ENRIQUES INVOLUTIONS ON SINGULAR K3 SURFACES OF SMALL DISCRIMINANTS

ICHIRO SHIMADA AND DAVIDE CESARE VENIANI

Abstract. We classify Enriques involutions on a K3 surface, up to conjugation in the automorphism group, in terms of lattice theory. We enumerate such involutions on singular K3 surfaces with transcendental lattice of discriminant smaller than or equal to 36. For 11 of these K3 surfaces, we apply Borcherds method to compute the automorphism group of the Enriques surfaces covered by them. In particular, we investigate the structure of the two most algebraic Enriques surfaces.

1. Introduction

1.1. **Background.** Let X be a complex K3 surface. We denote by $S_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ the lattice of numerical equivalence classes of divisors on X, and by T_X the orthogonal complement of S_X in $H^2(X, \mathbb{Z})$, which we call the *transcendental lattice* of X. Suppose that X is *singular*, that is, the Picard number $\operatorname{rank} S_X$ attains the possible maximum $h^{1,1}(X) = 20$. The *discriminant* of a singular K3 surface X is the determinant of a Gram matrix of T_X. Since T_X is an even positive definite lattice of rank 2, the discriminant d of X is a positive integer satisfying $d \equiv 0$ or $3 \bmod 4$. Note that T_X is naturally oriented by the Hodge structure. By the classical work of Shioda–Inose [32], we know that the isomorphism class of the oriented lattice T_X determines X up to $\mathbb{C}$-isomorphism.

An involution $\tilde{\varepsilon}\colon X \to X$ of a K3 surface X is called an *Enriques involution* if $\tilde{\varepsilon}$ acts freely on X. Sertöz [25] gave a simple criterion to determine whether a singular K3 surface has an Enriques involution or not (see Theorem 3.2.1 and also Lee [18]). On the other hand, Ohashi [22] showed that each complex K3 surface X (not necessarily singular) has only finitely many Enriques involutions up to conjugation in the automorphism group $\operatorname{Aut}(X)$ of X, and that there exists no universal bound for the number of conjugacy classes of Enriques involutions.

Ohashi also gave a lattice theoretic method to enumerate Enriques involutions on certain K3 surfaces. In a subsequent paper [23] he classified all Enriques involutions on the Kummer surface $\operatorname{Km}(\operatorname{Jac}(C))$ associated with the jacobian variety of a generic curve C of genus 2.

For some K3 surfaces X, the group $\operatorname{Aut}(X)$ can be calculated by Borcherds method ([3], [4]); for instance, Kondo [16] implemented it in order to compute $\operatorname{Aut}(\operatorname{Km}(\operatorname{Jac}(C)))$.

1.2. **Main results.** In this paper, we classify, up to conjugation in $\operatorname{Aut}(X)$, all Enriques involutions $\tilde{\varepsilon}$ on the singular K3 surfaces X whose discriminant d satisfies

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$d \leq 36$. The classification is given in Table 3.1 and builds on a refinement and generalization of Ohashi's method. Our main result, namely Theorem 3.1.9, applies to any K3 surface.

We then concentrate on 11 of these singular K3 surfaces, listed in Table 4.1, to which we can apply Borcherds method in order to compute the automorphism group. We first write the action of $\mathrm{Aut}(X)$ on the nef chamber of X explicitly. Building on this data, we re-enumerate all Enriques involutions up to conjugation. Using also a result of the preprint [6] (see Section 2.9), we are able to calculate the automorphism group of the Enriques surfaces covered by these K3 surfaces. The results are given in Theorem 5.4.1 and Table 5.1.

Note that the enumeration of Enriques involutions by Ohashi's method and by Borcherds method are carried out independently. The results are, of course, consistent. We hope that these methods will be applied to many other K3 surfaces (with smaller Picard number) and Enriques surfaces covered by them, and that in these works, our general results on a K3 surface admitting an Enriques involution (Lemma 3.1.7 and Proposition 3.1.8) will be useful.

Recently, many studies on the automorphism groups $\mathrm{Aut}(Y)$ of Enriques surfaces Y have appeared ([1], [19], [30]). Our result gives a description of $\mathrm{Aut}(Y)$ in terms of its action on the lattice S_Y of numerical equivalence classes of divisors on Y. We expect that this description is helpful in the search for a more geometric description of $\mathrm{Aut}(Y)$, that is, for writing elements of $\mathrm{Aut}(Y)$ as birational self-maps on some projective model of Y.

Computations were carried out using `GAP` [9] and `sage` on SageMath [33]. Further computational data is provided on the web page [31].

As a corollary of our calculations, we obtain the following. For $d = 3, 4$ or 7, there exists exactly one singular K3 surface X_d of discriminant d up to $\mathbb{C}$-isomorphism. The K3 surfaces X_3, X_4, also known as "the two most algebraic K3 surfaces", were studied by Vinberg [37]. Neither X_3 nor X_4 admits any Enriques involution, but X_7 does; following Vinberg, we call the Enriques surfaces covered by X_7 the *most algebraic Enriques surfaces.*

Theorem 1.2.1. *The singular K3 surface X_7 of discriminant 7 has exactly two Enriques involutions $\tilde{\varepsilon}_{\mathrm{I}}$ and $\tilde{\varepsilon}_{\mathrm{II}}$ up to conjugation in $\mathrm{Aut}(X_7)$. Let Y_{I} and Y_{II} be the quotient Enriques surfaces corresponding to $\tilde{\varepsilon}_{\mathrm{I}}$ and $\tilde{\varepsilon}_{\mathrm{II}}$, respectively. Then $\mathrm{Aut}(Y_{\mathrm{I}})$ is finite of order 8, and $\mathrm{Aut}(Y_{\mathrm{II}})$ is finite of order 24.*

Nikulin [21] and Kondo [15] classified all complex Enriques surfaces whose automorphism group is finite. It turns out that these Enriques surfaces are divided into 7 classes I, II, ..., VII, which we call *Nikulin-Kondo type.* See Kondo [15] for the properties of these Enriques surfaces.

Corollary 1.2.2. *The most algebraic Enriques surfaces have finite automorphism groups and their Nikulin-Kondo types are* I *and* II.

Mukai (private communication) had already realized this result previously. Answering a question by G. Kapustka, in Section 6 we give explicit models of the most algebraic Enriques surfaces Y_{I} and Y_{II} as Enriques sextic surfaces.

1.3. **Contents.** This paper is organized as follows. In Section 2, we recall basic facts about lattices, K3 surfaces and Enriques surfaces, and fix notions and notation. In Section 3, we classify all Enriques involutions on singular K3 surfaces with discriminant ≤ 36 by a generalization of Ohashi's method. In Section 4, we recall Borcherds method, and apply it to the 11 singular K3 surfaces whose transcendental lattices are listed in Table 4.1. Recently, many geometric studies of singular K3 surfaces of small discriminant have appeared (see, for example, [2], [10], [17], [35]). We summarize the computational data for these 11 singular K3 surfaces in Table 4.2. In Section 5, we explain an algorithm to calculate Enriques involutions and the automorphism groups of the Enriques surfaces from the data obtained by Borcherds method, and apply this method to the 11 singular K3 surfaces. In Section 6, we study the most algebraic Enriques surfaces Y_{I} and Y_{II}.

2. Preliminaries

2.1. **Lattices.** A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a $\mathbb{Z}$-valued non-degenerate symmetric form $\langle\ ,\ \rangle$. The *determinant* $\det L$ of L is the determinant of any Gram matrix of L. A lattice L is *unimodular* if $\det L = \pm 1$. A lattice with the same underlying $\mathbb{Z}$-module as L and symmetric form $n \cdot \langle\ ,\ \rangle$ is denoted by $L(n)$. The group of isometries of L is denoted $\mathrm{O}(L)$. We let $\mathrm{O}(L)$ act on L *from the right*. A vector v of a lattice L is called an *n-vector* if $\langle v, v\rangle = n$. We denote by $\mathcal{R}_L$ the set of (-2)-vectors of a lattice L.

A lattice L is *even* if $\langle v, v\rangle \in 2\mathbb{Z}$ for all $v \in L$; otherwise, it is *odd*. The *signature* of a lattice L is the signature of $L \otimes \mathbb{R}$. Analogously, we say that L is *positive definite*, *negative definite* or *indefinite* if $L \otimes \mathbb{R}$ is. A lattice L of rank $n > 1$ is *hyperbolic* if the signature is $(1, n-1)$. A *positive cone* of a hyperbolic lattice L is one of the two connected components of $\{v \in L \otimes \mathbb{R} \mid \langle v, v\rangle > 0\}$. For a hyperbolic lattice L and a positive cone $\mathcal{P}_L$ of L, we denote by $\mathrm{O}(L, \mathcal{P}_L)$ the group of isometries of L that preserves $\mathcal{P}_L$.

The standard positive definite lattices associated to Dynkin graphs will be denoted A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, E_8.

2.2. **Surfaces.** Let Z be a K3 surface or an Enriques surface. We denote by S_Z the lattice of numerical equivalence classes of divisors on Z, and call it the *Néron–Severi lattice* of Z. Then S_Z is an even hyperbolic lattice, provided that $\operatorname{rank} S_Z > 1$. Let $\mathcal{P}_Z$ denote the positive cone of S_Z that contains an ample class, and let $\mathcal{R}_Z$ be the set of (-2)-vectors of S_Z. For simplicity, we denote by $\operatorname{aut}(Z)$ the the image of the natural representation

$$\rho_Z \colon \operatorname{Aut}(Z) \to \mathrm{O}(S_Z, \mathcal{P}_Z). \tag{2.1}$$

We put

$$N_Z := \{x \in \mathcal{P}_Z \mid \langle x, [\Gamma]\rangle \geq 0 \text{ for all curves } \Gamma \text{ on } Z\},$$

and call it the *nef chamber* of Z. It is obvious that the action of $\operatorname{aut}(Z)$ on $\mathcal{P}_Z$ preserves N_Z.

2.3. **Finite bilinear and quadratic forms.** A *finite quadratic form* is a finite abelian group G together with a function $q \colon G \to \mathbb{Q}/2\mathbb{Z}$ which satisfies

$$q(n\alpha) = n^2 q(\alpha) \text{ for every } \alpha \in G \text{ and } n \in \mathbb{Z}$$

such that the function $b(q) \colon G \times G \to \mathbb{Q}/\mathbb{Z}$ defined by

$$(\alpha, \beta) \mapsto \frac{q(\alpha+\beta) - q(\alpha) - q(\beta)}{2}$$

is a finite symmetric bilinear form. For the sake of simplicity, we will denote by q also the underlying finite abelian group G. The *length*, i.e. the minimal number of generators, of G (resp. of the p-torsion part of G) is denoted by $\ell(G)$ (resp. $\ell_p(G)$). A subgroup $\Gamma \subset G$ is called *isotropic* if $q|\Gamma = 0$, where $q|\Gamma$ denotes the restriction of q to Γ. Given an isotropic subgroup Γ, the quadratic form q descends to the quotient group $\Gamma^\perp/\Gamma$, where

$$\Gamma^\perp := \{\alpha \in G \mid b(q)(\alpha,\gamma) = 0 \text{ for every } \gamma \in \Gamma\};$$

we denote the resulting finite quadratic form by $q|\Gamma^\perp/\Gamma$.

If L is a lattice, then the group $L^\vee/L$, where $L^\vee := \mathrm{Hom}(L,\mathbb{Z}) \subset L \otimes \mathbb{Q}$, is a finite abelian group of order $|\det L|$. The *discriminant bilinear form* of a lattice L is the finite symmetric bilinear form induced by $\langle\ ,\ \rangle$

$$b(L)\colon L^\vee/L \times L^\vee/L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, the *discriminant quadratic form* of L is the finite quadratic form induced by $\langle\ ,\ \rangle$

$$q(L)\colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z}.$$

Let $\mathrm{O}(q(L))$ denote the automorphism group of the finite quadratic form $q(L)$, which we let act on $q(L)$ *from the right*. There is a natural homomorphism

$$\mathrm{O}(L) \to \mathrm{O}(q(L)), \qquad g \mapsto q(g).$$

Let $C_n(e)$ be the cyclic group of order n generated by e. For $k \geq 1$, we denote by u_k (resp. v_k) the finite quadratic form with underlying group $C_{2^k}(e) \times C_{2^k}(f)$ such that $\langle e,e\rangle = \langle f,f\rangle = 0$ (resp. $\langle e,e\rangle = \langle f,f\rangle = 1$) and $\langle e,f\rangle = \frac{1}{2^k}$. For $a,b \in \mathbb{Z}$ prime to each other, we denote by $\langle \frac{a}{b}\rangle$ the finite quadratic form with underlying group $C_b(e)$ such that $\langle e,e\rangle = \frac{a}{b}$.

2.4. **Genera.** Given a pair of non-negative integers (s_+,s_-) and a non-degenerate finite quadratic (resp. bilinear) form h, the *genus* $\mathfrak{g}(s_+,s_-,h)$ is the set of isometry classes of even (resp. odd) lattices of signature (s_+,s_-) with discriminant quadratic (resp. bilinear) form isomorphic to h. If a genus contains only the isometry class of a lattice L, we say that L *is unique in its genus*.

In general, enumerating all isometry classes in a given genus is a non-trivial problem. It is computationally easier to find lattices of smaller determinant, so the following elementary lemma can be very useful.

Lemma 2.4.1. *Given a lattice L and a prime number p, then $\ell_p(L^\vee/L) = \mathrm{rank}\, L$ if and only if $L = L'(p)$ for some lattice L'. In this case, and if moreover L is even and $p = 2$, then L' is odd if and only if $q(L) = \langle\frac{1}{2}\rangle \oplus q'$ or $q(L) = \langle\frac{3}{2}\rangle \oplus q'$ for some finite quadratic form q'.* ∎

Remark 2.4.2. Suppose q is a finite quadratic form admitting an isotropic subgroup Γ. In order to enumerate all isometry classes of even lattices in $\mathfrak{g}(s_+,s_-,q)$, we can take advantage of Proposition 1.4.1 in [20]: first we enumerate all lattices in $\mathfrak{g}(s_+,s_-,q|\Gamma^\perp/\Gamma)$, then we inspect all sublattices of index $|\Gamma|$.

Given a finite (bilinear or quadratic) form h and $s \in \mathbb{N}$, the following algorithm, suggested by Degtyarev, finds all (odd or even) lattices in $\mathfrak{g}(s,0,h)$. If h is quadratic we put $b = b(h)$, otherwise we put $b = h$.

Algorithm 2.4.3. Let r be the smallest possible rank for which there exists an (odd or even) positive definite lattice M of rank r and discriminant bilinear form $-b$. By results of Nikulin [20], for each $N \in \mathfrak{g}(s,0,h)$ there exists a primitive embedding $\iota\colon M \hookrightarrow L$ into some positive definite unimodular lattice L of rank $r+s$ such that $[\iota]^\perp \cong N$. Taking advantage of the classification of positive definite unimodular lattices of small rank (see, for instance, Table 16.7 in [7]), we list all such lattices L. Using `GAP` and the function `ShortestVectors`, we list all primitive embeddings $\iota\colon M \hookrightarrow L$ for all $M \in \mathfrak{g}'$ and all L. Then, we compute the lattices $[\iota]^\perp$ and select those ones which belong to $\mathfrak{g}(s,0,h)$. In order to eliminate pairs of isomorphic lattices, one can use the attribute `is_globally_equivalent_to` of the class `QuadraticForm` in `sage`. ∎

The algorithm works provided that $r+s$ is small enough and that we can find a lattice M explicitly. In order to find M, we can apply the algorithm recursively to $\mathfrak{g}(r,0,-b)$. If $r = 1$ or 2, this genus can be enumerated a priori (see, for instance, Chapter 15 in [7]).

Remark 2.4.4. Another well-known way to enumerate lattices in a given genus is Kneser's neighboring method [14]. This method has been implemented in `sage` by Brandhorst ([5] and private communication).

2.5. **Primitive embeddings.** Given an embedding of lattices $\iota\colon M \hookrightarrow S$, we denote by $[\iota]$ its image and by $[\iota]^\perp$ the orthogonal complement of $[\iota]$ in S. An embedding $\iota\colon M \hookrightarrow S$ is called *primitive* if $S/[\iota]$ is a torsion-free group. All primitive embeddings are considered up to the action of $\mathrm{O}(M)$.

Proposition 2.5.1 (Proposition 1.15.1 in [20])**.** *If $\iota\colon M \hookrightarrow S$ is a primitive embedding of even lattices, then there exist a subgroup $H \subset M^\vee/M$ and an isomorphism of finite quadratic forms $\beta\colon q([\iota])|H \to q(S)|\beta(H)$ such that*

$$q([\iota]^\perp) \cong (-q([\iota])) \oplus q(S)|\Gamma_\beta^\perp/\Gamma_\beta,$$

where Γ_β is the push-out of β in $(-q([\iota])) \oplus q(S)$. ∎

Given a primitive embedding $\iota\colon M \hookrightarrow S$, we put

$$\mathrm{O}(S,[\iota]) := \{g \in \mathrm{O}(S) \mid [\iota]^g = [\iota]\},$$

and we denote by $\mathrm{O}(q(S),[\iota])$ its image in $\mathrm{O}(q(S))$ by the natural homomorphism $\mathrm{O}(S) \to \mathrm{O}(q(S))$.

Fix now two even lattices M, N and consider the set $I(S,M,N)$ of primitive embeddings $\iota\colon M \hookrightarrow S$ such that $[\iota]^\perp \cong N$. The group $\mathrm{O}(S)$ acts on $I(S,M,N)$ in a natural way.

Consider also the set of pairs (H,γ), where $H \subset M^\vee/M$ is a subgroup and $\gamma\colon q(M)|H \to -q(N)|\gamma(H)$ is an isomorphism of finite quadratic forms such that

$$q(M) \oplus q(N)|\Gamma_\gamma^\perp/\Gamma_\gamma \cong q(S), \tag{2.2}$$

where Γ_γ is the push-out of γ in $q(M)\oplus q(N)$. We say that two such pairs (H,γ) and (H',γ') are *equivalent* if there exist $\varphi \in \mathrm{O}(M)$ and $\psi \in \mathrm{O}(N)$ such that $H^{q(\varphi)} = H'$ and

$$\gamma' \circ q(\varphi) = q(\psi) \circ \gamma. \tag{2.3}$$

Proposition 2.5.2 (Proposition 1.5.1 in [20])**.** *In the above notation, there is a one-to-one correspondence between the elements of $I(S,M,N)$ modulo the action of $\mathrm{O}(S)$ and the set of pairs (H,γ) modulo equivalence.* ∎

Proposition 2.5.3 (Proposition 1.5.2 in [20])**.** *For a fixed pair (H, γ) corresponding to the orbit of a primitive embedding $\iota\colon M \hookrightarrow S$, the subgroup $\mathrm{O}(q(S), [\iota])$ consists of those elements $\xi \in \mathrm{O}(q(S))$ for which there exist $\varphi \in \mathrm{O}(M)$ and $\psi \in \mathrm{O}(N)$ such that $H^{q(\varphi)} = H$, equation (2.3) holds, and ξ corresponds under the isomorphism (2.2) to the automorphism induced by φ and ψ on $\Gamma_\gamma^\perp/\Gamma_\gamma$.* ∎

2.6. **Chambers and their faces.** Let V be a $\mathbb{Q}$-vector space of dimension $n > 1$ with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon V \times V \to \mathbb{Q}$ such that $V \otimes \mathbb{R}$ is of signature $(1, n-1)$. Let $\mathcal{P}_V$ be one of the two connected components of $\{x \in V \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$. For $v \in V$ with $\langle v, v\rangle < 0$, we put

$$(v)^\perp := \{x \in \mathcal{P}_V \mid \langle x, v\rangle = 0\},$$

which is a hyperplane of $\mathcal{P}_V$. For a set $\mathcal{V}$ of vectors $v \in V$ with $\langle v, v\rangle < 0$, we denote by $\mathcal{V}^\perp$ the family of hyperplanes $\{(v)^\perp \mid v \in \mathcal{V}\}$.

Let $\mathcal{V}$ be a set of vectors $v \in V$ with $\langle v, v\rangle < 0$ such that the family of hyperplanes $\mathcal{V}^\perp$ is locally finite. A $\mathcal{V}^\perp$*-chamber* is the closure in $\mathcal{P}_V$ of a connected component of the complement

$$\mathcal{P}_V \setminus \bigcup_{H \in \mathcal{V}^\perp} H.$$

Let $\overline{\mathcal{P}}_V$ be the closure of $\mathcal{P}_V$ in $V \otimes \mathbb{R}$, and $\partial\,\overline{\mathcal{P}}_V$ the boundary $\overline{\mathcal{P}}_V \setminus \mathcal{P}_V$ of $\overline{\mathcal{P}}_V$. Let C be a $\mathcal{V}^\perp$-chamber, and $\overline{C}$ the closure of C in $V \otimes \mathbb{R}$. We say that C is *quasi-finite* if $\overline{C} \cap \partial\,\overline{\mathcal{P}}_V$ is contained in a union of at most countably many real half-lines of $V \otimes \mathbb{R}$.

Let C be a quasi-finite $\mathcal{V}^\perp$-chamber. Suppose that we are given a set U_C of vectors $v \in V$ with $\langle v, v\rangle < 0$ such that

$$C = \{x \in \mathcal{P}_V \mid \langle x, v\rangle \geq 0 \ \text{ for all } \ v \in U_C\}.$$

A *wall* of C is a closed subset w of C for which there exists a hyperplane $H \in \mathcal{V}^\perp$ with $w = C \cap H$ such that w contains a non-empty open subset of H. Let w be a wall of C. A vector $v \in V$ with $\langle v, v\rangle < 0$ is said to *define* w if w is equal to $C \cap (v)^\perp$ and $\langle x, v\rangle > 0$ holds for all interior points x of C. A vector $v_0 \in U_C$ defines a wall of C if and only if there exists a point $y \in \mathcal{P}_V$ such that $\langle y, v_0\rangle < 0$ and that $\langle y, v'\rangle > 0$ holds for all $v' \in U_C$ with $(v')^\perp \neq (v_0)^\perp$. Therefore, if U_C is finite, we can calculate the set of walls of C by means of linear programming.

A *face* is a closed subset of C that is the intersection of a finite number of walls of C. Let f be a face of C. We denote by $\langle f\rangle$ the minimal linear subspace of V containing f. The *dimension of* f is the dimension of $\langle f\rangle$. Suppose that $m := \dim f$ is ≥ 2. Since f contains a non-empty open subset of $\langle f\rangle$, the linear space $\langle f\rangle$ contains a vector v with $\langle v, v\rangle > 0$, and hence the restriction of $\langle\ ,\ \rangle$ to $\langle f\rangle$ is of signature $(1, m-1)$. We denote by

$$\iota_{\langle f\rangle}\colon \langle f\rangle \hookrightarrow V \ \text{ and } \ \mathrm{pr}_{\langle f\rangle}\colon V \twoheadrightarrow \langle f\rangle$$

the inclusion and the orthogonal projection, respectively, and let $\mathcal{P}_{\langle f\rangle}$ be the positive cone of $\langle f\rangle$ that is mapped into $\mathcal{P}_V$ by $\iota_{\langle f\rangle}$. We put

$$\iota_{\langle f\rangle}^* \mathcal{V}^\perp := \{\iota_{\langle f\rangle}^{-1}(H) \mid H \in \mathcal{V}^\perp \text{ such that } \iota_{\langle f\rangle}^{-1}(H) \text{ is a hyperplane of } \mathcal{P}_{\langle f\rangle}\},$$

which is a locally finite family of hyperplanes of $\mathcal{P}_{\langle f\rangle}$. Note that $\iota_{\langle f\rangle}^* \mathcal{V}^\perp$ is equal to $(\mathrm{pr}_{\langle f\rangle}^* \mathcal{V})^\perp$, where

$$\mathrm{pr}_{\langle f\rangle}^* \mathcal{V} := \{\mathrm{pr}_{\langle f\rangle}(v) \mid v \in \mathcal{V} \ \text{ such that } \ \langle \mathrm{pr}_{\langle f\rangle}(v), \mathrm{pr}_{\langle f\rangle}(v)\rangle < 0\}.$$

Then the face f of C is an $\iota^*_{\langle f\rangle}\mathcal{V}^\perp$-chamber in $\mathcal{P}_{\langle f\rangle}$, and is equal to

$$\{z \in \mathcal{P}_{\langle f\rangle} \mid \langle z, \mathrm{pr}_{\langle f\rangle}(v)\rangle \ge 0 \text{ for all } v \in U_C \text{ with } \langle \mathrm{pr}_{\langle f\rangle}(v), \mathrm{pr}_{\langle f\rangle}(v)\rangle < 0\}.$$

Therefore, if U_C is finite, we can calculate the set of walls of the $\iota^*_{\langle f\rangle}\mathcal{V}^\perp$-chamber f, and hence we can calculate the set of all faces of C by descending induction on the dimension of faces.

Let w be a wall of C. Then there exists a unique $\mathcal{V}^\perp$-chamber C' such that $C \cap C' = w$. This $\mathcal{V}^\perp$-chamber C' is said to be *adjacent to C across the wall w*.

2.7. **Induced chambers.** Let L be an even hyperbolic lattice. We apply the above definitions to $L \otimes \mathbb{Q}$. Let $\mathcal{P}_L$ be a positive cone of L, and let $\mathcal{V}$ be a set of vectors $v \in L \otimes \mathbb{Q}$ with $\langle v, v\rangle < 0$ such that the family $\mathcal{V}^\perp$ of hyperplanes of $\mathcal{P}_L$ is locally finite. Suppose that we have a primitive embedding

$$\iota_S \colon S \hookrightarrow L$$

of an even hyperbolic lattice S of rank $m < n$, and let $\mathcal{P}_S$ be the positive cone of S that is mapped into $\mathcal{P}_L$ by ι_S. We use the same letter ι_S to denote the inclusion $\mathcal{P}_S \hookrightarrow \mathcal{P}_L$. We denote the orthogonal projection by $\mathrm{pr}_S \colon L \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$, and put

$$\iota_S^*\mathcal{V}^\perp := \{\iota_S^{-1}(H) \mid H \in \mathcal{V}^\perp \text{ such that } \iota_S^{-1}(H) \text{ is a hyperplane of } \mathcal{P}_S\},$$
$$\mathrm{pr}_S^*\mathcal{V} := \{\mathrm{pr}_S(v) \mid v \in \mathcal{V} \text{ with } \langle \mathrm{pr}_S(v), \mathrm{pr}_S(v)\rangle < 0\}.$$

Then $\iota_S^*\mathcal{V}^\perp = (\mathrm{pr}_S^*\mathcal{V})^\perp$ is a locally finite family of hyperplanes of $\mathcal{P}_S$. A $\mathcal{V}^\perp$-chamber $C \subset \mathcal{P}_L$ is said to be *non-degenerate* with respect to ι_S if the closed subset $\iota_S^{-1}(C)$ of $\mathcal{P}_S$ contains a non-empty open subset of $\mathcal{P}_S$. Suppose that C is non-degenerate with respect to ι_S. Then $\iota_S^{-1}(C)$ is an $\iota_S^*\mathcal{V}^\perp$-chamber, which we call the chamber *induced by C*. If C is quasi-finite, then so is the induced chamber $\iota_S^{-1}(C)$.

2.8. **Vinberg chambers and Conway chambers.** Let L be as above. Note that the family $\mathcal{R}_L^\perp$ of hyperplanes is locally finite, where $\mathcal{R}_L$ is the set of (-2)-vectors. Each $r \in \mathcal{R}_L$ defines a reflection $x \mapsto x + \langle x, r\rangle r$. Let $W(L)$ be the subgroup of $\mathcal{O}(L, \mathcal{P}_L)$ generated by reflections with respect to (-2)-vectors. Then each $\mathcal{R}_L^\perp$-chamber is a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

For $n = 10$ and $n = 26$, let L_n be an even unimodular hyperbolic lattice of rank n, which is unique up to isomorphism. We denote by $\mathcal{P}_n$ a positive cone of $L_n \otimes \mathbb{R}$, and by $\mathcal{R}_n$ the set of (-2)-vectors of L_n.

An $\mathcal{R}_{10}^\perp$-chamber in $\mathcal{P}_{10}$ is called a *Vinberg chamber*. It is known that a Vinberg chamber is quasi-finite.

Theorem 2.8.1 (Vinberg [36])**.** *A Vinberg chamber has exactly* 10 *walls.* ∎

An $\mathcal{R}_{26}^\perp$-chamber in $\mathcal{P}_{26}$ is called a *Conway chamber*. It is known that a Conway chamber is quasi-finite. A non-zero primitive vector $\mathbf{w} \in L_{26} \cap \partial\mathcal{P}_{26}$ is called a *Weyl vector* if the negative definite lattice $[\mathbf{w}]^\perp/[\mathbf{w}]$ is isomorphic to the negative definite Leech lattice, where $[\mathbf{w}]^\perp := \{v \in L_{26} \mid \langle v, \mathbf{w}\rangle = 0\}$.

Theorem 2.8.2 (Conway [36])**.** *For each Conway chamber C, there exists a unique Weyl vector $\mathbf{w}_C$ such that the walls of C are defined by (-2)-vectors $r \in \mathcal{R}_{26}$ satisfying $\langle \mathbf{w}, r\rangle = 1$.* ∎

2.9. **Primitive embeddings of $L_{10}(2)$ into L_{26}.** In [6], we classified all primitive embeddings of $L_{10}(2)$ into L_{26}. It turns out that, up to the action of $\mathrm{O}(L_{10}(2)) = \mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$, there exist exactly 17 primitive embeddings, which are named as being of type

$$\mathtt{12A}, \mathtt{12B},\ \mathtt{20A}, \dots, \mathtt{20A}, \dots, \mathtt{20F},\ \mathtt{40A}, \dots, \mathtt{40E},\ \mathtt{96A}, \dots, \mathtt{96C},\ \mathtt{infty}.$$

Let $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ be a primitive embedding. Identifying positive cones of $L_{10}(2)$ with positive cones of L_{10} and replacing ι with $-\iota$ if necessary, we assume that ι maps $\mathcal{P}_{10}$ into $\mathcal{P}_{26}$. Then $\mathcal{P}_{10}$ is covered by $\iota^*\mathcal{R}_{26}^{\perp}$-chambers. Since Conway chambers are quasi-finite, every $\iota^*\mathcal{R}_{26}^{\perp}$-chambers are quasi-finite. In [6], we have proved the following:

Theorem 2.9.1. *Suppose that ι is not of type* `infty`*. Let D and D' be $\iota^*\mathcal{R}_{26}^{\perp}$-chambers. Then there exists an isometry $g \in \mathrm{O}^+(L_{10})$ that preserves the set of $\iota^*\mathcal{R}_{26}^{\perp}$-chambers and maps D to D'. Each $\iota^*\mathcal{R}_{26}^{\perp}$-chamber has only a finite number of walls, and each wall is defined by a (-2)-vector. If $D \cap (r)^{\perp}$ is a wall of D with $r \in \mathcal{R}_{10}$, then the $\iota^*\mathcal{R}_{26}^{\perp}$-chamber adjacent to D across the wall $D \cap (r)^{\perp}$ is the image of the reflection of D into the hyperplane $(r)^{\perp}$.* ∎

Remark 2.9.2. If a primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ is of type `infty`, then the $\iota^*\mathcal{R}_{26}^{\perp}$-chamber has infinitely many walls. The embedding ι is of type `infty` if and only if $[\iota]^{\perp}$ contains no (-2)-vectors.

Let Y be an Enriques surface. Then the Néron-Severi lattice S_Y is isomorphic to L_{10}. It is known that the nef chamber N_Y is bounded by hyperplanes $(r)^{\perp}$ defined by (-2)-vectors $r \in \mathcal{R}_Y$. In [6], we have proved the following:

Theorem 2.9.3. *Let $[\sigma, \tau]$ be one of the pairs*

$$[\mathtt{12A}, \mathrm{I}],\ [\mathtt{12B}, \mathrm{II}],\ [\mathtt{20A}, \mathrm{V}],\ [\mathtt{20B}, \mathrm{III}],\ [\mathtt{20C}, \mathrm{VII}],\ [\mathtt{20D}, \mathrm{VII}],\ [\mathtt{20E}, \mathrm{VI}],\ [\mathtt{20F}, \mathrm{IV}].$$

Then every $\iota^*\mathcal{R}_{26}^{\perp}$-chamber D for a primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ of type σ is equal to the nef chamber N_Y of an Enriques surface Y with finite automorphism group of Nikulin-Kondo type τ under an isomorphism $L_{10} \cong S_Y$. ∎

2.10. **K3 surfaces.** Let X be a complex projective K3 surface with transcendental lattice T_X. Then the nef chamber N_X is an $\mathcal{R}_X^{\perp}$-chamber, and each wall of N_X is defined by the class of a smooth rational curve on X. We put

$$\mathrm{O}(S_X, N_X) := \{ g \in \mathrm{O}(S_X) \mid N_X^g = N_X \}.$$

Recall that $W_X := W(S_X)$ is the subgroup of $\mathrm{O}(S_X, \mathcal{P}_X)$ generated by reflections with respect to (-2)-vectors. The following relations hold (see [22]):

$$\mathrm{O}(S_X, \mathcal{P}_X) = W_X \rtimes \mathrm{O}(S_X, N_X), \tag{2.4}$$

$$W_X \subset \ker(\mathrm{O}(S_X) \to \mathrm{O}(q(S_X))). \tag{2.5}$$

Let $\mathrm{O}(T_X, \omega_X)$ be the group of isometries of T_X that preserves the 1-dimensional subspace $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$, and let $\mathrm{O}(q(T_X), \omega_X)$ be the image of $\mathrm{O}(T_X, \omega_X)$ by the natural homomorphism $\mathrm{O}(T_X) \to \mathrm{O}(q(T_X))$. The even unimodular overlattice $H^2(X, \mathbb{Z})$ of the orthogonal direct sum $S_X \oplus T_X$ induces an anti-isometry between the discriminant forms of S_X and of T_X (see [20]), and hence induces an isomorphism $\mathrm{O}(q(S_X)) \cong \mathrm{O}(q(T_X))$. Let $\mathrm{O}(q(S_X), \omega_X)$ be the image of $\mathrm{O}(q(T_X), \omega_X)$ through this isomorphism. We say that an isometry $g \in \mathrm{O}(S_X)$ *satisfies the period condition* if $q(g) \in \mathrm{O}(q(S_X), \omega_X)$. Let $\mathrm{O}(S_X, \omega_X)$ denote the group of isometries satisfying the period condition. Recall that $\mathrm{aut}(X) \subset \mathrm{O}(S_X, \mathcal{P}_X)$ is the image of $\mathrm{Aut}(X)$ by (2.1). The Torelli theorem for complex K3 surfaces asserts that

$$\mathrm{aut}(X) = \mathrm{O}(S_X, N_X) \cap \mathrm{O}(S_X, \omega_X). \tag{2.6}$$

In particular, if $g \in \mathrm{O}(S_X, \omega_X)$ maps an interior point of N_X to an interior point of N_X, then g belongs to $\mathrm{aut}(X)$.

Remark 2.10.1. By the Torelli theorem, the kernel of $\rho_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)$ is isomorphic to the kernel of the natural homomorphism $\mathrm{O}(T_X, \omega_X) \to \mathrm{O}(q(T_X))$.

2.11. **Singular K3 surfaces.** Let X be a singular K3 surface. Its transcendental lattice T_X admits a basis with respect to which the Gram matrix is of the form

$$[a, b, c] := \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

with $0 \leq 2b \leq a \leq c$. We write $X(T)$ for the K3 surface corresponding to an oriented positive definite even lattice T of rank 2. The lattice $\overline{T} = [a, -b, c]$ defines a distinct oriented isomorphism class if and only if $0 < 2b < a < c$.

Remark 2.11.1. If X is a singular K3 surface, the subgroup $\mathrm{O}(T_X, \omega_X)$ can be identified with the subgroup consisting of isometries of T_X of positive determinant. Its image $\mathrm{O}(q(T_X), \omega_X)$ depends only on the genus of T_X.

3. Classification of Enriques involutions up to conjugation

Let X be a complex projective K3 surface. We are interested in classifying the images ε of Enriques involutions $\tilde{\varepsilon}$ in $\mathrm{aut}(X)$ through the natural representation (2.1) up to conjugation in $\mathrm{aut}(X)$. The image $\varepsilon \in \mathrm{aut}(X)$ is also call an Enriques involution. This is essentially the same problem by the following observation due to Ohashi.

Proposition 3.0.1 (Ohashi [22]). *Let $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 \colon X \to X$ be two Enriques involutions. Then the quotients $Y_i := X/\langle \tilde{\varepsilon}_i \rangle$, $i = 1, 2$, are isomorphic over $\mathbb{C}$ if and only if ε_1, ε_2 are conjugate in* $\mathrm{aut}(X)$. ∎

In this section, after recalling part of Ohashi's work, we refine and generalize his main Theorem 2.3 in [22].

3.1. **Main result.** Given an Enriques involution $\varepsilon \in \mathrm{aut}(X)$, we put

$$S_X^{\varepsilon=1} := \{ v \in S_X \mid v^{\varepsilon} = v \}.$$

We have the following criterion by Keum.

Theorem 3.1.1 (Keum [12]). *An involution $\varepsilon \in \mathrm{aut}(X)$ is an Enriques involution if and only if the following holds: the sublattice $S_X^{\varepsilon=1}$ is isomorphic to $L_{10}(2)$ and its orthogonal complement in S_X contains no (-2)-vectors.* ∎

Let I_X be the set of primitive embeddings $\iota \colon L_{10}(2) \hookrightarrow S_X$ such that the orthogonal complement $[\iota]^{\perp}$ of the image of ι in S_X contains no (-2)-vectors. The group $\mathrm{O}(S_X)$ acts on I_X in a natural way.

Proposition 3.1.2 (Proposition 2.2 in [22]). *For every $\iota \in I_X$ and $g \in \mathrm{O}(S_X)$ such that $[\iota]^g$ intersects the interior of N_X, there exists a unique $\varepsilon \in \mathrm{aut}(X)$ such that $S_X^{\varepsilon=1} = [\iota]^g$.* ∎

Corollary 3.1.3. *Let $\varepsilon_1, \varepsilon_2 \in \mathrm{aut}(X)$ be two Enriques involutions. Then, there exists $\gamma \in \mathrm{aut}(X)$ such that $\varepsilon_2 = \gamma \circ \varepsilon_1 \circ \gamma^{-1}$ if and only if $(S_X^{\varepsilon_1=1})^\gamma = S_X^{\varepsilon_2=1}$.* ∎

Proposition 3.1.4 (Step 1 of Theorem 2.3 in [22])**.** *For every $\iota \in I_X$ there exists $h \in \mathrm{O}(S_X)$ such that $[\iota]^h$ intersects the interior of N_X.* ∎

Lemma 3.1.5 (Step 2 of Theorem 2.3 in [22])**.** *Suppose $[\iota]$ intersects the interior of N_X. If there exist an Enriques involution $\varepsilon \in \mathrm{aut}(X)$ and $g \in \mathrm{O}(S_X)$ such that $S_X^{\varepsilon=1} = [\iota]^g$, then there exists $\tilde{g} \in \mathrm{O}(S_X, N_X)$ such that $S_X^{\varepsilon=1} = [\iota]^{\tilde{g}}$.* ∎

Proposition 3.1.6. *Given $\iota \in I_X$, let $\varepsilon_1, \varepsilon_2 \in \mathrm{aut}(X)$ be two Enriques involutions with $S_X^{\varepsilon_1=1} = [\iota]^{g_1}$ and $S_X^{\varepsilon_2=1} = [\iota]^{g_2}$ for some $g_1, g_2 \in \mathrm{O}(S_X, N_X)$. Then the Enriques involutions ε_1 and ε_2 are conjugate in $\mathrm{aut}(X)$ if and only if the natural images $q(g_1), q(g_2) \in \mathrm{O}(q(S_X))$ belong to the same double coset with respect to $\mathrm{O}(q(S_X), [\iota])$ and $\mathrm{O}(q(S_X), \omega_X)$.*

Proof. Let $\iota_i := g_i \circ \iota$ for $i = 1, 2$. Suppose there exists $\gamma \in \mathrm{aut}(X)$ with $\varepsilon_2 = \gamma \circ \varepsilon_1 \circ \gamma^{-1}$. Let $\varphi := g_2^{-1} \circ \gamma \circ g_1$, so that $\varphi \in \mathrm{O}(S_X, [\iota])$. Indeed, by Corollary 3.1.3,

$$[\iota]^\varphi = [\gamma \circ \iota_1]^{g_2^{-1}} = [\iota_2]^{g_2^{-1}} = [\iota].$$

As $g_1 = \varphi \circ g_2 \circ \gamma^{-1}$ and $\gamma \in \mathrm{O}(S_X, \omega_X)$, the automorphisms $q(g_1)$, $q(g_2)$ of $q(S_X)$ belong to the same double coset.

Conversely, assume that there exist $\varphi \in \mathrm{O}(S_X, [\iota])$ and $\gamma' \in \mathrm{O}(S_X, \omega_X)$ such that $q(g_2) = q(\varphi \circ g_1 \circ \gamma')$ in $\mathrm{O}(q(S_X))$. Without loss of generality, we can suppose $\varphi \in \mathrm{O}(S_X, N_X)$. In fact, we can first exchange φ with $-\varphi$ if necessary and suppose that $\varphi \in \mathrm{O}(S_X, \mathcal{P}_X)$. By (2.4) and (2.5), we can write $\varphi = w \circ \varphi'$, with $w \in W_X$ and $\varphi' \in \mathrm{O}(S_X, N_X)$ and exchange φ with φ' if necessary. Define now $\gamma := g_2 \circ \varphi^{-1} \circ g_1^{-1}$. Then $\gamma \in \mathrm{O}(S_X, N_X)$ and $q(\gamma) = q(\gamma')$, so $\gamma \in \mathrm{O}(S_X, \omega_X)$. The Torelli Theorem (2.6) implies that $\gamma \in \mathrm{aut}(X)$. Furthermore, we have

$$[\iota_1]^\gamma = ([\iota]^{\varphi^{-1}})^{g_2} = [\iota_2],$$

so ε_1 and ε_2 are conjugate in $\mathrm{aut}(X)$ by Corollary 3.1.3. □

Lemma 3.1.7. *If a K3 surface X admits at least one Enriques involution, then the lattice S_X is unique in its genus and the natural homomorphism $\mathrm{O}(S_X) \to \mathrm{O}(q(S_X))$ is surjective.*

Proof. Let $\iota \colon L_{10}(2) \hookrightarrow S_X$ be a primitive embedding. Then $q(S_X) \cong (q([\iota]) \oplus q([\iota]^\perp))|\Gamma^\perp/\Gamma$ for some isotropic subgroup Γ of $q([\iota]) \oplus q([\iota]^\perp)$. Since $q([\iota]) \cong q(L_{10}(2)) \cong u_1^{\oplus 5}$, this implies that

$$\ell_p(S_X^\vee/S_X) \leq \mathrm{rank}[\iota]^\perp = \mathrm{rank}\, S_X - 10$$

for every odd prime p. Moreover, if $\ell_2(S_X^\vee/S_X) = \mathrm{rank}\, S_X$, then $q(S_X) = q([\iota]) \oplus q'$ for some finite quadratic form q'. Therefore, we can conclude by Theorem 1.14.2 in [20]. □

Combining Lemma 3.1.7 and the same argument as in Step 5 of Theorem 2.3 in [22], we prove the following proposition.

Proposition 3.1.8. *If a K3 surface X admits at least one Enriques involution, then $\mathrm{O}(S_X, N_X) \to \mathrm{O}(q(S_X))$ is surjective.* □

Our main result is the following theorem.

Theorem 3.1.9. *Let X be a K3 surface and $\iota_1, \ldots, \iota_r \in I_X$ be a complete set of representatives for the action of* $\mathrm{O}(S_X)$ *on* I_X*. Then there exists a bijection between the set of Enriques involutions up to conjugation in* $\mathrm{aut}(X)$ *and the disjoint union of the sets of double cosets*

$$\mathrm{O}(q(S_X), [\iota_i]) \backslash \mathrm{O}(q(S_X)) / \mathrm{O}(q(S_X), \omega_X), \quad i = 1, \ldots, r.$$

Proof. Let $G = \mathrm{O}(S_X)$, $H_i = \mathrm{O}(q(S_X), [\iota_i])$ and $K = \mathrm{O}(q(S_X), \omega_X)$. For each $i = 1, \ldots, r$, fix $h_i \in G$ such that $[\iota_i]^{h_i}$ intersects the interior of N_X (Proposition 3.1.4). As exchanging ι_i with $h_i \circ \iota$ replaces H_i with a conjugate subgroup, we can suppose without loss of generality that $[\iota_i]$ intersects the interior of N_X. For each Enriques involution $\varepsilon \in \mathrm{aut}(X)$ there exists a unique $i \in \{1, \ldots, r\}$ such that there exists $g \in G$ with $S_X^{\varepsilon=1} = [\iota_i]^g$. Moreover, by Lemma 3.1.5, we can suppose that $g \in \mathrm{O}(S_X, N_X)$. We map such an ε to the double coset $H_i q(g) K \in H_i \backslash G / K$. This function is trivially well-defined and injective by Proposition 3.1.6.

To show surjectivity, take $i \in \{1, \ldots, r\}$ and $H_i \xi K \in H_i \backslash G / K$, with $\xi \in G$. By Proposition 3.1.8, $\xi = q(g)$ for some $g \in \mathrm{O}(S_X, N_X)$. As $[\iota_i]^g$ also intersects the interior of N_X, by Proposition 3.1.2 there is an Enriques involution $\varepsilon \in \mathrm{aut}(X)$ which maps to $H_i \xi K$. This concludes the proof. □

Corollary 3.1.10. *The number of Enriques involutions of a singular K3 surface X up to conjugation in* $\mathrm{aut}(X)$ *only depends on the genus of the transcendental lattice T_X.*

Proof. The lattice S_X is unique in its genus by Lemma 3.1.7, so it is completely determined by the genus of T_X. The subgroup $\mathrm{O}(q(S_X), \omega_X)$ is also determined by the genus of T_X when X is singular (see Remark 2.11.1). The subgroups $\mathrm{O}(q(S_X), [\iota])$ for $\iota \in I_X$ only depend on S_X, so in turn they depend only on the genus of T_X. □

Remark 3.1.11. Schütt [24] described a relation of two singular K3 surfaces whose transcendental lattices are in the same genus. See also [26].

3.2. **Table 3.1.** Table 3.1 contains the list of all singular K3 surfaces X of discriminant d with $d \leq 36$, given by their respective transcendental lattices T_X, together with the list of the Enriques involutions that they admit, up to conjugation in $\mathrm{aut}(X)$. We will illustrate presently how this table was compiled.

The following theorem by Sertöz builds on work by Keum [12] and characterizes singular K3 surfaces without Enriques quotients.

Theorem 3.2.1 (Sertöz [25]; see also [11])**.** *Let X be a singular K3 surface of discriminant d. Then X has no Enriques involution if and only if $d \equiv 3\,(8)$ or $T_X \in \{[2,0,2], [2,0,4], [2,0,8]\}$.* ∎

In all other cases, we determined the set of conjugacy classes of all Enriques involutions in $\mathrm{aut}(X)$ by means of Theorem 3.1.9. The item $|\mathrm{Enr}|$ in Table 3.1 indicates the number of such conjugacy classes.

First of all, one must determine a complete set of representatives for the action of $\mathrm{O}(S_X)$ on I_X. Given a positive definite even lattice N of rank 10 without 2-vectors (see Theorem 3.1.1), we put

$$I_X(N) := \{\iota \in I_X \mid [\iota]^\perp \cong N(-1)\}.$$

Clearly, the sets $I_X(N)$ form a partition of I_X which respects the $\mathrm{O}(S_X)$-action, so we reduce the problem to computing a complete set of representatives for the action of $\mathrm{O}(S_X)$ on $I_X(N)$, for each N such that $I_X(N) \neq \emptyset$.

Table 3.1. Enriques involutions up to conjugation of singular K3 surfaces of discriminant $d \leq 36$ (see Section 3.2).

d	T_X	$\lvert\mathrm{Enr}\rvert$	$q(N)$	N	$\lvert I_X(N)\rvert$
3	$[2,1,2]$	0	–	–	–
4	$[2,0,2]$	0	–	–	–
7	$[2,1,4]$	2	$u_1^{\oplus 5} \oplus \langle \frac{2}{7} \rangle$	$N_{10,7}^{144}(2)$ $N_{10,7}^{242}(2)$	1 1
8	$[2,0,4]$	0	–	–	–
11	$[2,1,6]$	0	–	–	–
12	$[2,0,6]$	1	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{6} \rangle$	$M_{10,3}^{144}(2)$	1
12	$[4,2,4]$	3	$u_1^{\oplus 4} \oplus v_1 \oplus \langle \frac{4}{3} \rangle$	$N_{10,3}^{246}(2)$	3×1
15	$[2,1,8]$	5	$u_1^{\oplus 5} \oplus \langle \frac{2}{15} \rangle$	$N_{10,15}^{90}(2)$ $N_{10,15}^{132}(2)$ $N_{10,15}^{144}(2)$ $N_{10,15}^{240}(2)$	1 1 2 1
15	$[4,1,4]$	4	$u_1^{\oplus 5} \oplus \langle \frac{4}{15} \rangle$	$N_{10,15}^{92}(2)$ $N_{10,15}^{112}(2)$ $N_{10,15}^{242}(2)$	1 1 2
16	$[2,0,8]$	0	–	–	–
16	$[4,0,4]$	9	$u_1^{\oplus 4} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{4} \rangle$	$D_{10}(2)$ $N_{10,4}^{244}(2)$	3×1 5×1
			$u_1^{\oplus 3} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{4} \rangle$	$N_{10,1024}^{0,308}$	1
19	$[2,1,10]$	0	–	–	–
20	$[2,0,10]$	1	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{10} \rangle$	$M_{10,5}^{132}(2)$	1
20	$[4,2,6]$	2	$u_1^{\oplus 4} \oplus \langle \frac{3}{2} \rangle \oplus \langle \frac{3}{10} \rangle$	$M_{10,5}^{92}(2)$ $M_{10,5}^{242}(2)$	1 1
23	$[2,1,12]$, $[4,\pm 1,6]$	7	$u_1^{\oplus 5} \oplus \langle \frac{2}{23} \rangle$	$N_{10,23}^{74}(2)$ $N_{10,23}^{84}(2)$ $N_{10,23}^{112}(2)$ $N_{10,23}^{132}(2)$ $N_{10,23}^{144}(2)$ $N_{10,23}^{240}(2)$ $N_{10,23}^{242}(2)$	1 1 1 1 1 1 1
24	$[2,0,12]$	1	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{12} \rangle$	$M_{10,6}^{90}(2)$	1
24	$[4,0,6]$	1	$u_1^{\oplus 4} \oplus \langle \frac{3}{2} \rangle \oplus \langle \frac{11}{12} \rangle$	$M_{10,6}^{242}(2)$	1
27	$[2,1,14]$	0	–	–	–

Continued on next page

Table 3.1 – continued from previous page

d	T	$\|\mathrm{Enr}\|$	$q(N)$	N	$\|I_X(N)\|$
27	$[6,3,6]$	0	–	–	–
28	$[2,0,14]$	1	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{14} \rangle$	$M_{10,7}^{112}(2)$	1
28	$[4,2,8]$	24	$u_1^{\oplus 5} \oplus \langle \frac{2}{7} \rangle$	$N_{10,7}^{144}(2)$	$3 \times 1 + 4 \times 2$
				$N_{10,7}^{242}(2)$	$4 \times 1 + 4 \times 2$
			$u_1^{\oplus 4} \oplus \langle \frac{2}{7} \rangle$	$N_{10,1792}^{0,274}$	1
31	$[2,1,16]$, $[4,\pm 1,8]$	9	$u_1^{\oplus 5} \oplus \langle \frac{2}{23} \rangle$	$N_{10,31}^{60}(2)$	1
				$N_{10,31}^{72}(2)$	1
				$N_{10,31}^{86}(2)$	1
				$N_{10,31}^{90}(2)$	1
				$N_{10,31}^{112}(2)$	1
				$N_{10,31}^{128}(2)$	1
				$N_{10,31}^{144}(2)$	1
				$N_{10,31}^{240}(2)$	1
				$N_{10,31}^{242}(2)$	1
32	$[2,0,16]$	1	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{16} \rangle$	$M_{10,8}^{84}(2)$	1
32	$[4,0,8]$	33	$u_1^{\oplus 4} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{8} \rangle$	$N_{10,8}^{138}(2)$	$2 \times 1 + 4 \times 2$
				$N_{10,8}^{146}(2)$	$3 \times 1 + 2 \times 2$
				$N_{10,8}^{242}(2)$	$3 \times 1 + 5 \times 2$
			$u_1^{\oplus 3} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{8} \rangle$	$N_{10,2048}^{0,210}$	1
				$N_{10,2048}^{0,250}$	1
				$N_{10,2048}^{0,274}$	1
32	$[6,2,6]$	3	$u_1^{\oplus 4} \oplus \langle \frac{3}{2} \rangle \oplus \langle \frac{3}{16} \rangle$	$M_{10,8}^{112}(2)$	1
				$M_{10,8}^{144}(2)$	1
				$M_{10,8}^{240}(2)$	1
35	$[2,1,18]$	0	–	–	–
35	$[6,1,6]$	0	–	–	–
36	$[2,0,18]$	3	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{18} \rangle$	$M_{10,9}^{74}(2)$	1
				$M_{10,9}^{90}(2)$	1
				$M_{10,9}^{128}(2)$	1
36	$[4,2,10]$	2	$u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{5}{18} \rangle$	$M_{10,9}^{80}(2)$	1
				$M_{10,9}^{242}(2)$	1
36	$[6,0,6]$	3	$u_1^{\oplus 4} \oplus \langle \frac{1}{6} \rangle \oplus \langle \frac{1}{6} \rangle$	$M_{10,9}^{60}(2)$	1
				$M_{10,9}^{132}(2)$	1
				$M_{10,9}^{240}(2)$	1

We find all such lattices in the following way. Using Proposition 2.5.1, we list all possible finite quadratic forms q, such that $q \cong q(N)$. For each form q, we determine all lattices N in the genus $\mathfrak{g}(10, 0, q)$ without 2-vectors (see Algorithm 2.4.3).

All possible finite quadratic forms $q = q(N)$ and orthogonal complements N have been listed in Table 3.1 in the items $q(N)$ and N. The name $N_{r,d}^{\rho_2,\rho_4}$ (resp. $M_{r,d}^{\rho_2,\rho_4}$) denotes a positive definite even (resp. odd) lattice of rank r, determinant d, with ρ_2 2-vectors and ρ_4 4-vectors (ρ_4 omitted if not needed to distinguish two lattices). A Gram matrix for each of these lattices can be found in [31].

Since $I_X(N) = I(S_X, L_{10}(2), N(-1))$ as defined in Section 2.5, a complete set of representatives $\iota_1, \dots, \iota_r$ up to the action of $\mathrm{O}(S_X)$ on $I_X(N)$ can be enumerated using Proposition 2.5.2. For each $i \in \{1, \dots, r\}$, the subgroup $H_i = \mathrm{O}(q(S_X), [\iota_i])$ of $G = \mathrm{O}(q(S_X))$ can be determined using Proposition 2.5.3. On the other hand, the subgroup $K = \mathrm{O}(q(S_X), \omega_X)$ can be computed using Remark 2.11.1.

Remark 3.2.2. In order to apply Proposition 2.5.2, it is worth mentioning that for $L = L_{10}(2)$ the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(q(L))$ is surjective and that, up to the action of $\mathrm{O}(q(L))$, there are only two subgroups of $L^\vee/L$ of order 2.

On the other hand, since N is positive definite, we can compute $\mathrm{O}(N)$ by the attribute `automorphism_group` of the class `QuadraticForm` in `sage`; hence, we can compute its image in $\mathrm{O}(q(N))$.

The item $|I_X(N)|$ gives the cardinalities of the sets of double cosets $H_i\backslash G/K$. For instance, the entry "$3\times 1+4\times 2$" means that $r = 7$, $|H_i\backslash G/K| = 1$ for $i = 1, 2, 3$ and $|H_i\backslash G/K| = 2$ for $i = 4, \dots, 7$. Note that the item $|\mathrm{Enr}|$ is the sum of the items $|I_X(N)|$ over the lattices N.

4. Automorphism groups of singular K3 surfaces

4.1. **Borcherds method.** We explain Borcherds method ([3], [4]) to calculate $\mathrm{aut}(X)$ of a K3 surface X and its action on N_X. The details of the algorithms in the computation below are explained in [27]. Suppose that we have a primitive embedding

$$\iota_X : S_X \hookrightarrow L_{26}.$$

We assume that ι_X maps $\mathcal{P}_X$ to the positive cone $\mathcal{P}_{26}$ of L_{26}, and consider the decomposition of $\mathcal{P}_X$ by $\iota_X^*\mathcal{R}_{26}^\perp$-chambers, that is, by chambers induced by Conway chambers non-degenerate with respect to ι_X. Since ι_X maps $\mathcal{R}_X$ to $\mathcal{R}_{26}$, every $\mathcal{R}_X^\perp$-chamber is a union of $\iota_X^*\mathcal{R}_{26}^\perp$-chambers. In particular, the nef chamber N_X is a union of $\iota_X^*\mathcal{R}_{26}^\perp$-chambers. Since a Conway chamber is quasi-finite, every $\iota_X^*\mathcal{R}_{26}^\perp$-chamber is quasi-finite.

The orthogonal complement $[\iota_X]^\perp$ of the image of ι_X is an even negative definite lattice. The even unimodular overlattice L_{26} of $S_X \oplus [\iota_X]^\perp$ induces an *anti*-isometry $q(S_X) \cong -q([\iota_X]^\perp)$, and hence an isomorphism $\mathrm{O}(q(S_X)) \cong \mathrm{O}(q([\iota_X]^\perp))$. We assume the following condition:

(A) the image of $\mathrm{O}(q(S_X), \omega_X)$ by the isomorphism $\mathrm{O}(q(S_X)) \cong \mathrm{O}(q([\iota_X]^\perp))$ above is contained in the image of the natural homomorphism $\mathrm{O}([\iota_X]^\perp) \to \mathrm{O}(q([\iota_X]^\perp))$.

Since $\mathrm{O}([\iota_X]^\perp)$ and $\mathrm{O}(q(S_X), \omega_X)$ are finite, we can determine whether this condition is fulfilled or not. Suppose that Condition (A) is satisfied. Then every isometry

$g \in \mathrm{O}(S_X, \omega_X) \cap \mathrm{O}(S_X, \mathcal{P}_X)$ extends to an isometry $\tilde{g} \in \mathrm{O}(L_{26}, \mathcal{P}_{26})$, which preserves the set of Conway chambers. Therefore every isometry of S_X satisfying the period condition preserves the set of $\iota_X^* \mathcal{R}_{26}^{\perp}$-chambers.

We also assume the following condition:

(B) $[\iota_X]^{\perp}$ cannot be embedded into the negative definite Leech lattice.

For example, if $[\iota_X]^{\perp}$ contains a (-2)-vector, then this condition is fulfilled. Condition (B) implies that each $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber D in $\mathcal{P}_X$ has only a finite number of walls (see [27]). More precisely, if D is induced by a Conway chamber C, then the set of vectors defining walls of D can be calculated from the Weyl vector $\mathbf{w}_C$ corresponding to C by Theorem 2.8.2. By this finiteness, we can calculate, for two $\iota_X^* \mathcal{R}_{26}^{\perp}$-chambers D and D', the set of all isometries $g \in \mathrm{O}(S_X)$ such that $D^g = D'$. In particular, the group

$$\mathrm{O}(S_X, D) := \{g \in \mathrm{O}(S_X) \mid D^g = D\}$$

is finite, and can be calculated explicitly. If $D \subset N_X$, then

$$\mathrm{aut}(X, D) := \mathrm{O}(S_X, D) \cap \mathrm{O}(S_X, \omega_X)$$

is contained in $\mathrm{aut}(X)$, and can be calculated explicitly.

Definition 4.1.1. Let D be an $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber contained in N_X. A wall $D \cap (v)^{\perp}$ of D is called an *outer wall* if it is defined by a (-2)-vector, that is, if there exists a rational number λ such that $-2/\langle v, v\rangle = \lambda^2$ and $\lambda v \in S_X$. Otherwise, we say that $D \cap (v)^{\perp}$ is an *inner wall*.

A wall $D \cap (v)^{\perp}$ is an outer wall if and only if $N_X \cap (v)^{\perp}$ is a wall of N_X. The $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber D'adjacent to D across a wall $D \cap (v)^{\perp}$ of D is contained in N_X if and only if $D \cap (v)^{\perp}$ is an inner wall.

Let D be an $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber, and let $\mathbf{w}_C$ be the Weyl vector corresponding to a Conway chamber C inducing $D = \iota_X^{-1}(C)$. Let $D \cap (v)$ be a wall of D, and let D' be the $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber adjacent to D across $D \cap (v)^{\perp}$. Then we can calculate the Weyl vector $\mathbf{w}_{C'}$ corresponding to a Conway chamber C' inducing $D' = \iota_X^{-1}(C')$ (see [27]), and hence we can calculate the set of walls of D', which is again finite. Therefore we can determine whether there exists an isometry $g \in \mathrm{O}(S_X, \omega_X)$ that maps D to D'.

Definition 4.1.2. Let $D \cap (v)^{\perp}$ be an inner wall of an $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber D contained in N_X. An isometry $g \in \mathrm{O}(S_X, \omega_X)$ is said to be an *extra automorphism* associated with $D \cap (v)^{\perp}$ if g maps D to the $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber adjacent to D across $D \cap (v)^{\perp}$.

Let g be an extra automorphism as above. Since g satisfies the period condition, Condition (A) implies that g preserves the set of $\iota_X^* \mathcal{R}_{26}^{\perp}$-chambers. Moreover g maps an interior point of N_X to the interior of N_X, and hence $g \in \mathrm{aut}(X)$. We consider the following condition:

(IX) There exists an $\iota_X^* \mathcal{R}_{26}^{\perp}$-chamber D_0 contained in N_X such that every inner wall of D_0 has an extra automorphism.

Definition 4.1.3. We say that an embedding ι_X satisfying Conditions (A), (B) and (IX) is of *simple Borcherds type*.

Theorem 4.1.4 ([27])**.** *Suppose that ι_X is of simple Borcherds type.*

Table 4.1. The 11 K3 surfaces to which we can apply Borcherds method (see Section 4.2).

No.	T_X	`root type`	m_1	m_2	m_3	m_4	k_1	k_2
1	$[2,1,2]$	E_6	12	6	6	3	103680	2
2	$[2,0,2]$	D_6	8	4	4	2	46080	2
3	$[2,1,4]$	A_6	4	2	2	1	10080	2
4	$[2,0,4]$	D_5+A_1	4	2	2	1	7680	2
5	$[2,0,6]$	A_5+A_1	4	2	2	1	2880	2
6	$[4,2,4]$	D_4+A_2	12	6	1	1	13824	12
7	$[2,1,8]$	A_4+A_2	4	2	2	1	2880	4
8	$[4,0,4]$	$2A_3$	8	4	1	1	4608	8
9	$[4,2,6]$	A_4+2A_1	4	2	1	1	1920	4
10	$[2,0,12]$	$A_3+A_2+A_1$	4	2	2	1	1152	4
11	$[6,0,6]$	$2A_2+2A_1$	8	4	1	1	2304	16

(1) *For any point v of N_X, there exists an automorphism g of X such that $v^g \in D_0$.*

(2) *Let $o_1, \dots, o_m$ be the orbits of the action of* $\mathrm{aut}(X, D_0)$ *on the set of inner walls of D_0, and, for $i = 1, \dots, m$, let $g(o_i)$ be an extra automorphism associated with an inner wall $D_0 \cap (v_i)^\perp$ belonging to o_i. Then* $\mathrm{aut}(X)$ *is generated by* $\mathrm{aut}(X, D_0)$ *and the extra automorphisms $g(o_1), \dots, g(o_m)$.* ∎

4.2. **Application to certain singular K3 surfaces.** We consider singular K3 surfaces with transcendental lattice $T_X = [a, b, c]$ in Table 4.1. These transcendental lattices are characterized among all even binary positive definite lattices by the following properties: there exists a primitive embedding $\iota_X \colon S_X \hookrightarrow L_{26}$ of simple Borcherds type such that the orthogonal complement $[\iota_X]^\perp$ is generated by (-2)-vectors. In particular, Condition (B) is satisfied. The column `root type` in Table 4.1 indicates the ADE-type of the standard fundamental root system of $[\iota_X]^\perp$. For these cases, the natural homomorphism $\mathrm{O}([\iota_X]^\perp) \to \mathrm{O}(q([\iota_X]^\perp))$ is surjective and hence Condition (A) is satisfied. The following data are also given in Table 4.1.

- m_1 is the order of $\mathrm{O}(T_X)$, m_2 is the order of $\mathrm{O}(T_X, \omega_X)$, m_3 is the order of the kernel K of the homomorphism $\mathrm{O}(T_X) \to \mathrm{O}(q(T_X))$, and m_4 is the order of $\mathrm{O}(T_X, \omega_X) \cap K$. Then m_4 is the order of the kernel of ρ_X by Remark 2.10.1, and the order of $\mathrm{O}(q(T_X), \omega_X) \cong \mathrm{O}(q(S_X), \omega_X)$ is m_2/m_4.
- k_1 is the order of $\mathrm{O}([\iota_X]^\perp)$, and k_2 is the order of $\mathrm{O}(q(T_X)) \cong \mathrm{O}(q(S_X)) \cong \mathrm{O}(q([\iota_X]^\perp))$.

We have a Conway chamber C_0 that induces an $\iota_X^* \mathcal{R}_{26}^\perp$-chamber D_0 contained in N_X. Let $\mathbf{w} \in L_{26}$ be the Weyl vector corresponding to C_0, and let $\mathbf{w}_S \in S_X \otimes \mathbb{Q}$ be the image of $\mathbf{w}$ by the orthogonal projection $\mathrm{pr}_S \colon L_{26} \otimes \mathbb{Q} \to S_X \otimes \mathbb{Q}$. For each of the 11 cases, we can confirm that $\mathbf{w}_S$ belongs to the interior of D_0 and that $\mathbf{w}_S$ is invariant under the action of $\mathrm{aut}(X, D_0)$. Let o be an orbit of the action of $\mathrm{aut}(X, D_0)$ on the set of walls of D_0, and let $D_0 \cap (v)^\perp$ be a member of o. We choose the defining vector v of this wall in such a way that v is *primitive* in $S_X^\vee$. Then v is unique. The values $n := \langle v, v \rangle$ and $a := \langle v, \mathbf{w}_S \rangle$ are independent of the choice of the wall $D_0 \cap (v)^\perp \in o$. Suppose that the orbit o consists of inner walls. Then we can find an extra automorphism $g \in \mathrm{aut}(X)$ associated with $D_0 \cap (v)^\perp$ by a direct calculation. Hence ι_X is of simple Borcherds type. The degree $d_g := \langle \mathbf{w}_S^g, \mathbf{w}_S \rangle$ is also independent of the choice of $D_0 \cap (v)^\perp$ and g. Table 4.2 contains the data of walls and extra automorphisms of D_0. If $D_0 \cap (v)^\perp$ is an inner wall, the (-2)-vectors r of L_{26} such that $(r)^\perp$ passes through $\iota_X(D_0 \cap (v)^\perp) \subset \mathcal{P}_{26}$ form a root system, whose ADE-type is also given below.

Remark 4.2.1. Almost all results in Table 4.2 have already appeared in previous works. See Vinberg [37] for Nos. 1 and 2 of Table 4.1, Ujikawa [34] for No. 3, Keum and Kondo [13] for Nos. 6 and 8, [27] for Nos. 4, 5 and 6, [28] for Nos. 7, 9 and 11.

Remark 4.2.2. In Table 4.2, the order of the finite group $\mathrm{aut}_0 := \mathrm{aut}(X, D_0)$ is given. The list of all elements of aut_0 is given in [31].

Table 4.2. Walls and extra automorphisms of D_0.

T_X	$\lvert\mathrm{aut}_0\rvert$	$\langle \mathbf{w}_S, \mathbf{w}_S \rangle$	No.	$\lvert o\rvert$		n	a	d_g	root type
$[2,1,2]$	72	78		6	outer	-2	1		
				18	outer	-2	1		
			1	12	inner	$-2/3$	9	321	E_7
$[2,0,2]$	120	55		10	outer	-2	1		
				15	outer	-2	1		
				20	outer	$-1/2$	$17/2$		
			1	5	inner	-1	6	127	D_7
$[2,1,4]$	336	28		28	outer	-2	1		
			1	14	inner	$-8/7$	4	56	A_7
			2	28	inner	$-4/7$	6	154	D_7
			3	56	inner	$-2/7$	7	371	E_7
$[2,0,4]$	48	$61/2$		6	outer	-2	1		
				8	outer	-2	1		
				12	outer	-2	1		
				2	outer	$-1/2$	$11/2$		
			1	3	inner	$-3/2$	$3/2$	$67/2$	A_2+D_5
			2	4	inner	-1	5	$161/2$	A_1+D_6
			3	6	inner	-1	5	$161/2$	A_1+D_6
			4	8	inner	$-3/4$	6	$253/2$	A_1+E_6
			5	24	inner	$-3/4$	6	$253/2$	A_1+E_6
			6	8	inner	$-1/4$	$13/2$	$737/2$	E_7
$[2,0,6]$	144	18		12	outer	-2	1		
				18	outer	-2	1		
				12	outer	$-1/2$	$11/2$		
				36	outer	$-1/2$	$11/2$		
			1	4	inner	$-3/2$	$3/2$	21	A_2+A_5
			2	24	inner	$-7/6$	$7/2$	39	A_1+A_6
			3	6	inner	$-2/3$	4	66	A_7
			4	24	inner	$-2/3$	5	93	A_1+D_6
			5	36	inner	$-2/3$	5	93	A_1+D_6
			6	24	inner	$-1/6$	$11/2$	381	E_7
$[4,2,4]$	1152	16		32	outer	-2	1		
			1	8	inner	$-4/3$	2	22	A_3+D_4

Continued on next page

Table 4.2 – continued from previous page

T_X	$\lvert\mathrm{aut}_0\rvert$	$\langle \mathbf{w}_S, \mathbf{w}_S \rangle$	No.	$\lvert o\rvert$		n	a	d_g	root type
			2	72	inner	-1	4	48	$A_2 + D_5$
			3	96	inner	$-1/3$	5	166	D_7
$[2, 1, 8]$	720	12		36	outer	-2	1		
			1	12	inner	$-4/3$	2	18	$A_3 + A_4$
			2	40	inner	$-6/5$	3	27	$A_2 + A_5$
			3	90	inner	$-4/5$	4	52	$A_2 + D_5$
			4, 5	30	inner	$-8/15$	4	72	A_7
			6, 7	120	inner	$-2/15$	5	387	E_7
$[4, 0, 4]$	3840	10		40	outer	-2	1		
			1	64	inner	$-5/4$	$5/2$	20	$A_3 + A_4$
			2	40	inner	-1	3	28	$A_3 + D_4$
			3	160	inner	$-1/2$	4	74	A_7
			4	320	inner	$-1/4$	$9/2$	172	D_7
$[4, 2, 6]$	120	11		5	outer	-2	1		
				30	outer	-2	1		
			1, 2	6	inner	$-3/2$	$3/2$	14	$A_1 + A_2 + A_4$
			3	20	inner	$-6/5$	3	26	$2A_1 + A_5$
			4	30	inner	$-6/5$	3	26	$2A_1 + A_5$
			5	1	inner	-1	2	19	$A_3 + A_4$
			6	30	inner	$-4/5$	4	51	$2A_1 + D_5$
			7	40	inner	$-4/5$	4	51	$2A_1 + D_5$
			8	60	inner	$-4/5$	4	51	$2A_1 + D_5$
			9, 10	20	inner	$-7/10$	$7/2$	46	$A_1 + A_6$
			11, 12	20	inner	$-3/10$	$9/2$	146	$A_1 + E_6$
			13, 14	60	inner	$-3/10$	$9/2$	146	$A_1 + E_6$
			15	10	inner	$-1/5$	4	171	D_7
$[2, 0, 12]$	720	$15/2$		45	outer	-2	1		
				45	outer	$-1/2$	$7/2$		
			1	10	inner	$-3/2$	$3/2$	$21/2$	$2A_2 + A_3$
			2	30	inner	$-4/3$	2	$27/2$	$A_1 + 2A_3$
			3	72	inner	$-5/4$	$5/2$	$35/2$	$A_1 + A_2 + A_4$
			4	60	inner	-1	3	$51/2$	$A_1 + A_2 + D_4$
			5	12	inner	$-5/6$	$5/2$	$45/2$	$A_3 + A_4$
			6	40	inner	$-3/4$	3	$63/2$	$A_2 + A_5$
			7, 8	120	inner	$-7/12$	$7/2$	$99/2$	$A_1 + A_6$
			9	120	inner	$-1/3$	4	$207/2$	$A_1 + D_6$
			10	180	inner	$-1/3$	4	$207/2$	$A_1 + D_6$
			11, 12	120	inner	$-1/12$	4	$783/2$	E_7
$[6, 0, 6]$	1440	5		60	outer	-2	1		
			1	40	inner	$-3/2$	$3/2$	8	$A_1 + 3A_2$
			2	180	inner	$-4/3$	2	11	$2A_1 + A_2 + A_3$
			3	10	inner	-1	2	13	$2A_2 + A_3$
			4, 5	144	inner	$-5/6$	$5/2$	20	$A_1 + A_2 + A_4$
			6	240	inner	$-2/3$	3	32	$2A_1 + A_5$
			7	360	inner	$-2/3$	3	32	$2A_1 + A_5$
			8	180	inner	$-1/3$	3	59	$A_2 + D_5$
			9, 10	240	inner	$-1/6$	$7/2$	152	$A_1 + E_6$
			11, 12	720	inner	$-1/6$	$7/2$	152	$A_1 + E_6$

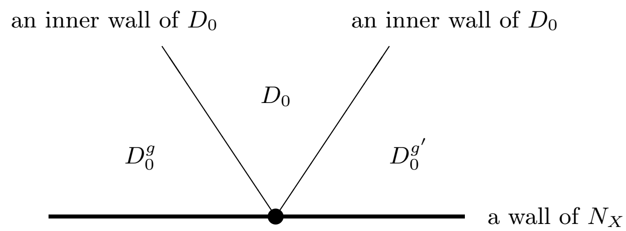

FIGURE 5.1. A D_0-inner face that is not N_X-inner.

5. ENRIQUES INVOLUTIONS AND BORCHERDS METHOD

In this section, we assume that X is a complex K3 surface admitting a primitive embedding $\iota_X\colon S_X \hookrightarrow L_{26}$ of simple Borcherds type and, in addition, that

(C) the natural homomorphism $\rho_X\colon \operatorname{Aut}(X) \to \mathrm{O}(S_X, \mathcal{P}_X)$ is injective.

5.1. **Inner faces.** Let D_0 be an $\iota_X^*\mathcal{R}_{26}^\perp$-chamber contained in N_X. Let $w_1, \dots, w_k$ be the inner walls of D_0. For each w_i, we calculate an extra automorphism $g_i \in \operatorname{aut}(X)$ associated with w_i (see Definition 4.1.2).

Definition 5.1.1. A face f of D_0 is said to be D_0-*inner* if f is not contained in any outer wall of D_0, whereas f is said to be N_X-*inner* if f is not contained in any wall of N_X.

Remark 5.1.2. An N_X-inner face is always D_0-inner. The converse is, however, not true in general as illustrated in Figure 5.1, in which a black circle indicates a D_0-inner face of codimension 2 that is not N_X-inner.

Let f be a D_0-inner face of dimension > 0. We put

$$\begin{aligned}\mathcal{D}(f) &:= \{D \mid D \text{ is an } \iota_X^*\mathcal{R}_{26}^\perp\text{-chamber contained in } N_X \text{ and containing } f\},\\ \mathcal{A}(X, f) &:= \{g \in \operatorname{aut}(X) \mid D_0^g \in \mathcal{D}(f)\} = \{g \in \operatorname{aut}(X) \mid f \subset D_0^g\},\\ \operatorname{aut}(X, f) &:= \{g \in \operatorname{aut}(X) \mid f^g = f\}.\end{aligned}$$

The set $\mathcal{D}(f)$ is calculated by the following method.

Algorithm 5.1.3. We set $\mathcal{D} = [D_0]$, $\gamma_0 = \mathrm{id}$, $\mathit{\Gamma} = [\gamma_0]$, and $i = 0$. During the calculation, the ordered set $\mathcal{D}$ is a subset of $\mathcal{D}(f)$, and the $(i+1)$st member γ_i of $\mathit{\Gamma}$ is an element of $\operatorname{aut}(X)$ that maps D_0 to the $(i+1)$st member D_i of $\mathcal{D}$. While $i < |\mathcal{D}|$, we execute the following. We calculate the set $\{w_{\nu(1)}, \dots, w_{\nu(m)}\}$ of inner walls $w_{\nu(j)}$ of D_0 such that $f \subset w_{\nu(j)}^{\gamma_i}$. Let $g_{\nu(j)} \in \operatorname{aut}(X)$ be an extra automorphism associated with $w_{\nu(j)}$. For each $j = 1, \dots, m$, we calculate the induced chamber $D' := D_0^{g_{\nu(j)}\gamma_i}$, which is adjacent to $D_i = D_0^{\gamma_i}$ across $w_{\nu(j)}^{\gamma_i}$ and contains f. If D' has not yet been added to $\mathcal{D}$, we add D' to $\mathcal{D}$ and $g_{\nu(j)}\gamma_i$ to $\mathit{\Gamma}$. Then we increment i to $i+1$. ∎

When this algorithm terminates, the list $\mathcal{D}$ is equal to $\mathcal{D}(f)$. Moreover, we have calculated $\mathit{\Gamma} = \{g_D \mid D \in \mathcal{D}(f)\}$, where $g_D \in \operatorname{aut}(X)$ maps D_0 to $D \in \mathcal{D}(f)$. Note that the action of $g_D \in \mathit{\Gamma}$ preserves the walls of N_X. The following is obvious from the definition.

Criterion 5.1.4. The D_0-inner face f is N_X-inner if and only if, for any $g_D \in \mathit{\Gamma}$ and any outer wall $D_0 \cap (r)^\perp$ of D_0, the wall $(D_0 \cap (r)^\perp)^{g_D}$ of $D = D_0^{g_D}$ does not contain f. ∎

Suppose that f is N_X-inner and D is an element of $\mathcal{D}(f)$. Note that the set of all elements $g \in \mathrm{aut}(X)$ that maps D_0 to D is equal to $\mathrm{aut}(X, D_0) \cdot g_D$. Therefore we can calculate $\mathcal{A}(X, f)$ by

$$\mathcal{A}(X, f) = \bigsqcup_{D \in \mathcal{D}(f)} \mathrm{aut}(X, D_0) \cdot g_D.$$

The subgroup $\mathrm{aut}(X, f)$ of $\mathrm{aut}(X)$ is contained in the finite set $\mathcal{A}(X, f)$, and thus we can calculate $\mathrm{aut}(X, f)$.

Definition 5.1.5. Let f and f' be N_X-inner faces of D_0. We say that f and f' are $\mathrm{aut}(X)$*-equivalent* (resp. $\mathrm{aut}(X, D_0)$*-equivalent*) if there exists an element $g \in \mathrm{aut}(X)$ (resp. $g \in \mathrm{aut}(X, D_0)$) such that $f^g = f'$.

Even though $\mathrm{aut}(X)$ is infinite in general, we can calculate the $\mathrm{aut}(X)$-equivalence classes by the following:

Criterion 5.1.6. The faces f and f' are $\mathrm{aut}(X)$-equivalent if and only if there exists an element $g \in \mathcal{A}(X, f')$ such that $f^g = f'$. ∎

5.2. **An algorithm to classify all Enriques involutions.** Let $\tilde{\varepsilon}\colon X \to X$ be an Enriques involution, and $\pi\colon X \to Y := X/\langle\tilde{\varepsilon}\rangle$ the quotient morphism to the Enriques surface Y. Let $\varepsilon \in \mathrm{aut}(X)$ denote the image of $\tilde{\varepsilon}$ by the natural homomorphism (2.1). Then π induces a primitive embedding $\pi^*\colon S_Y(2) \hookrightarrow S_X$. We have canonical identifications $S_Y(2) \otimes \mathbb{R} = S_Y \otimes \mathbb{R}$ and $\mathrm{O}(S_Y(2)) = \mathrm{O}(S_Y)$. In particular, we regard the positive cone $\mathcal{P}_Y$ of S_Y as a positive cone of $S_Y(2)$. The embedding π^* induces an embedding

$$\pi^*\colon \mathcal{P}_Y \hookrightarrow \mathcal{P}_X.$$

Henceforth, we regard $S_Y(2)$ as a primitive sublattice of S_X and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by π^*. Note that $S_Y(2)$ is equal to $\{v \in S_X \mid v^\varepsilon = v\}$, and $\mathcal{P}_Y$ is equal to $\{x \in \mathcal{P}_X \mid x^\varepsilon = x\}$.

Proposition 5.2.1. *We have $N_Y = N_X \cap \mathcal{P}_Y$. Let y be a point of N_Y. Then y is an interior point of N_Y if and only if y is an interior point of N_X.*

Proof. The first equality is obvious. By Theorem 3.1.1, the orthogonal complement of $S_Y(2)$ in S_X contains no (-2)-vectors, and a line bundle of Y is ample if and only if its pull-back to X is ample. □

Let y be a sufficiently general point of N_Y. By Theorem 4.1.4, there exists an automorphism $g \in \mathrm{aut}(X)$ such that $y^g \in D_0$, and hence $D_0 \cap N_Y^g$ contains a non-empty open subset of $\mathcal{P}_Y^g$. Therefore, replacing ε by $g^{-1}\varepsilon g$, we can assume that

$$E_0 := D_0 \cap N_Y$$

contains a non-empty open subset of $\mathcal{P}_Y$. Consider the composite

$$\iota_Y := \iota_X \circ \pi^*\colon S_Y(2) \hookrightarrow L_{26}$$

of primitive embeddings. Then $\mathcal{P}_Y$ is decomposed into the union of $\iota_Y^*\mathcal{R}_{26}^\perp$-chambers. Since every wall of N_Y is defined by a (-2)-vector, it follows that N_Y is decomposed into a union of $\iota_Y^*\mathcal{R}_{26}^\perp$-chambers. Note that E_0 is one of the $\iota_Y^*\mathcal{R}_{26}^\perp$-chambers in N_Y.

Definition 5.2.2. For a closed subset A of D_0, the *minimal face* of D_0 for A is the face of D_0 containing A with the minimal dimension.

Let f_ε be the minimal face of D_0 for E_0. Since the orthogonal complement of $S_Y(2)$ in S_X contains no (-2)-vector, the face f_ε is N_X-inner. Moreover, the involution $\varepsilon \in \mathrm{aut}(X)$ belongs to $\mathrm{aut}(X, f_\varepsilon)$. Let ε' be an Enriques involution such that $f_{\varepsilon'}$ is a face of D_0. If ε' is conjugate to ε, then f_ε is $\mathrm{aut}(X)$-equivalent to $f_{\varepsilon'}$. If $f_\varepsilon = f_{\varepsilon'}$, then ε and ε' are conjugate if and only if ε and ε' are conjugate in $\mathrm{aut}(X, f_\varepsilon)$.

We calculate all N_X-inner faces of D_0 of dimension ≥ 10 by descending induction of the dimension of faces (see Section 2.6), and compute a complete set of representatives of the $\mathrm{aut}(X)$-equivalence classes. For each representative f, we calculate $\mathrm{aut}(X, f)$. We then calculate the set of Enriques involutions ε contained in $\mathrm{aut}(X, f)$ such that $f_\varepsilon = f$ by Keum's criterion (Theorem 3.1.1), and thus we obtain a set of complete representatives of Enriques involutions in $\mathrm{aut}(X)$ modulo conjugation.

5.3. **Computation of** $\mathrm{Aut}(Y)$**.** Let ε be a representative of $\mathrm{aut}(X)$-conjugacy classes of Enriques involutions obtained by the method above. In particular, we have an $\iota_Y^* \mathcal{R}_{26}^{\perp}$-chamber $E_0 = D_0 \cap N_Y$, the minimal face f_ε of D_0 for E_0, and the associated data $\mathcal{D}(f_\varepsilon)$, $\mathcal{A}(X, f_\varepsilon)$, $\mathrm{aut}(X, f_\varepsilon)$. We put

$$\mathrm{aut}(X, \varepsilon) := \{ g_X \in \mathrm{aut}(X) \mid \varepsilon g_X = g_X \varepsilon \} = \{ g_X \in \mathrm{aut}(X) \mid S_Y(2)^{g_X} = S_Y(2) \},$$

where the second equality follows from $S_Y(2) = \{ v \in S_X \mid v^\varepsilon = v \}$. We have a natural restriction homomorphism $\mathrm{aut}(X, \varepsilon) \to \mathrm{O}(S_Y)$, which is denoted by $g_X \mapsto g_X | S_Y$. By Condition (C), we have a natural identification

$$\mathrm{Aut}(Y) \cong \mathrm{aut}(X, \varepsilon)/\langle \varepsilon \rangle. \tag{5.1}$$

Under the identification (5.1), the homomorphism $\rho_Y \colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$ is identified with the homomorphism $g_X \bmod \langle \varepsilon \rangle \mapsto g_X | S_Y$. The method below, when it works, gives us a finite set of generators of $\mathrm{aut}(X, \varepsilon)$, and hence a finite set of generators of $\mathrm{Aut}(Y)$.

Recall that $\mathrm{aut}(Y)$ is the image of $\mathrm{Aut}(Y)$ by ρ_Y. We put

$$\mathrm{aut}(Y, E_0) := \{ g \in \mathrm{aut}(Y) \mid E_0^g = E_0 \},$$

and let $\mathrm{Aut}(Y, E_0)$ denote the inverse image of $\mathrm{aut}(Y, E_0)$ by ρ_Y.

Proposition 5.3.1. *The action of* $\mathrm{aut}(Y)$ *on* N_Y *preserves the set of* $\iota_Y^* \mathcal{R}_{26}^{\perp}$*-chambers contained in* N_Y*.*

Proof. Let g be an element of $\mathrm{aut}(Y)$. Then g extends to $g_X \in \mathrm{aut}(X, \varepsilon)$. By Condition (A), this isometry $g_X \in \mathrm{O}(S_X, \omega_X) \cap \mathrm{O}(S_X, \mathcal{P}_X)$ extends to an isometry $\tilde{g}_X$ of L_{26}, which preserves the set of Conway chambers. Hence its restriction g to $S_Y(2)$ preserves the set of chambers induced by Conway chambers. □

We put

$$\mathrm{aut}(X, \varepsilon, f_\varepsilon) := \mathrm{aut}(X, \varepsilon) \cap \mathrm{aut}(X, f_\varepsilon).$$

Proposition 5.3.2. *The identification* (5.1) *induces* $\mathrm{Aut}(Y, E_0) \cong \mathrm{aut}(X, \varepsilon, f_\varepsilon)/\langle \varepsilon \rangle$*.*

Proof. Note that $E_0 = f_\varepsilon \cap N_Y$. Since E_0 contains an interior point of the face f_ε, an element g_X of $\mathrm{aut}(X, \varepsilon)$ fixes E_0 if and only if g_X fixes f_ε. □

Corollary 5.3.3. *By the identification* (5.1)*, the kernel of* $\rho_Y \colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$ *is equal to*

$$\{ g_X \in \mathrm{aut}(X, \varepsilon, f_\varepsilon) \mid g_X | S_Y = \mathrm{id} \}/\langle \varepsilon \rangle.$$

Recall from Section 2.9 that we have classified primitive embeddings of $S_Y(2) \cong L_{10}(2)$ into L_{26}. The $\iota_Y^*\mathcal{R}_{26}^{\perp}$-chamber E_0 has only finitely many walls. By Remark 2.9.2, the primitive embedding $\iota_Y\colon S_Y(2) \hookrightarrow L_{26}$ is not of type `infty`. By Theorem 2.9.1, every $\iota_Y^*\mathcal{R}_{26}^{\perp}$-chamber E has only a finite number of walls, and each wall of E is defined by a (-2)-vector $r \in \mathcal{R}_Y$.

Definition 5.3.4. A wall w of E_0 is said to be *outer* if w is contained in a wall of N_Y. Otherwise w is said to be *inner*.

There are several criteria to determine whether a given wall w of E_0 is outer or inner.

Criterion 5.3.5. Suppose that the wall w of E_0 is defined by $r \in \mathcal{R}_Y$. Then w is outer if and only if there exists a (-2)-vector u in the orthogonal complement $[\pi^*]^{\perp}$ of $S_Y(2)$ in S_X such that $(u + r)/2 \in S_X$. ∎

Indeed, the condition in the statement is equivalent to the condition that r is the class of an effective divisor of Y (see [21]).

Criterion 5.3.6. Let $f_\varepsilon(w)$ be the minimal face of D_0 for the closed subset w of D_0. Then w is inner if and only if $f_\varepsilon(w)$ is N_X-inner. ∎

Indeed, by minimality of $f_\varepsilon(w)$, there exists an interior point y of w that is an interior point of $f_\varepsilon(w)$. Then the statement follows from Proposition 5.2.1.

When E_0 has no inner walls, we have $E_0 = N_Y$ and $|\operatorname{Aut}(Y)| < \infty$, and the Nikulin-Kondo type of Y is obtained by comparing the configuration of (-2)-vectors defining the walls of E_0 with the dual graphs of smooth rational curves given in [15].

We consider $\operatorname{Aut}(Y)$ when E_0 has an inner wall. Let I_0 denote the set of inner walls of E_0. For each $w = E_0 \cap (r)^{\perp} \in I_0$ with $r \in \mathcal{R}_Y$, we put $E(w) := E_0^{s_r}$, where $s_r\colon \mathcal{P}_Y \to \mathcal{P}_Y$ is the reflection into the hyperplane $(r)^{\perp} \subset \mathcal{P}_Y$. Theorem 2.9.1 implies that $E(w)$ is the $\iota_Y^*\mathcal{R}_{26}^{\perp}$-chamber adjacent to E_0 across w. Recall that $\mathcal{A}(X, f_\varepsilon(w))$ is the set of $g_X \in \operatorname{aut}(X)$ such that $D_0^{g_X}$ contains $f_\varepsilon(w)$. If the restriction $g_X|S_Y$ to $S_Y(2)$ of $g_X \in \operatorname{aut}(X, \varepsilon)$ maps E_0 to $E(w)$, then $g_X \in \mathcal{A}(X, f_\varepsilon(w))$ holds.

Definition 5.3.7. An element g_X of $\operatorname{aut}(X, \varepsilon) \cap \mathcal{A}(X, f_\varepsilon(w))$ is an *extra automorphism* for the inner wall $w \in I_0$ if the restriction $g_X|S_Y$ of g_X to $S_Y(2)$ maps E_0 to $E(w)$.

Since $\mathcal{A}(X, f_\varepsilon(w))$ is finite, we can determine the existence of an extra automorphism for each inner wall of E_0.

Theorem 5.3.8. *Suppose that Condition* (C) *is satisfied. Suppose also that the following holds:*

(IY) *there exists an extra automorphism $g_X(w)$ for each inner wall $w \in I_0$.*

Then $\operatorname{aut}(X, \varepsilon)$ *is generated by the finite subgroup* $\operatorname{aut}(X, \varepsilon, f_\varepsilon)$ *and the extra automorphisms $g_X(w)$ $(w \in I_0)$.*

Proof. Let Γ denote the subgroup of $\operatorname{aut}(X, \varepsilon)$ generated by the extra automorphisms $g_X(w)$ $(w \in I_0)$. First we prove the following claim. For any $\iota_Y^*\mathcal{R}_{26}^{\perp}$-chamber E contained in N_Y, there exists an element $\gamma \in \Gamma$ such that $\gamma|S_Y$ maps E_0 to E. There exists a chain $E_0, E_1, \dots, E_m = E$ of $\iota_Y^*\mathcal{R}_{26}^{\perp}$-chambers contained in N_Y such that E_{i-1} and E_i is adjacent for $i = 1, \dots, m$. We prove the claim by induction on the length m of the chain with the case $m = 0$ being trivial. Suppose that $m > 0$. There exists an element $\gamma' \in \Gamma$ such that $\gamma'|S_Y$ maps E_0 to E_{m-1}. Let E' be the $\iota_Y^* \mathcal{R}_{26}^{\perp}$-chamber that is mapped to E_m by $\gamma'|S_Y$. Then E' is adjacent to E_0. Note that $\gamma'|S_Y \in \mathrm{aut}(Y)$ preserves N_Y. Therefore E' is contained in N_Y. In particular, the wall w between E_0 and E' is inner, and hence there exists an extra automorphism $g_X(w)$ such that $g_X(w)|S_Y$ maps E_0 to E'. We put $\gamma := g_X(w) \cdot \gamma' \in \Gamma$. Then $\gamma|S_Y$ maps E_0 to E_m.

Next we show that Γ and $\mathrm{aut}(X, \varepsilon, f_\varepsilon)$ generate $\mathrm{aut}(X, \varepsilon)$. Let g be an arbitrary element of $\mathrm{aut}(X, \varepsilon)$. We apply the claim above to the $\iota_Y^* \mathcal{R}_{26}^{\perp}$-chamber $E_0^{g|S_Y}$, and obtain an element $\gamma \in \Gamma$ such that $(g\gamma^{-1})|S_Y$ is an element of $\mathrm{aut}(Y, E_0)$. By Proposition 5.3.2, we have $g\gamma^{-1} \in \mathrm{aut}(X, \varepsilon, f_\varepsilon)$. □

Definition 5.3.9. We say that a triple $(X, \iota_X, \varepsilon)$ of a K3 surface X, a primitive embedding $\iota_X \colon S_X \hookrightarrow L_{26}$, and an Enriques involution ε of X is of *simple Borcherds type* if X satisfies Condition (C), (X, ι_X) is of simple Borcherds type in the sense of Definition 4.1.3, and ε satisfies Condition (IY).

Remark 5.3.10. The notion of simple Borcherds type was introduced in [29] for K3 surfaces. We hope that we can find a bound on the degrees of polarizations similar to that of [29] for Enriques surfaces.

5.4. **Enriques involutions of the 11 singular K3 surfaces.** We apply the method in the previous section to the singular K3 surfaces in Section 4.2. First remark that Condition (C) holds for the 11 cases except for the cases $T_X = [2, 1, 2]$ and $T_X = [2, 0, 2]$ (see Remark 2.10.1 and Table 4.1). Note that in these two cases, and also in the case $T_X = [2, 0, 4]$, there exist no Enriques involutions by Theorem 3.2.1.

Our main result is as follows.

Theorem 5.4.1. *Let X be one of the singular K3 surfaces of No. $\neq 1, 2, 4$ in Table 4.1, and let $\iota_X \colon S_X \hookrightarrow L_{26}$ be the primitive embedding given in Section 4.2. Then the Enriques involutions of X modulo conjugation in $\mathrm{Aut}(X) \cong \mathrm{aut}(X)$ are given in Table 5.1. For each Enriques involution ε on X, the triple $(X, \iota_X, \varepsilon)$ is of simple Borcherds type.*

We explain the contents of Table 5.1. The item ι_Y is the type of the primitive embedding $\iota_Y \colon S_Y(2) \hookrightarrow L_{26}$ given in [6]. The item NK is the Nikulin-Kondo type of the $\iota_Y^* \mathcal{R}_{26}^{\perp}$-chamber E_0 (see Theorem 2.9.3). The item `m4` is the number of (-4)-vectors in the orthogonal complement of $S_Y(2)$ in S_X. The item |ws| is the number of walls of E_0. The item $|G_\varepsilon|$ is the order of

$$G_\varepsilon := \mathrm{aut}(X, \varepsilon, f_\varepsilon).$$

The item $|I_0|$ is the number of inner walls of E_0.

Remark 5.4.2. For the Enriques involution No. 24 on X with $T_X = [6, 0, 6]$, the $\iota_Y^* \mathcal{R}_{26}^{\perp}$-chamber E_0 has 40 walls and the configuration of the walls is not of Nikulin-Kondo type. The dual graph is too complicated to be presented here. See [31] for the matrix presentation of this configuration.

The item $|K_\rho|$ is the order of the kernel of $\rho_Y \colon \mathrm{Aut}(Y) \to \mathrm{aut}(Y)$, and the item |aut| is the order of $\mathrm{aut}(Y)$. The fact that $\mathrm{aut}(Y)$ is infinite when I_0 is non-empty was confirmed by selecting elements of $\mathrm{aut}(Y)$ randomly by means of the finite

Table 5.1. Enriques involutions of the 11 singular K3 surfaces (see Section 5.4).

No.	T_X	$\dim f_\varepsilon$	ι_Y	NK	m4	$\lvert\mathrm{ws}\rvert$	$\lvert G_\varepsilon\rvert$	$\lvert I_0\rvert$	$\lvert K_\rho\rvert$	$\lvert\mathrm{aut}\rvert$
1	$[2,1,4]$	19	12B	II	144	12	48	0	1	24
2		18	12A	I	242	12	16	0	2	4
3	$[2,0,6]$	19	12B	II	144	12	48	0	1	24
4	$[4,2,4]$	18	12A	I	246	12	16	0	2	4
5		18	20B	III	246	20	64	4	2	∞
6		17	20A	V	246	20	96	0	2	24
7	$[2,1,8]$	19	20D	VII	90	20	120	5	1	∞
8		19	12B	II	144	12	48	0	1	24
9		19	12B	II	144	12	48	0	1	24
10		18	12A	I	240	12	8	2	2	∞
11		17	20A	V	132	20	48	4	1	∞
12	$[4,0,4]$	20	20F	IV	180	20	640	0	1	320
13		19	20D	VII	180	20	120	5	1	∞
14		19	12B	II	180	12	48	0	1	24
15		18	12A	I	244	12	16	0	2	4
16		18	12A	I	244	12	16	2	4	∞
17		18	20B	III	244	20	64	8	2	∞
18		18	20B	III	244	20	64	4	2	∞
19		18	20B	III	308	20	256	0	2	64
20		17	20A	V	244	20	32	4	2	∞
21	$[4,2,6]$	19	20D	VII	92	20	240	0	1	120
22		18	12A	I	242	12	16	0	2	4
23	$[2,0,12]$	19	20D	VII	90	20	120	5	1	∞
24	$[6,0,6]$	20	40E		60	40	1440	10	1	∞
25		18	12A	I	240	12	16	2	4	∞
26		17	20A	V	132	20	48	4	1	∞

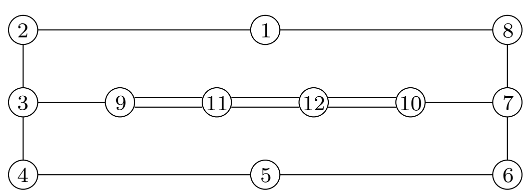

Figure 5.2. Configuration of Nikulin-Kondo type I

generating set of $\mathrm{aut}(Y)$ obtained by Theorem 5.3.8 and finding a matrix of infinite order among these sample elements.

Remark 5.4.3. Consider the Enriques involutions of Nos. 10, 16 and 25, that is, the cases where the Nikulin-Kondo type is I and $\mathrm{Aut}(Y)$ is infinite. In these cases, we

Table 5.2. N_X-inner faces corresponding to Enriques involutions.

T_X	dim	`numb`	`pws`	$\|\mathcal{D}\|$	$\|\mathrm{aut}(X, f)\|$	ε
$[2, 1, 4]$	19	14	1^1	2	48	No. 1
	18	$42 + 84$	$1^2, 1^1 2^1$	6	16	No. 2
$[2, 0, 6]$	19	6	3^1	2	48	No. 3
$[4, 2, 4]$	18	288×2	$1^1 2^1, 1^1 3^1$	8	16	No. 4
	18	12	1^2	6	576	No. 5
	17	144	$1^2 2^1$	12	96	No. 6
$[2, 1, 8]$	19	12	1^1	2	120	No. 7
	19	30	4^1	2	48	No. 8
	19	30	5^1	2	48	No. 9
	18	180×4	$1^1 4^1, 1^1 5^1, 3^1 4^1, 3^1 5^1$	8	8	No. 10
	17	90×2	$1^2 3^1$	12	48	No. 11
$[4, 0, 4]$	20	1		1	3840	No. 12
	19	64	1^1	2	120	No. 13
	19	160	3^1	2	48	No. 14
	18	960×2	$1^1 2^1, 1^1 4^1$	8	16	No. 15
	18	960×2	$2^1 3^1$	8	16	No. 16
	18	60	2^2	4	256	Nos. 17, 18, 19
	17	$480 + 960$	$1^2 2^1, 1^1 2^2$	12	32	No. 20
$[4, 2, 6]$	19	1	5^1	2	240	No. 21
	18	30×2	$4^1 5^1, 4^1 15^1$	8	16	No. 22
$[2, 0, 12]$	19	12	5^1	2	120	No. 23
$[6, 0, 6]$	20	1		1	1440	No. 24
	18	360×2	$7^1 8^1$	8	16	No. 25
	17	180×2	$2^2 8^1$	12	48	No. 26

have $|I_0| = 2$. The configuration of Nikulin-Kondo type I is as in Figure 5.2, and the inner walls are defined by the (-2)-vectors ⑪ and ⑫.

See [31] for the inner walls of E_0 for the other Enriques involutions. The finite generating sets of $\mathrm{aut}(X, \varepsilon)$ and of $\mathrm{aut}(Y)$ are also given explicitly in [31].

Table 5.2 is a list of N_X-inner faces of D_0 that corresponds to Enriques involutions. Note that an $\mathrm{aut}(X)$-equivalence class of N_X-inner faces is a union of orbits of the action of $\mathrm{aut}(X, D_0)$ on the set of N_X-inner faces.

The item `numb` gives the number of faces in the $\mathrm{aut}(X)$-equivalence class. The formula in this column shows the decomposition of the $\mathrm{aut}(X)$-equivalence class into a union of $\mathrm{aut}(X, D_0)$-orbits. The item `pws` indicates the types of inner walls of D_0 passing through the face. The type of an inner wall of D_0 is given by No. in Table 4.2.

For example, take the case $T_X = [2, 1, 4]$. For a face f in the $\mathrm{aut}(X)$-equivalence class corresponding to the Enriques involution No. 2, there exist exactly two inner walls of D_0 passing through f, and they are both of type 1, whereas for another face f' in this $\mathrm{aut}(X)$-equivalence class, there exist exactly two inner walls of D_0 passing through f', and they are of type 1 and 2.

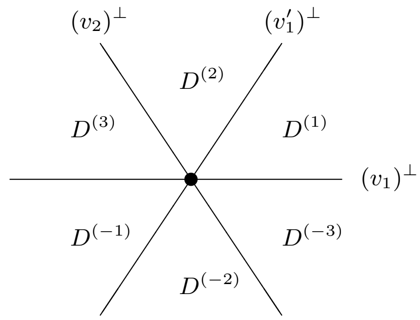

FIGURE 5.3. The N_X-inner face f.

We explain how the data `pws` depends on the choice of a representative of an $\mathrm{aut}(X)$-equivalence class. Let f be a face in this $\mathrm{aut}(X)$-equivalence class. Then there exist exactly three members $(v_1)^\perp, (v'_1)^\perp, (v_2)^\perp$ in the family $\iota_X^* \mathcal{R}_{26}^\perp$ of hyperplanes that pass through f, where v_1, v'_1, v_2 are primitive vectors of $S_X^\vee$ such that $\langle v_1, v_1\rangle = \langle v'_1, v'_1\rangle = -8/7$ and $\langle v_2, v_2\rangle = -4/7$. See Figure 5.3. If D_0 is located in the region $D^{(\pm 1)}$, then the data `pws` for f is 1^2, whereas if D_0 is located in the region $D^{(\pm 2)}$ or $D^{(\pm 3)}$, then the data `pws` for f is $1^1 2^1$.

The item $|\mathcal{D}|$ is the size of $\mathcal{D}(f)$ and $|\mathrm{aut}(X, f)|$ is the order of the group $\mathrm{aut}(X, f)$. The item ε shows the Nos. of the Enriques involutions given in Table 5.1.

6. The two most algebraic Enriques surfaces

In this section, we study the two most algebraic Enriques surfaces, that is, Enriques surfaces covered by the singular K3 surface X_7 of discriminant 7.

We recall that the Néron–Severi lattice and the automorphism group of X_7 were determined by Ujikawa [34]. Elliptic fibrations on X_7 were studied by Harrache–Lecacheux [10] and Lecacheux [17].

6.1. **Conjugagy classes of Enriques involutions.** We exemplify Theorem 3.1.9 for the case X_7. Let $T = T_{X_7} = [2, 1, 4]$ and $S = S_{X_7}$. Let $\iota \in I_{X_7}$ and put $N := [\iota]^\perp(-1)$. Let $q := q(T) = \langle \frac{2}{7} \rangle$, so that $q(S) \cong -q \cong \langle \frac{6}{7} \rangle$. In the notation of Proposition 2.5.1, the subgroup $H \subset q([\iota])$ must be trivial, so N is an even lattice of genus $\mathfrak{g}(10, 0, u_1^{\oplus 5} \oplus q)$. By Lemma 2.4.1, $N \cong N'(2)$, with N' an even lattice of genus $\mathfrak{g}(10, 0, q)$.

Lemma 6.1.1. *The genus $\mathfrak{g}(10, 0, q)$ contains exactly two isomorphism classes, namely $N_{10,7}^{242}$ and $N_{10,7}^{144}$ (see [31]).*

Proof. Let N' be a lattice in this genus. The smallest lattice with bilinear form $b = -b(q)$ is the odd lattice $M_{3,7} := [2, 1, 2, 1, 1, 3]$, which is unique in its genus. Thus, by [20], $N' \cong [\iota]^\perp$ for some primitive embedding $\iota\colon M_{3,7} \hookrightarrow L$ into a unimodular lattice L of rank 13. Inspecting all such embeddings, we find exactly two non-isomorphic even orthogonal complements. □

By Proposition 2.5.2, for both $N = N_{10,7}^{242}(2)$ and $N = N_{10,7}^{144}(2)$, the set $I_{X_7}(N)$ has exactly one $\mathrm{O}(S)$-orbit. Thus, $r = 2$ in Theorem 3.1.9. Since $\mathrm{O}(q(S), \omega_{X_7}) = \mathrm{O}(q(S))$, there is exactly 1 double coset in both cases. Hence, X_7 admits exactly two Enriques involutions up to conjugation in $\mathrm{aut}(X)$. The two involutions can be distinguished by the number of (-4)-vectors in the orthogonal complements of their fixed lattices.

6.2. **Models of the two Enriques quotients.** By the results of Section 5.4, the two quotients Y_{I} and Y_{II} of X_7 have Nikulin-Kondo type I and II. Kondo [15] gives two explicit 1-dimensional families containing all Enriques surfaces of Nikulin-Kondo type I and II. Each family depend on one parameter α; in this section we determine which values of α give Y_{I} and Y_{II}. We first summarize Kondo's construction.

Let ϕ be the involution on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$([u_0, u_1], [v_0, v_1]) \mapsto ([u_0, -u_1], [v_0, -v_1]),$$

and consider the curves $L_1\colon u_0 = u_1$, $L_2\colon u_0 = -u_1$, $L_3\colon v_0 = v_1$, $L_4\colon v_0 = -v_1$. Let C be a curve of bidegree $(2, 2)$, defined by a polynomial $f(u_0, u_1, v_0, v_1)$, which is invariant with respect to ϕ, and consider the divisor $B = C + \sum_{i=1}^{4} L_i$.

Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the minimal resolution of the double covering ramified over B. In Kondo's families, C is chosen so that X is a K3 surface and ϕ lifts to an Enriques involution $\tilde{\varepsilon}$ of X. We let Y be the quotient of X by $\tilde{\varepsilon}$.

For $i = 1, 2$, the composite morphism $\pi_i = \mathrm{pr}_i \circ \pi\colon X \to \mathbb{P}^1$ is an elliptic fibration on X, which induces an elliptic fibration $\bar{\pi}_i$ on Y. There is a third elliptic fibration $\pi_3\colon X \to \mathbb{P}^1$, one of whose fiber is the strict transform of C on X. The half pencils of $\bar{\pi}_3\colon Y \to \mathbb{P}^1$ correspond to the fibers over C and over $\sum L_i$.

For $i = 1, 2, 3$, we choose coordinates so that the half-pencils of $\bar{\pi}_i$ are mapped to $[0, 1], [1, 0] \in \mathbb{P}^1$. The image of the morphism $\pi_1 \times \pi_2 \times \pi_3\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is then defined by the tridegree $(2, 2, 2)$ polynomial

$$(u_0^2 - u_1^2)(v_0^2 - v_1^2)w_0^2 = f(u_0, u_1, v_0, v_1)w_1^2.$$

Consider the Segre embedding $\Sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$, defined by

$$\begin{aligned}([u_0, u_1], [v_0, v_1], [w_0, w_1]) &\mapsto [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] = \\ &= [u_0v_0w_0, u_0v_1w_1, u_1v_0w_1, u_1v_1w_0, u_0v_0w_1, u_0v_1w_0, u_1v_0w_0, u_1v_1w_1].\end{aligned}$$

The involution on $\mathbb{P}^7$ given by $[x_0, \dots, x_7] \mapsto [x_0, \dots, x_3, -x_4, \dots, -x_7]$ induces the Enriques involution $\tilde{\varepsilon}$ on X. Hence, we have the following commuting diagram

$$\begin{array}{ccccc} X & \xrightarrow{\pi_1\times\pi_2\times\pi_3} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xhookrightarrow{\ \Sigma\ } & \mathbb{P}^7 \\ \downarrow & & & & \downarrow{\scriptstyle \mathrm{pr}_{0123}} \\ Y & & \longrightarrow & & \mathbb{P}^3 \end{array}$$

where pr_{0123} is the projection $[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \mapsto [x_0, x_1, x_2, x_3]$. Note that the half-pencils on Y are mapped onto the coordinate tetrahedron in $\mathbb{P}^3$, so the image of Y in $\mathbb{P}^3$ is defined by an *Enriques sextic surface*, i.e. a non-normal surface of degree 6 in $\mathbb{P}^3$ that passes doubly through the edges of the coordinate tetrahedron (see [8]).

6.2.1. *Nikulin-Kondo type* I. For $\alpha \in \mathbb{C} \setminus \{1, \frac{1}{2}, \frac{3}{2}\}$, let C be the curve defined by

$$C\colon (2\,u_0^2 - u_1^2)(v_0^2 - v_1^2) = (2\,\alpha v_0^2 + (1 - 2\,\alpha)v_1^2)(u_0^2 - u_1^2).$$

Put $B = C + \sum_{i=1}^{4} L_i$. Then, the minimal resolution of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over B is a K3 surface X endowed with an Enriques involution $\tilde{\varepsilon}$ such that the quotient $X/\langle\tilde{\varepsilon}\rangle$ has Nikulin-Kondo type I.

Consider the curves

$$Q_1\colon u_1 v_0 + u_0 v_1 = 0; \quad Q_2\colon u_1 v_0 - u_0 v_1 = 0;$$
$$Z\colon (u_0 + 3\,u_1)\,v_0^2 + (3\,u_0 + u_1)\,v_1^2 = 0.$$

The curve Z intersects Q_1 and Q_2 in one point with multiplicity 3, and intersects C with even multiplicities if and only if

$$\alpha = \frac{15}{16} \quad \text{or} \quad \alpha = \frac{17}{16}.$$

(The two cases differ only by a relabeling of the variables.)

In these cases, consider the sublattice $S' \subset S_X$ generated by the classes of the strict transforms of $C, L_1, \dots, L_4, Q_1, Q_2, Z$ and of the exceptional divisors. Then, $\operatorname{rank} S' = 20$ and $\det S' = 7$, hence the same holds for S_X. This implies that X is isomorphic to X_7, so the quotient $X/\langle\tilde{\varepsilon}\rangle$ is isomorphic to Y_{I}.

An Enriques sextic model for Y_{I} is given by

$$\begin{aligned}(2\,\alpha - 2)x_0^2x_1^2x_2^2 + x_0^2x_1^2x_3^2 + x_0^2x_2^2x_3^2 + (2\,\alpha - 2)x_1^2x_2^2x_3^2 = \\ = x_0x_1x_2x_3\left(x_0^2 + (2\,\alpha - 3)x_1^2 + (2\,\alpha - 1)x_2^2 + x_3^2\right).\end{aligned}$$

6.2.2. *Nikulin-Kondo type* II. For $\alpha \in \mathbb{C} \setminus \{0, -1\}$, let C be the curve defined by

$$C\colon (v_0^2 - v_1^2)u_0^2 - (v_0^2 + \alpha v_1^2)u_1^2 = 0.$$

Put $B = C + \sum_{i=1}^{4} L_i$. Then, the minimal resolution of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over B is a K3 surface X endowed with an Enriques involution $\tilde{\varepsilon}$ such that the quotient $X/\langle\tilde{\varepsilon}\rangle$ has Nikulin-Kondo type II.

Consider the curves

$$F_1\colon u_1 = 0; \quad F_2\colon v_1 = 0;$$
$$Z\colon (u_0 - u_1)v_0 + (u_0 + 3\,u_1)v_1 = 0$$

The curve Z intersects C in a third point of multiplicity 2 exactly when

$$\alpha = 63.$$

In this case, consider the sublattice $S' \subset S_X$ generated by the classes of the strict transforms of $C, L_1, \dots, L_4, F_1, F_2, Z$ and of the exceptional divisors. Then, $\operatorname{rank} S' = 20$ and $\det S' = 7$, hence the same holds for S_X. This implies that X is isomorphic to X_7, so the quotient $X/\langle\tilde{\varepsilon}\rangle$ is isomorphic to Y_{II}.

An Enriques sextic model for Y_{II} is given by

$$-x_0^2x_1^2x_2^2 + x_0^2x_1^2x_3^2 + x_0^2x_2^2x_3^2 + \alpha x_1^2x_2^2x_3^2 = x_0x_1x_2x_3\left(x_0^2 - x_1^2 - x_2^2 + x_3^2\right).$$

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