ON CHARACTERISTIC POLYNOMIALS OF AUTOMORPHISMS OF ENRIQUES SURFACES

SIMON BRANDHORST, SŁAWOMIR RAMS, AND ICHIRO SHIMADA

Abstract. Let f be an automorphism of a complex Enriques surface Y and let p_f denote the characteristic polynomial of the isometry f^* of the numerical Néron-Severi lattice of Y induced by f. We apply a modification of McMullen's method to prove that the modulo-2 reduction $(p_f(x) \bmod 2)$ is a product of modulo-2 reductions of (some of) the five cyclotomic polynomials Φ_m, where $m \leq 9$ and m is odd. We study Enriques surfaces that realize modulo-2 reductions of Φ_7, Φ_9 and show that each of the five polynomials $(\Phi_m(x) \bmod 2)$ is a factor of the modulo-2 reduction $(p_f(x) \bmod 2)$ for a complex Enriques surface.

1. Introduction

The subject of this note are isometries of the numerical Néron-Severi lattices induced by automorphisms of Enriques surfaces. To state our results, let Y (resp. X) be a complex Enriques surface (resp. its K3 cover) and let $\mathrm{Num}(Y)$ be the numerical Néron-Severi lattice of Y (i.e. $\mathrm{Num}(Y) := \mathrm{NS}(Y)/\mathrm{Tors}$). Each automorphism $f \in \mathrm{Aut}(Y)$ induces an isometry $f^* \in \mathrm{O}(\mathrm{Num}(Y))$. It is natural to study the properties of the characteristic polynomial of the latter.

In this note we prove the following refinement of [14, Theorem 1.2].

Theorem 1.1. *Let f be an automorphism of a complex Enriques surface Y and let p_f be the characteristic polynomial of the isometry $f^* : \mathrm{Num}(Y) \to \mathrm{Num}(Y)$.*

a) The modulo-2 reduction $(p_f(x) \bmod 2)$ is a product of (some of) the following polynomials:

$$F_1(x) = x + 1, \quad F_3(x) = x^2 + x + 1, \quad F_5(x) = x^4 + x^3 + x^2 + x + 1,$$
$$F_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \quad F_9(x) = x^6 + x^3 + 1\,.$$

b) Each of the five polynomials F_1, F_3, F_5, F_7, F_9 does appear in the factorization of the modulo-2 reduction $(p_f(x) \bmod 2)$ for an automorphism f

---

Date: 23rd of September 2019.

2010 *Mathematics Subject Classification.* Primary: 14J28; 14J50 Secondary: 37B40.

S. B. is supported by SFB-TRR 195 "Symbolic Tools in Mathematics and their Application" of the German Research Foundation(DFG). S. R. is partially supported by the Polish National Science Centre (NCN) OPUS grant 2017/25/B/ST1/00853. I. S. is supported by JSPS KAKENHI Grant Number 15H05738, 16H03926, and 16K13749.

of a complex Enriques surface. Any realization of F_7 and F_9 is by a semi-symplectic automorphism.

Recall that the proof of [14, Theorem 1.2] shows that each factor of $(p_f(x) \bmod 2)$ either equals one of the five polynomials listed in Thm 1.1, or it is the modulo-2 reduction F_{15} of the cyclotomic polynomial $\Phi_{15} \in \mathbb{Z}[x]$. Moreover, examples with factors F_1, F_3, F_5 were given in [9] (see also [14, Example 3.1]), whereas the question if F_7, F_9 and F_{15} can appear in the factorization of the modulo-2 reduction of p_f for an automorphism $f \in \mathrm{Aut}(Y)$ was left open (c.f. [14, Example 3.1.b]). This question is answered in Theorem 1.1.

To state the next theorem, we introduce some notation. Let us denote the covering involution of the double étale cover $\pi : X \to Y$ by ε. Moreover, we put $\tilde{f} \in \mathrm{Aut}(X)$ to denote a (non-unique) lift of an automorphism $f \in \mathrm{Aut}(Y)$. Let $N := (H^2(X,\mathbb{Z})^{\varepsilon})^{\perp}$ be the orthogonal complement of the ε-invariant sublattice $H^2(X,\mathbb{Z})^{\varepsilon}$ in the lattice $H^2(X,\mathbb{Z})$. Recall that N is stable under the cohomological action $\tilde{f}^*$ and the restriction $f_N := \tilde{f}^*|_N$ is of finite order. Using Theorem 1.1, we can sharpen [14, Theorem 1.1] as well.

Theorem 1.2. *Let Y be a complex Enriques surface and let f be an automorphism of Y. Then, the order of f_N is a divisor of at least one of the following five integers:*

$$36, 48, 56, 84, 120.$$

Among the 28 numbers that satisfy the above condition, at least the following 14 *integers*

$$1, \dots, 10, 12, 14, 15, 20$$

are realized as orders.

Remark 1.3. We note that if the order of f_N is 7 or 9, then the cyclic subgroup generated by f_N is unique up to conjugacy in the orthogonal group $\mathrm{O}(N)$. For the remaining 14 integers

$$16, 18, 21, 24, 28, 30, 36, 40, 42, 48, 56, 60, 84, 120,$$

we do not know whether they arise as orders of f_N for some $f \in \mathrm{Aut}(Y)$.

Originally, our interest in the subject of this note was motivated by the question what constraints on the dynamical spectra of Enriques surfaces result from the existence of the double étale K3 cover (c.f. [22, Theorem 1.2]). Indeed, Theorem 1.1.a yields a new constraint on the Salem numbers that appear as the dynamical degrees of automorphisms of Enriques surfaces (e.g. it implies that none of the Salem numbers given as # 3, 13, 16, 34, 35 in the table in [14, Appendix] can be the dynamical degree of an automorphisms of a complex Enriques surface), whereas Theorem 1.1.b shows that the above constraint cannot be strengthened.

It should be mentioned that automorphism groups of Enriques surfaces remain a subject of intensive research. Much is known in the case of Enriques surfaces with finite automorphism groups (even in positive characteristic) and unnodal Enriques surfaces, but a general picture is still missing. In this context both the constraints given by Theorem 1.2 and the geometry of the families of Enriques surfaces discussed in Propositions 5.3, 4.2, 4.7 are of separate interest. Still, such considerations exceed the scope of this paper.

We sketch our strategy for the proof of Theorem 1.1. The argument in [14] is based on criteria for a polynomial to be the characteristic polynomial of an isometry of a lattice. Unfortunately, all the six polynomials $F_1, \dots, F_9, F_{15}$ do appear as factors of modulo-2 reductions of characteristic polynomials of isometries of the lattice $U \oplus E_8(-1)$ and the lattice N. Thus we need to take Hodge structures and the ample cone into account as well. In this note we apply a modification of McMullen's method (see [16], [17]) to obtain constraints on automorphisms of Enriques surfaces that can realize the factors F_7, F_9, F_{15}. In particular, we can rule out the existence of the highest-degree factor F_{15} (Prop. 3.1), and derive properties of the K3 covers of Enriques surfaces which realize F_7 (Prop. 5.2) and F_9 (Section 4). Then an algorithm based on Borcherd's method ([1], [2]) and the ideas from [28] and [4] allow us to find abstract Enriques surfaces realizing F_7 and F_9. For the readers convenience, the algorithm is presented in Section 6 in pseudocode. We close this section with a related open question. For an Enriques surface Y we call the order of the image of bi-canonical representation

$$\mathrm{Aut}(Y) \to \mathrm{GL}\left(H^0(Y, K_Y^{\otimes 2})\right)$$

of the automorphism group the transcendental index $I(Y)$ of Y.

Question 1.4. What are the possible transcendental indices of complex Enriques surfaces?

Note that all realizations of F_7 and F_9 must be by semi-symplectic automorphisms. Hence, we know that 7 and 9 do not divide $I(Y)$.

Notation: In this note, we work over the field of complex numbers $\mathbb{C}$. Given a prime p, $\mathbb{Z}_p$ denotes the ring of p-adic integers. For a ring R, we denote by $R^\times$ its group of units. For a group G and a prime p, G_p is the p-Sylow subgroup of G.

2. Preliminaries

Basic notation. We maintain the notation of the previous section. In particular, $\pi : X \to Y$ is the K3 cover of Y and ε is the covering involution of π. Moreover, we have the finite index sublattice

$$M \oplus N \subseteq H^2(X, \mathbb{Z}) \tag{2.1}$$

where $M := H^2(X, \mathbb{Z})^\varepsilon$ coincides with the pullback of $H^2(Y, \mathbb{Z})$ by π and $N := M^\perp$ (see e.g. [20]). In particular, we have $M \simeq U(2) \oplus E_8(-2)$ and $N \simeq U \oplus U(2) \oplus E_8(-2)$, where U (resp. E_8) denotes the unimodular hyperbolic plane (resp. the unique even unimodular positive-definite lattice of rank 8). Let f be an automorphism of Y. The sublattices M and N are preserved by the isometry $\tilde{f}^* \in \mathrm{Aut}(H^2(X,\mathbb{Z}))$, so as in [14] we can define the maps

$$f_M := \tilde{f}^*|_M \text{ and } f_N := \tilde{f}^*|_N$$

and let p_N, p_M (resp. μ_N, μ_M) denote their characteristic (resp. minimal) polynomials. Then, (see e.g. [14, §3]) we have

$$p_M \equiv p_f \bmod 2 \text{ and } (p_M \bmod 2) \mid (p_N \bmod 2). \tag{2.2}$$

As we already mentioned, f_N is a map of finite order (see e.g. [22, Lemma 4.2]), so p_N is a product of cyclotomic polynomials.

Recall that (see [24, Prop 2.2], [15, Thm 1.1])

$$N \cap \mathrm{NS}(X) \text{ contains no vectors of square } (-2). \tag{2.3}$$

Indeed, suppose to the contrary. By Riemann-Roch, a vector of square (-2) in $N \cap \mathrm{NS}(X)$ or its negative is the class of an effective divisor $C \in \mathrm{NS}(X)$ such that $\langle \pi^*(D), C\rangle = 0$ for every $D \in \mathrm{NS}(Y)$. This is impossible by the Nakai-Moishezon criterion, because we can choose D so that $\pi^*(D)$ is ample.

For an automorphism f and an integer $k \in \mathbb{N}$ we define two lattices

$$N_k := \ker(\Phi_k(f_N)) \quad \text{ and } \quad M_k := \ker(\Phi_k(f_M))\,. \tag{2.4}$$

where $\Phi_k(x)$ stands for the k-th cyclotomic polynomial. Finally, to simplify our notation we put

$$F_k(x) := (\Phi_k(x) \bmod 2)\,.$$

An automorphism f of an Enriques surface is called *semi-symplectic*, if it acts trivially on the global sections $H^0(Y, K_Y^{\otimes 2})$ of the bi-canonical bundle. This is the case if and only both lifts $\tilde{f}$ and $\tilde{f}\circ\varepsilon$ of f act on $H^0(X,\Omega_X^2)$ as ± 1. We denote by $\mathrm{Aut}_s(Y)$ the subgroup of semi-symplectic automorphisms.

Lattice. Let $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ and K be the fraction field of R. An R-*lattice* is a finitely generated free R-module equipped with a non-degenerate symmetric K-valued bilinear form b. If the form is R valued, we call the lattice *integral*. If further $b(x,x) \in 2R$ for every $x \in L$, the lattice is called *even*. The *dual lattice* of L is

$$L^\vee = \{x \in L \mid b(x,L) \subseteq R\}.$$

If L is integral, then $L \subseteq L^\vee$ and we call the quotient $L^\vee/L$ the *discriminant group* of L. For $r \in R$, an R-lattice L is called r-*modular* if $rL^\vee = L$. If $r = 1$, we call the lattice *unimodular*. The Gram matrix $G = (G_{ij})$ with respect to an R-basis $(e_1, \dots e_n)$ of L is defined by $G_{ij} = b(e_i, e_j)$. The determinant $\det L \in R/R^{\times 2}$ of L is the determinant of any Gram matrix. For $R = \mathbb{Z}$ we have $|L^\vee/L| = |\det L|$. The discriminant group carries the discriminant bilinear form induced by $b(x,y) \mod R$ for $x, y \in L^\vee$. If L is an even lattice, its discriminant group moreover carries a torsion quadratic form induced by $x \mapsto b(x,x) \mod 2R$, called *discriminant form*. We say that two R-lattices (L,b), (L',b') are isomorphic if there is an R-linear isomorphism $\phi : L \to L'$ such that $b(x,x) = b'(\phi(x),\phi(x))$. For $r \in R$ we denote by $L(r)$

the lattice with the same underlying free module as L but with bilinear form rb.

Let L, L', L'' be lattices. The orthogonal direct sum of two lattices is denoted by $L \oplus L'$. A sublattice $L' \subseteq L$ is called *primitive* if L/L' is torsion free. This is equivalent to $(L' \otimes K) \cap L = L'$. We call

$$L' \oplus L'' \subseteq L$$

a *primitive extension* if L', L'' are primitive sublattices of L and $\operatorname{rank} L' + \operatorname{rank} L'' = \operatorname{rank} L$. The finite group $L''/(L \oplus L')$ is the *glue* of the primitive extension. For any prime p dividing its order, we say that L and L' are glued above/over p. The signature (pair) (s_+, s_-) of a $\mathbb{Z}$-lattice L is the signature of $L \otimes \mathbb{R}$ where s_+ is the number of positive and s_- is the number of negative eigenvalues of a Gram matrix. We denote by U the even unimodular lattice of signature $(1,1)$. By A_n $(n \in \mathbb{N})$, D_n $(n \geq 4)$, E_6, E_7, E_8 the positive definite root lattice with the respective Dynkin diagram.

Genus. Two $\mathbb{Z}$-lattices L and L' are in the same *genus* if $L \otimes \mathbb{R} \cong L' \otimes \mathbb{R}$ and for all prime numbers p we have $L \otimes \mathbb{Z}_p \cong L' \otimes \mathbb{Z}_p$. The genus is an effectively computable invariant and has a compact description in terms of the so called *genus symbols* introduced by Conway and Sloane [8, Chapter 15]. In what follows we give a short account.

Let p be an odd prime. A p-adic unimodular lattice L_1 is determined up to isometry by its rank n_1 and the p-adic square class $\epsilon_1 \in \{\pm 1\}$ of its determinant. This is denoted by the symbol $1^{\epsilon_1 n_1}$. Let $q = p^k$, and recall that a lattice is q-modular if it is of the form $L(q)$ for some unimodular L. A q-modular p-adic lattice is given up to isomorphism by its scale q, its rank n_q and the square class of the unit part of its determinant $\epsilon_q(L(q)) := \epsilon_1(L)$. This is denoted by the symbol $q^{\epsilon_q n_q}$.

A p-adic lattice L admits a so called *Jordan decomposition*

$$L = L_1 \oplus L_p \oplus \cdots \oplus L_{p^k}$$

into p^i-modular lattices L_{p^i}. The latter are called the Jordan constituents. The decomposition is not unique. Nevertheless we can compute the isomorphism class of L from it. It is uniquely determined by the collection of $(\epsilon_q, n_q)_q$ for the q-modular lattices L_q as q runs through the powers of p. This collection is called the *p-adic symbol* of L. We introduce the notation for p-adic symbols with an example. The 3-adic symbol

$$1^2 3^{-2} 27^5$$

denotes a $\mathbb{Z}_3$-lattice

$$L = L_1 \oplus L_3 \oplus L_{27}$$

such that L_1 is unimodular of rank 2 and determinant a square, L_3 is 3 modular of rank 2 and unit part of the determinant a non-square, that is the determinant is $2 \cdot 3$ and the unit part is 2 which is a 3-adic non-square unit, L_{27} is 27 modular of rank 5 and the unit part of its determinant is a square. We see that L is isomorphic to $\mathbb{Z}_3^9$ with diagonal Gram matrix given by

$$\mathrm{diag}(1, 1, 6, 3, 27, 27, 27, 27, 27).$$

An *even* unimodular 2-adic lattice L_1 is determined by its rank n_1 and $\epsilon_1 \in \{\pm 1\}$ which is 1 if the determinant is congruent to 1 or 7 modulo 8 and -1 if it is congruent to 3 or 5. This is denoted by $1^{\epsilon_1 n_1}$. As before we obtain symbols for q-modular lattices and have a Jordan decomposition. A Jordan constituent is called *even* if it is the twist of an even unimodular 2-adic lattice. A 2-adic lattice all whose Jordan constituents are even is called *completely even.* Two completely even lattices are isomorphic if and only if they have the same invariants (ϵ_q, n_q) for all powers q of 2. If the lattices in question are not completely even, the classification involves an additional quantity called the oddity. However, in this note all lattices considered are completely even.

To describe a genus it is enough to give the signature pair and the local symbols at primes dividing twice the determinant. This is collected in a single symbol called the (Conway-Sloane) genus symbol. For example A_2 is even of rank 2 and has determinant 3. In particular it is 2-adically unimodular and has the 2-adic symbol 1^{-2}. To compute the 3-adic symbol, we note that it is 3-adically equivalent to the lattice $\mathrm{diag}(2, 6)$ with 3-adic symbol $1^{-1}3^{-1}$. Together this gives

$$\mathrm{II}_{(2,0)} 1^{-1} 3^{-1}.$$

Here the II indicates that this lattice is even and the index $(2, 0)$ that it is positive definite of rank 2. Finally, the unimodular Jordan constituents can be reconstructed from the determinant. Thus they are omitted and the symbol is abbreviated to $\mathrm{II}_{(2,0)} 3^{-1}$.

Note that Conway and Sloane give necessary and sufficient conditions on when a collection of local symbols defines a non-empty genus [8, Thm 15.11 on p. 383].

Remark 2.1. The genus symbols and their relation with discriminant forms are implemented in sageMath [26] by the first author. It is possible to compute all classes in a genus using Kneser's neighboring algorithm [27] and Siegel's mass formula. Similarly roots can be found using short vector enumerators [6, §.2.7.3]. We used the implementation provided by PARI [25] via sageMath.

In the following we relate the genus symbols with primitive extensions and isometries.

Lemma 2.2. *Let L and L' be completely even p-adic lattices with symbols $(\epsilon_q, n_q)_q$ respectively $(\epsilon'_q, n'_q)_q$ then $L \oplus L'$ has symbol $(\epsilon_q \epsilon'_q, n_q + n'_q)$.*

Proof. If $\bigoplus L_q$ and $\bigoplus L'_q$ are the respective Jordan decompositions, then $\bigoplus (L_q \oplus L'_q)$ is a Jordan decomposition of the sum. Finally the square class is multiplicative and the rank is additive. □

Lemma 2.3. *Let L and L' be completely even p-adic lattices with symbols $(\epsilon_q, n_q)_q$ and $(\epsilon'_q, n'_q)_q$. Then there is a primitive extension $L \oplus L' \subseteq L''$ with L'' unimodular if and only if for all $q > 1$ $n'_q = n_q$ and $\epsilon'_q = \delta^{n_q}\epsilon_q$ where* $\delta = \begin{cases} 1 & \text{for } p \equiv 1, 2 \mod 4 \\ -1 & \text{for } p \equiv 3 \mod 4. \end{cases}$

Proof. From [21, Cor. 1.6.2] we know that the existence of a unimodular primitive extension is equivalent to existence of an anti isometry of the discriminant forms of L and L'. Since the lattices are completely even, this means precisely that the Jordan constituents of scale $q > 1$ are anti isomorphic. If L is a q-modular lattice with symbol (n_q, ϵ_q), then $L(-1)$ has determinant $(-1)^{n_q} \det L$. Hence the symbol of $L(-1)$ is $(n_q, \delta^{n_q}\epsilon_q)_q$ where for $p \neq 2$, δ is 1 or -1 according to -1 being a p-adic square or not. If $p = 2$, then $\delta = 1$. □

In the sequel we will apply the following lemma.

Lemma 2.4. *Let L be a $\mathbb{Z}$-lattice and let $g \in \mathrm{O}(L)$ be an isometry with minimal polynomial Φ_3. Then L is completely even and the 2-adic symbols of the genus of L are of the form*

$$q_i^{\epsilon_i n_i} \qquad \textit{where } q_i = 2^i,\ n_i \textit{ is even and } \epsilon_i = (-1)^{n_i/2}.$$

Proof. This is a special case of [12, Prop. 2.17, Kor. 2.36]. □

In particular, when L is a rank-2 (resp. rank-4) lattice of discriminant at most 4 (resp. 16) its 2-adic symbols are 1^{-2}, 2^{-2} (resp. 1^4, $1^{-2}2^{-2}$, 2^4, $1^{-2}4^{-2}$)

$\Phi_n(x)$-lattices. In the sequel we need the notion of a $\Phi_n(x)$-lattice. The reader can consult [10], [17, § 5] for a concise and more general exposition of the facts we briefly sketch below.
Recall that a $\Phi_n(x)$-lattice is defined to be a pair (L, f) where L is an integral lattice and $f \in \mathrm{O}(L)$ is an isometry with characteristic polynomial $\Phi_n(x)$. Let $n > 2$, the principal $\Phi_n(x)$-lattice $(L_0, \langle\cdot,\cdot\rangle_0, f_0)$ is defined as the $\mathbb{Z}$-module $L_0 := \mathbb{Z}[\zeta_n]$ equipped with the scalar product

$$\langle g_1, g_2 \rangle_0 = \mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}[\zeta_n]} \left(\frac{g_1 \overline{g_2}}{r'_n(\zeta_n + \zeta_n)} \right)$$

where Tr is the field trace of $\mathbb{Q}[\zeta_n]/\mathbb{Q}$, $r_n \in \mathbb{Q}[x]$ is the minimal polynomial of $\zeta_n + \zeta_n^{-1}$, and r'_n is its derivative. Finally, $f_0 \colon L_0 \to L_0$, $x \mapsto \zeta_n \cdot x$, is an isometry with minimal polynomial Φ_n. One can show that L_0 is an even lattice and

$$\det(L_0) = |\Phi_n(1)\Phi_n(-1)|. \tag{2.5}$$

Given a pair (L, f) as above and an element $a \in \mathbb{Z}[f + f^{-1}] \subset \mathrm{End}(L)$ one can define another inner product on L by the formula $\langle g_1, g_2 \rangle_a := \langle a g_1, g_2 \rangle_0$.

We say that the resulting lattice is the twist of L by a and denote it by $L(a)$. Recall, that by [17, Thm 5.2]

(2.6) every $\Phi_n(x)$-lattice is a twist of the principal lattice $(L_0, \langle\cdot,\cdot\rangle_0, f_0)$.

The genus symbols of Φ_n-lattices are computed in [12, Satz 2.57].

Equivariant gluing. We note the following well known Lemma for later use.

Lemma 2.5. *If $A \oplus B \subseteq C$ is a primitive extension, then*

$$\det A \det B = [C : A \oplus B]^2 \cdot \det C$$

and

$$\det A \mid [C : A \oplus B] \cdot \det C.$$

Moreover, if p is a prime such that $p \nmid [C : A \oplus B]$, then

$$C \otimes \mathbb{Z}_p = (A \otimes \mathbb{Z}_p) \oplus (B \otimes \mathbb{Z}_p).$$

Let $a \in \mathrm{O}(A), b \in \mathrm{O}(B), c \in \mathrm{O}(C)$ be isometries. We call $(A, a) \oplus (B, b) \subseteq (C, c)$ an equivariant primitive extension if the restriction $c|_{A\oplus B} = a \oplus b$.

Lemma 2.6. *Let $(A, a) \oplus (B, b) \hookrightarrow (C, c)$ be an equivariant primitive extension with characteristic polynomials p_A, p_B. Then any prime dividing the index $[C : A \oplus B]$ divides the resultant $\mathrm{res}(p_A, p_B)$.*

Proof. Apply [17, Prop. 4.2] to $G = C/(A \oplus B)$. □

Lemma 2.7. *Let $(A, a) \oplus (B, b) \hookrightarrow (C, c)$ be an equivariant primitive extension. Suppose that the characteristic polynomial p_a of a is $\Phi_n(x)$. Then the glue $G = C/(A \oplus B)$ satisfies*

$$|G| \mid \mathrm{res}(\Phi_n, \mu)$$

where $\mu = \mu_b$ is the minimal polynomial of b.

Proof. Let G_A denote the orthogonal projection of G to $A^\vee/A$ and $\overline{a}$ the automorphism on G_A induced by a. Since G_A is a finite $\mathbb{Z}[\zeta_n]$-module generated by one element, we have $G_A = \mathbb{Z}[\zeta_n]/I$ where I is the kernel of the map $\mathbb{Z}[\zeta_n] \mapsto \mathrm{End}\, G_A$ that sends the root of unity ζ_n to $\overline{a}$. This yields:

$$\mu(\overline{a}) = 0 \text{ thus } \mu(\zeta_n) \in I$$

and

$$|G| = |G_A| = |\mathcal{O}_K/I| = N(I) \mid N(\mu(\zeta_n)) = \prod_{(k,n)=1} \mu(\zeta_n^k) = \mathrm{res}(\phi_n, \mu_b)$$

where $N(I)$ is the norm of the ideal I. □

3. Ruling out the factor F_{15}

The main aim of this section is to prove the following proposition.

Proposition 3.1. *Let f be an automorphism of an Enriques surface Y and let p_f be the minimal polynomial of the map $f^* : \mathrm{Num}(Y) \to \mathrm{Num}(Y)$. Then the modulo-2 reduction $(p_f(x) \bmod 2)$ is never divisible by the polynomial*

$$F_{15} = x^8 + x^7 + x^5 + x^4 + x^3 + x + 1$$

i.e. by the modulo-2 reduction of the cyclotomic polynomial $\Phi_{15}(x) \in \mathbb{Z}[x]$.

Recall (see e.g. [5]) that p_f is a product of cyclotomic polynomials and at most one Salem factor. Since p_f is reciprocal, $(p_f(x) \bmod 2)$ is divisible by an irreducible factor of F_{15} if and only if it is divisible by the whole F_{15} (c.f. [14]).

Proof of Prop. 3.1 Assume that $F_{15} \mid (p_f \bmod 2)$. Combined with [14, Remark 2.4], this implies that

$$(p_M \bmod 2) = F_{15} \cdot F_1^2 \quad \text{and} \quad (F_{15} \cdot F_1^2) \mid (p_N \bmod 2). \tag{3.1}$$

By [14, Lemma 2.1] and [14, Lemma 2.5] the charateristic polynomial p_N is a product of cyclotomic polynomials of degree at most 8. Computing modulo-2 reductions of all such cyclotomic polynomials, one infers that either $\Phi_{15} \mid p_N$ or $\Phi_{30} \mid p_N$. Replacing $\tilde{f}$ by a power coprime to 15 we can assume that p_N is a product of the Φ_k for $k \in \{1, 3, 5, 15\}$. Together with (3.1) this leaves us with the two possibilities

$$p_N = \Phi_{15} \cdot \Phi_1^4 \quad \text{or} \quad p_N = \Phi_{15} \cdot \Phi_3 \cdot \Phi_1^2. \tag{3.2}$$

We consider the (primitive) f_N-invariant sublattice N_{15} (see (2.4)) and denote its orthogonal complement in N by $N_{15}^{\perp}$. Since $\Phi_{15}(x)$ has no real roots, the signature of N_{15} is of the form $(2k, 2(4-k))$ with $k \in \{1, 2, 3, 4\}$. Recall that N is of signature $(2, 10)$ and contains N_{15}. Thus the signature of N_{15} is either $(0, 8)$ or $(2, 6)$.

By definition

$$N_{15} \oplus N_{15}^{\perp} \subset N$$

is a primitive extension. Let $G = N/(N_{15} \oplus N_{15}^{\perp})$ be the glue between N_{15} and $N_{15}^{\perp}$. Then by Lemma 2.7 we have

$$|G| \mid \mathrm{res}(\Phi_{15}, \mu_{f|_{N_{15}^{\perp}}})$$

But we have

$$\mathrm{res}(\Phi_{15}, \Phi_1) = 1 \text{ and } \mathrm{res}(\Phi_{15}, \Phi_3) = 25\,. \tag{3.3}$$

In particular, if $|G| > 1$ then

$$p_N = \Phi_{15} \cdot \Phi_3 \cdot \Phi_1^2. \tag{3.4}$$

In what follows we treat the cases whether G is trivial or not separately.

The case when G is trivial. Assume that the glue G is trivial, i.e.

$$N_{15} \oplus N_{15}^{\perp} = N \in \mathrm{II}_{(2,10)} 2^{10}. \tag{3.5}$$

Let (ϵ_q, n_q) be the 2-adic genus symbol of N_{15} and (ϵ'_q, n'_q) the symbol of $N_{15}^{\perp}$. From Lemma 2.2 we infer that $10 = n_2 + n'_2$. Further $n'_2 \leq \operatorname{rank} N_{15}^{\perp} = 4$ and $n_2 \leq \operatorname{rank} N_{15} = 8$. Thus we obtain $6 \leq n_2 \leq 8$. Since N_{15} is a Φ_{15}-lattice, we can calculate all Φ_{15}-lattices matching this condition. There is exactly one such lattice up to isometry:

$$N_{15} \cong E_8(-2) \in \mathrm{II}_{(0,8)} 2^{8}. \tag{3.6}$$

Using Lemma 2.2 once more, we calculate the genus symbol of $N_{15}^{\perp}$ from those of N and N_{15} and see that

$$N_{15}^{\perp} \cong U \oplus U(2) \in \mathrm{II}_{(2,2)} 2^{2} \tag{3.7}$$

is the unique class in its genus. From (3.6), (3.7) and [23, Lemma 7.7] we infer that the spectral radius of f_M is one (i.e. f has trivial entropy). Thus p_M is not divisible by a Salem polynomial and must be a product of cyclotomic polynomials. A direct computation of modulo-2 reductions of all cyclotomic polynomials of degree at most 8 shows that either Φ_{30} or Φ_{15} divides p_M. By replacing $\tilde{f}$ with its iteration (i.e. by $\tilde{f}^2$ or $\tilde{f}^4$) we can assume that

$$p_M = \Phi_{15} \cdot \Phi_1^2 .$$

We consider the rank 2 lattice M_1 and the rank 8 lattice M_{15} (see (2.4)). Because Φ_{15} has no real roots, M_{15} has signature $(2k, 2(4-k))$. But M is of signature $(1,9)$, so the lattice M_{15} is negative-definite. Since the resultant $\operatorname{res}(\Phi_{15}, \Phi_1)$ is trival, there is no glue between M_1 and M_{15} which leaves us with $M_{15} \cong E_8(-2)$ and $M_1 \cong U(2)$. We observe that

$$H_{15} := \ker \Phi_{15}(\tilde{f}^*) = \overline{M_{15} \oplus N_{15}}$$

is the primitive closure of $M_{15} \oplus N_{15}$ in $H^2(X, \mathbb{Z})$. Since the resultant $\operatorname{res}(\Phi_{15}, \Phi_1\Phi_3) = 25$ is odd, there is no glue over 2 between H_{15} and $H_{15}^{\perp}$. As further $\det(M_{15} \oplus N_{15})$ and hence $\det(H_{15})$ is not divisible by 3, there is no glue above 3 either. Thus H_{15} is a direct summand of the unimodular lattice $H^2(X, \mathbb{Z})$. In particular, it is an even negative-definite, unimodular lattice of rank 16. Such a lattice is either the direct sum of two copies of $E_8(-1)$ or it is the even negative-definite, unimodular lattice Γ_{16} whose root sublattice is $D_{16}(-1)$ (see e.g. [7, Table 1]). Each of those lattices has roots, so we can find a root in $\mathrm{NS}(X) \supseteq H_{15}$. By Riemann-Roch such a root defines an effective divisor $C \in \mathrm{NS}(X)$ such that

$$C + \tilde{f}^* C + \ldots + (\tilde{f}^*)^{14} C = 0 \in H^2(X, \mathbb{Z})$$

and we arrive at a contradiction because $\mathrm{NS}(X)$ contains an ample class (c.f. [17, §2]). Hence the glue G cannot be trivial.

The case when G is non trivial. Assume that the glue G is non-trivial. In particular, the characteristic polynomial p_N satisfies (3.4).
By Lemma 2.7 and (3.3) we have either $|G| = 5$ or $|G| = 25$. Thus Lemma 2.5 with [3, Prop. 5.1] implies that $|G| = 25$ and

$$\det(N_{15}) \cdot \det(N_{15}^{\perp}) = \det(N) \cdot |G|^2 = 2^{10} \cdot 5^4$$

Observe that $\det(N_{15}) = 2^{8k}r$ for some $k, r \in \mathbb{N}$, so we have

$$\det(N_{15}) = 2^8 \cdot 5^2. \tag{3.8}$$

One computes that the genus of a Φ_{15}-lattice with this determinant and signature either $(2, 6)$ or $(0, 8)$ is unique. It is given by

$$N_{15} \in \mathrm{II}_{(2,6)} 2^8 5^{-2} \text{ and thus } N_{15}^{\perp} \in \mathrm{II}_{(0,4)} 2^2 5^{-2}$$

(using Lemmas 2.2 and 2.3). Since $\mathrm{res}(\Phi_3, \Phi_1) = 3$ is odd, we know that

$$N_{15}^{\perp} \otimes \mathbb{Z}_2 = (N_1 \otimes \mathbb{Z}_2) \oplus (N_3 \otimes \mathbb{Z}_2).$$

The rank of N_3 is 2, so by Lemma 2.4 the 2-adic symbol of N_3 is q^{-2} for $q = 2^i$. The 2-adic symbol of $N_{15}^{\perp}$ is $1^2 2^2$. By Lemma 2.2, $i \leq 1$, and if $i = 0, 1$, then the sign is wrong. Hence $N_3 \otimes \mathbb{Z}_2$ cannot be a direct summand of $N_{15}^{\perp} \otimes \mathbb{Z}_2$ which is a contradiction. □

4. The factor F_9

In this section we maintain the notation of previous sections and prove Theorems 1.1, 1.2. We assume that $f \in \mathrm{Aut}(Y)$ satisfies the condition

$$F_9 \mid (p_f \bmod 2)\,. \tag{4.1}$$

After replacing $\tilde{f}$ by some power co-prime to 3 we may assume that f_N is of order 9. Since $F_9 F_1^2$ divides p_N, we can rule out $p_N = \Phi_9^2$. This leaves us with the three possibilities

$$p_N = \Phi_9 \Phi_3^k \Phi_1^{6-2k} \qquad k \in \{0, 1, 2\}. \tag{4.2}$$

As usual we set $N_9 := \ker(\Phi_9(f_N))$ and denote by $N_9^{\perp}$ the orthogonal complement of N_9 in $N \in \mathrm{II}_{(2,10)} 2^{10}$. By Lemma 2.7 $\det N_9 \mid 2^6 \,\mathrm{res}(\Phi_9, \Phi_3 \Phi_1) = 2^6 \cdot 3^3$. Using the description of N_9 as Φ_9-lattice, we enumerate the possibilities for N_9. This yields 4 cases and with Lemmas 2.2 and 2.3 we calculate the corresponding genus of $N_9^{\perp}$.

$$N_9 \in \mathrm{II}_{(0,6)} 2^{-6} 3^1 \text{ and } N_9^{\perp} \in \mathrm{II}_{(2,4)} 2^{-4} 3^{-1} \tag{4.3}$$

$$N_9 \in \mathrm{II}_{(0,6)} 2^{-6} 3^{-3} \text{ and } N_9^{\perp} \in \mathrm{II}_{(2,4)} 2^{-4} 3^3 \tag{4.4}$$

$$N_9 \in \mathrm{II}_{(2,4)} 2^{-6} 3^{-1} \text{ and } N_9^{\perp} \in \mathrm{II}_{(0,6)} 2^{-4} 3^1 \tag{4.5}$$

$$N_9 \in \mathrm{II}_{(2,4)} 2^{-6} 3^3 \text{ and } N_9^{\perp} \in \mathrm{II}_{(0,6)} 2^{-4} 3^{-3} \tag{4.6}$$

We can rule out the cases (4.5) and (4.6) since in each case the genus of $N_9^\perp$ consists of a single class (see Remark 2.1) which contains roots. We continue by determining the characteristic polynomial.

Lemma 4.1. *Let $g \in \mathrm{O}(N)$ be an isometry of order 9, then the characteristic polynomial p_N of g is not of the form*

$$p_N = \Phi_9\Phi_3^2\Phi_1^2.$$

Proof. Suppose that $p_N = \Phi_9\Phi_3^2\Phi_1^2$. Recall that by Lemmas 2.5 and 2.6.

$$N \otimes \mathbb{Z}_2 = (N_9 \otimes \mathbb{Z}_2) \oplus (N_3 \otimes \mathbb{Z}_2) \oplus (N_1 \otimes \mathbb{Z}_2).$$

We see that $N_3 \otimes \mathbb{Z}_2$ is of rank 4 and has maximal scale of a 2-adic Jordan component equal to 2. By Lemma 2.4 the possible 2-adic symbols of N_3 are 1^4, $1^{-2}2^{-2}$ and 2^4.
In all cases (4.3) - (4.6) the 2-adic symbol of $N_9^\perp$ is $1^2 2^{-4}$. Therefore $N_3 \otimes \mathbb{Z}_2$ cannot be a direct summand of $N_9^\perp \otimes \mathbb{Z}_2$. Indeed, in the first case 1^4 the unimodular part is too big. In the second case $1^{-2}2^{-2}$ the unimodular part has the wrong determinant, and finally in the last case 2^4 the 2-modular part has wrong determinant. This contradiction completes the proof. □

If $p_N = \Phi_9\Phi_1^6$, then we must be in case (4.3) and $N_9^\perp = N_1$. Since the signature of N_1 is $(2,4)$, it contains the transcendental lattice. In particular, f is semi-symplectic. Choosing the covering K3 surface general enough, we may assume that N_1 is its transcendental lattice. This situation is analyzed in the next

Proposition 4.2. *Let Y be an Enriques surface such that its covering K3 surface X has transcendental lattice*

$$T(X) \cong U \oplus U(2) \oplus A_2(-2) \in \mathrm{II}_{(2,4)} 2^{-4} 3^{-1}$$

and satisfies the condition

$$N \cap \mathrm{NS}(X) \cong E_6(-2) \in \mathrm{II}_{(0,6)} 2^{-6} 3^1.$$

Then the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y)) \otimes \mathbb{F}_2$ generates a group isomorphic to $\mathcal{S}_5$.

Proof. The image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y))$ can be calculated with Algorithm 6.6. It is generated by 64 explicit matrices (see [31]). Their mod 2 reductions generate a group isomorphic to $\mathcal{S}_5$. The latter can be checked with help of [11]. □

Since $\mathcal{S}_5$ does not contain an element of order 9, we are left with

$$p_N = \Phi_9\Phi_3\Phi_1^4.$$

We derive further restrictions.

Lemma 4.3. *Let $g \in \mathrm{O}(N)$ be an isometry with characteristic polynomial*

$$p_N = \Phi_9\Phi_3^1\Phi_1^4.$$

Then $N_3 = A_2(n)$ with $n \in \{\pm 2, \pm 6\}$.

Proof. One can easily see that A_2 is the principal Φ_3-lattice. By (2.6) $N_3 = A_2(n)$ for some $n \in \mathbb{Z}$. In the following we show that $n \in \{\pm 2, \pm 6\}$ by bounding the determinant of N_3. By Lemma 2.7 we have

$$\det N_3 \mid 2^2 \operatorname{res}(\Phi_3, \Phi_9\Phi_1) = 2^2 3^3.$$

By Lemma 2.4 the 2-adic symbol of N_3 is either 1^{-2} or 2^{-2}. The first one is not a direct summand of $N_9^\perp \otimes \mathbb{Z}_2$ (see Lemma 2.2), so we are left with the second. Hence $|n| \neq 1$. □

Lemma 4.4. *Let $f \in \operatorname{Aut}(Y)$ be an automorphism of an Enriques surface such that $p_N = \Phi_9\Phi_3^1\Phi_1^4$ and (4.3) holds. Then $N_3 \cong A_2(-2)$ and $N_1 \cong U(2) \oplus U$.*

Proof. By assumption (4.3) $\det N_9^\perp = 2^4 3$, and Lemma 2.7 yields $\det N_3 \mid 2^2 9$. Thus by Lemma 4.3, we are left with $N_3 = A_2(\pm 2)$. We see that $\det N_1 \mid 2^2 3^2$. Suppose that $N_3 = A_2(2) \in \mathrm{II}_{(2,0)} 2^{-2} 3^1$. There is a single genus of signature $(0, 4)$, 2-adic symbol $1^2 2^2$ and determinant dividing $2^2 3^2$, namely $N_1 \in \mathrm{II}_{(0,4)} 2^2 3^2$. It consists of a single class which has roots. Thus $N_3 \cong A_2(-2)$. We calculate the possible genus symbols of N_1 as $\mathrm{II}_{(2,2)} 2^2$ and $\mathrm{II}_{(2,2)} 2^2 9^{\pm 1}$. In the second case N_1 and N_3 must be glued non-trivially over 3. This is impossible, as the only possibility for the glue groups are $(N_3^\vee/N_3)_3$ whose discriminant form is non-degenerate and $3(N_1^\vee/N_1)_3$ whose discriminant form is degenerate. Thus $N_1 \in \mathrm{II}_{(2,2)} 2^2$ which implies $N_1 \cong U(2) \oplus U$ since it is unique in this genus. □

If the transcendental lattice is $U \oplus U(2)$, then as before we see that the spectral radius of $\tilde{f}$ is one. Since M_1 is of rank 2 and $f_M|M_1$ has spectral radius zero, it is of finite order. Since $M_1^\perp$ is definite f_M is of finite order there as well. Thus $\tilde{f}$ is an automorphism of order 9 on a complex Enriques surface. However no such isomorphism exists (cf. [19]). We are left with case (4.4) and $p_N = \Phi_9\Phi_3\Phi_1^4$.

Lemma 4.5. *Let $f \in \operatorname{Aut}(Y)$ be an automorphism of an Enriques surface such that $p_N = \Phi_9\Phi_3^1\Phi_1^4$ and (4.4) holds. Then $N_3 \cong A_2(-6)$ and $N_1 \in \mathrm{II}_{(2,2)} 2^{-2} 9^1$. Moreover $N_1^\perp \cong A_8(-2)$.*

Proof. Recall that $\zeta_9 \cdot x := g(x)$ defines a $\mathbb{Z}[\zeta_9]$-module structure on N_9 and its discriminant group. Thus $N_9^\vee/N_9 \cong \mathbb{Z}[\zeta_9]/I$ for some ideal I. Since we are in case (4.4), I is of norm $\det N_9 = 2^6 3^3$. There is only one such ideal, namely $2(1-\zeta_9)^3$. We see that the action of g on the 3-primary part $(N_9^\vee/N_9)_3 \cong \mathbb{Z}[\zeta_9]/(1-\zeta_9)^3$ has minimal polynomial $(x-1)^3 = x^3 - 1$. In particular it has order 3. Thus the order of g on

$$\left(N_9^{\perp\vee}/N_9^\perp\right)_3 \cong (N_9^\vee/N_9)_3$$

is 3 as well. This is only possible if the order of g on $(N_3^\vee/N_3)_3 \cong \mathbb{Z}[\zeta_3]/(1-\zeta_3)^i$ is 3 (this group is a subquotient of $(N_3 \oplus N_1)^\vee/(N_3 \oplus N_1)$). This implies that $i \geq 2$, i.e. that $\det N_3$ is divisible by 9. From Lemma 4.3 we see that $N_3 = A_2(\pm 6)$. Now that we know the determinant of N_3 and $N_9^\perp$, we can estimate that of N_1 to be a divisor of $2^2 3^2$. Since N_3 has a 3-adic Jordan component of scale 9 and $N_9^\perp$ not, N_3 cannot be a direct summand of $N_9^\perp$. Thus N_3 and N_1 are glued non-trivially over 3. Consequently the determinant of N_1 is $2^2 3^2$.

Suppose that $N_3 \cong A_2(6)$, then the signature of N_1 is $(0,4)$. There is only one genus with 2-adic genus symbol $1^2 2^2$, signature $(0,4)$ and determinant $2^2 3^2$: $\mathrm{II}_{(0,4)} 2^2 3^2$ it consists of a single class which has roots.

Suppose now that $N_3 \cong A_2(-6)$. Then we obtain 3 possibilities for the genus of N_1:

(1) $\mathrm{II}_{(2,2)} 2^2 3^{-2}$; There is only one possibility to glue N_3 and N_1 equivariantly over 3 (up to isomorphism). It results in $\mathrm{II}_{(2,4)} 2^{-4} 3^1 9^1$ which is not what we need;
(2) $\mathrm{II}_{(2,2)} 2^2 9^{-1}$; the full 3-adic symbol is $1^{-3} 9^{-1}$. But that has the wrong sign at scale 1.
(3) $\mathrm{II}_{(2,2)} 2^2 9^1$ indeed there is a unique possibility to glue N_3 and N_1 equivariantly over 3. It yields the correct result.

□

Corollary 4.6. *If F_9 divides $(p_f \bmod 2)$, then $F_1^2 F_3 F_9$ divides $(p_f \bmod 2)$.*

Proof. By the previous proposition $(N_3^\vee / N_3)_2 \cong \mathbb{F}_2^2$. Hence F_3 divides p_N mod 2. Since $p_f \bmod 2 \mid p_N \bmod 2 = F_9 F_3 F_1^4$, the corollary is proven. □

We have determined the Néron-Severi lattice of the K3 cover of a generic Enriques surface admitting an automorphism with F_9 dividing p_f mod 2. This allows us to compute the semi-symplectic part of the automorphism group and locate f in there.

Proposition 4.7. *Let Y be an Enriques surface such that its K3 cover X satisfies the condition*

$$\mathrm{NS}(X) \cap N \cong A_8(-2) \in \mathrm{II}_{(0,8)} 2^8 9^1$$

and has the transcendental lattice given by

$$N_1 \in \mathrm{II}_{(2,2)} 2^{-2} 9^1.$$

Then, the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y) \otimes \mathbb{F}_2)$ generates a group isomorphic to $\mathcal{S}_9$.
In particular, the polynomials F_7 and F_9 do appear as factors of modulo-2 reductions of characteristic polynomials of isometries induced by some automorphisms of the Enriques surface Y.

Proof. The proof is a direct computation with the help of Algorithm 6.6 (c.f. proof of Prop. 4.2). The existence of the factors F_7 and F_9 follows since the symmetric group $\mathcal{S}_9$ has elements of order 7 and 9. □

Finally we can give the proofs of the main results of this note.

Proof of Theorem 1.1 a) One can repeat verbatim the proof of [14, Theorem 1.2] to see that the modulo-2 reduction ($p_N(x) \bmod 2$) is the product of some of the polynomials $F_1, F_3, F_5, F_7, F_9, F_{15}$. By (2.2) the same holds for ($p_f(x) \bmod 2$). The claim follows from Prop. 3.1.
b) follows from Prop. 4.7. □

Proof of Theorem 1.2 If the order of f_N is $90, 45, 72$, then F_9 divides $p_N \bmod 2$. Hence, by the previous corollary, $p_N \bmod 2$ is divisible by $F_1^2 F_3 F_9$. In particular, p_N (of degree 12) cannot be divided by Φ_5 as well. This excludes orders 45 and 90. If the order is 72, then the characteristic polynomial must be divisible by Φ_8 and by one of $\Phi_{3a}\Phi_{9b}$ with $a, b \in \{1, 2\}$. From the previous considerations we know that $N_8 \in \mathrm{II}_{(2,2)}2^{-2}9^1$. This is impossible, as can be seen using the description of N_8 as a twist of the principal Φ_8-lattice. □

5. The factor F_7

The main aim of this section is to study Enriques surfaces Y with an automorphism $f \in \mathrm{Aut}(Y)$ such that

$$F_7 \mid (p_f \bmod 2)\,. \tag{5.1}$$

The existence of such surfaces follows from Prop. 4.7. Here we derive a lattice-theoretic constraint given by (5.1) and show that it indeed defines Enriques surfaces with the desired property. We maintain the notation of the previous sections. Recall (see (2.1)) that

$$N \in \mathrm{II}_{(2,10)}2^{10}.$$

In the sequel we will need the following lemma.

Lemma 5.1. *Let $g \in \mathrm{O}(N)$ be an isometry such that its characteristic polynomial is the product $\Phi_7(x)\Phi_1(x)^6$. Then there are two possibilities for the genera of the lattices $N_7 := \ker \Phi_7(g)$ and $N_1 := \ker \Phi_1(g)$; either*

$$N_7 \in \mathrm{II}_{(2,4)}2^6 7^{-1} \quad \text{and} \quad N_1 \in \mathrm{II}_{(0,6)}2^4 7^1$$

or

$$N_7 \in \mathrm{II}_{(0,6)}2^6 7^1 \quad \text{and} \quad N_1 \in \mathrm{II}_{(2,4)}2^4 7^{-1}.$$

In either case the genus of N_1 contains a single class. In the first case the class of N_1 has roots.

Proof. Since $\mathrm{res}(\Phi_1, \Phi_7) = 7$, Lemma 2.7 implies that the index $[N : N_7 \oplus N_1]$ divides 7. But in any case $7 = |\Phi_7(1)\Phi_7(-1)|$ divides $\det N_7$ (see (2.5) and (2.6)). Thus we obtain

$$[N : N_7 \oplus N_1] = 7.$$

Consequently, for all $p \neq 7$, $N \otimes \mathbb{Z}_p = (N_7 \otimes \mathbb{Z}_p) \oplus (N_1 \otimes \mathbb{Z}_p)$. In particular for $p = 2$. Using the description of N_7 as a twist of the principal Φ_7-lattice

we compute the two possibilities for the genus of N_7 (see Remark 2.1).
It remains to determine the genus of N_1. Since we have

$$N \otimes \mathbb{Z}_2 = (N_7 \otimes \mathbb{Z}_2) \oplus (N_1 \otimes \mathbb{Z}_2)\,,$$

the 2-adic symbol of N_1 must be 2^4. To compute the 7-adic symbol note that $N \otimes \mathbb{Z}_7$ is unimodular, thus Lemma 2.3 applies. As (-1) is a non-square in $\mathbb{Z}_7$ this means that the signs ϵ_7 of the 7-modular Jordan constituents of N_7 and N_1 must be different. The claim that N_1 is unique in its genus in the first case is checked with a computer algebra system (see Remark 2.1). In the second case N_1 is indefinite and we can use [8, Thm.15.19]. □

Recall that X (resp. $\tilde{f} \in \mathrm{Aut}(X)$) stands for the K3-cover of an Enriques surface Y (resp. for a lift of an automorphism $f \in \mathrm{Aut}(Y)$).

Proposition 5.2. *Let Y be an Enriques surfaces with an automorphism $f \in \mathrm{Aut}(Y)$ such that* (5.1) *holds. Then* $\mathrm{NS}(X)$ *contains a primitive $\tilde{f}^*$-invariant sublattice which belongs to the genus* $\mathrm{II}_{(1,15)}2^4 7^1$ *and* $N \cap \mathrm{NS}(X)$ *contains the $\tilde{f}^*$-invariant sublattice $A_6(-2) \cong N_7 \in \mathrm{II}_{(0,6)}2^6 7^1$ primitively.*

Proof. Since F_7 divides p_f, (2.2) implies that the characteristic polynomial p_N is divisible by the cyclotomic polynomial Φ_7. Moreover, after replacing f by f^k with $k \in \mathbb{N}$ coprime to 7, we may assume that

$$p_N = \Phi_7(x)\Phi_1(x)^6.$$

Now we can apply Lemma 5.1. The first case is impossible as then N_1 is contained in $\mathrm{NS}(X) \cap N$ and contains roots (see (2.3)). Thus we are left with the second case. Since $N_1 \subseteq N$ is of signature $(2, 4)$ it must contain the transcendental lattice (and f is semi-symplectic). Thus the orthogonal complement of N_1 in $H^2(X, \mathbb{Z})$ is the sought for $\tilde{f}^*$ invariant sublattice of $\mathrm{NS}(X)$. □

Finally, we apply Algorithm 6.6 to check that the condition of Prop. 5.2 indeed gives Enriques surfaces such that (5.1) holds.

Proposition 5.3. *If the K3 cover X of an Enriques surface Y satisfies the following conditions:*

- *(a)* $\mathrm{NS}(X) \in \mathrm{II}_{(1,15)}2^4 7^1$ *and*
- *(b)* $N \cap \mathrm{NS}(X) \cong A_6(-2) \in \mathrm{II}_{(0,6)}2^6 7^1$.

then the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y)) \otimes \mathbb{F}_2$ *generates a group isomorphic to $\mathcal{S}_7$. In particular, the Enriques surface Y admits an automorphism $f \in$* $\mathrm{Aut}(Y)$ *such that the modulo-*2 *reduction* $(p_f(x) \bmod 2)$ *is divisible by the polynomial F_7.*

Proof. Apply Algorithm 6.6 and [11] as in the proof of Prop. 4.2. □

6. Appendix: an algorithm to calculate generators

In this appendix, we present an algorithm to calculate a finite generating set of the image of the natural homomorphism from the automorphism group of an Enriques surface to the orthogonal group of the numerical Néron-Severi lattice of the Enriques surface. Our algorithm is based on Borcherds' method [1, 2] with the result in [4].

6.1. **Borcherds' method.** We use the notation and terminologies in [4]. In particular, we denote by Y an Enriques surface, $\pi\colon X \to Y$ the universal covering of Y, and S_X and S_Y the numerical Néron-Severi lattices of X and of Y, respectively (that is, $S_X = \mathrm{NS}(X)$ and $S_Y = \mathrm{Num}(Y)$ in the notation of previous sections.) Let $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$) be the positive cone of $S_X \otimes \mathbb{R}$ (resp. $S_Y \otimes \mathbb{R}$) containing an ample class. Let N_X (resp. N_Y) be the cone consisting of all $x \in \mathcal{P}_X$ (resp. all $x \in \mathcal{P}_Y$) such that $\langle x, [\Gamma]\rangle \geq 0$ for any curve Γ on X (resp. on Y). We let the orthogonal group $\mathrm{O}(L)$ of a $\mathbb{Z}$-lattice L act on the lattice from the *right*. Suppose that L is even. A vector $r \in L$ is a (-2)-*vector* if $\langle r, r\rangle = -2$. Let $W(L)$ denote the subgroup of $\mathrm{O}(L)$ generated by the reflections $s_r\colon x \mapsto x + \langle x, r\rangle r$ with respect to (-2)-vectors r of L. For a subset A of $L \otimes \mathbb{R}$, we denote by A^g the image of A under the action of $g \in \mathrm{O}(L)$ (*not* the fixed locus of g in A), and put

$$\mathrm{O}(L, A) := \{\, g \in \mathrm{O}(L) \mid A = A^g \,\}.$$

We have natural homomorphisms

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X, \mathcal{P}_X), \quad \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y).$$

We denote by $\mathrm{aut}(X)$ and $\mathrm{aut}(Y)$ the images of these homomorphisms. Recall that $\mathrm{Aut}_s(Y)$ consists of the semi-symplectic automorphisms, i.e. those that act trivially on $H^0(Y, \omega_Y^{\otimes 2})$. We denote by $\mathrm{Aut}_s(X)$ the subgroup consisting of those automorphisms acting as ± 1 on $H^0(X, \Omega_X^2) \cong H^{2,0}(X)$. The subgroups $\mathrm{aut}_s(X) \subseteq \mathrm{aut}(X)$ and $\mathrm{aut}_s(Y) \subseteq \mathrm{aut}(Y)$ are defined as the respective images. Our goal is to calculate a finite generating set of $\mathrm{aut}_s(Y)$.

Remark 6.1. We note that $\mathrm{Aut}_s(Y)$ is of finite index in $\mathrm{Aut}(Y)$. This index is one if the only isometries of T_X that preserve $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$ are ± 1, where T_X is the transcendental lattice of X.

We have the primitive embedding

$$\pi^*\colon S_Y(2) \hookrightarrow S_X,$$

which induces $\mathcal{P}_Y \hookrightarrow \mathcal{P}_X$. We regard S_Y as a submodule of S_X and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by π^*. Then we have

$$N_Y = N_X \cap \mathcal{P}_Y. \tag{6.1}$$

If $\alpha \in S_Y$ is ample on Y, then $\pi^*(\alpha)$ is ample on X. Hence we have $N_Y^\circ = N_X^\circ \cap \mathcal{P}_Y$, where N_Y° and N_X° are the interiors of N_Y and N_X, respectively. Let Q denote the orthogonal complement of the sublattice $S_Y(2)$ in S_X.

Since Q is negative-definite, the group $\mathrm{O}(Q)$ is finite. We consider the following assumptions for an element g of $\mathrm{O}(S_Y, \mathcal{P}_Y)$:

- (i) There exists an isometry $h \in \mathrm{O}(Q)$ such that the action of $g \oplus h$ on $S_Y(2) \oplus Q$ preserves the overlattice S_X of $S_Y(2) \oplus Q$ and the action of $(g \oplus h)|S_X$ on the discriminant group $S_X^\vee/S_X$ of S_X is ± 1.
- (ii-a) There exists an ample class $\alpha \in S_Y$ of Y such that there exist no vectors $r \in S_X$ with $\langle r, r\rangle = -2$ satisfying $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^g), r\rangle < 0$.
- (ii-b) For an arbitrary ample class $\alpha \in S_Y$ of Y, there exist no vectors $r \in S_X$ with $\langle r, r\rangle = -2$ satisfying $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^g), r\rangle < 0$.

Proposition 6.2. *Let g be an element of $\mathrm{O}(S_Y, \mathcal{P}_Y)$. Then g is in $\mathrm{aut}_s(Y)$ if* (i) *and* (ii-a) *hold. If g is in $\mathrm{aut}_s(Y)$, then* (i) *and* (ii-b) *hold.*

Proof. An element g of $\mathrm{O}(S_Y, \mathcal{P}_Y)$ is in $\mathrm{aut}_s(Y)$ if and only if there exists an element $\tilde{g} \in \mathrm{aut}_s(X)$ that preserves $S_Y \subset S_X$ and satisfies $\tilde{g}|S_Y = g$. By the Torelli theorem, we see that an element $\tilde{g}'$ of $\mathrm{O}(S_X, \mathcal{P}_X)$ is in $\mathrm{aut}_s(X)$ if and only if the action of $\tilde{g}'$ on $S_X^\vee/S_X$ is ± 1 and $\tilde{g}'$ preserves N_X. Since N_X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$ (see Example 1.5 of [4]), we have

$$N_X^\circ \cap N_X^h \neq \emptyset \implies N_X = N_X^h$$

for any $h \in \mathrm{O}(S_X, \mathcal{P}_X)$. Therefore both of (ii-a) and (ii-b) are equivalent to the condition that $N_X^{\tilde{g}} = N_X$ for any $\tilde{g} \in \mathrm{O}(S_X, \mathcal{P}_X)$ satisfying $S_Y^{\tilde{g}} = S_Y$ and $\tilde{g}|S_Y = g$. □

Suppose that we have a primitive embedding

$$\iota_X \colon S_X \hookrightarrow L_{26},$$

where L_{26} is an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. (A more standard notation is $\mathrm{II}_{1,25}$.) Composing π^* and ι_X, we obtain a primitive embedding

$$\iota_Y \colon S_Y(2) \hookrightarrow L_{26}.$$

Let $\mathcal{P}_{26}$ be the positive cone of L_{26} into which $\mathcal{P}_Y$ is mapped. We regard S_Y as a primitive submodule of L_{26}, and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_{26}$ by ι_Y. Recall from [4] that a Conway chamber is a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$. The tessellation of $\mathcal{P}_{26}$ by Conway chambers induces a tessellation of $\mathcal{P}_Y$ by induced chambers.

Proposition 6.3. *The action of $\mathrm{aut}_s(Y)$ on $\mathcal{P}_Y$ preserves the tessellation of $\mathcal{P}_Y$ by induced chambers.*

Proof. Let g be an element of $\mathrm{aut}_s(Y)$. By the proof of Proposition 6.2, there exists an isometry $\tilde{g} \in \mathrm{O}(S_X, \mathcal{P}_X)$ such that $S_Y^{\tilde{g}} = S_Y$, $\tilde{g}|S_Y = g$ and the action of $\tilde{g}$ on $S_X^\vee/S_X$ is ± 1. By the last condition, we see that $\tilde{g}$ further extends to an isometry $g_{26} \in \mathrm{O}(L_{26}, \mathcal{P}_{26})$. Since the action of g_{26} on

$\mathcal{P}_{26}$ preserves the tessellation by Conway chambers, the action of g on $\mathcal{P}_Y$ preserves the tessellation by induced chambers. □

Let L_{10} be an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism. In [4], we have classified all primitive embeddings of $S_Y(2) \cong L_{10}(2)$ into L_{26}, and studied the tessellation of $\mathcal{P}_Y$ by induced chambers. It turns out that, up to the action of $\mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$, there exist exactly 17 primitive embeddings $L_{10}(2) \hookrightarrow L_{26}$, and except for one primitive embedding named as "infty", the associated tessellation of $\mathcal{P}_Y$ by induced chambers has the following properties:

- Each induced chamber D is bounded by a finite number of walls, and each wall is defined by a (-2)-vector.
- If a (-2)-vector r defines a wall $w = D \cap (r)^\perp$ of an induced chamber D, then the reflection $s_r \colon x \mapsto x + \langle x, r\rangle r$ into the mirror $(r)^\perp$ maps D to the induced chamber adjacent to D across the wall w.

In particular, the tessellation of $\mathcal{P}_Y$ by induced chambers is *simple* in the sense of [30].

6.2. **Main Algorithm.** Suppose that the primitive embedding ι_Y is not of type "infty". Suppose also that we have calculated the walls of an induced chamber $D_0 \subset \mathcal{P}_Y$ contained in N_Y.

Before starting the main algorithm, we calculate the finite groups $\mathrm{O}(Q)$ and $\mathrm{O}(S_Y, D_0)$. We also fix an ample class α that is contained in the interior of D_0. In the following, an induced chamber D is expressed by an element $\tau_D \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ such that $D = {D_0}^{\tau_D}$. Note that τ_D is uniquely determined by D up to left multiplications of elements of $\mathrm{O}(S_Y, D_0)$.

Then we have the following auxiliary algorithms.

Algorithm 6.4. Given an induced chamber D, we can determine whether $D \subset N_Y$ or not. Indeed, by (6.1), we have $D \subset N_Y$ if and only if there exist no (-2)-vectors r of S_X such that $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^{\tau_D}), r\rangle < 0$. The set of such (-2)-vectors can be calculated by the algorithm in Section 3.3 of [29]. ▮

Suppose that $D \subset N_Y$. A wall $D \cap (r)^\perp$ of D is said to be *inner* if the induced chamber D^{s_r} adjacent to D across $D \cap (r)^\perp$ is contained in N_Y. Otherwise, we say that $D \cap (r)^\perp$ is *outer*.

Algorithm 6.5.

Input: An embedding $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$, the groups $\mathrm{O}(S_Y, D_0)$, $\mathrm{O}(Q)$ and two induced chambers $D, D' \subset N_Y$ represented by $\tau_D, \tau_{D'}$.
Output: The set $\{\gamma \in \mathrm{aut}_s(Y) \mid D' = D^\gamma\}$.

Compute $\mathrm{Isom}(D, D') := \tau_D^{-1}\mathrm{O}(S_Y, D_0)\tau_{D'}$.
 This is the set of all isometries $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ that satisfy $D' = D^g$.
Initialize $\mathcal{I} := \{\}$
for $g \in \mathrm{Isom}(D, D')$ **do**
 Use $\mathrm{O}(Q)$ and Proposition 6.2 to check

4: **if** $g \in \mathrm{aut}_s(Y)$ **then**
5: add g to $\mathcal{I}$.
6: Return $\mathcal{I}$.

Note that since both D and D' are contained in N_Y, condition (ii-a) of Proposition 6.2 is always satisfied in line 4. For $D = D'$, Algorithm 6.5 calculates the group

$$\mathrm{aut}_s(Y, D) := \mathrm{O}(S_Y, D) \cap \mathrm{aut}_s(Y).$$

Two induced chambers D and D' in N_Y are said to be $\mathrm{aut}_s(Y)$-*equivalent* if there exists an element $\gamma \in \mathrm{aut}_s(Y)$ such that $D' = D^\gamma$.

Algorithm 6.6.

Input: An embedding $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$
and an induced chamber $D_0 \subset N_Y$.
Output: A list $\mathcal{R}$ of representatives of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers contained in N_Y and a generating set $\mathcal{G}$ of $\mathrm{aut}_s(Y)$.

Initialize $\mathcal{R} := [D_0]$, $\mathcal{G} := \{\}$ and $i := 0$.
while $i \leq |\mathcal{R}|$ **do**
Let D_i be the $(i+1)$st element of $\mathcal{R}$.
Replace $\mathcal{G}$ by $\mathcal{G} \cup \mathrm{aut}_s(Y, D_i)$.
Let $\mathcal{W}$ be the set of walls of D_i.
Compute orbit representatives of $\mathcal{W}$ under the action of $\mathrm{aut}_s(Y, D_i)$.
for each representative wall w of $\mathcal{W}/\mathrm{aut}_s(Y, D_i)$ **do**
Let r be the (-2)-vector of S_Y defining the wall $w = D \cap (r)^\perp$.
Let s_r be the reflection $x \mapsto x + \langle x, r\rangle r$.
Let $D_w = D_i^{s_r}$ be the induced chamber adjacent to D_i across w.
Set $\tau_{D_w} := \tau_{D_i} s_r$.
if $D_w \not\subset N_Y$ **then**
continue with the next representative wall.
Set $f :=$ true.
for each $D \in \mathcal{R}$ **do**
if D is $\mathrm{aut}_s(Y)$-equivalent to D_w **then**
Let $\gamma \in \mathrm{aut}_s(Y)$ be an element such that $D_w = D^\gamma$.
Add γ to $\mathcal{G}$.
Replace f by false.
Break the for loop.
if $f =$ true **then**
Add D_w to $\mathcal{R}$.
Increment i.
Return $\mathcal{R}$ and $\mathcal{G}$.

Proof. This Algorithm is proved in the same way as the proof of Proposition 6.3 of [28]. □

6.3. **Examples.** The details of the following computations are available at [31].

6.3.1. *The Enriques surface in Proposition 5.3.* The Picard number of the covering $K3$ surface is 16, and the orthogonal complement Q of $S_Y(2)$ in S_X is $A_6(-2)$. Therefore $\mathrm{O}(Q)$ is of order 10080. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is $8A_1+2D_4$. Hence the embedding ι_Y is of type 40B in the notation of [4]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 2. Let D_0 and D_1 be the representatives of $\mathrm{aut}_s(Y)$-equivalence classes. For $i=0,1$, the group $\mathrm{aut}_s(Y,D_i)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and the 40 walls of D_i are decomposed into 10 orbits under the action of $\mathrm{aut}_s(Y,D_i)$. Among the 40 walls, exactly $3\times 4=12$ walls are outer walls. For each inner wall w, the two induced chambers containing w are not $\mathrm{aut}_s(Y)$-equivalent, that is, one is $\mathrm{aut}_s(Y)$-equivalent to D_0 and the other is $\mathrm{aut}_s(Y)$-equivalent to D_1.

6.3.2. *The Enriques surface in Proposition 4.2.* The Picard number of the covering $K3$ surface is 16, and the orthogonal complement Q of $S_Y(2)$ in S_X is $E_6(-2)$. Therefore $\mathrm{O}(Q)$ is of order 103680. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is D_4+D_5. Hence the embedding ι_Y is of type 20A, which means that D_0 is bounded by walls defined by (-2)-vectors that form the dual graph of Nikulin-Kondo's type V [13]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 20. They are decomposed into the following three types.

Type	$\lvert\mathrm{aut}_s(Y,D)\rvert$	outer walls	inner walls	number
a	1	1×7	1×13	2
b	1	1×5	1×15	6
c	2	$1\times 2+2\times 2$	$1\times 2+2\times 6$	12.

For example, there exist twelve $\mathrm{aut}_s(Y)$-equivalence classes of type c. If D is an induced chamber of type c, then $\mathrm{aut}_s(Y,D)$ is $\mathbb{Z}/2\mathbb{Z}$, and D has 6 outer walls and 14 inner walls. Under the action of $\mathrm{aut}_s(Y,D)$, the 6 outer walls are decomposed into 4 orbits of size $1,1,2,2$, and the 14 inner walls are decomposed into 8 orbits of size $1,1,2,\dots,2$.

6.3.3. *The Enriques surface in Proposition 4.7.* The Picard number of the covering $K3$ surface is 18, and the orthogonal complement Q of $S_Y(2)$ in S_X is $A_8(-2)$. Therefore $\mathrm{O}(Q)$ is of order 725760. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is A_3+A_4. Hence the embedding ι_Y is of type 20D, which means that D_0 is bounded by walls defined by (-2)-vectors that form the dual graph of Nikulin-Kondo's type VII [13]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 1. The group $\mathrm{aut}_s(Y,D_0)$ is isomorphic to $\mathfrak{S}_3$, and the 20 walls of D_0 are decomposed into 6 orbits, each of which consists of

6 outer, 3 outer, 3 outer, 3 inner, 3 inner, 2 inner.

References

[1] Borcherds, R., Automorphism groups of Lorentzian lattices. Journal of Algebra **111**, no. 1 (1987): 133–53.

[2] Borcherds, R., Coxeter groups, Lorentzian lattices, and K3 surfaces. International Mathematics Research Notices **1998** no. 19 (1998): 1011–31.

[3] Bayer-Fluckiger, E. Isometries of quadratic spaces, J. Eur. Math. Soc. **17** (2015), no. 7, 1629–1656.

[4] Brandhorst, S. and Shimada, I. Borcherds' method for Enriques surfaces, 2019. Preprint, arXiv:1903.01087.

[5] Cantat, S., Dynamics of automorphisms of compact complex surfaces, Frontiers in Complex Dynamics: In celebration of John Milnor's 80th birthday, 463–514, Princeton Mathematical Series, Princeton University Press

[6] Cohen, H., A course in computational algebraic number theory, vol. 138 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1993.

[7] Conway, J. H., Sloane. N.J.A. Low-dimensional lattices. I. Quadratic forms of small determinant. Proc. Roy. Soc. London Ser. A **418** (1988), no. 1854, 17–41

[8] Conway, J. H., Sloane. N.J.A. Sphere Packings, Lattices and Groups. vol. 290 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, New York, 1999.

[9] Dolgachev, I., Salem numbers and Enriques surfaces, Experimental Mathematics, DOI: 10.1080/10586458.2016.1261743

[10] Gross, Benedict H. and McMullen, Curtis T., Automorphisms of even unimodular lattices and unramified Salem numbers, J. Algebra, **257** (2002), no. 2, 265–290.

[11] The GAP Group. *GAP - Groups, Algorithms, and Programming.* Version 4.8.6; 2016 (`http://www.gap-system.org`).

[12] Höppner, S. Lokale Eigenschaften von Gittern mit einem Automorphismus. PH.D. thesis, Dortmund 2016, available as http://hdl.handle.net/2003/34892 DOI:10.17877/DE290R-16940

[13] Kondō, S. Enriques surfaces with finite automorphism groups. Japan. J. Math. (N.S.) **12** (1986), 191–282.

[14] Matsumoto, Y., Ohashi, H., Rams, S. On automorphisms of Enriques surfaces and their entropy. Math. Nachr. **291** (2018), no. 13, 2084–2098.

[15] Keum, J.-H. Every algebraic Kummer surface is the K3-cover of an Enriques surface., Nagoya Math. J., **118** (1990), 99–110.

[16] McMullen, Curtis T., $K3$ surfaces, entropy and glue., J. Reine Angew. Math., **658** (2011), 1–25.

[17] McMullen, Curtis T., Automorphisms of projective $K3$ surfaces with minimum entropy., Invent. Math., **203** (2016), no. 1, 179–215.

[18] Mukai, S. and Ohashi, H., The automorphism groups of Enriques surfaces covered by symmetric quartic surfaces., Recent advances in algebraic geometry, 307–320, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015.

[19] Mukai, S. and Ohashi, H., Finite groups of automorphisms of Enriques surfaces and the Mathieu group M_{12}, preprint, arXiv: 1410.7535 (math.AG)

[20] Namikawa, Y., Periods of Enriques surfaces, Math. Ann. **270** (1985), no. 2, 201–222.

[21] Nikulin, V. V., Integral symmetric bilinear forms and some of their applications (English translation), Math. USSR Izv., **14** (1980), 103–167.

[22] Oguiso, K., The third smallest Salem number in automorphisms of $K3$ surfaces, Algebraic geometry in East Asia-Seoul 2008, 331–360.

[23] Oguiso, K., Yu, Minimum positive entropy of complex Enriques surface automorphisms preprint, arxiv: 1807.09452v1 (math.AG)

[24] Ohashi, H., On the number of Enriques quotients of a K3 surface, Publ. Res. Inst. Math. Sci. **43** (2007), no. 1, 181–200.

[25] The PARI Group, Univ. Bordeaux. *PARI/GP version* `2.11.1`, 2018. available from `http://pari.math.u-bordeaux.fr/`.

[26] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.6)*, 2019. `https://www.sagemath.org`.

[27] Rudolf Scharlau and Boris Hemkemeier. Classification of integral lattices with large class number. *Math. Comp.*, 67(222):737–749, 1998.

[28] Ichiro Shimada. An algorithm to compute automorphism groups of K3 surfaces and an application to singular K3 surfaces. Int. Math. Res. Not. IMRN **22** (2015), 11961–12014.

[29] Ichiro Shimada. Projective models of the supersingular $K3$ surface with Artin invariant 1 in characteristic 5. *J. Algebra*, 403:273–299, 2014.

[30] Ichiro Shimada. Holes of the Leech lattice and the projective models of $K3$ surfaces. *Math. Proc. Cambridge Philos. Soc.*, 163(1):125–143, 2017.

[31] Ichiro Shimada. On characteristic polynomials of automorphisms of Enriques surfaces: computational data, 2019. http://www.math.sci.hiroshima-u.ac.jp/ shimada/K3andEnriques.html.

Fachbereich Mathematik, Saarland University, Campus E2.4 Zi. 222, 66123 Saarbrücken, Germany

Email address: `brandhorst@math.uni-sb.de`

Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland

Email address: `slawomir.rams@uj.edu.pl`

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

Email address: `ichiro-shimada@hiroshima-u.ac.jp`