ON BALLICO-HEFEZ CURVES AND ASSOCIATED SUPERSINGULAR SURFACES

HOANG THANH HOAI AND ICHIRO SHIMADA

Abstract. Let p be a prime integer, and q a power of p. The Ballico-Hefez curve is a non-reflexive nodal rational plane curve of degree $q+1$ in characteristic p. We investigate its automorphism group and defining equation. We also prove that the surface obtained as the cyclic cover of the projective plane branched along the Ballico-Hefez curve is unirational, and hence is supersingular. As an application, we obtain a new projective model of the supersingular $K3$ surface with Artin invariant 1 in characteristic 3 and 5.

1. Introduction

We work over an algebraically closed field k of positive characteristic $p > 0$. Let $q = p^\nu$ be a power of p.

In positive characteristics, algebraic varieties often possess interesting properties that are not observed in characteristic zero. One of those properties is the failure of reflexivity. In [4], Ballico and Hefez classified irreducible plane curves X of degree $q+1$ such that the natural morphism from the conormal variety $C(X)$ of X to the dual curve $X^\vee$ has inseparable degree q. The Ballico-Hefez curve in the title of this note is one of the curves that appear in their classification. It is defined in Fukasawa, Homma and Kim [8] as follows.

Definition 1.1. The *Ballico-Hefez curve* is the image of the morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$[s : t] \mapsto [s^{q+1} : t^{q+1} : st^q + s^q t].$$

Theorem 1.2 (Ballico and Hefez [4], Fukasawa, Homma and Kim [8])**.** (1) *Let B be the Ballico-Hefez curve. Then B is a curve of degree $q+1$ with $(q^2-q)/2$ ordinary nodes, the dual curve $B^\vee$ is of degree 2, and the natural morphism $C(B) \to B^\vee$ has inseparable degree q.*

(2) *Let $X \subset \mathbb{P}^2$ be an irreducible singular curve of degree $q+1$ such that the dual curve $X^\vee$ is of degree > 1 and the natural morphism $C(X) \to X^\vee$ has inseparable degree q. Then X is projectively isomorphic to the Ballico-Hefez curve.*

Recently, geometry and arithmetic of the Ballico-Hefez curve have been investigated by Fukasawa, Homma and Kim [8] and Fukasawa [7] from various points of view, including coding theory and Galois points. As is pointed out in [8], the Ballico-Hefez curve has many properties in common with the Hermitian curve; that

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2000 *Mathematics Subject Classification.* primary 14H45, secondary 14J25.

Partially supported by JSPS Grant-in-Aid for Challenging Exploratory Research No.23654012 and JSPS Grants-in-Aid for Scientific Research (C) No.25400042 .

is, the Fermat curve of degree $q+1$, which also appears in the classification of Ballico and Hefez [4]. In fact, we can easily see that the image of the line

$$x_0 + x_1 + x_2 = 0$$

in $\mathbb{P}^2$ by the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by

$$[x_0 : x_1 : x_2] \mapsto [x_0^{q+1} : x_1^{q+1} : x_2^{q+1}]$$

is projectively isomorphic to the Ballico-Hefez curve. Hence, up to linear transformation of coordinates, the Ballico-Hefez curve is defined by an equation

$$x_0^{\frac{1}{q+1}} + x_1^{\frac{1}{q+1}} + x_2^{\frac{1}{q+1}} = 0$$

in the style of "Coxeter curves" (see Griffith [9]).

In this note, we prove the the following:

Proposition 1.3. *Let B be the Ballico-Hefez curve. Then the group*

$$\mathrm{Aut}(B) := \{\, g \in \mathrm{PGL}_3(k) \mid g(B) = B \,\}$$

of projective automorphisms of $B \subset \mathbb{P}^2$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_q)$.

Proposition 1.4. *The Ballico-Hefez curve is defined by the following equations:*

- *When $p = 2$,*

$$x_0^q x_1 + x_0 x_1^q + x_2^{q+1} + \sum_{i=0}^{\nu-1} x_0^{2^i} x_1^{2^i} x_2^{q+1-2^{i+1}} = 0, \quad \textit{where } q = 2^\nu.$$

- *When p is odd,*

$$2(x_0^q x_1 + x_0 x_1^q) - x_2^{q+1} - (x_2^2 - 4x_1x_0)^{\frac{q+1}{2}} = 0.$$

Remark 1.5. In fact, the defining equation for $p = 2$ has been obtained by Fukasawa in an apparently different form (see Remark 3 of [6]).

Another property of algebraic varieties peculiar to positive characteristics is the failure of Lüroth's theorem for surfaces; a non-rational surface can be unirational in positive characteristics. A famous example of this phenomenon is the Fermat surface of degree $q+1$. Shioda [18] and Shioda-Katsura [19] showed that the Fermat surface F of degree $q+1$ is unirational (see also [16] for another proof). This surface F is obtained as the cyclic cover of $\mathbb{P}^2$ with degree $q+1$ branched along the Fermat curve of degree $q+1$, and hence, for any divisor d of $q+1$, the cyclic cover of $\mathbb{P}^2$ with degree d branched along the Fermat curve of degree $q+1$ is also unirational.

We prove an analogue of this result for the Ballico-Hefez curve. Let d be a divisor of $q+1$ larger than 1. Note that d is prime to p.

Proposition 1.6. *Let $\gamma : S_d \to \mathbb{P}^2$ be the cyclic covering of $\mathbb{P}^2$ with degree d branched along the Ballico-Hefez curve. Then there exists a dominant rational map $\mathbb{P}^2 \cdots\to S_d$ of degree $2q$ with inseparable degree q.*

Note that S_d is not rational except for the case $(d, q+1) = (3, 3)$ or $(2, 4)$.

A smooth surface X is said to be *supersingular* (in the sense of Shioda) if the second l-adic cohomology group $H^2(X)$ of X is generated by the classes of curves. Shioda [18] proved that every smooth unirational surface is supersingular. Hence we obtain the following:

Corollary 1.7. *Let $\rho : \tilde{S}_d \to S_d$ be the minimal resolution of S_d. Then the surface $\tilde{S}_d$ is supersingular.*

We present a finite set of curves on $\tilde{S}_d$ whose classes span $H^2(\tilde{S}_d)$. For a point P of $\mathbb{P}^1$, let $l_P \subset \mathbb{P}^2$ denote the line tangent at $\phi(P) \in B$ to the branch of B corresponding to P. It was shown in [8] that, if P is an $\mathbb{F}_{q^2}$-rational point of $\mathbb{P}^1$, then l_P and B intersect only at $\phi(P)$, and hence the strict transform of l_P by the composite $\tilde{S}_d \to S_d \to \mathbb{P}^2$ is a union of d rational curves $l_P^{(0)}, \dots, l_P^{(d-1)}$.

Proposition 1.8. *The cohomology group $H^2(\tilde{S}_d)$ is generated by the classes of the following rational curves on $\tilde{S}_d$; the irreducible components of the exceptional divisor of the resolution $\rho : \tilde{S}_d \to S_d$ and the rational curves $l_P^{(i)}$, where P runs through the set $\mathbb{P}^1(\mathbb{F}_{q^2})$ of $\mathbb{F}_{q^2}$-rational points of $\mathbb{P}^1$ and $i = 0, \dots, d-1$.*

Note that, when $(d, q+1) = (4, 4)$ and $(2, 6)$, the surface $\tilde{S}_d$ is a $K3$ surface. In these cases, we can prove that the classes of rational curves given in Proposition 1.8 generate the Néron-Severi lattice $\mathrm{NS}(\tilde{S}_d)$ of $\tilde{S}_d$, and that the discriminant of $\mathrm{NS}(\tilde{S}_d)$ is $-p^2$. Using this fact and the result of Ogus [13, 14] and Rudakov-Shafarevic [15] on the uniqueness of a supersingular $K3$ surface with Artin invariant 1, we prove the following:

Proposition 1.9. (1) *If $p = q = 3$, then $\tilde{S}_4$ is isomorphic to the Fermat quartic surface*

$$w^4 + x^4 + y^4 + z^4 = 0.$$

(2) *If $p = q = 5$, then $\tilde{S}_2$ is isomorphic to the Fermat sextic double plane*

$$w^2 = x^6 + y^6 + z^6.$$

Recently, many studies on these supersingular $K3$ surfaces with Artin invariant 1 in characteristics 3 and 5 have been carried out. See [10, 12] for characteristic 3 case, and [11, 17] for characteristic 5 case.

Thanks are due to Masaaki Homma and Satoru Fukasawa for their comments. We also thank the referee for his/her suggestion on the first version of this paper.

2. Basic properties of the Ballico-Hefez curve

We recall some properties of the Ballico-Hefez curve B. See Fukasawa, Homma and Kim [8] for the proofs.

It is easy to see that the morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ is birational onto its image B, and that the degree of the plane curve B is $q+1$. The singular locus $\mathrm{Sing}(B)$ of B consists of $(q^2 - q)/2$ ordinary nodes, and we have

$$\phi^{-1}(\mathrm{Sing}(B)) = \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q).$$

In particular, the singular locus $\mathrm{Sing}(S_d)$ of S_d consists of $(q^2 - q)/2$ ordinary rational double points of type A_{d-1}. Therefore, by Artin [1, 2], the surface S_d is not rational if $(d, q+1) \neq (3, 3), (2, 4)$.

Let t be the affine coordinate of $\mathbb{P}^1$ obtained from $[s : t]$ by putting $s = 1$, and let (x, y) be the affine coordinates of $\mathbb{P}^2$ such that $[x_0 : x_1 : x_2] = [1 : x : y]$. Then the morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ is given by

$$t \mapsto (t^{q+1}, t^q + t).$$

For a point $P = [1 : t]$ of $\mathbb{P}^1$, the line l_P is defined by

$$x - t^q y + t^{2q} = 0.$$

Suppose that $P \notin \mathbb{P}^1(\mathbb{F}_{q^2})$. Then l_P intersects B at $\phi(P) = (t^{q+1}, t^q + t)$ with multiplicity q and at the point $(t^{q^2+q}, t^{q^2} + t^q) \neq \phi(P)$ with multiplicity 1. In particular, we have $l_P \cap \mathrm{Sing}(B) = \emptyset$.

Suppose that $P \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$. Then l_P intersects B at the node $\phi(P)$ of B with multiplicity $q+1$. More precisely, l_P intersects the branch of B corresponding to P with multiplicity q, and the other branch transversely.

Suppose that $P \in \mathbb{P}^1(\mathbb{F}_q)$. Then $\phi(P)$ is a smooth point of B, and l_P intersects B at $\phi(P)$ with multiplicity $q + 1$. In particular, we have $l_P \cap \mathrm{Sing}(B) = \emptyset$.

Combining these facts, we see that $\phi(\mathbb{P}^1(\mathbb{F}_q))$ coincides with the set of smooth inflection points of B. (See [8] for the definition of inflection points.)

3. Proof of Proposition 1.3

We denote by $\phi_B : \mathbb{P}^1 \to B$ the birational morphism $t \mapsto (t^{q+1}, t^q + t)$ from $\mathbb{P}^1$ to B. We identify $\mathrm{Aut}(\mathbb{P}^1)$ with $\mathrm{PGL}_2(k)$ by letting $\mathrm{PGL}_2(k)$ act on $\mathbb{P}^1$ by

$$[s : t] \mapsto [as + bt : cs + dt] \quad \text{for} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(k).$$

Then $\mathrm{PGL}_2(\mathbb{F}_q)$ is the subgroup of $\mathrm{PGL}_2(k)$ consisting of elements that leave the set $\mathbb{P}^1(\mathbb{F}_q)$ invariant. Since ϕ_B is birational, the projective automorphism group $\mathrm{Aut}(B)$ of B acts on $\mathbb{P}^1$ via ϕ_B. The subset $\phi_B(\mathbb{P}^1(\mathbb{F}_q))$ of B is projectively characterized as the set of smooth inflection points of B, and we have $\mathbb{P}^1(\mathbb{F}_q) = \phi_B^{-1}(\phi_B(\mathbb{P}^1(\mathbb{F}_q)))$. Hence $\mathrm{Aut}(B)$ is contained in the subgroup $\mathrm{PGL}_2(\mathbb{F}_q)$ of $\mathrm{PGL}_2(k)$. Thus, in order to prove Proposition 1.3, it is enough to show that every element

$$g := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{F}_q$$

of $\mathrm{PGL}_2(\mathbb{F}_q)$ is coming from the action of an element of $\mathrm{Aut}(B)$. We put

$$\tilde{g} := \begin{bmatrix} a^2 & b^2 & ab \\ c^2 & d^2 & cd \\ 2ac & 2bd & ad + bc \end{bmatrix},$$

and let the matrix $\tilde{g}$ act on $\mathbb{P}^2$ by the left multiplication on the column vector ${}^t[x_0 : x_1 : x_2]$. Then we have

$$\phi \circ g = \tilde{g} \circ \phi,$$

because we have $\lambda^q = \lambda$ for $\lambda = a, b, c, d \in \mathbb{F}_q$. Therefore $g \mapsto \tilde{g}$ gives an isomorphism from $\mathrm{PGL}_2(\mathbb{F}_q)$ to $\mathrm{Aut}(B)$.

4. Proof of Proposition 1.4

We put

$$F(x, y) := \begin{cases} x + x^q + y^{q+1} + \sum_{i=0}^{\nu-1} x^{2^i} y^{q+1-2^{i+1}} & \text{if } p = 2 \text{ and } q = 2^\nu, \\ 2x + 2x^q - y^{q+1} - (y^2 - 4x)^{\frac{q+1}{2}} & \text{if } p \text{ is odd}, \end{cases}$$

that is, F is obtained from the homogeneous polynomial in Proposition 1.4 by putting $x_0 = 1, x_1 = x, x_2 = y$. Since the polynomial F is of degree $q + 1$ and the plane curve B is also of degree $q + 1$, it is enough to show that $F(t^{q+1}, t^q + t) = 0$.

Suppose that $p = 2$ and $q = 2^\nu$. We put

$$S(x,y) := \sum_{i=0}^{\nu-1} \left(\frac{x}{y^2}\right)^{2^i}.$$

Then $S(x,y)$ is a root of the Artin-Schreier equation

$$s^2 + s = \left(\frac{x}{y^2}\right)^q + \frac{x}{y^2}.$$

Hence $S_1 := S(t^{q+1}, t^q + t)$ is a root of the equation $s^2 + s = b$, where

$$b := \left[\frac{t^{q+1}}{(t^q+t)^2}\right]^q + \frac{t^{q+1}}{(t^q+t)^2} = \frac{t^{2q^2+q+1} + t^{q^2+3q} + t^{q^2+q+2} + t^{3q+1}}{(t^q+t)^{2q+2}}.$$

We put

$$S'(x,y) := \frac{x + x^q + y^{q+1}}{y^{q+1}}.$$

We can verify that $S_2 := S'(t^{q+1}, t^q + t)$ is also a root of the equation $s^2 + s = b$. Hence we have either $S_1 = S_2$ or $S_1 = S_2 + 1$. We can easily see that both of the rational functions S_1 and S_2 on $\mathbb{P}^1$ have zero at $t = \infty$. Hence $S_1 = S_2$ holds, from which we obtain $F(t^{q+1}, t^q + t) = 0$.

Suppose that p is odd. We put

$$S(x,y) := 2x + 2x^q - y^{q+1}, \quad S_1 := S(t^{q+1}, t^q + t), \quad \text{and}$$
$$S'(x,y) := (y^2 - 4x)^{\frac{q+1}{2}}, \quad S_2 := S'(t^{q+1}, t^q + t).$$

Then it is easy to verify that both of S_1^2 and S_2^2 are equal to

$$t^{2q^2+2q} - 2t^{2q^2+q+1} + t^{2q^2+2} - 2t^{q^2+3q} + 4t^{q^2+2q+1} - 2t^{q^2+q+2} + t^{4q} - 2t^{3q+1} + t^{2q+2}.$$

Therefore either $S_1 = S_2$ or $S_1 = -S_2$ holds. Comparing the coefficients of the top-degree terms of the polynomials S_1 and S_2 of t, we see that $S_1 = S_2$, whence $F(t^{q+1}, t^q + t) = 0$ follows.

5. Proof of Propositions 1.6 and 1.8

We consider the universal family

$$L := \{\ (P, Q) \in \mathbb{P}^1 \times \mathbb{P}^2 \ \mid\ Q \in l_P\ \}$$

of the lines l_P, which is defined by

$$x - t^q y + t^{2q} = 0$$

in $\mathbb{P}^1 \times \mathbb{P}^2$, and let

$$\pi_1 : L \to \mathbb{P}^1, \quad \pi_2 : L \to \mathbb{P}^2$$

be the projections. We see that $\pi_1 : L \to \mathbb{P}^1$ has two sections

$$\sigma_1 \quad : \quad t \mapsto (t, x, y) = (t, t^{q+1}, t^q + t),$$
$$\sigma_q \quad : \quad t \mapsto (t, x, y) = (t, t^{q^2+q}, t^{q^2} + t^q).$$

For $P \in \mathbb{P}^1$, we have $\pi_2(\sigma_1(P)) = \phi(P)$ and $l_P \cap B = \{\pi_2(\sigma_1(P)), \pi_2(\sigma_q(P))\}$. Let $\Sigma_1 \subset L$ and $\Sigma_q \subset L$ denote the images of σ_1 and σ_q, respectively. Then Σ_1 and Σ_q are smooth curves, and they intersect transversely. Moreover, their intersection points are contained in $\pi_1^{-1}(\mathbb{P}^1(\mathbb{F}_{q^2}))$.

We denote by $\overline{M}$ the fiber product of $\gamma : S_d \to \mathbb{P}^2$ and $\pi_2 : L \to \mathbb{P}^2$ over $\mathbb{P}^2$. The pull-back $\pi_2^* B$ of B by π_2 is equal to the divisor $q\Sigma_1 + \Sigma_q$. Hence $\overline{M}$ is defined by

$$\begin{cases} z^d = (y - t^q - t)^q (y - t^{q^2} - t^q), \\ x - t^q y + t^{2q} = 0. \end{cases} \tag{5.1}$$

We denote by $M \to \overline{M}$ the normalization, and by

$$\alpha : M \to L, \quad \eta : M \to S_d$$

the natural projections. Since d is prime to q, the cyclic covering $\alpha : M \to L$ of degree d branches exactly along the curve $\Sigma_1 \cup \Sigma_q$. Moreover, the singular locus $\mathrm{Sing}(M)$ of M is located over $\Sigma_1 \cap \Sigma_q$, and hence is contained in $\alpha^{-1}(\pi_1^{-1}(\mathbb{P}^1(\mathbb{F}_{q^2})))$.

Since η is dominant and $\rho : \tilde{S}_d \to S_d$ is birational, η induces a rational map

$$\eta' : M \cdots \to \tilde{S}_d.$$

Let A denote the affine open curve $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^2})$. We put

$$L_A := \pi_1^{-1}(A), \quad M_A := \alpha^{-1}(L_A).$$

Note that M_A is smooth. Let $\pi_{1,A} : L_A \to A$ and $\alpha_A : M_A \to L_A$ be the restrictions of π_1 and α, respectively. If $P \in A$, then l_P is disjoint from $\mathrm{Sing}(B)$, and hence $\eta(\alpha^{-1}(\pi_1^{-1}(P))) = \gamma^{-1}(l_P)$ is disjoint from $\mathrm{Sing}(S_d)$. Therefore the restriction of η' to M_A is a morphism. It follows that we have a proper birational morphism

$$\beta : \tilde{M} \to M$$

from a smooth surface $\tilde{M}$ to M such that β induces an isomorphism from $\beta^{-1}(M_A)$ to M_A and that the rational map η' extends to a morphism $\tilde{\eta} : \tilde{M} \to \tilde{S}_d$. Summing up, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
M_A & \hookrightarrow & \tilde{M} & \xrightarrow{\tilde{\eta}} & \tilde{S}_d \\
\| & \square & \downarrow \beta & & \downarrow \rho \\
M_A & \hookrightarrow & M & \xrightarrow{\eta} & S_d \\
\alpha_A \downarrow & \square & \downarrow \alpha & & \downarrow \gamma \\
L_A & \hookrightarrow & L & \xrightarrow{\pi_2} & \mathbb{P}^2 \\
\pi_{1,A} \downarrow & \square & \downarrow \pi_1 & & \\
A & \hookrightarrow & \mathbb{P}^1 & . &
\end{array} \tag{5.2}$$

Since the defining equation $x - t^q y + t^{2q} = 0$ of L in $\mathbb{P}^1 \times \mathbb{P}^2$ is a polynomial in $k[x, y][t^q]$, and its discriminant as a quadratic equation of t^q is $y^2 - 4x \neq 0$, the projection π_2 is a finite morphism of degree $2q$ and its inseparable degree is q. Hence η is also a finite morphism of degree $2q$ and its inseparable degree is q. Therefore, in order to prove Proposition 1.6, it is enough to show that M is rational. We denote by $k(M) = k(\overline{M})$ the function field of M. Since $x = t^q y - t^{2q}$ on $\overline{M}$, the field $k(M)$ is generated over k by y, z and t. Let c denote the integer $(q + 1)/d$, and put

$$\tilde{z} := \frac{z}{(y - t^q - t)^c} \in k(M).$$

Then, from the defining equation (5.1) of $\overline{M}$, we have

$$\tilde{z}^d = \frac{y - t^{q^2} - t^q}{y - t^q - t}.$$

Therefore we have

$$y = \frac{\tilde{z}^d(t^q + t) - (t^{q^2} + t^q)}{\tilde{z}^d - 1},$$

and hence $k(M)$ is equal to the purely transcendental extension $k(\tilde{z}, t)$ of k. Thus Proposition 1.6 is proved.

We put

$$\Xi := \tilde{M} \setminus M_A = \beta^{-1}(\alpha^{-1}(\pi_1^{-1}(\mathbb{P}^1(\mathbb{F}_{q^2})))).$$

Since the cyclic covering $\alpha : M \to L$ branches along the curve $\Sigma_1 = \sigma_1(\mathbb{P}^1)$, the section $\sigma_1 : \mathbb{P}^1 \to L$ of π_1 lifts to a section $\tilde{\sigma}_1 : \mathbb{P}^1 \to M$ of $\pi_1 \circ \alpha$. Let $\tilde{\Sigma}_1$ denote the strict transform of the image of $\tilde{\sigma}_1$ by $\beta : \tilde{M} \to M$.

Lemma 5.1. *The Picard group* $\mathrm{Pic}(\tilde{M})$ *of* $\tilde{M}$ *is generated by the classes of* $\tilde{\Sigma}_1$ *and the irreducible components of* Ξ.

Proof. Since $\Sigma_1 \cap \Sigma_q \cap L_A = \emptyset$, the morphism

$$\pi_{1,A} \circ \alpha_A : M_A \to A$$

is a smooth $\mathbb{P}^1$-bundle. Let D be an irreducible curve on $\tilde{M}$, and let e be the degree of

$$\pi_1 \circ \alpha \circ \beta|_D : D \to \mathbb{P}^1.$$

Then the divisor $D - e\tilde{\Sigma}_1$ on $\tilde{M}$ is of degree 0 on the general fiber of the smooth $\mathbb{P}^1$-bundle $\pi_{1,A} \circ \alpha_A$. Therefore $(D - e\tilde{\Sigma}_1)|_{M_A}$ is linearly equivalent in M_A to a multiple of a fiber of $\pi_{1,A} \circ \alpha_A$. Hence D is linearly equivalent to a linear combination of $\tilde{\Sigma}_1$ and irreducible curves in the boundary $\Xi = \tilde{M} \setminus M_A$. □

The rational curves on $\tilde{S}_d$ listed in Proposition 1.8 are exactly equal to the irreducible components of

$$\rho^{-1}(\gamma^{-1}(\bigcup_{P \in \mathbb{P}^1(\mathbb{F}_{q^2})} l_P)).$$

Let $V \subset H^2(\tilde{S}_d)$ denote the linear subspace spanned by the classes of these rational curves. We will show that $V = H^2(\tilde{S}_d)$.

Let $h \in H^2(\tilde{S}_d)$ denote the class of the pull-back of a line of $\mathbb{P}^2$ by the morphism $\gamma \circ \rho : \tilde{S}_d \to \mathbb{P}^2$. Suppose that $P \in \mathbb{P}^1(\mathbb{F}_q)$. Then l_P is disjoint from $\mathrm{Sing}(B)$. Therefore we have

$$h = [(\gamma \circ \rho)^*(l_P)] = [l_P^{(0)}] + \cdots + [l_P^{(d-1)}] \in V.$$

Let $\tilde{B}$ denote the strict transform of B by $\gamma \circ \rho$. Then $\tilde{B}$ is written as $d \cdot R$, where R is a reduced curve on $\tilde{S}_d$ whose support is equal to $\tilde{\eta}(\tilde{\Sigma}_1)$. On the other hand, the class of the total transform $(\gamma \circ \rho)^* B$ of B by $\gamma \circ \rho$ is equal to $(q+1)h$. Since the difference of the divisors $d \cdot R$ and $(\gamma \circ \rho)^* B$ is a linear combination of exceptional curves of ρ, we have

$$\tilde{\eta}_*([\tilde{\Sigma}_1]) \in V. \tag{5.3}$$

By the commutativity of the diagram (5.2), we have

$$\tilde{\eta}(\Xi) \quad \subset \quad \rho^{-1}(\gamma^{-1}(\bigcup_{P\in\mathbb{P}^1(\mathbb{F}_{q^2})} l_P)).$$

Hence, for any irreducible component Γ of Ξ, we have

$$\tilde{\eta}_*([\Gamma]) \in V. \tag{5.4}$$

Let C be an arbitrary irreducible curve on $\tilde{S}_d$. Then we have

$$\tilde{\eta}_*\tilde{\eta}^*([C]) = 2q[C].$$

By Lemma 5.1, there exist integers $a, b_1, \dots, b_m$ and irreducible components $\Gamma_1, \dots, \Gamma_m$ of Ξ such that the divisor η^*C of $\tilde{M}$ is linearly equivalent to

$$a\tilde{\Sigma}_1 + b_1\Gamma_1 + \cdots + b_m\Gamma_m.$$

By (5.3) and (5.4), we obtain

$$[C] = \frac{1}{2q}\tilde{\eta}_*\tilde{\eta}^*([C]) \in V.$$

Therefore $V \subset H^2(\tilde{S}_d)$ is equal to the linear subspace spanned by the classes of all curves. Combining this fact with Corollary 1.7, we obtain $V = H^2(\tilde{S}_d)$.

6. Supersingular $K3$ surfaces

In this section, we prove Proposition 1.9. First, we recall some facts on supersingular $K3$ surfaces. Let Y be a supersingular $K3$ surface in characteristic p, and let $\mathrm{NS}(Y)$ denote its Néron-Severi lattice, which is an even hyperbolic lattice of rank 22. Artin [3] showed that the discriminant of $\mathrm{NS}(Y)$ is written as $-p^{2\sigma}$, where σ is a positive integer ≤ 10. This integer σ is called the *Artin invariant* of Y. Ogus [13, 14] and Rudakov-Shafarevic [15] proved that, for each p, a supersingular $K3$ surface with Artin invariant 1 is unique up to isomorphisms. Let X_p denote the supersingular $K3$ surface with Artin invariant 1 in characteristic p. It is known that X_3 is isomorphic to the Fermat quartic surface, and that X_5 is isomorphic to the Fermat sextic double plane. (See, for example, [12] and [17], respectively.) Therefore, in order to prove Proposition 1.9, it is enough to prove the following:

Proposition 6.1. *Suppose that $(d, q+1) = (4, 4)$ or $(2, 6)$. Then, among the curves on $\tilde{S}_d$ listed in Proposition 1.8, there exist 22 curves whose classes together with the intersection pairing form a lattice of rank 22 with discriminant $-p^2$.*

Proof. Suppose that $p = q = 3$ and $d = 4$. We put $\alpha := \sqrt{-1} \in \mathbb{F}_9$, so that $\mathbb{F}_9 := \mathbb{F}_3(\alpha)$. Consider the projective space $\mathbb{P}^3$ with homogeneous coordinates $[w : x_0 : x_1 : x_2]$. By Proposition 1.4, the surface S_4 is defined in $\mathbb{P}^3$ by an equation

$$w^4 = 2(x_0^3x_1 + x_0x_1^3) - x_2^4 - (x_2^2 - x_1x_0)^2.$$

Hence the singular locus $\mathrm{Sing}(S_4)$ of S_4 consists of the three points

$$\begin{aligned}
Q_0 \quad &:= \quad [0 : 1 : 1 : 0] \quad (\text{located over } \phi([1 : \alpha]) = \phi([1 : -\alpha]) \in B),\\
Q_1 \quad &:= \quad [0 : 1 : 2 : 1] \quad (\text{located over } \phi([1 : 1 + \alpha]) = \phi([1 : 1 - \alpha]) \in B),\\
Q_2 \quad &:= \quad [0 : 1 : 2 : 2] \quad (\text{located over } \phi([1 : 2 + \alpha]) = \phi([1 : 2 - \alpha]) \in B),
\end{aligned}$$

and they are rational double points of type A_3. The minimal resolution $\rho : \tilde{S}_4 \to S_4$ is obtained by blowing up twice over each singular point Q_a ($a \in \mathbb{F}_3$). The rational curves $l_P^{(i)}$ on $\tilde{S}_4$ given in Proposition 1.8 are the strict transforms of the following 40 lines $\bar{L}_\tau^{(\nu)}$ in $\mathbb{P}^3$ contained in S_4, where $\nu = 0, \dots, 3$:

$$\begin{aligned}
\bar{L}_0^{(\nu)} &:= \{x_1 = w - \alpha^\nu x_2 = 0\},\\
\bar{L}_1^{(\nu)} &:= \{x_0 + x_1 - x_2 = w - \alpha^\nu (x_2 + x_0) = 0\},\\
\bar{L}_2^{(\nu)} &:= \{x_0 + x_1 + x_2 = w - \alpha^\nu (x_2 - x_0) = 0\},\\
\bar{L}_\infty^{(\nu)} &:= \{x_0 = w - \alpha^\nu x_2 = 0\},\\
\bar{L}_{\pm\alpha}^{(\nu)} &:= \{-x_0 + x_1 \pm \alpha x_2 = w - \alpha^\nu x_2 = 0\},\\
\bar{L}_{1\pm\alpha}^{(\nu)} &:= \{\pm\alpha x_0 + x_1 + (-1 \pm \alpha) x_2 = w - \alpha^\nu (x_2 + x_0) = 0\},\\
\bar{L}_{2\pm\alpha}^{(\nu)} &:= \{\mp\alpha x_0 + x_1 + (1 \pm \alpha) x_2 = w - \alpha^\nu (x_2 - x_0) = 0\}.
\end{aligned}$$

We denote by $L_\tau^{(\nu)}$ the strict transform of $\bar{L}_\tau^{(\nu)}$ by ρ. Note that the image of $\bar{L}_\tau^{(\nu)}$ by the covering morphism $S_4 \to \mathbb{P}^2$ is the line $l_{\phi([1:\tau])}$. Note also that, if $\tau \in \mathbb{F}_3 \cup \{\infty\}$, then $\bar{L}_\tau^{(\nu)}$ is disjoint from $\mathrm{Sing}(S_4)$, while if $\tau = a + b\alpha \in \mathbb{F}_9 \setminus \mathbb{F}_3$ with $a \in \mathbb{F}_3$ and $b \in \mathbb{F}_3 \setminus \{0\} = \{\pm 1\}$, then $\bar{L}_\tau^{(\nu)} \cap \mathrm{Sing}(S_4)$ consists of a single point Q_a. Looking at the minimal resolution ρ over Q_a explicitly, we see that the three exceptional (-2)-curves in $\tilde{S}_4$ over Q_a can be labeled as $E_{a-\alpha}, E_a, E_{a+\alpha}$ in such a way that the following hold:

- $\langle E_{a-\alpha}, E_a\rangle = \langle E_a, E_{a+\alpha}\rangle = 1$, $\langle E_{a-\alpha}, E_{a+\alpha}\rangle = 0$.
- Suppose that $b \in \{\pm 1\}$. Then $L_{a+b\alpha}^{(\nu)}$ intersects $E_{a+b\alpha}$, and is disjoint from the other two irreducible components E_a and $E_{a-b\alpha}$.
- The four intersection points of $L_{a+b\alpha}^{(\nu)}$ $(\nu = 0, \dots, 3)$ and $E_{a+b\alpha}$ are distinct.

Using these, we can calculate the intersection numbers among the $9+40$ curves E_τ and $L_{\tau'}^{(\nu)}$ $(\tau \in \mathbb{F}_9,\ \tau' \in \mathbb{F}_9 \cup \{\infty\},\ \nu = 0, \dots, 3)$. From among them, we choose the following 22 curves:

$$\begin{gathered}
E_{-\alpha}, E_0, E_\alpha, E_{1-\alpha}, E_1, E_{1+\alpha}, E_{2-\alpha}, E_2, E_{2+\alpha},\\
L_0^{(0)}, L_0^{(1)}, L_0^{(2)}, L_0^{(3)}, L_1^{(0)}, L_1^{(1)}, L_2^{(0)}, L_2^{(1)}, L_\infty^{(1)},\\
L_{-\alpha}^{(0)}, L_{-\alpha}^{(1)}, L_{1-\alpha}^{(2)}, L_{2-\alpha}^{(0)}.
\end{gathered}$$

Their intersection numbers are calculated as in Table 6.1. We can easily check that this matrix is of determinant -9.

The proof for the case $p = q = 5$ and $d = 2$ is similar. We put $\alpha := \sqrt{2}$ so that $\mathbb{F}_{25} = \mathbb{F}_5(\alpha)$. In the weighted projective space $\mathbb{P}(3,1,1,1)$ with homogeneous coordinates $[w : x_0 : x_1 : x_2]$, the surface S_2 for $p = q = 5$ is defined by

$$w^2 = 2(x_0^5 x_1 + x_0 x_1^5) - x_2^6 - (x_2^2 + x_0 x_1)^3.$$

The singular locus $\mathrm{Sing}(S_2)$ consists of ten ordinary nodes

$$Q_{\{a+b\alpha, a-b\alpha\}} \quad (a \in \mathbb{F}_5, b \in \{1, 2\})$$

located over the nodes $\phi([1 : a + b\alpha]) = \phi([1 : a - b\alpha])$ of the branch curve B. Let $E_{\{a+b\alpha, a-b\alpha\}}$ denote the exceptional (-2)-curve in $\tilde{S}_2$ over $Q_{\{a+b\alpha, a-b\alpha\}}$ by the minimal resolution. As the 22 curves, we choose the following eight exceptional

$$\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -2
\end{bmatrix}$$

TABLE 6.1. Gram matrix of $\mathrm{NS}(\tilde{S}_4)$ for $q = 3$

$$\begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 3 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -2
\end{bmatrix}$$

TABLE 6.2. Gram matrix of $\mathrm{NS}(\tilde{S}_2)$ for $q = 5$

(-2)-curves

$$E_{\{-\alpha,\alpha\}},\quad E_{\{-2\,\alpha,2\,\alpha\}},\quad E_{\{1-\alpha,1+\alpha\}},\quad E_{\{1-2\,\alpha,1+2\,\alpha\}},$$
$$E_{\{2-\alpha,2+\alpha\}},\quad E_{\{3-2\,\alpha,3+2\,\alpha\}},\quad E_{\{4-\alpha,4+\alpha\}},\quad E_{\{4-2\,\alpha,4+2\,\alpha\}},$$

and the strict transforms of the following 14 curves on S_2:

$$\begin{array}{l}
\{\ x_1 \ = \ w-2\,\alpha x_2{}^3 \ = \ 0\ \},\\
\{\ x_1 \ = \ w+2\,\alpha x_2{}^3 \ = \ 0\ \},\\
\{\ x_0+x_1+4\,x_2 \ = \ w+2\,\alpha\,(3\,x_0+x_2)^3 \ = \ 0\ \},\\
\{\ 3\,x_0+x_1+3\,\alpha x_2 \ = \ w-2\,\alpha x_2{}^3 \ = \ 0\ \},\\
\{\ 2\,x_0+x_1+4\,\alpha x_2 \ = \ w+2\,\alpha x_2{}^3 \ = \ 0\ \},\\
\{\ 3\,x_0+x_1+2\,\alpha x_2+3\,x_0 \ = \ w-2\,\alpha x_2{}^3 \ = \ 0\ \},\\
\{\ (3+3\,\alpha)\,x_0+x_1+(4+\alpha)\,x_2 \ = \ w+2\,\alpha\,(3\,x_0+x_2)^3 \ = \ 0\ \},\\
\{\ (4+\alpha)\,x_0+x_1+(4+2\,\alpha)\,x_2 \ = \ w+2\,\alpha\,(3\,x_0+x_2)^3 \ = \ 0\ \},\\
\{\ (2+3\,\alpha)\,x_0+x_1+(3+3\,\alpha)\,x_2 \ = \ w-2\,\alpha\,(x_0+x_2)^3 \ = \ 0\ \},\\
\{\ (1+\alpha)\,x_0+x_1+(3+\alpha)\,x_2 \ = \ w-2\,\alpha\,(x_0+x_2)^3 \ = \ 0\ \},\\
\{\ (1+\alpha)\,x_0+x_1+(2+4\,\alpha)\,x_2 \ = \ w-2\,\alpha\,(x_2+4\,x_0)^3 \ = \ 0\ \},\\
\{\ (2+3\,\alpha)\,x_0+x_1+(2+2\,\alpha)\,x_2 \ = \ w+2\,\alpha\,(x_2+4\,x_0)^3 \ = \ 0\ \},\\
\{\ (3+3\,\alpha)\,x_0+x_1+(1+4\,\alpha)\,x_2 \ = \ w-2\,\alpha\,(x_2+2\,x_0)^3 \ = \ 0\ \},\\
\{\ (4+4\,\alpha)\,x_0+x_1+(1+2\,\alpha)\,x_2 \ = \ w-2\,\alpha\,(x_2+2\,x_0)^3 \ = \ 0\ \}.
\end{array}$$

Their intersection matrix is given in Table 6.2. It is of determinant -25. □

Remark 6.2. In the case $q = 5$, the Ballico-Hefez curve B is one of the sextic plane curves studied classically by Coble [5].

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Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

E-mail address: `hoangthanh2127@yahoo.com`

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

E-mail address: `shimada@math.sci.hiroshima-u.ac.jp`