}$ is the conjugate over $\mathbb{R}$. Then $\widetilde{\mathrm{T}}(\overline{S})$ is the reverse of $\widetilde{\mathrm{T}}(S)$; that is, $\varphi^{-1}(\varphi(\widetilde{\mathrm{T}}(S))) = \{\widetilde{\mathrm{T}}(S), \widetilde{\mathrm{T}}(\overline{S})\}$. Therefore $\widetilde{\mathrm{T}}(S)$ is real if and only if S and $\overline{S}$ are isomorphic. In particular, if S is defined over $\mathbb{R}$, then $\widetilde{\mathrm{T}}(S)$ is real.

Remark 4.2. It is known that every element $\widetilde{T}$ of $\widetilde{\mathcal{L}}$ is represented by a unique matrix $[2a, b, 2c] \in \mathcal{M}$ with

$$-a < b \leq a \leq c, \quad \text{with } b \geq 0 \text{ if } a = c,$$

and $\widetilde{T}$ is *not* real if and only if $0 < |b| < a < c$ holds. See [4, Chapter 15].

By the method of Yang [24] and Degtyarev [5], we see the following facts. (See also [19].) There are four connected components in the moduli space of complex maximizing sextics of type $A_{10} + A_9$. The members of two of them are irreducible sextics, and their oriented transcendental lattices are

$$\widetilde{L}[10, 0, 22] \quad \text{and} \quad \widetilde{L}[2, 0, 110] \qquad \text{(both are real)}.$$

The members of the other two are reducible. Each of them is a union of a line and an irreducible quintic, and their oriented transcendental lattices are

$$\widetilde{L}[8, 3, 8] \quad \text{and} \quad \widetilde{L}[2, 1, 28] \qquad \text{(both are real)}.$$

We will consider these reducible sextics C_0, whose defining equation is given by (1.1).

For simplicity, we write $C^\pm$ for $C_0^{\sigma\pm}$, $Y^\pm$ for $Y_{C_0^{\sigma\pm}}$ and $X^\pm$ for $X_{C_0^{\sigma\pm}}$. Let $D^\pm \subset Y^\pm$ be the pull-back of the union of the lines

$$x = 0 \quad \text{and} \quad z = 0$$

on $\mathbb{P}^2$. Since the singular points

$$[0:0:1] \;\; (A_{10}) \quad \text{and} \quad [1:0:0] \;\; (A_9)$$

of $C^\pm$ are on the union of these two lines, the curve $D^\pm$ satisfies the conditions (D1) and (D2) in §3 for $S = Y^\pm$ and $S^\sim = X^\pm$. We denote by $W^\pm$ the complement of $D^\pm$ in $Y^\pm$. (We have denoted $W^\pm$ by S^0 in §3.) Let $\mathbb{A}^2_{(y,z)}$ be the affine part of $\mathbb{P}^2$ given by $x \neq 0$ with the affine coordinates (y, z) obtained from $[x : y : z]$ by putting

$x = 1$, and let $L \subset \mathbb{A}^2_{(y,z)}$ be the affine line defined by $z = 0$. Then $W^\pm$ is the double cover of $\mathbb{A}^2_{(y,z)} \setminus L$ branching exactly along the union of L and the smooth affine quintic curve $Q^\pm \subset \mathbb{A}^2_{(y,z)}$ defined by

$$f^\pm(y, z) := G(1, y, z) \pm \sqrt{5} \cdot H(1, y, z) = 0.$$

Note that $Q^\pm$ intersects L only at the origin, and the intersection multiplicity is 5. Let

$$\pi^\pm \,:\, W^\pm \to \mathbb{A}^2_{(y,z)} \setminus L$$

be the double covering. We consider the projection

$$p \,:\, \mathbb{A}^2_{(y,z)} \to \mathbb{A}^1_z$$

defined by $p(y, z) := z$ onto an affine line with an affine coordinate z, and the composite

$$q^\pm \,:\, W^\pm \to \mathbb{A}^2_{(y,z)} \setminus L \to U := \mathbb{A}^1_z \setminus \{0\}$$

of $\pi^\pm$ and p, which serves as the surjective morphism ϕ in §3. Calculating the discriminant of $f^\pm(y, z)$ with respect to y, we see that there are four critical points of the finite covering

$$p|Q^\pm \,:\, Q^\pm \to \mathbb{A}^1_z$$

of degree 5. Three of them $R^\pm, S^\pm, \overline{S}^\pm$ are simple critical values, where

$$\begin{aligned} R^+ &= 0.42193..., & S^+ &= 0.23780... + 0.24431... \cdot \sqrt{-1}, \quad \text{and} \\ R^- &= 0.12593..., & S^- &= 27.542... + 45.819... \cdot \sqrt{-1}. \end{aligned}$$

The value $\overline{S}^\pm$ is the complex conjugate of $S^\pm$. The critical point over $0 \in \mathbb{A}^1_z$ is of multiplicity 5. The critical values of $q^\pm : W^\pm \to U$ are therefore $R^\pm, S^\pm, \overline{S}^\pm$, and the fiber of $q^\pm$ over each of them has only one ordinary node. We choose a sufficiently small positive real number b as a base point on U, and define the loop λ and the paths $\gamma^\pm_R$, $\gamma^\pm_S$, $\gamma^\pm_{\bar{S}}$ on U as in Figure 4.1. For $z \in U$, we put

$$\mathcal{Q}^\pm(z) := (p|Q^\pm)^{-1}(z) = p^{-1}(z) \cap Q^\pm,$$

and investigate the movement of the points $\mathcal{Q}^\pm(z)$ when z moves on U along the loop λ and the paths $\gamma^\pm_R$, $\gamma^\pm_S$, $\gamma^\pm_{\bar{S}}$. We put

$$\mathbb{A}^1_y := p^{-1}(b), \quad F^\pm := q^{\pm -1}(b) = \pi^{\pm -1}(\mathbb{A}^1_y) \subset W^\pm.$$

Note that the morphism

$$\pi^\pm|F^\pm \,:\, F^\pm \to \mathbb{A}^1_y$$

is the double covering branching exactly at the five points $\mathcal{Q}^\pm(b) \subset \mathbb{A}^1_y$. These branching points $\mathcal{Q}^\pm(b)$ are depicted as big dots in Figure 4.2. Hence $F^\pm$ is a Riemann surface of genus 2 minus one point. We choose a system of oriented simple closed curves

$$a_\nu \,:\, \mathbb{S}^1 \hookrightarrow F^\pm \qquad (\nu = 1, \ldots, 5)$$

in such a way that their images by the double covering $\pi^\pm|F^\pm : F^\pm \to \mathbb{A}^1_y$ are given in Figure 4.2, and that their intersection numbers on $F^\pm$ are equal to

$$([a_\nu], [a_{\nu+1}])_F = -([a_{\nu+1}], [a_\nu])_F = 1$$

for $\nu = 1, \ldots, 5$, where $a_6 := a_1$. (Note that $([a_\nu], [a_{\nu'}])_F = 0$ except for the case

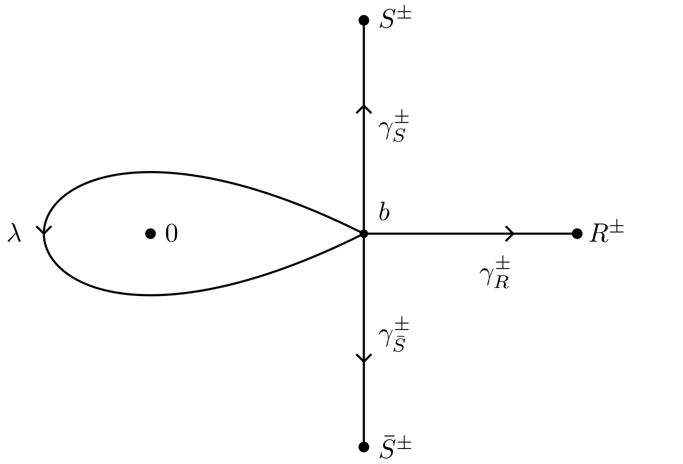

FIGURE 4.1. The loop λ and the paths $\gamma^{\pm}_{R}$, $\gamma^{\pm}_{S}$, $\gamma^{\pm}_{\bar{S}}$

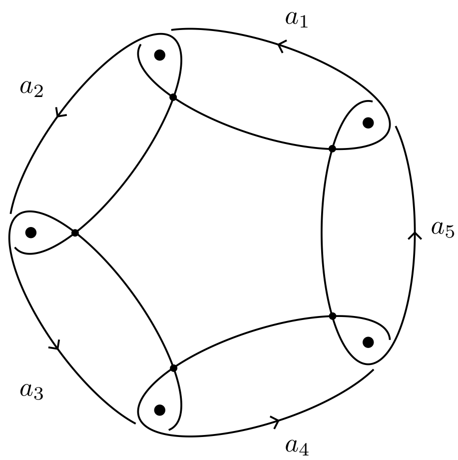

FIGURE 4.2. The system of simple closed curves on F_b

where $|\nu - \nu'| = 1$ or $\{\nu, \nu'\} = \{1, 5\}$.) Then $a_1 \cup \cdots \cup a_4$ is a strong deformation retract of $F^{\pm}$, and $[a_1], \ldots, [a_4]$ form a basis of $\mathrm{H}_1(F^{\pm}, \mathbb{Z})$. Moreover we have

$$[a_5] = -[a_1] - [a_2] - [a_3] - [a_4].$$

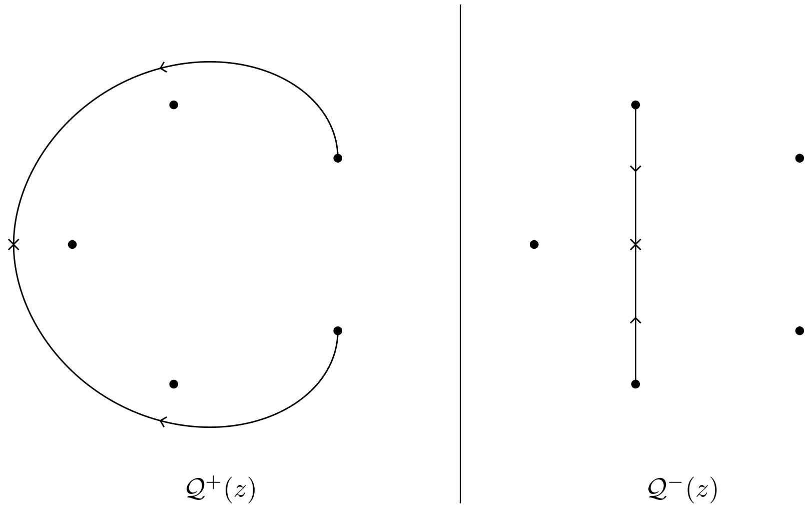

FIGURE 4.3. The movement of the branching points along $\gamma_R^\pm$

Since $Q^\pm$ is smooth at the origin and intersects L with multiplicity 5 at the origin, the movement of the branching points $\mathcal{Q}^\pm(z)$ along the loop λ is homotopically equivalent to the rotation around the origin of the angle $2\pi/5$. Hence the monodromy on the simple closed curves is given by $a_\nu \mapsto a_{\nu+1}$. Let

$$\Lambda_\nu \ : \ \mathbb{S}^1 \times I \to W^\pm$$

be the topological 2-chain over λ that connects a_ν and $a_{\nu+1}$. We have

$$\partial[\Lambda_\nu] = [a_{\nu+1}] - [a_\nu].$$

The movement of the branching points $\mathcal{Q}^\pm(z)$ when z moves from b to $R^\pm$ along the path $\gamma_R^\pm$ is homotopically equivalent to the movement depicted in Figure 4.3. Let

$$\Gamma_R^\pm \ : \ \Theta \to W^\pm$$

be the thimble over $\gamma_R^\pm$ corresponding to the critical point of $q^\pm : W^\pm \to U$ in the fiber over $R^\pm$. The vanishing cycle $\sigma_R^+ = -\partial\Gamma_R^+$ is depicted by a thick line in Figure 4.4. We choose the orientation of σ_R^+ as in Figure 4.4. Then we have

$$([\sigma_R^+], [a_1])_F = 1, \quad ([\sigma_R^+], [a_2])_F = ([\sigma_R^+], [a_3])_F = 0, \quad ([\sigma_R^+], [a_4])_F = 1,$$

and hence

$$[\sigma_R^+] = [a_1] - [a_2] + [a_3] - [a_4].$$

In the same way, we see that the homology class of the vanishing cycle $\sigma_R^- = -\partial\Gamma_R^-$ is equal to

$$[\sigma_R^-] = [a_2] + [a_3]$$

under an appropriate choice of orientation. The movement of the points $\mathcal{Q}^\pm(z)$ when z moves from b to $S^\pm$ along the path $\gamma_S^\pm$ is homotopically equivalent to the movement depicted in Figure 4.5. We choose the orientations of the thimbles

$$\Gamma_S^\pm \ : \ \Theta \to W^\pm$$

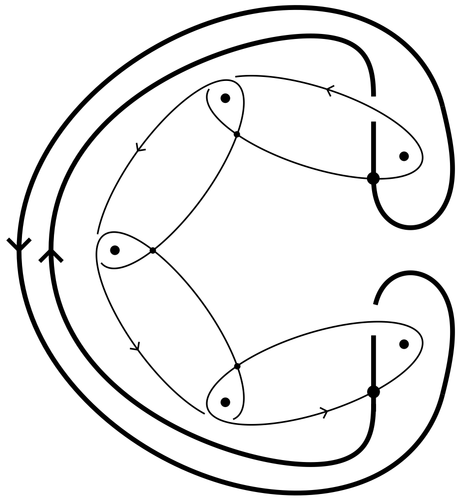

FIGURE 4.4. The vanishing cycle $\sigma_R^+ = -\partial\Gamma_R^+$

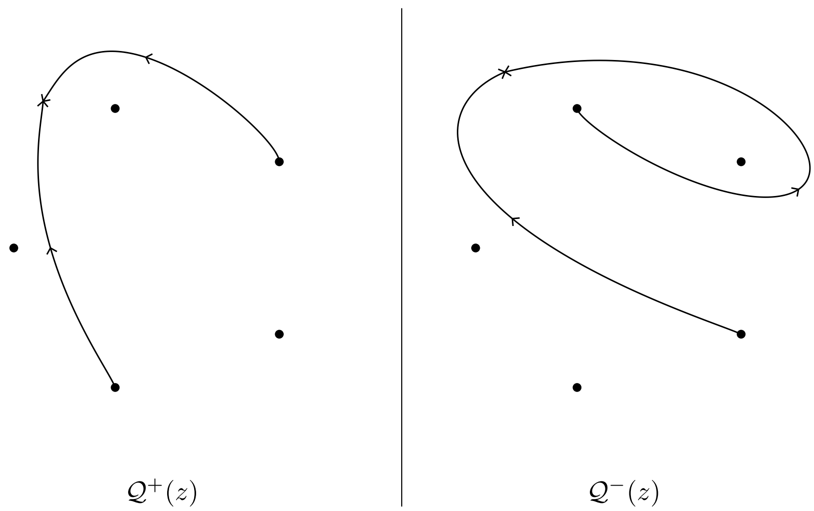

FIGURE 4.5. The movement of the branching points along $\gamma_S^{\pm}$

over $\gamma_S^{\pm}$ in such a way that the homology classes of the vanishing cycles $\sigma_S^{\pm} = -\partial\Gamma_S^{\pm}$ are

$$\begin{aligned} [\sigma_S^+] &= [a_1] - [a_2] - [a_3] \qquad \text{and} \\ [\sigma_S^-] &= 2[a_1] - [a_2] - [a_3] - [a_4]. \end{aligned}$$

The movement of the points $\mathcal{Q}^{\pm}(z)$ for the path $\gamma_{\bar{S}}^{\pm}$ is obtained from Figure 4.5 by the conjugation $^- : \mathbb{C} \to \mathbb{C}$ over $\mathbb{R}$. We choose the orientations of the thimbles $\Gamma_{\bar{S}}^{\pm}$

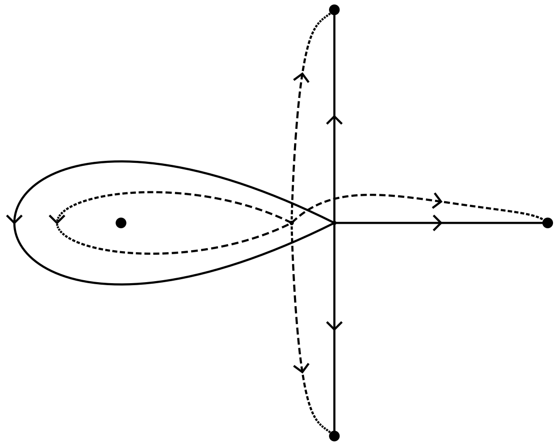

Figure 4.6. The perturbation

in such a way that $[\sigma^{\pm}_{\bar{S}}] = -\partial[\Gamma^{\pm}_{\bar{S}}]$ are equal to

$$\begin{array}{rcll} [\sigma^{+}_{\bar{S}}] & = & -[a_2]-[a_3]+[a_4] & \text{and} \\ [\sigma^{-}_{\bar{S}}] & = & -[a_1]-[a_2]-[a_3]+2[a_4]. & \end{array}$$

Now we can calculate the kernel $\mathrm{H}_2(W^{\pm},\mathbb{Z})$ of the homomorphism

$$\partial \;:\; \bigoplus_{\nu=1}^{4}\mathbb{Z}[\Lambda_\nu]\oplus\mathbb{Z}[\Gamma^{\pm}_{R}]\oplus\mathbb{Z}[\Gamma^{\pm}_{S}]\oplus\mathbb{Z}[\Gamma^{\pm}_{\bar{S}}] \;\to\; \bigoplus_{\nu=1}^{4}\mathbb{Z}[a_\nu].$$

We see that $\mathrm{H}_2(W^{+},\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 3 generated by

$$\begin{array}{rcl} S_1^{+} & := & -[\Lambda_1]-[\Lambda_3]+[\Gamma^{+}_{R}], \\ S_2^{+} & := & -6[\Lambda_1]-2[\Lambda_2]+2[\Lambda_3]+[\Lambda_4]+5[\Gamma^{+}_{S}], \\ S_3^{+} & := & [\Lambda_1]+[\Lambda_2]+[\Lambda_3]-[\Gamma^{+}_{S}]+[\Gamma^{+}_{\bar{S}}], \end{array}$$

while $\mathrm{H}_2(W^{-},\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 3 generated by

$$\begin{array}{rcl} S_1^{-} & := & -4[\Lambda_1]-3[\Lambda_2]-2[\Lambda_3]+[\Gamma^{-}_{R}]+2[\Gamma^{-}_{S}], \\ S_2^{-} & := & -11[\Lambda_1]-7[\Lambda_2]-3[\Lambda_3]+[\Lambda_4]+5[\Gamma^{-}_{S}], \\ S_3^{-} & := & 3[\Lambda_1]+3[\Lambda_2]+3[\Lambda_3]-[\Gamma^{-}_{S}]+[\Gamma^{-}_{\bar{S}}]. \end{array}$$

We deform the loop λ and the paths $\gamma^{\pm}_{R}$, $\gamma^{\pm}_{S}$ and $\gamma^{\pm}_{\bar{S}}$ as in Figure 4.6. The deformed loop λ' and paths $\gamma'^{\pm}_{R}$, $\gamma'^{\pm}_{S}$, $\gamma'^{\pm}_{\bar{S}}$ are depicted by the dotted curves. Then the intersection numbers of the topological 2-chains $T = \Lambda_\nu, \Gamma^{\pm}_{R}, \Gamma^{\pm}_{S}, \Gamma^{\pm}_{\bar{S}}$ and $T' = \Lambda'_\nu, \Gamma'^{\pm}_{R}, \Gamma'^{\pm}_{S}, \Gamma'^{\pm}_{\bar{S}}$ are calculated as in Table 4.1. Remark that the local intersection number $(t,t')_q$ of the underlying paths t of T and t' of T' is -1 for any intersection point q contained in the small open neighborhood Δ of b and b'. Therefore the

$T'\backslash T$	Λ_ν	$\Gamma_R^{\pm}$	$\Gamma_S^{\pm}$	$\Gamma_{\bar{S}}^{\pm}$
Λ'_ν	0	0	0	0
$\Gamma'^{\pm}_R$	$([a_\nu],[\sigma_R^{\pm}])_F$	-1	$([\sigma_S^{\pm}],[\sigma_R^{\pm}])_F$	0
$\Gamma'^{\pm}_S$	$([a_\nu],[\sigma_S^{\pm}])_F$	0	-1	0
$\Gamma'^{\pm}_{\bar{S}}$	$([a_{\nu+1}],[\sigma_{\bar{S}}^{\pm}])_F$	0	0	-1

TABLE 4.1. The intersection numbers of T and T'

intersection matrix of $\mathrm{H}_2(W^+,\mathbb{Z})$ is calculated as follows:

	S_1^+	S_2^+	S_3^+
S_1^+	0	0	0
S_2^+	0	40	-5
S_3^+	0	-5	2

.

Then $I(W^+)$ is generated by S_1^+, where $I(W^+)\subset \mathrm{H}_2(W^+)$ is the submodule defined by (3.1). Thus $\mathrm{T}(X^+)\cong \mathrm{H}_2(W^+)/I(W^+)$ is generated by $S_2^+ + I(W^+)$ and $S_3^+ + I(W^+)$, and $\mathrm{T}(X^+)$ is isomorphic to

$$L[40,-5,2]\cong L[2,1,28].$$

The intersection matrix of $\mathrm{H}_2(W^-,\mathbb{Z})$ is calculated as follows:

	S_1^-	S_2^-	S_3^-
S_1^-	22	55	-22
S_2^-	55	140	-55
S_3^-	-22	-55	22

.

Then $I(W^-)$ is generated by $S_1^- - S_3^-$. Therefore $\mathrm{T}(X^-)\cong \mathrm{H}_2(W^-)/I(W^-)$ is generated by $S_2^- + I(W^-)$ and $S_3^- + I(W^-)$, and $\mathrm{T}(X^-)$ is isomorphic to

$$L[140,-55,22]\cong L[8,3,8].$$

Thus Proposition 1.6 is proved.

Remark 4.3. For the algorithm to determine whether given two lattices of rank 2 are isomorphic or not, see [4, Chapter 15].

5. THE EQUATIONS

In this section, we construct homogeneous polynomials of degree 6 defining complex projective plane curves that have singular points of type A_{10} and of type A_9. Two of such polynomials are as follows:

$$\begin{gathered}10y^4z^2-20xy^2z^3+10x^2z^4-(-108\pm 40\sqrt{5})\,c_0xy^3z^2+\\ (-64\pm 20\sqrt{5}\,)c_0x^2yz^3+(-44\pm 20\sqrt{5}\,)c_0y^5z-(-58\pm 30\sqrt{5}\,)c_0{}^2x^3z^3+\\ (-48\pm 30\sqrt{5}\,)c_0{}^2x^2y^2z^2+(-14\pm 10\sqrt{5}\,)c_0{}^3x^3yz^2+(-9\pm 5\sqrt{5}\,)c_0{}^4x^4z^2,\end{gathered}\tag{5.1}$$

with $c_0\in\mathbb{C}^\times$. We explain how to obtain these equations.

First we prove a lemma.

Lemma 5.1. *Let $f(x,y)=0$ be a defining equation of a complex affine plane curve of degree 6 that has a singular point of type A_{10} at the origin with the tangent $x=0$. Then, after appropriate coordinate change of the form $(x,y)\mapsto(x,ay)$ with $a\neq 0$, f is equal to one of the following polynomials* (5.2) *or* (5.3) *up to multiplicative constant:*

$$\begin{aligned}(5.2)\qquad & x^2-2xy^2+y^4+a_{0,5}y^5+a_{0,6}y^6+a_{1,3}xy^3+a_{1,4}xy^4\\ & +a_{1,5}xy^5+a_{2,1}x^2y+a_{2,2}x^2y^2+a_{2,3}x^2y^3+a_{2,4}x^2y^4+a_{3,0}x^3\\ & +a_{3,1}x^3y+a_{3,2}x^3y^2+a_{3,3}x^3y^3+a_{4,0}x^4+a_{4,1}x^4y+a_{4,2}x^4y^2\\ & +a_{5,0}x^5+a_{5,1}x^5y+a_{6,0}x^6,\end{aligned}$$

where

$$\begin{aligned}
a_{3,0}= &\ {c_0}^2-a_{0,6}-a_{1,4}-a_{2,2},\\
a_{2,1}= &\ -2c_0+a_{0,5},\\
a_{1,3}= &\ 2c_0-2a_{0,5},\\
a_{4,0}= &\ -\tfrac{1}{2}(-3{c_0}^4+6{c_0}^2c_1-2{c_1}^2+3{c_0}^2a_{0,6}-4c_0a_{1,5}-{c_0}^2a_{2,2}\\
&\ -2c_0a_{2,3}+2a_{2,4}+2a_{3,2}),\\
a_{3,1}= &\ -2{c_0}^3+2c_0c_1+{c_0}^2a_{0,5}-a_{1,5}-a_{2,3},\\
a_{1,4}= &\ -\tfrac{1}{2}({c_0}^2-2c_1-2c_0a_{0,5}+3a_{0,6}+a_{2,2}),\\
a_{5,0}= &\ -\tfrac{1}{2}(-16{c_0}^6+45{c_0}^4c_1-33{c_0}^2{c_1}^2+2{c_1}^3-12{c_0}^3c_2+16c_0c_1c_2-2{c_2}^2)\\
&\ -(2{c_0}^5-3{c_0}^3c_1+c_0{c_1}^2)a_{0,5}-\tfrac{1}{2}(8{c_0}^4-9{c_0}^2c_1+3{c_1}^2)a_{0,6}\\
&\ -(-3{c_0}^3+2c_0c_1)a_{1,5}-\tfrac{1}{2}({c_0}^2c_1-{c_1}^2)a_{2,2}-(3{c_0}^2-2c_1)a_{2,4}\\
&\ -({c_0}^2-c_1)a_{3,2}+c_0a_{3,3}-a_{4,2},\\
a_{4,1}= &\ -\tfrac{1}{2}(15{c_0}^5-30{c_0}^3c_1+12c_0{c_1}^2+8{c_0}^2c_2-4c_1c_2)\\
&\ -(-4{c_0}^4+4{c_0}^2c_1-{c_1}^2)a_{0,5}-\tfrac{1}{2}{c_0}^3a_{0,6}-{c_0}^2a_{1,5}\\
&\ +\tfrac{1}{2}{c_0}^3a_{2,2}+2c_0a_{2,4}+c_0a_{3,2}-a_{3,3},\\
a_{2,3}= &\ 3{c_0}^3-4c_0c_1+2c_2-(3{c_0}^2-2c_1)a_{0,5}+3c_0a_{0,6}-2a_{1,5}-c_0a_{2,2},\\
a_{5,1}= &\ \tfrac{1}{2}(26{c_0}^7-83{c_0}^5c_1+74{c_0}^3{c_1}^2-12c_0{c_1}^3+25{c_0}^4c_2-40{c_0}^2c_1c_2\\
&\ +4{c_1}^2c_2+6c_0{c_2}^2-2c_3)-(2{c_0}^6-9{c_0}^4c_1+9{c_0}^2{c_1}^2+3{c_0}^3c_2-6c_0c_1c_2\\
&\ +{c_2}^2)a_{0,5}-\tfrac{1}{2}(-34{c_0}^5+59{c_0}^3c_1-24c_0{c_1}^2-9{c_0}^2c_2+6c_1c_2)a_{0,6}\\
&\ -(9{c_0}^4-11{c_0}^2c_1+2c_0c_2)a_{1,5}-\tfrac{1}{2}(2{c_0}^5-7{c_0}^3c_1+6c_0{c_1}^2+{c_0}^2c_2\\
&\ -2c_1c_2)a_{2,2}-(-7{c_0}^3+8c_0c_1-2c_2)a_{2,4}-(-2{c_0}^3+3c_0c_1-c_2)a_{3,2}\\
&\ -(2{c_0}^2-c_1)a_{3,3}+c_0a_{4,2},
\end{aligned}$$

with $a_{i,j}$, $c_k\in\mathbb{C}$, or

$$\begin{aligned}(5.3)\qquad & x^2-2xy^3+y^6-2x^2yc_0+2xy^4c_0+x^2y^2\big({c_0}^2+2({c_0}^2-c_1)\big)\\ & -2xy^5({c_0}^2-c_1)+x^3\big(-2c_0({c_0}^2-c_1)-a_{2,3}\big)+x^2y^3a_{2,3}\\ & +x^2y^4a_{2,4}+x^3y(3{c_0}^4-4{c_0}^2c_1+{c_1}^2+c_0a_{2,3}-a_{2,4})+x^3y^3a_{3,3}\\ & +x^3y^2\big(-3{c_0}^5+6{c_0}^3c_1-3c_0{c_1}^2+c_2-({c_0}^2-c_1)a_{2,3}+c_0a_{2,4}\big)\\ & +x^4a_{4,0}+x^4ya_{4,1}+x^4y^2a_{4,2}+x^5a_{5,0}+x^5ya_{5,1}+x^6a_{6,0},\end{aligned}$$

with $a_{i,j}$, $c_k\in\mathbb{C}$.

Conversely, if $c_3\neq 0$, then the affine curve defined by the polynomial (5.2) *has a singular point of type A_{10} at the origin, and if $c_2\neq 0$, then the affine curve defined by the polynomial* (5.3) *has a singular point of type A_{10} at the origin.*

We will use the following method to determine the type of singularities from the form of the equation.

Definition 5.2. Let k be an algebraically closed field and let $w = (w_0, w_1) \in \mathbb{Q}^2_{\geq 0}$. Let $M = x^{e_0}y^{e_1} \in k[[x,y]]$ be a monomial. We define the *weight* of M by $w(M) := \sum e_i w_i$. A formal power series $f \in k[[x,y]]$ is said to be *semi-quasihomogeneous* with respect to the weight w if f is of the form $f = f_{w=1} + f_{w>1}$ such that

(i) every non-zero coefficient monomial M in $f_{w=1}$ satisfies $w(M) = 1$, and $f_{w=1}$ defines an isolated singularity, and
(ii) every non-zero coefficient monomial M in $f_{w>1}$ satisfies $w(M) > 1$.

A semi-quasihomogeneous f is said to be *quasihomogeneous* with respect to the weight w if $f_{w>1} = 0$.

Proposition 5.3 ([13], Proposition 2.3)**.** *A semi-quasihomogeneous $f \in k[[x,y]]$ with respect to the weight $\mathcal{A}_m = (\frac{1}{2}, \frac{1}{m+1})$, $\mathcal{D}_m = (\frac{1}{m-1}, \frac{m-2}{2(m-1)})$, $\mathcal{E}_6 = (\frac{1}{3}, \frac{1}{4})$, $\mathcal{E}_7 = (\frac{1}{3}, \frac{2}{9})$ and $\mathcal{E}_8 = (\frac{1}{3}, \frac{1}{5})$ defines a simple singularity if $f_{w=1}$ defines an isolated singularity at the origin. The type of the singular point is A_m, D_m, E_6, E_7 and E_8 respectively.*

Proof of Lemma 5.1. Let $f_0(x,y) = \sum b_{i,j}x^iy^j \in \mathbb{C}[x,y]$ be a polynomial of degree 6 with complex coefficients $b_{i,j}$. Suppose that the affine plane curve defined by f_0 has a singularity of type A_{10} at $(0,0)$ with the tangent $x = 0$. We can write

$$f_0 = x^2 + b_{3,0}x^3 + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + (\text{higher terms}).$$

Firstly, let $w = (\frac{1}{2}, \frac{1}{3})$. If $b_{0,3} \neq 0$, then $(f_0)_{w=1} = x^2 + b_{0,3}y^3$ would define an isolated singularity at the origin and hence $f_0 = 0$ would have a singularity of type A_2 at the origin by Proposition 5.3. Thus $b_{0,3}$ must be equal to 0.

Secondly, let $w = (\frac{1}{2}, \frac{1}{4})$. Then f_0 is semi-quasihomogeneous with respect to w, and hence the quasihomogeneous part $(f_0)_{w=1}$ must define a *non*-isolated singularity at the origin by Proposition 5.3. Hence there exists $b \in \mathbb{C}$ such that $(f_0)_{w=1}$ is equal to $x^2 - 2bxy^2 + b^2y^4$. We divide into two cases, the case where $b \neq 0$ and the case where $b = 0$.

Case 1 ($b \neq 0$). We change the coordinate via $\sqrt{b}\,y \mapsto y$. Then we have $(f_0)_{w=1} \mapsto x^2 - 2xy^2 + y^4$. Therefore, without loss of generality, we can write

$$\begin{aligned} f_0(x,y) = {} & x^2 - 2xy^2 + y^4 + a_{0,5}y^5 + a_{0,6}y^6 + a_{1,3}xy^3 + a_{1,4}xy^4 \\ & + a_{1,5}xy^5 + a_{2,1}x^2y + a_{2,2}x^2y^2 + a_{2,3}x^2y^3 + a_{2,4}x^2y^4 + a_{3,0}x^3 \\ & + a_{3,1}x^3y + a_{3,2}x^3y^2 + a_{3,3}x^3y^3 + a_{4,0}x^4 + a_{4,1}x^4y + a_{4,2}x^4y^2 \\ & + a_{5,0}x^5 + a_{5,1}x^5y + a_{6,0}x^6, \end{aligned} \tag{5.4}$$

with $a_{i,j} \in \mathbb{C}$.

Change the coordinate via $x \mapsto x + y^2$. Assume that this coordinate change transforms f_0 into f_1. An elementary calculation shows that

$$\begin{aligned} f_0 \mapsto f_1 = {} & x^2 + a_{2,1}x^2y + (a_{0,5} + a_{1,3} + a_{2,1})y^5 + (a_{1,3} + 2a_{2,1})xy^3 + a_{3,0}x^3 \\ & +(a_{0,6} + a_{1,4} + a_{2,2} + a_{3,0})y^6 + (a_{2,2} + 3a_{3,0})x^2y^2 + (a_{1,4} + 2a_{2,2} + 3a_{3,0})xy^4 \\ & +a_{3,1}x^3y + (a_{1,5} + a_{2,3} + a_{3,1})y^7 + (a_{2,3} + 3a_{3,1})x^2y^3 + (a_{1,5} + 2a_{2,3} + 3a_{3,1})xy^5 \\ & +a_{4,0}x^4 + (a_{2,4} + a_{3,2} + a_{4,0})y^8 + (a_{3,2} + 4a_{4,0})x^3y^2 + (2a_{2,4} + 3a_{3,2} + 4a_{4,0})xy^6 \\ & +(a_{2,4} + 3a_{3,2} + 6a_{4,0})x^2y^4 + a_{4,1}x^4y + (a_{3,3} + a_{4,1})y^9 + 3(a_{3,3} + 2a_{4,1})x^2y^5 \\ & +(a_{3,3} + 4a_{4,1})x^3y^3 + (3a_{3,3} + 4a_{4,1})xy^7 + a_{5,0}x^5 + (a_{4,2} + a_{5,0})y^{10} \\ & +(a_{4,2} + 5a_{5,0})x^4y^2 + 2(2a_{4,2} + 5a_{5,0})x^3y^4 + 2(3a_{4,2} + 5a_{5,0})x^2y^6 \\ & +(4a_{4,2} + 5a_{5,0})xy^8 + a_{5,1}x^5y + 5a_{5,1}x^4y^3 + 10a_{5,1}x^3y^5 + 10a_{5,1}x^2y^7 \\ & +5a_{5,1}xy^9 + a_{5,1}y^{11} + a_{6,0}x^6 + 6a_{6,0}x^5y^2 + 15a_{6,0}x^4y^4 + 20a_{6,0}x^3y^6 \\ & +15a_{6,0}x^2y^8 + 6a_{6,0}xy^{10} + a_{6,0}y^{12}. \end{aligned}$$

First, let $w = (\frac{1}{2}, \frac{1}{5})$. If there were the term y^5 with non-zero coefficient in f_1, the singularity of $f_1 = 0$ at the origin would be of type A_4 by Proposition 5.3. Next, let $w = (\frac{1}{2}, \frac{1}{6})$. By the same argument as above, the quasihomogeneous part $(f_1)_{w=1}$ of f_1 must define non-isolated singularities at the origin, because otherwise $f_1 = 0$ would have a singularity of type A_5. Thus there exists a complex number c_0 such that $(f_1)_{w=1} = x^2 + (a_{1,3} + 2a_{2,1})xy^3 + (a_{0,6} + a_{1,4} + a_{2,2} + a_{3,0})y^6$ is equal to $x^2 - 2c_0xy^3 + {c_0}^2y^6$. Consequently we have following conditions:

Step 1. $(x \mapsto x + y^2)$
$a_{0,5} + a_{1,3} + a_{2,1} = 0$,
$a_{0,6} + a_{1,4} + a_{2,2} + a_{3,0} = {c_0}^2$ and
$a_{1,3} + 2a_{2,1} = -2c_0$.

Then we change the coordinate via $x \mapsto x + c_0y^3$. Assume that this transformation takes f_1 to f_2. Let $w = (\frac{1}{2}, \frac{1}{7})$. The coefficient of y^7 is equal to 0, because otherwise the singularity at the origin would be of type A_6. Next, let $w = (\frac{1}{2}, \frac{1}{8})$. The quasihomogeneous part $(f_2)_{w=1}$ must define non-isolated singularities at the origin. Hence there exists $c_1 \in \mathbb{C}$ such that $(f_2)_{w=1}$ is equal to $x^2 - 2c_1xy^4 + {c_1}^2y^8$. Therefore we have

Step 2. $(x \mapsto x + c_0y^3)$
${c_0}^3 + {c_0}^2a_{0,5} - 3c_0a_{0,6} - 2c_0a_{1,4} + a_{1,5} - c_0a_{2,2} + a_{2,3} + a_{3,1} = 0$,
$3{c_0}^4 - 3{c_0}^2a_{0,6} - 3{c_0}^2a_{1,4} + c_0a_{1,5} - 2{c_0}^2a_{2,2} + 2c_0a_{2,3} + a_{2,4} + 3c_0a_{3,1} + a_{3,2} + a_{4,0} = {c_1}^2$ and
$-{c_0}^2 + 2c_0a_{0,5} - 3a_{0,6} - 2a_{1,4} - a_{2,2} = -2c_1$.

The coordinate change via $x \mapsto x + c_1y^4$ takes f_2 to f_3. The coefficient of y^9 in f_3 is equal to 0 and there exists $c_2 \in \mathbb{C}$ such that $(f_3)_{w=1} = x^2 - 2c_2xy^5 + {c_2}^2y^{10}$, where $w = (\frac{1}{2}, \frac{1}{10})$.

Step 3. $(x \mapsto x + c_1y^4)$
$\frac{3}{2}{c_0}^5 - 4c_1{c_0}^3 + 2{c_0}^4a_{0,5} - 3c_1{c_0}^2a_{0,5} + {c_1}^2a_{0,5} - \frac{11}{2}{c_0}^3a_{0,6} + 3c_1c_0a_{0,6} + 5{c_0}^2a_{1,5} + \frac{3}{2}{c_0}^3a_{2,2} - a_{2,2}c_1c_0 + 2{c_0}^2a_{2,3} - a_{2,3}c_1 - 2c_0a_{2,4} - c_0a_{3,2} + a_{3,3} + a_{4,1} = 0$,
$7{c_0}^6 - \frac{35}{2}c_1{c_0}^4 + \frac{15}{2}{c_1}^2{c_0}^2 + {c_1}^3 + {c_0}^5a_{0,5} + 3c_1{c_0}^3a_{0,5} - 3{c_1}^2c_0a_{0,5} - 9{c_0}^4a_{0,6} - \frac{9}{2}c_1{c_0}^2a_{0,6} + \frac{3}{2}{c_1}^2a_{0,6} + 11{c_0}^3a_{1,5} + 2c_1c_0a_{1,5} + 3{c_0}^4a_{2,2} + \frac{1}{2}c_1{c_0}^2a_{2,2} - \frac{1}{2}{c_1}^2a_{2,2} + 5{c_0}^3a_{2,3} - 5{c_0}^2a_{2,4} - 2c_1a_{2,4} - 3{c_0}^2a_{3,2} - c_1a_{3,2} + 3c_0a_{3,3} + 4c_0a_{4,1} +$

$a_{4,2}+a_{5,0}={c_2}^2$ and
$3{c_0}^3-4c_1c_0-3{c_0}^2a_{0,5}+2c_1a_{0,5}+3c_0a_{0,6}-2a_{1,5}-c_0a_{2,2}-a_{2,3}=-2c_2.$

Change the coordinate via $x\mapsto x+c_2y^5$. Suppose that this transformation takes f_3 to f_4. Let $w=(\frac{1}{2},\frac{1}{11})$. Then $(f_4)_{w=1}$ is equal to $x^2+c_3y^{11}$, where c_3 is given below. If $c_3\neq 0$, then $(f_4)_{w=1}=0$ defines an isolated singular point at the origin, and hence $f_4=0$ has a singular point of type A_{10} at the origin.

Step 4. $(x\mapsto x+c_2y^5)$
$13{c_0}^7-\frac{83}{2}c_1{c_0}^5+37{c_1}^2{c_0}^3-6{c_1}^3c_0+\frac{25}{2}c_2{c_0}^4-20c_2c_1{c_0}^2+c_2{c_1}^2+3{c_2}^2c_0+2{c_0}^6a_{0,5}-9c_1{c_0}^4a_{0,5}+9{c_1}^2{c_0}^2a_{0,5}+3c_2{c_0}^3a_{0,5}-6c_2c_1c_0a_{0,5}+{c_2}^2a_{0,5}-17{c_0}^5a_{0,6}+\frac{59}{2}c_1{c_0}^3a_{0,6}-12{c_1}^2c_0a_{0,6}-\frac{9}{2}c_2{c_0}^2a_{0,6}+3c_2c_1a_{0,6}+9{c_0}^4a_{1,5}-11c_1{c_0}^2a_{1,5}+{c_1}^2a_{1,5}+2c_2c_0a_{1,5}+{c_0}^5a_{2,2}-\frac{7}{2}c_1{c_0}^3a_{2,2}+3{c_1}^2c_0a_{2,2}+\frac{1}{2}c_2{c_0}^2a_{2,2}-c_2c_1a_{2,2}-7{c_0}^3a_{2,4}+8c_1c_0a_{2,4}-2c_2a_{2,4}-2{c_0}^3a_{3,2}+3c_1c_0a_{3,2}-c_2a_{3,2}+2{c_0}^2a_{3,3}-c_1a_{3,3}-c_0a_{4,2}+a_{5,1}=c_3.$

Solve the system of linear equations appearing in each step by choosing unknowns suitably. Then we have the solutions denoted in (5.2).

Case 2 $(b=0)$. In this case, without loss of generality we can write

$$\begin{aligned}f_0=&x^2-2b'xy^3+{b'}^2y^6+a_{1,4}xy^4+a_{1,5}xy^5+a_{2,1}x^2y+a_{2,2}x^2y^2\\&+a_{2,3}x^2y^3+a_{2,4}x^2y^4+a_{3,0}x^3+a_{3,1}x^3y+a_{3,2}x^3y^2+a_{3,3}x^3y^3\\&+a_{4,0}x^4+a_{4,1}x^4y+a_{4,2}x^4y^2+a_{5,0}x^5+a_{5,1}x^5y+a_{6,0}x^6.\end{aligned}$$

Assume that $b'=0$. If $a_{1,4}\neq 0$, then the polynomial f_0 is semi-quasihomogeneous with respect to the weight $w=(\frac{1}{2},\frac{1}{8})$ and $(f_0)_{w=1}$ defines an isolated singularity. Hence $f_0=0$ would have a singularity of type A_7 at the origin. If $a_{1,4}=0$ and $a_{1,5}\neq 0$, then f_0 is semi-quasihomogeneous with respect to $w=(\frac{1}{2},\frac{1}{10})$, and $(f_0)_{w=1}$ defines an isolated singularity at the origin, so that $f_0=0$ would have a singularity of type A_9 at the origin. If $a_{1,4}=a_{1,5}=0$, then f_0 defines non-isolated singularities at the origin. Therefore b' is not equal to zero. Furthermore, the coordinate change $\sqrt[3]{b'}\,y\mapsto y$ takes b' to 1. Therefore we can write

$$\begin{aligned}f_0=&x^2-2xy^3+y^6+a_{1,4}xy^4+a_{1,5}xy^5+a_{2,1}x^2y+a_{2,2}x^2y^2+a_{2,3}x^2y^3\\&+a_{2,4}x^2y^4+a_{3,0}x^3+a_{3,1}x^3y+a_{3,2}x^3y^2+a_{3,3}x^3y^3+a_{4,0}x^4\\&+a_{4,1}x^4y+a_{4,2}x^4y^2+a_{5,0}x^5+a_{5,1}x^5y+a_{6,0}x^6.\end{aligned}\tag{5.5}$$

By a similar argument as in Case 1, we have the following three steps:

Step 1. $(x\mapsto x+y^3)$
$a_{1,4}+a_{2,1}=0,$
$a_{1,5}+a_{2,2}+a_{3,0}={c_0}^2$ and
$a_{1,4}+2a_{2,1}=-2c_0,$
Step 2. $(x\mapsto x+c_0y^4)$
$-c_0a_{1,5}+a_{2,3}+a_{3,0}=0,$
${c_0}^4-{c_0}^2a_{1,5}+2c_0a_{2,3}+a_{2,4}+3c_0a_{3,0}+a_{3,1}={c_1}^2$ and
$-2{c_0}^2-a_{1,5}=-2c_1,$
Step 3. $(x\mapsto x+c_1y^5)$
$3{c_0}^5-6{c_0}^3c_1+3c_0{c_1}^2+{c_0}^2a_{2,3}-c_1a_{2,3}-c_0a_{2,4}+a_{3,2}=c_2,$

where $c_i \in \mathbb{C}$. Regard $a_{1,4}$, $a_{1,5}$, $a_{2,1}$, $a_{2,2}$, $a_{3,0}$, $a_{3,1}$ and $a_{3,2}$ as unknowns and solve the system of linear equations. The solutions are

$$\begin{array}{rcl} a_{2,1} & = & -2c_0, \\ a_{2,2} & = & {c_0}^2 - a_{1,5}, \\ a_{1,4} & = & 2c_0, \\ a_{1,5} & = & -2({c_0}^2 - c_1), \\ a_{3,0} & = & -2c_0({c_0}^2 - c_1) - a_{2,3}, \\ a_{3,1} & = & 3{c_0}^4 - 4{c_0}^2 c_1 + {c_1}^2 + c_0 a_{2,3} - a_{2,4} \text{ and} \\ a_{3,2} & = & -3{c_0}^5 + 6{c_0}^3 c_1 - 3c_0 {c_1}^2 + c_2 - ({c_0}^2 - c_1)a_{2,3} + c_0 a_{2,4}. \end{array}$$

Substituting them for the coefficient of (5.5), we obtain the polynomial (5.3). □

Claim 5.4. Let $F(x, y, z) \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree 6 that satisfies

$$F(x, y, 1) = f(x, y),$$

where f is the polynomial (5.2) in the statement of Lemma 5.1 with $c_3 \neq 0$. Let $g(y, z) := F(1, y, z)$. Then g is semi-quasihomogeneous with respect to the weight $w = (\frac{1}{10}, \frac{1}{2})$ if and only if

$$a_{0,6} = a_{1,5} = a_{2,4} = a_{3,2} = a_{3,3} = a_{4,2} = a_{6,0} = 0 \tag{5.6}$$

and

$$\left\{ \begin{array}{rcl} c_1 & = & \frac{1}{2}(5 \pm \sqrt{5}\,){c_0}^2, \\ a_{2,2} & = & 2(c_1 + c_0 a_{0,5}) - {c_0}^2, \\ a_{0,5} & = & \frac{2}{5}(-11 \pm 5\sqrt{5}\,)c_0, \\ c_3 & = & -\frac{6}{25}(-123 \pm 55\sqrt{5}\,){c_0}^7. \end{array} \right. \tag{5.7}$$

Moreover, if (5.6) and (5.7) hold, then $g_{w=1}$ defines an isolated singularity at the origin and hence $g = 0$ has a singular point of type A_9 at $(0, 0)$ by Proposition 5.3.

Proof. We write g in the form

$$g = g_{w<1} + g_{w=1} + g_{w>1},$$

where $w = (\frac{1}{10}, \frac{1}{2})$. The condition $g_{w<1} = 0$ is equivalent to (5.6) and

$$\left\{ \begin{array}{rcl} 0 & = & 2(c_1 + c_0 a_{0,5}) - {c_0}^2 - a_{2,2} \\ 0 & = & -4{c_0}^3 + 6c_1 c_0 + 5a_{0,5}{c_0}^2 - 2a_{0,5}c_1 - 2c_2 \\ 0 & = & 5{c_0}^4 - 5{c_0}^2 c_1 + {c_1}^2 \\ 0 & = & {a_{0,5}}^2(5{c_0}^2 - 2c_1)(15{c_0}^3 - 10c_0 c_1 + 5{c_0}^2 a_{0,5} - 2c_1 a_{0,5}) - 4c_3 \\ 0 & = & 12{c_0}^5 - 20{c_0}^3 c_1 + 8c_0{c_1}^2 + 25{c_0}^4 a_{0,5} - 20{c_0}^2 c_1 a_{0,5} + 4{c_1}^2 a_{0,5}. \end{array} \right.$$

Solving this system of equations, we get (5.7). Note that we have $c_0 \neq 0$ by the assumption $c_3 \neq 0$. Substituting (5.6) and (5.7) for coefficients of g, we have

$$g_{w=1} = \frac{-9 \pm 5\sqrt{5}}{10}{c_0}^4 z^2 + \frac{2(-11 \pm 5\sqrt{5}\,)}{5}c_0\, zy^5.$$

Since $c_0 \neq 0$, $g_{w=1}$ defines an isolated singularity. □

Note that $10F(x, y, z)$ is equal to (5.1) under the condition (5.6) and (5.7).

Finally, let $c_0 = 1$. The curve defined by the equation (5.1) has a singular point of type A_{10} at $(0 : 0 : 1)$, a singular point of type A_9 at $(1 : 0 : 0)$, and is smooth except for these two points.

Remark 5.5. In [2], a different method to obtain defining equations of sextic curves with big Milnor number is given.

References

[1] E. Artal-Bartolo. Sur les couples de Zariski. *J. Algebraic Geom.*, 3(2):223–247, 1994.

[2] E. Artal Bartolo, J. Carmona Ruber, and J. I. Cogolludo Agustín. On sextic curves with big Milnor number. In *Trends in singularities*, Trends Math., pages 1–29. Birkhäuser, Basel, 2002.

[3] E. Artal Bartolo, J. Carmona Ruber, and J. I. Cogolludo Agustín. Effective invariants of braid monodromy. *Trans. Amer. Math. Soc.*, 359(1):165–183 (electronic), 2007.

[4] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, third edition, 1999.

[5] A. Degtyarev. On deformations of singular plane sextics. *J. Algebraic Geom.*, 17(1):101–135, 2008.

[6] Carl Friedrich Gauss. *Disquisitiones arithmeticae*. Springer-Verlag, New York, 1986. Translated and with a preface by Arthur A. Clarke, Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst and with a preface by Waterhouse.

[7] K. Lamotke. The topology of complex projective varieties after S. Lefschetz. *Topology*, 20(1):15–51, 1981.

[8] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979. English translation: Math USSR-Izv. 14 (1979), no. 1, 103–167 (1980).

[9] M. Oka. Symmetric plane curves with nodes and cusps. *J. Math. Soc. Japan*, 44(3):375–414, 1992.

[10] U. Persson. Horikawa surfaces with maximal Picard numbers. *Math. Ann.*, 259(3):287–312, 1982.

[11] U. Persson. Double sextics and singular K-3 surfaces. In *Algebraic geometry, Sitges (Barcelona), 1983*, volume 1124 of *Lecture Notes in Math.*, pages 262–328. Springer, Berlin, 1985.

[12] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič. Torelli's theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:530–572, 1971. Reprinted in I. R. Shafarevich, Collected Mathematical Papers, Springer-Verlag, Berlin, 1989, pp. 516–557.

[13] M. Roczen. Recognition of simple singularities in positive characteristic. *Math. Z.*, 210(4):641–653, 1992.

[14] M. Schütt. Fields of definition of singular $K3$ surfaces. *Commun. Number Theory Phys.*, 1(2):307–321, 2007.

[15] I. Shimada. Fundamental groups of open algebraic varieties. *Topology*, 34(3):509–531, 1995.

[16] I. Shimada. A note on Zariski pairs. *Compositio Math.*, 104(2):125–133, 1996.

[17] I. Shimada. Picard-Lefschetz theory for the universal coverings of complements to affine hypersurfaces. *Publ. Res. Inst. Math. Sci.*, 32(5):835–927, 1996.

[18] I. Shimada. On the Zariski-van Kampen theorem. *Canad. J. Math.*, 55(1):133–156, 2003.

[19] I. Shimada. On arithmetic Zariski pairs in degree 6. *Adv. Geom.*, 8(2):205–225, 2008.

[20] I. Shimada. Transcendental lattices and supersingular reduction lattices of a singular $K3$ surface. *Trans. Amer. Math. Soc.*, 361(2):909–949, 2009.

[21] I. Shimada. Non-homeomorphic conjugate complex varieties, 2007. preprint, arXiv:math/0701115.

[22] T. Shioda and H. Inose. On singular $K3$ surfaces. In *Complex analysis and algebraic geometry*, pages 119–136. Iwanami Shoten, Tokyo, 1977.

[23] T. Shioda. $K3$ surfaces and sphere packings. *J. Math. Soc. Japan*, 60(4):1083–1105, 2008.

[24] Jin-Gen Yang. Sextic curves with simple singularities. *Tohoku Math. J. (2)*, 48(2):203–227, 1996.

[25] O. Zariski. On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve. *Amer. J. Math.*, 51(2):305–328, 1929.

[26] O. Zariski. The Topological Discriminant Group of a Riemann Surface of Genus p. *Amer. J. Math.*, 59(2):335–358, 1937.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, JAPAN

E-mail address: `arima@math.sci.hokudai.ac.jp`

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526, JAPAN

E-mail address: `shimada@math.sci.hiroshima-u.ac.jp`