ON AN ENRIQUES SURFACE ASSOCIATED WITH A QUARTIC HESSIAN SURFACE

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Dedicated to Professor Jonghae Keum on the occasion of his 60th birthday

Abstract. Let Y be a complex Enriques surface whose universal cover X is birational to a general quartic Hessian surface. Using the result on the automorphism group of X due to Dolgachev and Keum, we obtain a finite presentation of the automorphism group of Y. The list of elliptic fibrations on Y and the list of combinations of rational double points that can appear on a surface birational to Y are presented. As an application, a set of generators of the automorphism group of the generic Enriques surface is calculated explicitly.

1. Introduction

We work over the complex number field $\mathbb{C}$. An involution on a $K3$ surface is called an *Enriques involution* if it has no fixed-points. Let $\overline{X}$ be a general quartic Hessian surface, which means that $\overline{X}$ is the quartic surface in $\mathbb{P}^3$ defined by the equation

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) = 0,$$

where $F = F(x_1, \dots, x_4)$ is a general cubic homogeneous polynomial. Then $\overline{X}$ has ten ordinary nodes p_α as its only singularities, and contains exactly ten lines ℓ_β. Let A denote the set of subsets α of $\{1, 2, 3, 4, 5\}$ with $|\alpha| = 3$, and B the set of subsets β of $\{1, \dots, 5\}$ with $|\beta| = 2$. Then p_α and ℓ_β can be indexed by $\alpha \in A$ and $\beta \in B$, respectively, in such a way that $p_\alpha \in \ell_\beta$ if and only if $\alpha \supset \beta$. Let $X \to \overline{X}$ be the minimal resolution, let E_α be the exceptional curve over p_α, and let L_β be the strict transform of ℓ_β. It is classically known (see Dolgachev and Keum [7]) that the $K3$ surface X has an Enriques involution ε that interchanges E_α and $L_{\bar{\alpha}}$ for each $\alpha \in A$, where $\bar{\alpha} := \{1, \dots, 5\} \setminus \alpha$. We denote the quotient morphism by

$$\pi : X \to Y := X/\langle \varepsilon \rangle.$$

The first application of Borcherds method ([3], [4]) to the automorphism group of $K3$ surfaces was given by Kondo [15]. A set of generators of the automorphism group $\mathrm{Aut}(X)$ of the $K3$ surface X above was obtained in Dolgachev and Keum [7] by this method. On the other hand, we presented in [27] a computer algorithm for Borcherds method. Using this computational tool, we obtain an explicit description of $\mathrm{Aut}(X)$ and a fundamental domain D_X of its action on the cone

$$N(X) := \{\, x \in \mathcal{P}_X \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } X \,\}$$

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in $S_X \otimes \mathbb{R}$, where S_X is the Néron-Severi lattice of X with the intersection form $\langle\ ,\ \rangle$, $\mathcal{P}_X$ is the connected component of $\{x \in S_X \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$ containing an ample class, and $[C] \in S_X$ is the class of a curve $C \subset X$.

By analyzing this result, we obtain the following results on the automorphism group $\mathrm{Aut}(Y)$ of the Enriques surface Y. Let $\iota_\alpha \colon X \to X$ denote the involution of X induced by the double covering $\overline{X} \to \mathbb{P}^2$ obtained from the projection with the center $p_\alpha \in \overline{X}$. In [7], it was proved that ι_α commutes with ε. Hence ι_α induces an involution $j_\alpha \colon Y \to Y$ of Y.

Theorem 1.1. *The automorphism group* $\mathrm{Aut}(Y)$ *of* Y *is generated by the ten involutions* j_α*. The following relations form a set of defining relations of* $\mathrm{Aut}(Y)$ *with respect to these generators* j_α*;*

$$j_\alpha^2 = \mathrm{id}$$

for each ordinary node p_α*,*

$$(j_\alpha j_{\alpha'} j_{\alpha''})^2 = \mathrm{id}$$

for each triple $(p_\alpha, p_{\alpha'}, p_{\alpha''})$ *of distinct three ordinary nodes such that there exists a line in* $\overline{X}$ *passing through* $p_\alpha, p_{\alpha'}, p_{\alpha''}$*, and*

$$(j_\alpha j_{\alpha'})^2 = \mathrm{id}$$

for each pair $(p_\alpha, p_{\alpha'})$ *of distinct ordinary nodes such that the line in* $\mathbb{P}^3$ *passing through* p_α *and* $p_{\alpha'}$ *is not contained in* $\overline{X}$*.*

Remark 1.2. Recently Dolgachev [8] studied the group generated by the involutions j_α (that is, $\mathrm{Aut}(Y)$ by Theorem 1.1), and showed that this group is isomorphic to a subgroup of $\Gamma \rtimes \mathfrak{S}_5$, where Γ is a group isomorphic to the Coxeter group with the anti-Petersen graph as its Coxeter graph. See Corollary 4.4 of [8]. (This corollary was also known to Mukai.)

Remark 1.3. Mukai and Ohashi informed us that they also proved, without computer-aided calculations, that $\mathrm{Aut}(Y)$ is generated by j_α ($\alpha \in A$). See also [21].

Let S_Y denote the lattice of numerical equivalence classes of divisors on Y, which is isomorphic to $H^2(Y, \mathbb{Z})/(\text{torsion})$ equipped with the cup-product. By the result of [20], [14] and [19], we know that the action of $\mathrm{Aut}(Y)$ on S_Y is faithful. Theorem 1.1 is proved by investigating this faithful action. More precisely, let $\mathcal{P}_Y$ denote the connected component of $\{y \in S_Y \otimes \mathbb{R} \mid \langle y, y\rangle > 0\}$ containing an ample class. We put

$$N(Y) := \{\ y \in \mathcal{P}_Y \ \mid\ \langle y, [C]\rangle \ge 0 \ \text{ for all curves } C \text{ on } Y\ \}.$$

It is obvious that $\mathrm{Aut}(Y)$ acts on $N(Y)$. We give a description of a fundamental domain D_Y of the action of $\mathrm{Aut}(Y)$ on $N(Y)$. For $v \in S_Y \otimes \mathbb{R}$ with $\langle v, v\rangle < 0$, let $(v)^\perp$ denote the hyperplane in $\mathcal{P}_Y$ defined by $\langle v, x\rangle = 0$.

Theorem 1.4. *There exists a fundamental domain* D_Y *of the action of* $\mathrm{Aut}(Y)$ *on* $N(Y)$ *with the following properties.*

(1) *The fundamental domain* D_Y *is bounded by* $10 + 10$ *hyperplanes* $(\bar{u}_\alpha)^\perp$ *and* $(\bar{v}_\alpha)^\perp$*, where* α *runs through the set* A*.*
(2) *For each* $\alpha \in A$*, the vector* $\bar{u}_\alpha$ *is the class of the smooth rational curve* $\pi(E_\alpha) = \pi(L_{\bar{\alpha}})$ *on* Y*, and hence* $(\bar{u}_\alpha)^\perp$ *is a hyperplane bounding* $N(Y)$*.*
(3) *For each* $\alpha \in A$*, the involution* $j_\alpha \in \mathrm{Aut}(Y)$ *maps* D_Y *to the chamber adjacent to* D_Y *across the wall* $D_Y \cap (\bar{v}_\alpha)^\perp$ *of* D_Y*.*

$[R_{\text{full}}, R_{\text{half}}]$	number
$[\emptyset, A_4]$	1
$[A_5+A_1, \emptyset]$	10
$[D_5, \emptyset]$	5
$[E_6, \emptyset]$	5

TABLE 1.1. Elliptic fibrations on Y

ADE-type	number	ADE-type	number
E_6	1	A_3+A_1	1
A_5+A_1	5	$2A_2$	1
$3A_2$	1	A_2+2A_1	1
D_5	1	$4A_1$	5
A_5	1	A_3	1
A_4+A_1	1	A_2+A_1	1
A_3+2A_1	5	$3A_1$	2
$2A_2+A_1$	1	A_2	1
D_4	1	$2A_1$	1
A_4	1	A_1	1

TABLE 1.2. RDP-configurations on Y

Let Z be an Enriques surface. Then an elliptic fibration $\phi\colon Z \to \mathbb{P}^1$ has exactly two multiple fibers $2E_1$ and $2E_2$.

Theorem 1.5. *Up to the action of* $\mathrm{Aut}(Y)$*, the Enriques surface* Y *has exactly* $1+10+5+5$ *elliptic fibrations. Their* ADE*-types of reducible fibers are given in Table 1.1, in which* R_{full} *and* R_{half} *denote the* ADE*-types of non-multiple reducible fibers and of the half of the multiple reducible fibers, respectively.*

An RDP-*configuration* on an Enriques surface Z is the exceptional divisor of a birational morphism $Z \to \bar{Z}$, where $\bar{Z}$ has only rational double points as its singularities. The support of an RDP-configuration is an ADE-configuration of smooth rational curves.

Theorem 1.6. *Up to the action of* $\mathrm{Aut}(Y)$*, the Enriques surface* Y *has exactly* 33 *non-empty* RDP*-configurations. Their* ADE*-types are given in Table 1.2.* [1]

Remark 1.7. In [30], all RDP-configurations on complex Enriques surfaces are classified by some lattice theoretic equivalence relation.

The lattice S_Z of numerical equivalence classes of divisors on an Enriques surface Z is isomorphic to the even unimodular hyperbolic lattice L_{10} of rank 10, which is unique up to isomorphism. The group $\mathrm{O}^+(L_{10})$ of isometries of L_{10} preserving

[1]Added on 2020 August 22: In the previous version of this paper, the numbers of $\mathrm{Aut}(Y)$-equivalence classes of rational double points were calculated wrongly. See Section 7.6.

a positive cone $\mathcal{P}_{10}$ of $L_{10} \otimes \mathbb{R}$ is generated by the reflections with respect to the roots. Vinberg [33] determined the shape of a standard fundamental domain of the action of $\mathrm{O}^+(L_{10})$ on $\mathcal{P}_{10}$. (See Section 3.1.) Hence we call these fundamental domains *Vinberg chambers*.

In [2], Barth and Peters determined the automorphism group $\mathrm{Aut}(Z_{\mathrm{gen}})$ of a *generic* Enriques surface Z_{gen}. (See also Nikulin [23].) Let $\mathcal{P}_{Z_{\mathrm{gen}}}$ be the positive cone of $S_{Z_{\mathrm{gen}}} \otimes \mathbb{R}$ containing an ample class. We identify $S_{Z_{\mathrm{gen}}}$ with L_{10} by an isometry that maps $\mathcal{P}_{Z_{\mathrm{gen}}}$ to $\mathcal{P}_{10}$. Since Z_{gen} contains no smooth rational curves, we have $N(Z_{\mathrm{gen}}) = \mathcal{P}_{10}$. Note that the discriminant form $q_{L_{10}(2)}$ of $L_{10}(2)$ is a quadratic form over $\mathbb{F}_2$ with Witt defect 0, and hence its automorphism group $\mathrm{O}(q_{L_{10}(2)})$ is isomorphic to $\mathrm{GO}^+_{10}(2)$ in the notation of [1]. Moreover, the natural homomorphism

$$\rho\colon \mathrm{O}^+(L_{10}) \to \mathrm{O}(q_{L_{10}(2)}) \cong \mathrm{GO}^+_{10}(2)$$

is surjective. It was shown in [2] that the natural representation of $\mathrm{Aut}(Z_{\mathrm{gen}})$ on $S_{Z_{\mathrm{gen}}} \cong L_{10}$ identifies $\mathrm{Aut}(Z_{\mathrm{gen}})$ with the kernel of ρ. In particular, $\mathrm{Aut}(Z_{\mathrm{gen}})$ is isomorphic to a normal subgroup of $\mathrm{O}^+(L_{10})$ with index

$$|\mathrm{GO}^+_{10}(2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600.$$

By the following theorem, we can describe the way how the automorphism group changes in $\mathrm{O}^+(L_{10})$ under the specialization from Z_{gen} to Y (see Remark 7.18).

Theorem 1.8. *Under an isometry $S_Y \cong L_{10}$ that maps $\mathcal{P}_Y$ to $\mathcal{P}_{10}$, the fundamental domain D_Y in Theorem 1.4 is a union of following number of Vinberg chambers:*

$$2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 31 = 906608640.$$

This theorem also gives us a method to calculate a set of generators of $\mathrm{Aut}(Z_{\mathrm{gen}})$ explicitly. Using this set, we carry out an experiment on the entropies of automorphisms of Z_{gen}.

The moduli of quartic Hessian surfaces has been studied by several authors in order to investigate the moduli of cubic surfaces ([6], [13], [16]). In [6], Dardanelli and van Geemen studied several interesting subfamilies of this moduli. It seems to be an interesting problem to investigate the change of the automorphism group under specializations of Y to members of these subfamilies by the method given in this paper.

As is shown in Remark 4 (2) of [21], there exists a specialization from Y to the Enriques surface Y_{VI} with $\mathrm{Aut}(Y_{\mathrm{VI}}) \cong \mathfrak{S}_5$ that appeared in the Nikulin-Kondo's classification of Enriques surfaces with finite automorphism groups ([23], [14]). Kondo pointed out that the roots $\bar{u}_\alpha$, $\bar{v}_\alpha$ defining the walls of D_Y given in Theorem 1.4 have the same configuration as the smooth rational curves on Y_{VI}. (Compare (7.1), (7.2) and Figure 6.4 of [14].) It is also an interesting problem to investigate the change of the automorphism group under various generalizations from the seven Enriques surfaces with finite automorphism groups.

The first application of Borcherds method to the automorphism group of an Enriques surface was given in [29], in which we investigated an Enriques surface whose universal cover is a $K3$ surface of Picard number 20 with the transcendental lattice of discriminant 36.

This paper is organized as follows. In Section 2, we collect preliminaries about lattices and chambers. In Section 3, we recall the results on the even unimodular hyperbolic lattices due to Vinberg [33] and Conway [5]. In Section 4, we explain

Borcherds method and its application to $K3$ surfaces. In Section 5, we present some algorithms to study the geometry of an Enriques surface. In particular, we give an application of Borcherds method to an Enriques surface. In Section 6, we re-calculate, by the algorithm in [27], the results of Dolgachev and Keum [7] on the general quartic Hessian surface, and convert these results into machine-friendly format. With these preparations, the main results are proved in Section 7. In the last section, we calculate a set of generators of $\mathrm{Aut}(Z_{\mathrm{gen}})$, and search for elements of $\mathrm{Aut}(Z_{\mathrm{gen}})$ with small entropies.

For the computation, we used `GAP` [10]. The computational data are presented in the author's webpage [32]. In fact, once the basis of the Leech lattice (Table 3.1), the basis of S_X (6.3), the embedding $S_X \hookrightarrow L_{26}$ (Table 6.1), and the basis of S_Y (Table 6.4) are fixed, the other data can be derived by the algorithms in this paper.

2. Preliminaries

2.1. **Lattices.** A submodule M of a free $\mathbb{Z}$-module L is said to be *primitive* if L/M is torsion-free. A non-zero vector $v \in L$ is *primitive* if $\mathbb{Z}v \subset L$ is primitive.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle \,:\, L \times L \to \mathbb{Z}.$$

Let m be a non-zero integer. For an lattice $(L, \langle\ ,\ \rangle)$, we denote by $L(m)$ the lattice $(L, m\langle\ ,\ \rangle)$. Every vector of $L \otimes \mathbb{R}$ is written as a *row* vector, and the *orthogonal group* $\mathrm{O}(L)$ of L acts on L from the *right*. We put

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}), \quad L_\mathbb{Q} := L \otimes \mathbb{Q}, \quad L_\mathbb{R} := L \otimes \mathbb{R}.$$

Then we have natural inclusions $L \hookrightarrow L^\vee \hookrightarrow L_\mathbb{Q} \hookrightarrow L_\mathbb{R}$. The *discriminant group* of L is defined to be $L^\vee/L$. A lattice L is *unimodular* if $L^\vee/L$ is trivial. A lattice L of rank n is *hyperbolic* if $n > 1$ and the signature of $L_\mathbb{R}$ is $(1, n-1)$, whereas L is *negative-definite* if the signature is $(0, n)$.

A lattice L is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. Suppose that L is even. Then the *discriminant form*

$$q_L \,:\, L^\vee/L \to \mathbb{Q}/2\mathbb{Z}$$

of L is defined by $q_L(x \bmod L) := \langle x, x\rangle \bmod 2\mathbb{Z}$ for $x \in L^\vee$. See [22] for the basic properties of discriminant forms. We denote by $\mathrm{O}(q_L)$ the automorphism group of the finite quadratic form q_L. We regard $L^\vee$ as a submodule of $L_\mathbb{Q}$, and let $\mathrm{O}(L)$ act on $L^\vee$ from the right. We have a natural homomorphism

$$\eta_L \,:\, \mathrm{O}(L) \to \mathrm{O}(q_L).$$

A vector $r \in L$ with $\langle r, r\rangle = -2$ is called a *root*. A root $r \in L$ defines a *reflection*

$$s_r \,:\, x \mapsto x + \langle x, r\rangle r,$$

which belongs to $\mathrm{O}(L)$. The *Weyl group* $W(L)$ of L is defined to be the subgroup of $\mathrm{O}(L)$ generated by the reflections s_r with respect to all the roots r of L.

Let L be an even hyperbolic lattice. A *positive cone* of L is one of the two connected components of $\{x \in L_\mathbb{R} \mid \langle x, x\rangle > 0\}$. We fix a positive cone $\mathcal{P}$ of L. Let $\mathrm{O}^+(L)$ denote the stabilizer subgroup of $\mathcal{P}$ in $\mathrm{O}(L)$. Then $W(L)$ acts on $\mathcal{P}$. For a root $r \in L$, we put

$$(r)^\perp := \{\ x \in \mathcal{P} \ \mid\ \langle x, r\rangle = 0\ \}.$$

The following is obvious:

Proposition 2.1. *The family* $\{(r)^\perp \mid r$ *is a root of* $L\}$ *of hyperplanes of* $\mathcal{P}$ *is locally finite in* $\mathcal{P}$. □

A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \;\setminus\; \bigcup_r (r)^\perp,$$

where r runs through the set of all roots. Let D be one of the standard fundamental domains of $W(L)$. We put

$$\mathrm{aut}(D) := \{\, g \in \mathrm{O}^+(L) \;\mid\; D^g = D \,\}.$$

Then we have $\mathrm{O}^+(L) = W(L) \rtimes \mathrm{aut}(D)$.

2.2. $V_{\mathbb{R}}$**-chambers.** Let V be a $\mathbb{Q}$-vector space of dimension $n > 0$, and let V^* denote the dual $\mathbb{Q}$-vector space $\mathrm{Hom}(V, \mathbb{Q})$. We put $V_{\mathbb{R}} := V \otimes \mathbb{R}$. For a non-zero linear form $f \in V^* \setminus \{0\}$, we put

$$H_f := \{x \in V_{\mathbb{R}} \mid f(x) \ge 0\}, \qquad [f]^\perp := \{x \in V_{\mathbb{R}} \mid f(x) = 0\} = \partial H_f.$$

Definition 2.2. A closed subset $\overline{C}$ of $V_{\mathbb{R}}$ is called a $V_{\mathbb{R}}$*-chamber* if $\overline{C}$ contains a non-empty open subset of $V_{\mathbb{R}}$, and there exists a subset $\mathcal{F}$ of $V^* \setminus \{0\}$ such that

$$\overline{C} \;=\; \bigcap_{f \in \mathcal{F}} H_f.$$

When this is the case, we say that $\mathcal{F}$ *defines the* $V_{\mathbb{R}}$*-chamber* $\overline{C}$.

Suppose that a subset $\mathcal{F}$ of $V^* \setminus \{0\}$ defines a $V_{\mathbb{R}}$-chamber $\overline{C}$. We assume the following:

$$H_f \neq H_{f'} \;\text{ for distinct } f, f' \in \mathcal{F}. \tag{2.1}$$

We say that an element f of $V^* \setminus \{0\}$ *defines a wall of* $\overline{C}$ if $\overline{C}$ is contained in H_f and $\overline{C} \cap [f]^\perp$ contains a non-empty open subset of $[f]^\perp$. When this is the case, we call $\overline{C} \cap [f]^\perp$ the *wall of* $\overline{C}$ *defined by* f. By the assumption (2.1), we see that $f_0 \in \mathcal{F}$ defines a wall of $\overline{C}$ if and only if there exists a point $x \in V$ such that

$$f_0(x) < 0, \quad \text{and} \;\; f(x) \ge 0 \;\text{ for all } f \in \mathcal{F} \setminus \{f_0\}.$$

Hence we have the following.

Algorithm 2.3. Suppose that a $V_{\mathbb{R}}$-chamber $\overline{C}$ is defined by a finite subset $\mathcal{F}$ of $V^* \setminus \{0\}$ satisfying (2.1). Then an element $f_0 \in \mathcal{F}$ defines a wall of C if and only if the solution of the following problem of linear programing on V over $\mathbb{Q}$ is unbounded to $-\infty$: find the minimal value of $f_0(x)$ subject to the constraints $f(x) \ge 0$ for all $f \in \mathcal{F} \setminus \{f_0\}$. ∎

Let $\overline{C}$ and $\mathcal{F}$ be as above. We define the *faces of dimension* k of $\overline{C}$ for $k = n-1, \dots, 1$ by descending induction on k. The following is obvious.

Lemma 2.4. *Suppose that* $n > 1$*, and that* $f_0 \in \mathcal{F}$ *defines a wall of* $\overline{C}$*. For* $g \in V^*$*, let* $g|_{f_0^\perp} : f_0^\perp \to \mathbb{Q}$ *denote the restriction of* g *to the hyperplane* $f_0^\perp$ *of* V*. Then the wall* $\overline{C} \cap [f_0]^\perp$ *of* $\overline{C}$ *defined by* f_0 *is an* $[f_0]^\perp$*-chamber defined by*

$$\mathcal{F}|_{f_0^\perp} := \{\; g|_{f_0^\perp} \;\mid\; g \in \mathcal{F}, \;\; g|_{f_0^\perp} \neq 0 \,\}.$$

The faces of $\overline{C}$ of dimension $n-1$ are defined to be the walls of $\overline{C}$. Suppose that $0 < k < n-1$, and let F be a $(k+1)$-dimensional face of $\overline{C}$. Let $\langle F\rangle$ denote the minimal linear subspace of V containing F. We assume that (i) the linear space $\langle F\rangle$ is of dimension $k+1$, (ii) F is equal to the closed subset $\overline{C}\cap\langle F\rangle$ of $\overline{C}$, and (iii) F is an $(\langle F\rangle\otimes\mathbb{R})$-chamber defined by the subset

$$\mathcal{F}|_{\langle F\rangle} := \{\ g|_{\langle F\rangle}\ \mid\ g\in\mathcal{F},\ \ g|_{\langle F\rangle}\neq 0\ \}$$

of $\operatorname{Hom}(\langle F\rangle,\mathbb{Q})\setminus\{0\}$, where $g|_{\langle F\rangle}$ is the restriction of g to $\langle F\rangle$. Then the walls of the $(\langle F\rangle\otimes\mathbb{R})$-chamber F are defined. A face of dimension k of $\overline{C}$ is defined to be a wall of a $(k+1)$-dimensional face of $\overline{C}$. It is obvious that k-dimensional faces satisfy the assumptions (i), (ii), (iii), and hence the induction proceeds.

When $\mathcal{F}$ is finite, we can calculate all the faces of $\overline{C}$ by using Algorithm 2.3 iteratively.

Remark 2.5. At every step of iteration, we must remove redundant elements from $\mathcal{F}|_{\langle F\rangle}$ to obtain a subset $\mathcal{F}'_{\langle F\rangle}\subset\mathcal{F}|_{\langle F\rangle}$ that defines the walls of F and satisfies (2.1).

2.3. **Chambers.** Let V be as in the previous subsection. Suppose that $n>1$, and that V is equipped with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\,:\,V\times V\to\mathbb{Q}$$

such that $V_{\mathbb{R}}=V\otimes\mathbb{R}$ is of signature $(1,n-1)$. By $\langle\ ,\ \rangle$, we identify V and V^*. In particular, for a non-zero vector v of V, we put

$$H_v := \{x\in V_{\mathbb{R}}\mid\langle v,x\rangle\geq 0\},\qquad [v]^{\perp} := \{x\in V_{\mathbb{R}}\mid\langle v,x\rangle = 0\} = \partial H_v.$$

Let $\mathcal{P}_V$ be one of the two connected components of $\{x\in V_{\mathbb{R}}\mid\langle x,x\rangle>0\}$, and let $\overline{\mathcal{P}}_V$ denote the closure of $\mathcal{P}_V$ in $V_{\mathbb{R}}$. For a non-zero vector v of V, we put

$$(v)^{\perp} := [v]^{\perp}\cap\mathcal{P}_V,$$

which is non-empty if and only if $\langle v,v\rangle<0$.

Definition 2.6. A closed subset C of $\mathcal{P}_V$ is said to be a *chamber* if there exists a subset $\mathcal{F}$ of $V\setminus\{0\}$ with the following properties.

(i) The family $\{(v)^{\perp}\mid v\in\mathcal{F},\langle v,v\rangle<0\}$ of hyperplanes of the positive cone $\mathcal{P}_V$ is locally finite in $\mathcal{P}_V$.
(ii) Under the identification $V=V^*$, the set $\mathcal{F}$ defines a $V_{\mathbb{R}}$-chamber $\overline{C}$ such that

$$\overline{C}\subset\overline{\mathcal{P}}_V\ \text{ and }\ C=\mathcal{P}_V\cap\overline{C}. \tag{2.2}$$

When this is the case, we say that the chamber C is *defined by* $\mathcal{F}$.

Remark 2.7. A $V_{\mathbb{R}}$-chamber $\overline{C}$ satisfies $\overline{C}\subset\overline{\mathcal{P}}_V$ if and only if $\overline{C}\cap\mathcal{P}_V\neq\emptyset$ and $\overline{C}\cap\partial\,\overline{\mathcal{P}}_V$ is contained in the union of one-dimensional faces of $\overline{C}$.

Let C be a chamber defined by $\mathcal{F}\subset V\setminus\{0\}$. Let F be a k-dimensional face of $\overline{C}$. If $C\cap F\neq\emptyset$, we say that $C\cap F$ is a *face* of C of dimension k. Note that, by (2.2), if $k>1$, then $C\cap F$ is a face of C. In particular, since $n>1$, if $\overline{C}\cap[u]^{\perp}$ is a wall of $\overline{C}$ defined by $u\in V\setminus\{0\}$, then $C\cap(u)^{\perp}$ is called the *wall of C defined by u*.

When $k=1$, we may have $C\cap F=\emptyset$.

Definition 2.8. A one-dimensional face F of $\overline{C}$ contained in $\overline{C}\setminus C=\overline{C}\cap\partial\,\overline{\mathcal{P}}_V$ is called an *ideal face* of C. By abuse of language, an ideal face of C is also regarded as a face of C.

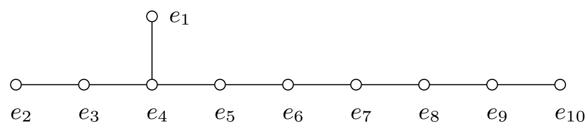


FIGURE 3.1. Walls of the Vinberg chamber D_{10}

2.4. **Chambers of a hyperbolic lattice.** Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$. Applying the above definition to $V = L_{\mathbb{Q}}$, we have the notion of chambers and their faces. We mean by a *chamber of* L a chamber of $L_{\mathbb{Q}}$.

Definition 2.9. We define the *automorphism group of a chamber* C *of* L by

$$\mathrm{aut}(C) := \{\ g \in \mathrm{O}^+(L)\ \mid\ C^g = C\ \}.$$

We put

$$L^\vee_{\mathrm{prim}} := \{\ v \in L^\vee\ \mid\ v \text{ is primitive in } L^\vee\ \}.$$

Then we have a canonical projection

$$L_{\mathbb{Q}} \setminus \{0\} \to L^\vee_{\mathrm{prim}}, \quad v \mapsto \tilde{v}$$

such that $H_v = H_{\tilde{v}}$ holds for all $v \in L_{\mathbb{Q}} \setminus \{0\}$.

Definition 2.10. Let $C \cap (u)^\perp$ be a wall of a chamber C of L defined by $u \in L_{\mathbb{Q}}$. A vector $v \in L^\vee$ is called the *primitive defining vector* of the wall $C \cap (u)^\perp$ if v is the vector of $L^\vee_{\mathrm{prim}}$ satisfying $H_v = H_u$. By definition, each wall $C \cap (u)^\perp$ of C has a unique primitive defining vector $\tilde{u}$.

If $\mathcal{F} \subset L_{\mathbb{Q}} \setminus \{0\}$ defines a chamber C, then so does the set

$$\tilde{\mathcal{F}} := \{\ \tilde{v}\ \mid\ v \in \mathcal{F}\ \}.$$

The assumption (2.1) holds automatically for $\tilde{\mathcal{F}}$. Hence converting $\mathcal{F}$ to $\tilde{\mathcal{F}}$ is a convenient method to achieve the property (2.1) when we use Algorithm 2.3 iteratively to determine the faces of a chamber (see Remark 2.5).

3. Even unimodular hyperbolic lattices L_{10} and L_{26}

For each positive integer n with $n \equiv 2 \bmod 8$, let L_n denote an even unimodular hyperbolic lattice of rank n, which is unique up to isomorphism. The lattice L_2 is denoted by U. We fix a basis f_1, f_2 of U such that the Gram matrix of U with respect to f_1, f_2 is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3.1. **The lattice L_{10}.** Let E_8 denote the *negative-definite* even unimodular lattice of rank 8 with the standard basis $e_1, \dots, e_8$, whose intersection numbers are given by the Dynkin diagram in Figure 3.1. We use $f_1, f_2, e_1, \dots, e_8$ as a basis of

$$L_{10} := U \oplus E_8,$$

and put

$$\begin{aligned} e_9 &:= [1, 0, -3, -2, -4, -6, -5, -4, -3, -2], \\ e_{10} &:= [-1, 1, 0, 0, 0, 0, 0, 0, 0, 0], \\ w_{10} &:= [31, 30, -68, -46, -91, -135, -110, -84, -57, -29]. \end{aligned}$$

We have $\langle e_9, e_9\rangle = \langle e_{10}, e_{10}\rangle = -2$, $\langle w_{10}, w_{10}\rangle = 1240$, and

$$\mathcal{W}_{10} := \{\ r \in L_{10}\ \mid\ \langle r, w_{10}\rangle = 1,\ \langle r, r\rangle = -2\ \} = \{e_1, \dots, e_{10}\}.$$

The roots $e_1, \dots, e_{10}$ form the Dynkin diagram in Figure 3.1. Let $\mathcal{P}_{10}$ be the positive cone of L_{10} containing w_{10}, and $\overline{\mathcal{P}}_{10}$ the closure of $\mathcal{P}_{10}$ in $L_{10} \otimes \mathbb{R}$. We put

$$\overline{D}_{10} := \{\ x \in L_{10} \otimes \mathbb{R}\ \mid\ \langle x, e_i\rangle \ge 0\ \text{ for } i = 1, \dots, 10\ \},\quad D_{10} := \overline{D}_{10} \cap \mathcal{P}_{10}.$$

Theorem 3.1 (Vinberg [33])**.** *The closed subset D_{10} of $\mathcal{P}_{10}$ is a chamber. The chamber D_{10} is a standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$. Each vector e_i of $\mathcal{W}_{10}$ defines a wall of D_{10}.*

Since the diagram of the walls of D_{10} in Figure 3.1 has no symmetries, the automorphism group $\mathrm{aut}(D_{10})$ of D_{10} is trivial. Hence $\mathrm{O}^+(L_{10})$ is equal to $W(L_{10})$, which is generated by the reflections with respect to the roots $e_1, \dots, e_{10}$.

Definition 3.2. A standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$ is called a *Vinberg chamber*.

3.2. **The lattice L_{26}.** Let Λ be the *negative-definite* Leech lattice; that is, the unique even *negative-definite* unimodular lattice of rank 24 with no roots. As a basis of Λ, we choose the row vectors $\lambda_1, \dots, \lambda_{24}$ of the matrix given in Table 3.1, and consider them as elements of the quadratic space $\mathbb{R}^{24}$ with the negative-definite inner product

$$(x, y) \mapsto -(x_1 y_1 + \cdots + x_{24} y_{24})/8.$$

This basis is constructed from the extended binary Golay code in the space of $\mathbb{F}_2$-valued functions on $\mathbb{P}^1(\mathbb{F}_{23}) = \{\infty\} \cup \mathbb{F}_{23}$, where the value at ∞ is at the first coordinate of each row vector (see Section 2.8 of [9]). We put

$$L_{26} := U \oplus \Lambda.$$

Then the vectors $f_1, f_2, \lambda_1, \dots, \lambda_{24}$ form a basis of L_{26}, which we will use throughout this paper. We put

$$w_{26} := f_1,\quad \mathcal{W}_{26} := \{\ r \in L_{26}\ \mid\ \langle r, w_{26}\rangle = 1,\ \langle r, r\rangle = -2\ \}.$$

Note that $\langle w_{26}, w_{26}\rangle = 0$. Let $\mathcal{P}_{26}$ be the positive cone of L_{26} that contains w_{26} in its closure $\overline{\mathcal{P}}_{26}$ in $L_{26} \otimes \mathbb{R}$. We then put

$$\overline{D}_{26} := \{\ x \in L_{26} \otimes \mathbb{R}\ \mid\ \langle x, r\rangle \ge 0\ \text{ for all } r \in \mathcal{W}_{26}\ \},\quad D_{26} := \overline{D}_{26} \cap \mathcal{P}_{26}.$$

Theorem 3.3 (Conway [5])**.** *The closed subset D_{26} of $\mathcal{P}_{26}$ is a chamber. The chamber D_{26} is a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$. Each vector r_λ of $\mathcal{W}_{26}$ defines a wall of D_{26}.*

Definition 3.4. A standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$ is called a *Conway chamber*.

A chamber D' in $\mathcal{P}_{26}$ is a Conway chamber if and only if D' is equal to D_{26}^g for some $g \in W(L_{26})$. The *Weyl vector* w' of a Conway chamber $D' = D_{26}^g$ is defined to be w_{26}^g, which is characterized by the following property:

$$D' = \{\ x \in \mathcal{P}_{26}\ \mid\ \langle x, r\rangle \ge 0 \text{ for all roots } r \in L_{26} \text{ satisfying } \langle w', r\rangle = 1\ \}.$$

$$
\begin{bmatrix}
0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
2 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 \\
1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
$$

Table 3.1. Basis of the Leech lattice Λ

4. Borcherds method for $K3$ surfaces

4.1. **Borcherds method.** We review the method due to Borcherds ([3], [4]) to calculate a standard fundamental domain of the Weyl group of an even hyperbolic lattice. See also [27] for the algorithmic description of Borcherds method.

Definition 4.1. In the following, a *tessellation* of a subset N of a positive cone $\mathcal{P}$ of a hyperbolic lattice means a decomposition of N into a union of chambers such that, if D and D' are distinct chambers in the decomposition, then the interiors of D and D' are disjoint.

Let S be an even hyperbolic lattice. Suppose that we have a primitive embedding

$$i\,:\, S \hookrightarrow L_{26}$$

of S into the even unimodular hyperbolic lattice L_{26} of rank 26. To simplify notation, we use the same letter i to denote $i\otimes\mathbb{R}\colon S\otimes\mathbb{R}\hookrightarrow L_{26}\otimes\mathbb{R}$. We put

$$\mathcal{P}_S := i^{-1}(\mathcal{P}_{26}),$$

where $\mathcal{P}_{26}$ is the fixed positive cone of L_{26}. Then $\mathcal{P}_S$ is a positive cone of S. We denote by

$$\mathrm{pr}_S\,:\, L_{26}\otimes\mathbb{R}\to S\otimes\mathbb{R}$$

the orthogonal projection to $S\otimes\mathbb{R}$. Note that $\mathrm{pr}_S(L_{26})\subset S^\vee$. Note also that, for $v\in L_{26}\otimes\mathbb{R}$, we have

$$i^{-1}((v)^\perp) = (\mathrm{pr}_S(r))^\perp = \{\, x\in\mathcal{P}_S \;\mid\; \langle x, \mathrm{pr}_S(v)\rangle = 0 \,\},$$

and hence $i^{-1}((v)^\perp)\neq\emptyset$ holds if and only if $\langle \mathrm{pr}_S(v), \mathrm{pr}_S(v)\rangle < 0$. We put

$$\mathcal{R}_S := \{\, \mathrm{pr}_S(r) \;\mid\; r \text{ is a root of } L_{26} \text{ such that } \langle \mathrm{pr}_S(r), \mathrm{pr}_S(r)\rangle < 0 \;\,\}.$$

Then the family of hyperplanes

$$\mathcal{H}_S := \{\, (\mathrm{pr}_S(r))^\perp \;\mid\; \mathrm{pr}_S(r)\in\mathcal{R}_S \,\}$$

is locally finite in $\mathcal{P}_S$ by Proposition 2.1. We investigate the tessellation of $\mathcal{P}_S$ obtained by this family of hyperplanes.

A Conway chamber $D' = D_{26}^g$ in $\mathcal{P}_{26}$ is said to be *non-degenerate* with respect to S if the closed subset $i^{-1}(D')$ of $\mathcal{P}_S$ contains a non-empty open subset of $\mathcal{P}_S$. When this is the case, the closed subset $i^{-1}(D')$ of $\mathcal{P}_S$ is a chamber in $\mathcal{P}_S$ defined by the following subset of $\mathcal{R}_S$, where w' is the Weyl vector of D';

$$\{ \operatorname{pr}_S(r) \mid r \text{ is a root of } L_{26} \text{ satisfying } \langle r, w' \rangle = 1 \text{ and } \langle \operatorname{pr}_S(r), \operatorname{pr}_S(r) \rangle < 0 \}.$$

Definition 4.2. A chamber of $\mathcal{P}_S$ is said to be an *induced chamber* if it is of the form $i^{-1}(D')$, where D' is a Conway chamber non-degenerate with respect to S.

Since $\mathcal{P}_{26}$ is tessellated by Conway chambers, the cone $\mathcal{P}_S$ is tessellated by induced chambers. Each induced chamber is the closure in $\mathcal{P}_S$ of a connected component of

$$\mathcal{P}_S \setminus \bigcup_{(v)^\perp \in \mathcal{H}_S} (v)^\perp.$$

The following is the main result of [27].

Proposition 4.3. *Suppose that the orthogonal complement of S in L_{26} cannot be embedded into the negative-definite Leech lattice Λ. Then each induced chamber $i^{-1}(D')$ has only a finite number of walls, and the set of walls can be calculated from the Weyl vector w' of D'.*

Every root r of S satisfies $r = \operatorname{pr}_S(r)$ and hence belongs to $\mathcal{R}_S$. Therefore we have the following:

Proposition 4.4. *Each standard fundamental domain of the action of $W(S)$ on $\mathcal{P}_S$ is tessellated by induced chambers.*

For each wall of an induced chamber $i^{-1}(D')$, there exists a unique induced chamber that shares the wall with $i^{-1}(D')$. More precisely, if $v \in \mathcal{R}_S$ defines a wall $i^{-1}(D') \cap (v)^\perp$ of $i^{-1}(D')$, then there exists a unique induced chamber $i^{-1}(D'')$ such that $-v \in \mathcal{R}_S$ defines a wall $i^{-1}(D'') \cap (-v)^\perp$ of $i^{-1}(D'')$, and that we have

$$i^{-1}(D') \cap (v)^\perp = i^{-1}(D'') \cap (-v)^\perp.$$

Definition 4.5. We call the chamber $i^{-1}(D'')$ satisfying the properties above the *induced chamber adjacent to $i^{-1}(D')$ across the wall $i^{-1}(D') \cap (v)^\perp$*.

Definition 4.6. We say that the primitive embedding $i\colon S \hookrightarrow L_{26}$ is *of simple Borcherds type* if, for any two induced chambers $i^{-1}(D')$ and $i^{-1}(D'')$, there exists an isometry $g \in \mathrm{O}^+(S)$ such that $i^{-1}(D'')^g = i^{-1}(D')$.

In order to prove that $i\colon S \hookrightarrow L_{26}$ is of simple Borcherds type, it is enough to choose an induced chamber $i^{-1}(D')$ and show that, for each wall $i^{-1}(D') \cap (v)^\perp$ of $i^{-1}(D')$, there exists an element $g \in \mathrm{O}^+(S)$ extending to an isometry of L_{26} such that $i^{-1}(D')^g$ is the induced chamber adjacent to $i^{-1}(D')$ across the wall $i^{-1}(D') \cap (v)^\perp$.

4.2. **Torelli theorem for $K3$ surfaces.** We recall how to apply Torelli theorem [25] for complex $K3$ surfaces to the study of the automorphism groups. Let X be a complex algebraic $K3$ surface. We denote by S_X the Néron-Severi lattice of X. Suppose that $\operatorname{rank} S_X > 1$. Then S_X is an even hyperbolic lattice. For a divisor D on X, we denote $[D] \in S_X$ the class of D. Let $\mathcal{P}_X$ be the positive cone of S_X containing an ample class. We put

$$N(X) := \{ x \in \mathcal{P}_X \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } X \}.$$

It is well-known that $N(X)$ is a chamber of S_X, that $N(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$, and that the correspondence $C \mapsto N(X) \cap ([C])^{\perp}$ gives a bijection from the set of smooth rational curves C on X to the set of walls of the chamber $N(X)$.

Let $\mathrm{Aut}(X)$ denote the automorphism group of X acting on X from the left. Let $\mathrm{aut}(X)$ be the image of the natural representation

$$\varphi_X : \ \mathrm{Aut}(X) \to \mathrm{O}^+(S_X)$$

defined by the *pullback* of the classes of divisors. Then we have $\mathrm{aut}(X) \subset \mathrm{aut}(N(X))$, where $\mathrm{aut}(N(X))$ is the automorphism group of the chamber $N(X)$.

Let T_X denote the orthogonal complement of $S_X = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ in the even unimodular lattice $H^2(X,\mathbb{Z})$ with the cup product, and let ω_X be a generator of the one-dimensional subspace $H^{2,0}(X)$ of $T_X \otimes \mathbb{C}$. We put

$$\mathrm{O}^{\omega}(T_X) := \{\ g \in \mathrm{O}(T_X)\ \mid\ \omega_X^g \in \mathbb{C}\,\omega_X\ \}.$$

By [22], there exists a unique isomorphism

$$\sigma_X \ :\ q_{S_X} \xrightarrow{\sim} -q_{T_X}$$

of finite quadratic forms such that the graph of σ_X is the image of $H^2(X,\mathbb{Z}) \subset S_X^{\vee} \oplus T_X^{\vee}$ by the natural projection to $(S_X^{\vee}/S_X) \oplus (T_X^{\vee}/T_X)$. We denote by

$$\sigma_{X*} \ :\ \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_{T_X})$$

the isomorphism induced by σ_X. Recall that $\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q_L)$ denotes the natural homomorphism. By [22] again, an element g of $\mathrm{O}(S_X)$ extends to an isometry of $H^2(X,\mathbb{Z})$ preserving the Hodge structure if and only if

$$\sigma_{X*}(\eta_{S_X}(g)) \in \eta_{T_X}(\mathrm{O}^{\omega}(T_X)). \tag{4.1}$$

By Torelli theorem for complex $K3$ surfaces [25], we have the following:

Theorem 4.7. (1) *An element g of* $\mathrm{aut}(N(X))$ *belongs to* $\mathrm{aut}(X)$ *if and only if g satisfies the condition* (4.1). (2) *The kernel of* $\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{aut}(X)$ *is isomorphic to the group*

$$\{\ g \in \mathrm{O}^{\omega}(T_X)\ \mid\ \eta_{T_X}(g) = 1\ \}.$$

Theorem 4.7 enables us to calculate $\mathrm{Aut}(X)$ from $\mathrm{aut}(N(X))$. In [15], Kondo studied the automorphism group of a generic Jacobian Kummer surface X by finding a primitive embedding $S_X \hookrightarrow L_{26}$ of simple Borcherds type and calculating $\mathrm{aut}(N(X))$. Since then, many authors have studied the automorphism groups of $K3$ surfaces by this method. See [27] and the references therein. On the other hand, in [11] and [27], this method was generalized to primitive embeddings $S_X \hookrightarrow L_{26}$ that are *not* of simple Borcherds type.

5. Computational study of geometry of an Enriques surface

5.1. **Borcherds method for an Enriques surface.** Suppose that a $K3$ surface X has an Enriques involution $\varepsilon \in \mathrm{Aut}(X)$, and let $g_{\varepsilon} := \varphi(\varepsilon) \in \mathrm{aut}(X)$ denote its action on S_X. Let $Y := X/\langle\varepsilon\rangle$ be the quotient of X by ε, and let $\pi \colon X \to Y$ denote the quotient morphism. Let S_Y denote the lattice of numerical equivalence classes of divisors on Y. It is well-known that S_Y is isomorphic to the even unimodular hyperbolic lattice L_{10} of rank 10. Let $\mathcal{P}_Y$ be the positive cone of S_Y containing an ample class. We put

$$N(Y) := \{\ y \in \mathcal{P}_Y \ \mid\ \langle y, [C]\rangle \ge 0 \ \text{ for all curves } C \text{ on } Y\ \}.$$

Let $\mathrm{aut}(Y)$ denote the image of the natural representation

$$\varphi_Y\ :\ \mathrm{Aut}(Y) \to \mathrm{O}^+(S_Y)$$

defined by the *pullback* of the classes of divisors. We have $\mathrm{aut}(Y) \subset \mathrm{aut}(N(Y))$. We consider the following two primitive sublattices of S_X:

$$S_X^+ := \{v \in S_X \mid v^{g_\varepsilon} = v\}, \qquad S_X^- := \{v \in S_X \mid v^{g_\varepsilon} = -v\}. \tag{5.1}$$

The homomorphism $\pi^* \colon S_Y \to S_X$ induces an isomorphism of lattices

$$\pi^*\ :\ S_Y(2) \xrightarrow{\sim} S_X^+,$$

by which we regard S_Y as a $\mathbb{Z}$-submodule of S_X. In particular, we have

$$\mathcal{P}_Y = (S_Y \otimes \mathbb{R}) \cap \mathcal{P}_X, \quad N(Y) = \mathcal{P}_Y \cap N(X).$$

(The second quality follows from the projection formula and the fact that π is finite.) We use $\langle\ ,\ \rangle_X$ and $\langle\ ,\ \rangle_Y$ to denote the intersection forms of S_X and S_Y, respectively, so that, for $y, y' \in S_Y \otimes \mathbb{R}$, we have $\langle y, y'\rangle_X = 2\,\langle y, y'\rangle_Y$.

For a group G and an element $g \in G$, we denote by $Z_G(g)$ the centralizer of g in G. We have a natural isomorphism

$$\mathrm{Aut}(Y) \cong Z_{\mathrm{Aut}(X)}(\varepsilon)/\langle\varepsilon\rangle.$$

Hence we have the following:

Proposition 5.1. *Suppose that the representation $\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{aut}(X)$ is an isomorphism. Then we have a surjective homomorphism*

$$\zeta\ :\ Z_{\mathrm{aut}(X)}(g_\varepsilon)/\langle g_\varepsilon\rangle \twoheadrightarrow \mathrm{aut}(Y)$$

defined by the commutative diagram

$$\begin{array}{ccc} Z_{\mathrm{Aut}(X)}(\varepsilon)/\langle\varepsilon\rangle & \cong & Z_{\mathrm{aut}(X)}(g_\varepsilon)/\langle g_\varepsilon\rangle \\ \wr\downarrow & & \downarrow\zeta \\ \mathrm{Aut}(Y) & \underset{\varphi_Y}{\twoheadrightarrow} & \mathrm{aut}(Y). \end{array}$$

The homomorphism ζ is defined as follows. If $g \in Z_{\mathrm{aut}(X)}(g_\varepsilon)$, then $S_Y^g = S_Y$ holds, and the restriction $g|_{S_Y} \in \mathrm{O}(S_Y)$ of g to S_Y gives $\zeta(g) \in \mathrm{aut}(Y)$.

We denote the orthogonal projection to $S_Y \otimes \mathbb{R} = S_X^+ \otimes \mathbb{R}$ by

$$\mathrm{pr}^+\ :\ S_X \otimes \mathbb{R} \to S_Y \otimes \mathbb{R}.$$

Suppose that we have a primitive embedding $i \colon S_X \hookrightarrow L_{26}$, and hence $\mathcal{P}_X$ is tessellated by induced chambers. The composite of the primitive embeddings $\pi^* \colon S_Y(2) \hookrightarrow S_X$ and $i \colon S_X \hookrightarrow L_{26}$ gives a primitive embedding of $S_Y(2)$ into L_{26}. By this embedding, the notion of induced chambers in $\mathcal{P}_Y$ is defined. Recall that $\mathrm{pr}_S \colon L_{26} \otimes \mathbb{R} \to S_X \otimes \mathbb{R}$ is the orthogonal projection. Let

$$\mathrm{pr}_S^+\ :\ L_{26} \otimes \mathbb{R} \to S_Y \otimes \mathbb{R}$$

denote the composite of pr_S and pr^+. Then the tessellation of $\mathcal{P}_Y$ by induced chambers is given by the locally finite family of hyperplanes

$$\{\ (\mathrm{pr}_S^+(r))^\perp\ \mid\ r \text{ is a root of } L_{26} \text{ such that } \langle \mathrm{pr}_S^+(r), \mathrm{pr}_S^+(r)\rangle_Y < 0\ \}. \tag{5.2}$$

This tessellation is the restriction to $\mathcal{P}_Y$ of the tessellation of $\mathcal{P}_X$ by induced chambers. Since $N(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$, Proposition 4.4 and $N(Y) = \mathcal{P}(Y) \cap N(X)$ imply the following:

Proposition 5.2. *The chamber $N(Y)$ is tessellated by induced chambers.*

Note that a chamber D_Y in $\mathcal{P}_Y$ is an induced chamber if and only if there exists an induced chamber D_X in $\mathcal{P}_X$ such that $D_Y = \mathcal{P}_Y \cap D_X$. More precisely, suppose that a subset $\mathcal{F}(D_X)$ of $S_X^\vee \setminus \{0\}$ defines an induced chamber D_X in $\mathcal{P}_X$;

$$D_X = \{\ x \in \mathcal{P}_X\ \mid\ \langle v, x\rangle_X \ge 0\ \text{ for all } v \in \mathcal{F}(D_X)\ \}.$$

Then we have

$$D_X \cap \mathcal{P}_Y = \{\ y \in \mathcal{P}_Y\ \mid\ \langle \mathrm{pr}^+(v), y\rangle_Y \ge 0\ \text{ for all } v \in \mathcal{F}(D_X)\ \}.$$

Hence, if $D_X \cap \mathcal{P}_Y$ contains a non-empty open subset of $\mathcal{P}_Y$, then $D_X \cap \mathcal{P}_Y$ is an induced chamber in $\mathcal{P}_Y$ defined by $\mathrm{pr}^+(\mathcal{F}(D_X)) \setminus \{0\}$.

5.2. **Smooth rational curves on an Enriques surface.** Let X and Y be as in the previous subsection. In particular, S_Y is regarded as a $\mathbb{Z}$-submodule of S_X. Let $h_Y \in S_Y$ be an ample class of Y. Then $h_Y \in S_X$ is ample on X. We explain a method to calculate the finite subset

$$\mathcal{R}_d := \{\ [C] \in S_Y\ \mid\ C \text{ is a smooth rational curve on } Y \text{ such that } \langle [C], h_Y\rangle_Y = d\ \}$$

of S_Y for each positive integer d by induction on d starting from $\mathcal{R}_0 = \emptyset$. Since the sublattice S_X^- of S_X is negative-definite, the set

$$\mathcal{T} := \{\ t \in S_X^-\ \mid\ \langle t, t\rangle_X = -4\ \}$$

is finite and can be calculated. We have the following:

Lemma 5.3 (Nikulin [23]). *Let $v \in S_Y$ be a vector such that $\langle v, v\rangle_Y = -2$ and $\langle v, h_Y\rangle_Y > 0$. Then the following conditions are equivalent.*

- (i) *The vector v is the class of an effective divisor D on Y such that π^*D is written as $[\Delta] + [\Delta]^\varepsilon$, where Δ is an effective divisor on X satisfying $\langle [\Delta], [\Delta]^\varepsilon\rangle_X = 0$.*
- (ii) *There exists an element $t \in \mathcal{T}$ satisfying $(v + t)/2 \in S_X$.* □

By the algorithm in Section 3 of [26], we compute the finite set

$$\mathcal{V}_d := \{\ v \in S_Y\ \mid\ \langle v, v\rangle_Y = -2,\ \langle v, h_Y\rangle_Y = d\ \}.$$

We then compute

$$\mathcal{V}'_d := \{\ v \in \mathcal{V}_d\ \mid\ v \text{ satisfies condition (ii) in Lemma 5.3}\ \}.$$

Lemma 5.3 implies that $\mathcal{R}_d \subset \mathcal{V}'_d$.

Lemma 5.4. *A vector v in $\mathcal{V}'_d$ fails to belong to $\mathcal{R}_d$ if and only if there exists a vector $r \in \mathcal{R}_{d'}$ with $0 < d' < d$ such that $\langle v, r\rangle_Y < 0$.*

Proof. Lemma 5.3 implies that there exists an effective divisor D on Y such that $v = [D]$. Then $v \in \mathcal{R}_d$ if and only if D is irreducible.

Suppose that there exists a smooth rational curve C on Y such that $\langle v, [C]\rangle_Y < 0$. Then D contains C. In particular, if $\langle [C], h_Y\rangle_Y < \langle v, h_Y\rangle_Y = d$, then D is not irreducible and hence $v \notin \mathcal{R}_d$. Conversely, suppose that D is reducible. Let $\Gamma_1, \dots, \Gamma_N$ be the distinct reduced irreducible components of D. If $\langle [D], [\Gamma_i]\rangle_Y \ge 0$ for all i, we would have $\langle v, v\rangle_Y \ge 0$. Therefore there exists an irreducible component Γ_i such that $\langle [D], [\Gamma_i]\rangle_Y < 0$. Then we have $\langle [\Gamma_i], [\Gamma_i]\rangle_Y < 0$ and hence Γ_i is a smooth rational curve. Since D is reducible, we see that $d' := \langle [\Gamma_i], h_Y\rangle_Y$ is smaller than d, and hence $r := [\Gamma_i] \in \mathcal{R}_{d'}$ satisfies $\langle v, r\rangle_Y < 0$. □

5.3. **An elliptic fibration on an Enriques surface.** Let Y be an Enriques surface with an ample class $h_Y \in S_Y$. Let $\phi\colon Y \to \mathbb{P}^1$ be an elliptic fibration. Then the class of a fiber of ϕ is written as $2f_\phi$, where f_ϕ is primitive in S_Y. For $p \in \mathbb{P}^1$, let E_p denote the divisor on Y such that

$$\phi^{-1}(p) = \begin{cases} E_p & \text{if } \phi^{-1}(p) \text{ is not a multiple fiber,} \\ 2E_p & \text{if } \phi^{-1}(p) \text{ is a multiple fiber.} \end{cases}$$

We give a method to calculate the reducible fibers of ϕ. Let $d_\phi := \langle 2f_\phi, h_Y\rangle_Y$ be the degree of a fiber of ϕ with respect to h_Y. Then we obviously have

$$\begin{aligned} \mathcal{R}(\phi) \quad &:= \quad \{[C] \in S_Y \mid C \text{ is a smooth rational curve on } Y \text{ contained in a fiber of } \phi\} \\ &= \quad \{[C] \in S_Y \mid [C] \in \mathcal{R}_d \text{ for some } d < d_\phi, \text{ and } \langle\, [C], f_\phi\,\rangle_Y = 0\}. \end{aligned}$$

We calculate the dual graph of the roots in $\mathcal{R}(\phi)$, and decompose $\mathcal{R}(\phi)$ into equivalence classes according to the connected components of the graph. Then there exists a canonical bijection between the set of these equivalence classes and the set of reducible fibers of ϕ. Let $\mathit{\Gamma}_p \subset \mathcal{R}(\phi)$ be one of the equivalence classes, and suppose that $\mathit{\Gamma}_p$ corresponds to a reducible fiber $\phi^{-1}(p)$. The roots in $\mathit{\Gamma}_p$ form an indecomposable extended Dynkin diagram, and its ADE-type is the ADE-type of the divisor E_p. We calculate

$$t(\mathit{\Gamma}_p) := \sum_{r \in \mathit{\Gamma}_p} m_r \cdot \langle r, h_Y\rangle_Y,$$

where m_r is the multiplicity in E_p of the irreducible component corresponding to $r \in \mathit{\Gamma}_p$ (see Figure 1.8 of [9]). We have $t(\mathit{\Gamma}_p) = \langle [E_p], h_Y\rangle_Y$, and hence $t(\mathit{\Gamma}_p)$ is either d_ϕ or $d_\phi/2$. Then $\phi^{-1}(p)$ is a multiple fiber if and only if $t(\mathit{\Gamma}_p) = d_\phi/2$.

6. General quartic Hessian surface

We review the results on the general quartic Hessian surface obtained in Dolgachev and Keum [7], and re-calculate these results in the form of vectors and matrices. *From now on, we denote by X the minimal resolution of the general quartic Hessian surface $\overline{X}$ defined in Introduction.*

It is known that the Néron-Severi lattice S_X of X is of rank 16, and $S_X^\vee/S_X$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/3\mathbb{Z})$. If two lines ℓ_β and $\ell_{\beta'}$ on $\overline{X}$ intersect, the intersection point is an ordinary node of $\overline{X}$. Hence we have

$$\langle [E_\alpha], [E_{\alpha'}]\rangle = (-2) \cdot \delta_{\alpha\alpha'}, \qquad \langle [L_\beta], [L_{\beta'}]\rangle = (-2) \cdot \delta_{\beta\beta'}, \tag{6.1}$$

where δ is Kronecker's delta symbol on $A \cup B$. Recall that the indexing of p_α and ℓ_β was done in such a way that the following holds:

$$\langle [E_\alpha], [L_\beta] \rangle = \begin{cases} 0 & \alpha \not\supset \beta, \\ 1 & \alpha \supset \beta. \end{cases} \tag{6.2}$$

The lattice S_X is generated by the classes $[E_\alpha]$ and $[L_\beta]$. More precisely, the classes of the following smooth rational curves form a basis of S_X:

$$\begin{aligned} &E_{123}, E_{124}, E_{125}, E_{134}, E_{135}, E_{145}, E_{234}, E_{235}, E_{245}, E_{345}, \\ &L_{45}, L_{35}, L_{34}, L_{25}, L_{24}, L_{13}. \end{aligned} \tag{6.3}$$

We fix, once and for all, this basis, and write elements of $S_X \otimes \mathbb{R}$ as *row* vectors. The Gram matrix of S_X with respect to this basis is readily calculated by (6.1) and (6.2). The fact that the classes of the 16 curves above form a basis of S_X can be confirmed by checking that the determinant of this Gram matrix is equal to $-2^4 \cdot 3$. Let $h_Q \in S_X$ denote the class of the pullback of a hyperplane section of $\overline{X} \subset \mathbb{P}^3$ by the minimal resolution $X \to \overline{X}$. Then we have

$$h_Q = [-1, 1, 1, -1, -1, 1, 1, 1, 3, 1, 2, 0, 0, 2, 2, -2].$$

Proposition 6.1. (1) *An element g of* $\mathrm{aut}(N(X))$ *belongs to* $\mathrm{aut}(X)$ *if and only if g satisfies $\eta_{S_X}(g) = \pm 1$, where $\eta_{S_X} : \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$ is the natural homomorphism.*
(2) *The representation $\varphi_X\colon \mathrm{Aut}(X) \to \mathrm{aut}(X)$ is an isomorphism.*

Proof. Since $\overline{X}$ is *general* in the family of quartic Hessian surfaces, we have

$$\mathrm{O}^\omega(T_X) = \{\pm 1\}.$$

Therefore the statements follow from Theorem 4.7. □

We then apply Borcherds method to S_X. Recall that we have fixed a basis $f_1, f_2, \lambda_1, \dots, \lambda_{24}$ of L_{26} in Section 3.2. Let M be the matrix given in Table 6.1. It is easy see that the homomorphism $i\colon S_X \to L_{26}$ given by $v \mapsto vM$ is a primitive embedding of S_X into L_{26} that maps $\mathcal{P}_X$ into $\mathcal{P}_{26}$. From now on, we regard S_X as a primitive sublattice of L_{26} by this embedding i.

Remark 6.2. This embedding $i\colon S_X \to L_{26}$ is equal to the embedding given by Dolgachev and Keum [7] up to the action of $\mathrm{O}^+(S_X)$ and $\mathrm{O}^+(L_{26})$. See [27] for a general method to embed the Néron-Severi lattice of a $K3$ surface into L_{26} in Borcherds method.

As in Section 4.1, we denote by $\mathrm{pr}_S\colon L_{26} \otimes \mathbb{R} \to S_X \otimes \mathbb{R}$ the orthogonal projection. We can calculate a Gram matrix of the orthogonal complement R of S_X in L_{26} explicitly, and confirm that R contains a root. Hence R cannot be embedded into the negative-definite Leech lattice Λ. Therefore Proposition 4.3 can be applied.

Let D_{26} be the Conway chamber with the Weyl vector $w_{26} = f_1 \in L_{26}$ (see Section 3.2). We put

$$D_X := i^{-1}(D_{26}).$$

Proposition 6.3. *The closed subset D_X of $\mathcal{P}_X$ contains the vector*

$$h_X := \mathrm{pr}_S(w_{26}) = [-3, 2, 2, -3, -3, 2, 2, 2, 7, 2, 5, 0, 0, 5, 5, -5]$$

of square-norm 20 *in its interior. Moreover h_X belongs to $N(X)$. Hence D_X is an induced chamber contained in $N(X)$.*

$$
\left[\begin{array}{cccccccccccccccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -6 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 6 & -2 & -2 & -2 & -1 & -2 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 2 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 5 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 2 \\
1 & 1 & -3 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -3 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 3 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & -1 & 2 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

TABLE 6.1. Primitive embedding of S_X into L_{26}

Proof. We can calculate the set

$$\mathcal{F}'(D_X) := \{\ \mathrm{pr}_S(r) \ \mid\ r \in \mathcal{W}_{26},\ \ \langle \mathrm{pr}_S(r), \mathrm{pr}_S(r)\rangle < 0\ \ \}$$

by the algorithm presented in Section 5 of [27]. By definition, we have

$$D_X = \{\ x \in \mathcal{P}_X \ \mid\ \langle x, v\rangle \geq 0 \ \text{ for all } v \in \mathcal{F}'(D_X)\ \}.$$

Hence we can confirm that h_X is an interior point of D_X. Therefore D_X is an induced chamber. Moreover, we can confirm that

$$\{\ r \in S_X \ \mid\ \langle r, h_X\rangle = 0,\ \langle r, r\rangle = -2\ \} = \emptyset$$

by the algorithm given in Section 3 of [26]. Since the singularities of $\overline{X}$ consist only of the ordinary nodes p_α $(\alpha \in A)$, we have

$$\{\ r \in S_X \ \mid\ \langle r, h_Q\rangle = 0,\ \langle r, r\rangle = -2\ \} = \{\ \pm[E_\alpha] \ \mid\ \alpha \in A\ \},$$

and the classes $[E_\alpha]$ define the walls of $N(X)$ that pass through the half-line $\mathbb{R}_{\geq 0}\, h_Q$. We see that $\langle [E_\alpha], h_X\rangle = 1 > 0$ for all $\alpha \in A$, and confirm that

$$\{\ r \in S_X \ \mid\ \langle r, h_Q\rangle > 0,\ \langle r, h_X\rangle < 0,\ \langle r, r\rangle = -2\ \} = \emptyset$$

type	$\langle v, h_X\rangle$	$\langle v, v\rangle$	number	
(a)	1	-2	20	outer
(b)	2	-1	10	inner
(c)	5	$-2/3$	24	inner
(d)	4	$-2/3$	30	inner

TABLE 6.2. Walls of D_X

by the algorithm given in Section 3 of [26]. These facts imply that h_X is an interior point of $N(X)$. By Proposition 4.4, $N(X)$ is tessellated by induced chambers. Hence D_X is contained in $N(X)$. □

Proposition 6.4. *The number of walls of D_X is $20 + 10 + 24 + 30 = 84$, among which 20 walls are walls of $N(X)$ and they are defined by the roots $[E_\alpha]$ and $[L_\beta]$, whereas the other $10 + 24 + 30$ walls are not walls of $N(X)$.*

Proof. In the proof of Proposition 6.3, the set $\mathcal{F}'(D_X)$ of vectors defining the chamber D_X is calculated. From $\mathcal{F}'(D_X)$, we can calculate the set $\mathcal{F}(D_X)$ of primitive defining vectors v of walls of D_X by Algorithm 2.3. The result is given in Table 6.2. The walls of D_X are divided into four types (a)–(d) according to the values of $\langle v, h_X\rangle$ and $\langle v, v\rangle$, where $v \in S_X^\vee$ is the primitive defining vector of the wall. It turns out that the 20 walls of type (a) are defined by $[E_\alpha]$ and $[L_\beta]$, and hence these walls are walls of $N(X)$. For each of the other walls, there exists no positive integer k such that $k^2\langle v, v\rangle = -2$. Hence these walls are not walls of $N(X)$. □

We call a wall of D_X an *outer wall* if it is a wall of $N(X)$, and call it an *inner wall* otherwise.

Proposition 6.5. *If $g \in \mathrm{aut}(X)$, then D_X^g is an induced chamber.*

Proof. Let R denote the orthogonal complement of S_X in L_{26}. By [22], we have an isomorphism $\sigma_R\colon q_R \xrightarrow{\sim} -q_{S_X}$ of finite quadratic forms given by the even unimodular overlattice L_{26} of $S_X \oplus R$. Let $\sigma_{R*}\colon \mathrm{O}(q_R) \xrightarrow{\sim} \mathrm{O}(q_{S_X})$ denote the isomorphism induced by σ_R. By Proposition 6.1, we have $\eta_{S_X}(g) = \pm 1$ and hence $\eta_{S_X}(g)$ belongs to the image of the composite homomorphism

$$\mathrm{O}(R) \xrightarrow{\eta_R} \mathrm{O}(q_R) \xrightarrow{\sigma_{R*}} \mathrm{O}(q_{S_X}).$$

By [22] again, there exists an element $\tilde{g}$ of $\mathrm{O}(L_{26})$ such that $\tilde{g}$ preserves the primitive sublattices S_X and R and that the restriction of $\tilde{g}$ to S_X is equal to g. Since $D_X^g = i^{-1}(D_{26}^{\tilde{g}})$, we see that D_X^g is an induced chamber. □

Proposition 6.6. *The automorphism group $\mathrm{aut}(D_X)$ of the chamber D_X is of order 240, and we have $h_X^g = h_X$ for all $g \in \mathrm{aut}(D_X)$.*

Proof. If $g \in \mathrm{O}^+(S_X)$ induces an automorphism of D_X, then g induces a permutation of the set $\{([E_\alpha])^\perp, ([L_\beta])^\perp\}$ of walls of type (a), and hence g induces a permutation of $A \cup B$ that preserves the intersection numbers (6.1) and (6.2). The permutations of $A \cup B$ preserving (6.1) and (6.2) form a group of order 240 generated by the permutations of $\{1, \ldots, 5\}$ and the switch $\alpha = \bar{\beta} \leftrightarrow \beta = \bar{\alpha}$. Conversely,

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 1 & 1 & -2 & -2 & 1 & 0 & 0 & 3 & 0 & 2 & -1 & -1 & 2 & 2 & -3 \\
-1 & 1 & 0 & -1 & -2 & 0 & 1 & 0 & 2 & 0 & 1 & -1 & 0 & 1 & 2 & -2 \\
-1 & 0 & 1 & -2 & -1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & -1 & 2 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 & -1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

TABLE 6.3. Matrix representation of the Enriques involution ε

each of these 240 permutations induces an isometry of S_X that preserves D_X. By direct calculation, we see that

$$h_X = \sum_{\alpha\in A} [E_\alpha] + \sum_{\beta \in B} [L_\beta]. \tag{6.4}$$

Hence we have $h_X^g = h_X$ for all $g \in \mathrm{aut}(D_X)$. □

Proposition 6.7. *The group* $\mathrm{aut}(D_X) \cap \mathrm{aut}(X)$ *is of order* 2*, and its non-trivial element* g_ε *is the image* $\varphi_X(\varepsilon)$ *of an Enriques involution* $\varepsilon \in \mathrm{Aut}(X)$ *by the natural representation* $\varphi_X\colon \mathrm{Aut}(X) \to \mathrm{aut}(X)$*. This Enriques involution* ε *switches* E_α *and* $L_{\bar\alpha}$ *for each* $\alpha \in A$*.*

Proof. By means of Proposition 6.1 (1), we can check by direct calculation that $\mathrm{aut}(D_X) \cap \mathrm{aut}(X)$ consists of the identity and the isometry that comes from the switch $\alpha \leftrightarrow \bar\alpha$. The matrix presentation of the isometry $g_\varepsilon \in \mathrm{aut}(D_X)$ induced by the switch is given in Table 6.3. As in (5.1), we put

$$S_X^+ := \{ v \in S_X \mid v^{g_\varepsilon} = v \}, \qquad S_X^- := \{ v \in S_X \mid v^{g_\varepsilon} = -v \}.$$

Then S_X^+ is of rank 10 generated by the row vectors $\eta_1, \dots, \eta_{10}$ of the matrix given in Table 6.4. We see that $S_X^+ \cong L_{10}(2)$. Indeed, we have chosen the basis $\eta_1, \dots, \eta_{10}$ of S_X^+ in such a way that the homomorphism from L_{10} to S_X^+ given by

$$f_1 \mapsto \eta_1, \quad f_2 \mapsto \eta_2, \quad e_1 \mapsto \eta_3, \quad \dots, \quad e_8 \mapsto \eta_{10} \tag{6.5}$$

induces an isometry $L_{10}(2) \xrightarrow{\sim} S_X^+$, where $f_1, f_2, e_1, \dots, e_8$ are the basis of L_{10} fixed in Section 3.1. On the other hand, we can confirm that the negative-definite even lattice S_X^- has no roots (see Proposition 7.11). Hence we conclude that g_ε is the image $\varphi_X(\varepsilon)$ of an Enriques involution ε on X by the criterion given in [12]. □

The walls of D_X of type (b) play an important role in the study of the Enriques surface $Y := X/\langle\varepsilon\rangle$. For $\alpha \in A$, let v_α be the unique vector of $S_X^\vee$ that satisfies

$$\langle v_\alpha, [E_{\alpha'}]\rangle = \delta_{\alpha\alpha'}, \quad \langle v_\alpha, [L_\beta]\rangle = \delta_{\bar\alpha\beta}. \tag{6.6}$$

Then the walls of D_X of type (b) are exactly the walls defined by v_α for some $\alpha \in A$. Since g_ε comes from the switch $\alpha \leftrightarrow \bar\alpha$, we have

$$v_\alpha^{g_\varepsilon} = v_\alpha. \tag{6.7}$$

$$\left[\begin{array}{cccccccccccccccc}
-1 & 1 & 0 & -1 & -1 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & -2 \\
-1 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 2 & 2 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 1 & 1 & -2 & -2 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & -1 & 2 & 1 & -3 \\
1 & -1 & -1 & 2 & 1 & -1 & 1 & -1 & -2 & 1 & -1 & 0 & 2 & -2 & -1 & 2 \\
-1 & 1 & 1 & -3 & -2 & 1 & 0 & 1 & 3 & -1 & 2 & -1 & -2 & 2 & 2 & -3 \\
1 & -1 & -1 & 2 & 3 & -1 & -1 & 0 & -3 & 1 & -2 & 2 & 1 & -2 & -2 & 3 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
-2 & 1 & 1 & -2 & -2 & 1 & 1 & 0 & 3 & -1 & 1 & -2 & -1 & 2 & 3 & -3 \\
1 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & -1 & 2
\end{array}\right]$$

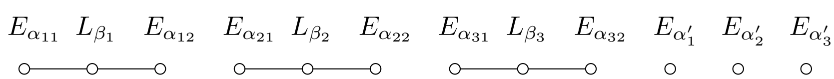
TABLE 6.4. Basis of S_X^+

Proposition 6.8. *For $\alpha \in A$, let $\bar{\pi}_\alpha \colon \overline{X} \to \mathbb{P}^2$ be the projection from the center $p_\alpha \in \overline{X}$, and let $\iota_\alpha \colon X \to X$ be the involution obtained from the double covering $\bar{\pi}_\alpha$. Then $g_\alpha := \varphi_X(\iota_\alpha) \in \mathrm{aut}(X)$ maps D_X to the induced chamber adjacent to D_X across the wall $D_X \cap (v_\alpha)^\perp$ of type* (b). *Moreover ι_α commutes with ε.*

Proof. Let $\pi_\alpha \colon X \to \mathbb{P}^2$ denote the composite of $\bar{\pi}_\alpha$ with the minimal resolution $X \to \overline{X}$. Then the class $\pi_\alpha^*([l]) \in S_X$ of the pullback of a line l on $\mathbb{P}^2$ is $h_Q - [E_\alpha]$. We calculate the finite set

$$\tilde{\Gamma}_\alpha := \{\ r \in S_X \ \mid\ \langle r, \pi_\alpha^*([l])\rangle = 0,\ \langle r, r\rangle = -2,\ \langle r, h_X\rangle > 0\ \}$$

by the algorithm given in Section 3 of [26]. From $\tilde{\Gamma}_\alpha$ and using the ample class h_X of X, we can calculate by the method described in Section 6.1 of [29] the set Γ_α of classes of smooth rational curves that are contracted by π_α. It turns out that the vectors in Γ_α form the following Dynkin diagram.



Here $\beta_1, \beta_2, \beta_3$ are the three elements of B contained in α, the three indices $\alpha, \alpha_{\nu 1}, \alpha_{\nu 2}$ are the three elements of A containing β_ν for $\nu = 1, 2, 3$, and $\alpha'_1, \alpha'_2, \alpha'_3$ are the three elements of A containing $\bar{\alpha} \in B$. In particular, the singular locus of the branch curve of $\bar{\pi}_\alpha \colon \overline{X} \to \mathbb{P}^2$ consists of 6 simple singular points, and its ADE-type is $3A_3 + 3A_1$. Then the eigenspace $V_1 := \mathrm{Ker}(g_\alpha - I_{16})$ of g_α in $S_X \otimes \mathbb{Q}$ is of dimension 10 spanned by the classes

$$h_Q - [E_\alpha], \quad [L_{\beta_\nu}], \quad [E_{\alpha_{\nu 1}}] + [E_{\alpha_{\nu 2}}], \quad [E_{\alpha'_\mu}] \qquad (\nu, \mu = 1, 2, 3),$$

and the eigenspace $V_{-1} := \mathrm{Ker}(g_\alpha + I_{16})$ is the orthogonal complement of V_1. Hence we can calculate the matrix representation of g_α. See the webpage [32] for these matrices. We can confirm by direct calculation of products of matrices that g_α and g_ε commute. Therefore ε and ι_α commute by Proposition 6.1 (2). By Proposition 6.5, we know that $D_X^{g_\alpha}$ is an induced chamber. We can confirm that $v_\alpha^{g_\alpha} = -v_\alpha$, and that $\langle h_X^{g_\alpha}, v\rangle > 0$ holds for all primitive defining vectors v of walls of D_X other than v_α. Hence $D_X^{g_\alpha}$ is adjacent to D_X across the inner wall $D_X \cap (v_\alpha)^\perp$ of D_X. □

In [7], Dolgachev and Keum also constructed automorphisms of X whose action on S_X maps D_X to the induced chamber adjacent to D_X across each inner wall of type (c) and type (d). Thus they obtained a set of generators of $\mathrm{Aut}(X)$. See [32] for the matrix representations of these automorphisms.

$$
\begin{aligned}
\bar{u}_{\{1,2,3\}} &= [0,0,1,0,1,2,1,1,0,0],\\
\bar{u}_{\{1,2,4\}} &= [0,0,1,1,1,2,2,2,1,1],\\
\bar{u}_{\{1,2,5\}} &= [0,1,-2,-1,-2,-3,-2,-1,-1,-1],\\
\bar{u}_{\{1,3,4\}} &= [0,0,1,0,0,0,0,0,0,0],\\
\bar{u}_{\{1,3,5\}} &= [0,0,1,0,1,2,2,1,1,0],\\
\bar{u}_{\{1,4,5\}} &= [1,0,-2,-1,-3,-4,-3,-2,-1,-1],\\
\bar{u}_{\{2,3,4\}} &= [0,0,1,1,2,2,1,1,1,1],\\
\bar{u}_{\{2,3,5\}} &= [0,1,-2,-1,-2,-3,-3,-3,-2,-1],\\
\bar{u}_{\{2,4,5\}} &= [1,1,-5,-3,-6,-9,-7,-5,-3,-1],\\
\bar{u}_{\{3,4,5\}} &= [1,0,-2,-1,-2,-4,-3,-3,-2,-1],\\
\bar{v}_{\{1,2,3\}} &= [0,1,-1,0,-1,-2,-1,-1,0,0],\\
\bar{v}_{\{1,2,4\}} &= [1,1,-4,-3,-5,-8,-7,-5,-3,-2],\\
\bar{v}_{\{1,2,5\}} &= [1,0,-1,-1,-2,-2,-2,-2,-1,0],\\
\bar{v}_{\{1,3,4\}} &= [1,0,-1,0,0,0,0,0,0,0],\\
\bar{v}_{\{1,3,5\}} &= [1,1,-4,-2,-5,-8,-7,-5,-4,-2],\\
\bar{v}_{\{1,4,5\}} &= [0,1,-1,-1,-1,-2,-1,-1,-1,0],\\
\bar{v}_{\{2,3,4\}} &= [1,1,-4,-3,-6,-8,-6,-5,-4,-2],\\
\bar{v}_{\{2,3,5\}} &= [1,0,-1,-1,-1,-2,-1,0,0,0],\\
\bar{v}_{\{2,4,5\}} &= [0,0,2,2,3,4,3,2,1,0],\\
\bar{v}_{\{3,4,5\}} &= [0,1,-1,-1,-2,-2,-2,-1,0,0]
\end{aligned}
$$

TABLE 7.1. Primitive defining vectors of the walls of D_Y

Lemma 6.9. *Let σ_α and σ_β denote the reflections of S_X with respect to the roots $[E_\alpha]$ and $[L_\beta]$, respectively. Then $D_X^{\sigma_\alpha}$ and $D_X^{\sigma_\beta}$ are induced chambers.*

Proof. Since $[E_\alpha]$ and $[L_\beta]$ are roots of L_{26}, the reflections σ_α and σ_β are the restrictions to S_X of reflections of L_{26}. $\square$

Combining this fact with the automorphisms of X constructed in [7], we obtain the following:

Corollary 6.10. *The embedding $i\colon S_X \hookrightarrow L_{26}$ is of simple Borcherds type.*

7. Geometry of the Enriques surface Y

From now on, we denote by Y the quotient of the $K3$ surface X by the Enriques involution ε given in Proposition 6.7.

7.1. **Chamber D_Y and generators of** $\mathrm{aut}(Y)$**.** As in Section 5.1, we identify the $\mathbb{Z}$-module S_Y with the $\mathbb{Z}$-submodule S_X^+ of S_X by π^* so that we have

$$\mathcal{P}_Y = (S_Y \otimes \mathbb{R}) \cap \mathcal{P}_X \quad \text{and} \quad N(Y) = N(X) \cap \mathcal{P}_Y.$$

We have fixed a basis $\eta_1, \dots, \eta_{10}$ of S_X^+ in such a way that the homomorphism (6.5) is an isometry $L_{10}(2) \xrightarrow{\sim} S_X^+$. We use $\eta_1, \dots, \eta_{10}$ as a basis of S_Y, and write elements of $S_Y \otimes \mathbb{R}$ as row vectors. Hence the Gram matrix of S_Y is equal to the standard Gram matrix of $L_{10} = U \oplus E_8$.

Recall that S_X is embedded in L_{26} by the matrix in Table 6.1. Let

$$\mathrm{pr}_S^+ \,:\, L_{26}\otimes\mathbb{R} \xrightarrow{\mathrm{pr}_S} S_X\otimes\mathbb{R} \xrightarrow{\mathrm{pr}^+} S_Y\otimes\mathbb{R}$$

denote the composite of the orthogonal projections pr_S and pr^+. We consider the tessellation of $\mathcal{P}_Y$ given by the locally finite family of hyperplanes (5.2). We put

$$D_Y := \mathcal{P}_Y \cap D_X.$$

By Proposition 6.6, we have $h_X \in S_Y$ and hence $h_X \in D_Y$. We put

$$h_Y := h_X.$$

Note that the class $h_Y \in S_Y$ is ample on Y, and that $\langle h_Y, h_Y\rangle_Y = 10$. With respect to the basis $\eta_1,\dots,\eta_{10}$ of S_Y, we have

$$h_Y = [3, 3, -8, -5, -10, -15, -12, -9, -6, -3].$$

Proposition 7.1. *The closed subset D_Y of $\mathcal{P}_Y$ is an induced chamber contained in $N(Y)$. The set of primitive defining vectors of walls of D_Y consists of* $10+10$ *vectors*

$$\bar{u}_\alpha := 2\,\mathrm{pr}^+([E_\alpha]) = 2\,\mathrm{pr}^+([L_{\bar\alpha}]), \quad \bar{v}_\alpha := 2\,\mathrm{pr}^+(v_\alpha)$$

of $S_Y^\vee$, where α runs through A, and $v_\alpha \in S_X^\vee$ is the primitive defining vector of an inner wall of D_X of type (b)*, which is characterized by* (6.6)*. The vector representations of $\bar{u}_\alpha$ and $\bar{v}_\alpha$ are given in Table 7.1.*

Proof. Since h_X is an interior point of D_X in $S_X\otimes\mathbb{R}$, the vector $h_Y = h_X \in D_Y$ is an interior point of D_Y in $S_Y\otimes\mathbb{R}$. Hence D_Y is a chamber. Since D_X is an induced chamber contained in $N(X)$, the chamber D_Y is an induced chamber contained in $N(Y) = \mathcal{P}_Y\cap N(X)$. By definition, the chamber D_Y is defined by the set of vectors

$$\{\ \mathrm{pr}^+(v)\ \mid\ v\in\mathcal{F}(D_X),\quad \langle \mathrm{pr}^+(v), \mathrm{pr}^+(v)\rangle_Y < 0\ \},$$

where $\mathcal{F}(D_X)$ is the set of primitive defining vectors of walls of D_X, which we have calculated in the proof of Proposition 6.4. Using Algorithm 2.3, we obtain the set of primitive defining vectors of walls of D_Y as Table 7.1. □

Remark 7.2. If $v\in\mathcal{F}(D_X)$ defines a wall of D_X of type (c) or (d), then we have $\langle \mathrm{pr}^+(v), \mathrm{pr}^+(v)\rangle_Y = 0$, and hence the hyperplane $(v)^\perp$ of $\mathcal{P}_X$ does not intersect $\mathcal{P}_Y \subset \mathcal{P}_X$.

Corollary 7.3. *The vector $\bar{u}_\alpha$ is the class of the smooth rational curve $\pi(E_\alpha) = \pi(L_{\bar\alpha})$ on Y. In particular, each of the walls $D_Y \cap (\bar{u}_\alpha)^\perp$ is a wall of $N(Y)$.*

The set of primitive defining vectors of walls of D_Y is denoted by

$$\mathcal{F}(D_Y)\ \ :=\ \ \{\bar{u}_\alpha \,|\, \alpha\in A\}\ \cup\ \{\bar{v}_\alpha \,|\, \alpha\in A\}.$$

We have

$$\text{(7.1)}\qquad \langle \bar{u}_\alpha, \bar{u}_{\alpha'}\rangle_Y = \begin{cases} -2 & \text{if } \alpha=\alpha', \\ 1 & \text{if } |\alpha\cap\alpha'| = 1, \\ 0 & \text{otherwise,}\end{cases} \quad \langle \bar{v}_\alpha, \bar{v}_{\alpha'}\rangle_Y = \begin{cases} -2 & \text{if } \alpha=\alpha', \\ 1 & \text{if } |\alpha\cap\alpha'| = 2, \\ 0 & \text{otherwise,}\end{cases}$$

and

$$\text{(7.2)}\qquad \langle \bar{u}_\alpha, \bar{v}_{\alpha'}\rangle_Y = 2\,\delta_{\alpha\alpha'}.$$

Moreover, we have

$$\langle \bar{u}_\alpha, h_Y\rangle_Y = 1, \quad \langle \bar{v}_\alpha, h_Y\rangle_Y = 2.$$

Proposition 7.4. *If $\bar{g} \in \mathrm{aut}(Y)$, then $D_Y^{\bar{g}}$ is an induced chamber.*

Proof. By Propositions 5.1 and 6.1, we have an element $g \in Z_{\mathrm{aut}(X)}(g_\varepsilon)$ such that $g|_{S_Y} = \bar{g}$. Since $D_Y^{\bar{g}} = \mathcal{P}_Y \cap D_X^g$, the statement follows from Proposition 6.5. □

Proposition 7.5. *The automorphism group $\mathrm{aut}(D_Y)$ of the chamber D_Y is isomorphic to the symmetric group of degree 5. Every element $\bar{g}$ of $\mathrm{aut}(D_Y)$ satisfies $h_Y^{\bar{g}} = h_Y$. The intersection $\mathrm{aut}(D_Y) \cap \mathrm{aut}(Y)$ is trivial.*

Proof. Let $\bar{g}$ be an element of $\mathrm{aut}(D_Y)$. Then $\bar{g}$ induces a permutation of $\mathcal{F}(D_Y)$ that preserves (7.1) and (7.2). Note that a vector $v \in \mathcal{F}(D_Y)$ belongs to $\{\bar{u}_\alpha \mid \alpha \in A\}$ (resp. to $\{\bar{v}_\alpha \mid \alpha \in A\}$) if there exist exactly three (resp. six) vectors $v' \in \mathcal{F}(D_Y)$ such that $\langle v, v' \rangle_Y = 1$. Hence the permutation of $\mathcal{F}(D_Y)$ induced by $\bar{g}$ induces a permutation of A preserving the set of pairs $\{\alpha, \alpha'\}$ satisfying $|\alpha \cap \alpha'| = 1$. Therefore this permutation comes from a permutation of $\{1, \dots, 5\}$. Conversely, it can be easily checked that a permutation of $\{1, \dots, 5\}$ induces an isometry of S_Y. Hence the first assertion is proved. From (6.4), we have $h_Y = \sum_{\alpha \in A} \bar{u}_\alpha$. Hence the second assertion follows. Suppose that $\bar{g} \in \mathrm{aut}(D_Y)$ belongs to $\mathrm{aut}(Y)$. By Propositions 5.1 and 6.1, there exists an element $g \in Z_{\mathrm{aut}(X)}(g_\varepsilon)$ such that $g|_{S_Y} = \bar{g}$. Since $h_X = h_Y$ and $h_Y^{\bar{g}} = h_Y$, we have $h_X^g = h_X$. Since $D_X^g \cap D_X$ contains an interior point h_X of D_X, we see that $D_X^g = D_X$. By Proposition 6.7, it follows that $g \in \{\mathrm{id}, g_\varepsilon\}$. Therefore $\bar{g}$ is the identity. □

The involution $\iota_\alpha \colon X \to X$ in Proposition 6.8 commutes with ε. Hence ι_α induces an involution $j_\alpha \colon Y \to Y$ of Y, whose representation on S_Y is

$$\bar{g}_\alpha := g_\alpha|_{S_Y},$$

where $g_\alpha \in \mathrm{aut}(X)$ is the representation of ι_α calculated in Proposition 6.8. We calculate the matrix representations of $\bar{g}_\alpha$. (See the webpage [32] for these matrices.)

Proposition 7.6. *The element $\bar{g}_\alpha \in \mathrm{aut}(Y)$ maps D_Y to the induced chamber adjacent to D_Y across the wall $D_Y \cap (\bar{v}_\alpha)^\perp$.*

Proof. By Proposition 7.4, we know that $D_Y^{\bar{g}_\alpha}$ is an induced chamber. It is easy to check that $v_\alpha \in S_X^+ \otimes \mathbb{R}$ and $v_\alpha^{g_\alpha} = -v_\alpha$, and hence $\bar{v}_\alpha^{\bar{g}_\alpha} = -\bar{v}_\alpha$. Therefore the vector $-\bar{v}_\alpha$ defines a wall of $D_Y^{\bar{g}_\alpha}$. Since $\langle h_Y^{\bar{g}_\alpha}, \bar{v} \rangle > 0$ holds for all vectors $\bar{v}$ of $\mathcal{F}(D_Y) \setminus \{\bar{v}_\alpha\}$, we see that $D_Y^{\bar{g}_\alpha}$ is adjacent to D_Y across the wall $D_Y \cap (\bar{v}_\alpha)^\perp$. □

Remark 7.7. Each involution $\bar{g}_\alpha$ has an eigenvalue 1 with multiplicity 6.

Proposition 7.8. *For $\alpha \in A$, let $\bar{\sigma}_\alpha \in \mathrm{O}(S_Y)$ denote the reflection of S_Y with respect to the root $\bar{u}_\alpha$. Then $\bar{\sigma}_\alpha$ maps D_Y to the induced chamber adjacent to D_Y across the wall $D_Y \cap (\bar{u}_\alpha)^\perp$.*

Proof. First remark that, since S_Y is embedded in L_{26} as $S_Y(2)$, the vector $\bar{u}_\alpha$ is not a root of L_{26}. Hence the argument of Lemma 6.9 does not work in this case. As was shown in Lemma 6.9, the chamber $D_X^{\sigma_\alpha \sigma_{\bar{\alpha}}}$ is an induced chamber in $\mathcal{P}_X$. By direct calculation, we see that $\sigma_\alpha \sigma_{\bar{\alpha}}$ commutes with g_ε and that the restriction of $\sigma_\alpha \sigma_{\bar{\alpha}}$ to S_Y is equal to $\bar{\sigma}_\alpha$. Hence $D_Y^{\bar{\sigma}_\alpha}$ is equal to $\mathcal{P}_Y \cap D_X^{\sigma_\alpha \sigma_{\bar{\alpha}}}$, and $D_Y^{\bar{\sigma}_\alpha}$ is an induced chamber. Since $\bar{\sigma}_\alpha$ is the reflection in the hyperplane $(\bar{u}_\alpha)^\perp$, it is obvious that $D_Y^{\bar{\sigma}_\alpha}$ is adjacent to D_Y across the wall $D_Y \cap (\bar{u}_\alpha)^\perp$. □

From these propositions, we obtain the following corollaries.

dim	1	2	3	4	5	6	7	8	9
outer faces	657	3420	7250	8525	6270	2940	840	135	10
$\mathrm{aut}(Y)$-classes	44	314	1077	1759	1669	1060	435	105	10
inner faces	0	0	0	0	0	60	90	45	10
$\mathrm{aut}(Y)$-classes	0	0	0	0	0	1	15	25	10

TABLE 7.2. Numbers of faces of D_Y and their $\mathrm{aut}(Y)$-equivalence classes

Corollary 7.9. *The primitive embedding $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ of $S_Y(2)$ into L_{26} is of simple Borcherds type.*

Corollary 7.10. *The group* $\mathrm{aut}(Y)$ *is generated by the* 10 *involutions* $\bar{g}_\alpha$. *The induced chamber D_Y is a fundamental domain of the action of* $\mathrm{aut}(Y)$ *on* $N(Y)$. *In particular, the mapping $\bar{g} \mapsto D_Y^{\bar{g}}$ gives rise to a bijection from* $\mathrm{aut}(Y)$ *to the set of induced chambers contained in* $N(Y)$.

7.2. **Smooth rational curves on Y.** We have the following:

Proposition 7.11 (Lemma 3.1 of [16]). *The lattice S_X^- is isomorphic to $E_6(2)$, where E_6 is the negative-definite root lattice of type E_6.*

Hence the set $\mathcal{T}$ of vectors of square-norm -4 in S_X^- consists of 72 elements. By the method in Section 5.2, we calculate the set $\mathcal{R}_d$ of the classes $[C]$ of smooth rational curves C on Y with $\langle [C], h_Y \rangle_Y = d$ for $d = 1, \dots, 46$.

Proposition 7.12. *Let d be a positive integer ≤ 46. If $d \not\equiv 1 \bmod 4$, then $\mathcal{R}_d$ is empty. If $d \equiv 1 \bmod 4$, then the cardinality of $\mathcal{R}_d$ is as follows.*

d	1	5	9	13	17	21	25	29	33	37	41	45
$\lvert\mathcal{R}_d\rvert$	10	10	60	180	480	750	1440	2880	4110	5640	9480	11280

7.3. **Faces of D_Y and defining relations of** $\mathrm{aut}(Y)$**.** For the sake of readability, we will use the following notation.

$$w(\alpha) := D_Y \cap (\bar{v}_\alpha)^\perp, \quad \bar{g}(\alpha) := \bar{g}_\alpha.$$

We say that a face F of D_Y is *inner* if a general point of F is an interior point of $N(Y)$, and that F is *outer* otherwise. (An ideal face is obviously outer.) In particular, the walls $D_Y \cap (\bar{u}_\alpha)^\perp$ are outer, and $D_Y \cap (\bar{v}_\alpha)^\perp$ are inner.

Definition 7.13. Let F and F' be faces of D_Y. We put $F \sim F'$ if there exists an element $\bar{g} \in \mathrm{aut}(Y)$ such that $F' = F^{\bar{g}}$; that is, the induced chambers D_Y and $D_Y^{\bar{g}}$ share the face F' and the face F of D_Y is mapped to the face F' of $D_Y^{\bar{g}}$ by $\bar{g}$. It is obvious that $\sim$ is an equivalence relation. When $F \sim F'$, we say that F and F' are $\mathrm{aut}(Y)$-*equivalent*.

Proposition 7.14. *The numbers of faces of D_Y and their* $\mathrm{aut}(Y)$*-equivalence classes are given in Table 7.2.*

Proof. The set of faces can be calculated from the set of walls of D_Y by using Algorithm 2.3 iteratively. A face F is outer if and only if there exists an outer wall of D_Y containing F. Therefore we can make the lists of all outer faces and of all inner faces.

The set of $\mathrm{aut}(Y)$-equivalence classes of faces is calculated by the following method. We put $F \overset{a}{\sim} F'$ if there exists an inner wall $w(\alpha)$ of D_Y containing the face F such that $F = F'^{\bar{g}(\alpha)}$. (The superscript a in the symbol $\overset{a}{\sim}$ is intended to mean "adjacent".) We show that the $\mathrm{aut}(Y)$-equivalence relation $\sim$ is the smallest equivalence relation containing the relation $\overset{a}{\sim}$. Indeed, it is obvious that $F \overset{a}{\sim} F'$ implies $F \sim F'$. Suppose that $F \sim F'$, and let $\bar{g} \in \mathrm{aut}(Y)$ be an element such that $F = F'^{\bar{g}}$. Looking at the tessellation of $N(Y)$ by induced chambers locally around a general point of F, we see that there exists a sequence of induced chambers

$$D_0 = D_Y,\ D_1,\ \dots,\ D_m = D_Y^{\bar{g}}$$

with the following properties:

(a) Each D_i contains F and is contained in $N(Y)$.
(b) For $i = 1, \dots, m$, the induced chambers D_{i-1} and D_i are adjacent across a wall containing F.

Let $\bar{g}_i$ be an element of $\mathrm{aut}(Y)$ such that $D_i = D_Y^{\bar{g}_i}$. Note that $\bar{g}_i$ is unique by Corollary 7.10. Let w_i be the wall between D_{i-1} and D_i. Since both of D_{i-1} and D_i are contained in $N(X)$, there exists an inner wall $w(\alpha_i)$ of D_Y that is mapped to w_i by $\bar{g}_{i-1}$. Then we have $\bar{g}_i = \bar{g}(\alpha_i)\bar{g}_{i-1}$, and hence

$$\bar{g}_i = \bar{g}(\alpha_i) \cdots \bar{g}(\alpha_1).$$

Let F_i be the face of D_Y that is mapped to the face F of D_i by $\bar{g}_i$. Since $\bar{g} = \bar{g}_m$, we have $F_m = F'$. Since $F_{i-1} = F_i^{\bar{g}(\alpha_i)}$, we have $F_{i-1} \overset{a}{\sim} F_i$. Therefore $F_0 = F$ and $F_m = F'$ are equivalent under the minimal equivalence relation containing $\overset{a}{\sim}$.

For each face F of D_Y, we can make the finite list of all faces F' of D_Y such that $F \overset{a}{\sim} F'$. From these lists, we calculate the set of $\mathrm{aut}(Y)$-equivalence classes of faces. □

We give a description of inner faces of D_Y of codimension 2. Let $w(\alpha) := D_Y \cap (\bar{v}_\alpha)^\perp$ be an inner wall. For any $\alpha' \in A \setminus \{\alpha\}$, the space $F_{\alpha'} := w(\alpha) \cap (\bar{v}_{\alpha'})^\perp$ contains a non-empty open subset of $(\bar{v}_\alpha)^\perp \cap (\bar{v}_{\alpha'})^\perp$. Indeed, the image $\mathrm{pr}(h_Y)$ of h_Y by the orthogonal projection to $(\bar{v}_\alpha)^\perp \cap (\bar{v}_{\alpha'})^\perp$ satisfies $\langle \mathrm{pr}(h_Y), \bar{u}_{\alpha''} \rangle_Y > 0$ for all $\alpha'' \in A$ and $\langle \mathrm{pr}(h_Y), \bar{v}_{\alpha''} \rangle_Y > 0$ for all $\alpha'' \in A \setminus \{\alpha, \alpha'\}$. Therefore the inner wall $w(\alpha)$ contains exactly 9 inner faces $F_{\alpha'}$ of codimension 2. Let x be a general point of $F_{\alpha'}$. If $\langle \bar{v}_\alpha, \bar{v}_{\alpha'} \rangle_Y = 0$, then $(\bar{v}_\alpha)^\perp$ and $(\bar{v}_{\alpha'})^\perp$ intersect perpendicularly at x and hence x is contained in exactly 4 induced chambers of $N(Y)$, while if $\langle \bar{v}_\alpha, \bar{v}_{\alpha'} \rangle_Y = 1$, then $(\bar{v}_\alpha)^\perp$ and $(\bar{v}_{\alpha'})^\perp$ intersect with angle $\pi/3$ at x and hence x is contained in exactly 6 induced chambers. These induced chambers lead to the relations among $\bar{g}(\alpha)$ in Proposition 7.16 below.

Proposition 7.15. *Let F be a non-ideal face of D_Y. Then the set*

$$\mathcal{G}(F) := \{\ \bar{g} \in \mathrm{aut}(Y)\ \mid\ F \subset D_Y^{\bar{g}}\ \}$$

is finite, and can be calculated explicitly.

Proof. Note that the family of hyperplanes (5.2) that gives the tessellation of $\mathcal{P}_Y$ by induced chambers is locally finite in $\mathcal{P}_Y$. Since F is not an ideal face, the number of induced chambers containing F is finite. Hence $\mathcal{G}(F)$ is finite.

The set $\mathcal{G}(F)$ can be calculated as follows. We set `G` $:= \{\mathrm{id}\}$, where id is the identity of $\mathrm{aut}(Y)$. Let `f` be the procedure that takes an element $\bar{g}$ of $\mathcal{G}(F)$ as an input and carries out the following task:

(a) Let F' be the face of D_Y such that $F'^{\bar{g}}$ is equal to F. We calculates the set $\{w(\alpha_1), \dots, w(\alpha_k)\}$ of inner walls of D_Y that contain F'.
(b) For each $j = 1, \dots, k$, we calculate $\bar{g}' := \bar{g}(\alpha_j)\bar{g}$, which is an element of $\mathcal{G}(F)$, and if $\bar{g}'$ is not yet in the set `G`, we add $\bar{g}'$ in `G` and input $\bar{g}'$ to the procedure `f`.

We input id to the procedure `f`. It is easy to see that, when the whole procedure terminates, the set `G` is equal to the set $\mathcal{G}(F)$. □

Proposition 7.16. *The following relations form a set of defining relations of* $\mathrm{aut}(Y)$ *with respect to the set of generators* $\{\bar{g}(\alpha) \mid \alpha \in A\}$*;*

$$\bar{g}(\alpha)^2 = \mathrm{id}$$

for each $\alpha \in A$*,*

$$(\,\bar{g}(\alpha)\,\bar{g}(\alpha')\,\bar{g}(\alpha'')\,)^2 = \mathrm{id}$$

for each triple $(\alpha, \alpha', \alpha'')$ *of distinct elements of* A *such that* $|\alpha \cap \alpha' \cap \alpha''| = 2$*, and*

$$(\,\bar{g}(\alpha)\,\bar{g}(\alpha')\,)^2 = \mathrm{id}$$

for each pair (α, α') *such that* $|\alpha \cap \alpha'| = 1$*.*

Proof. By the standard argument of the geometric group theory, there exists a one-to-one correspondence between a set of defining relations except for $\bar{g}(\alpha)^2 = \mathrm{id}$ and the set of 8-dimensional inner faces of D_Y. Let F be an 8-dimensional inner face of D_Y. Then there exist exactly two walls of D_Y containing F, and they are both inner. We put $D_0 := D_Y$, and choose an induced chamber D_1 from the two induced chambers that contain F and are adjacent to D_0. Then there exists a cyclic sequence

$$D_0, D_1, \dots, D_{m-1}, D_m = D_0$$

of induced chambers in $N(Y)$ with the following properties:

(a) For each $i \in \mathbb{Z}/m\mathbb{Z}$, D_{i-1} and D_{i+1} are the two induced chambers that contain F and are adjacent to D_i.
(b) If $i, j \in \mathbb{Z}/m\mathbb{Z}$ are distinct, then D_i and D_j are distinct.

We calculate the sequence of inner walls $w(\alpha_1), \dots, w(\alpha_m)$ of D_Y such that

$$D_i = D_Y^{\bar{g}(\alpha_i)\cdots\bar{g}(\alpha_1)}.$$

Then we have $\bar{g}(\alpha_m)\cdots\bar{g}(\alpha_1) = \mathrm{id}$, and this is the defining relation corresponding to the inner face F. The cycle $D_0, D_1, \dots, D_m$ for each F can be computed by Proposition 7.15. Thus we obtain a list of defining relations from the list of inner faces of D_Y that we have calculated in Proposition 7.14.

The $45 = 10 \times 3 + 15$ inner faces of dimension 8 are decomposed into $25 = 10 + 15$ $\mathrm{aut}(Y)$-equivalence classes. We see that, if $w(\alpha)$ and $w(\alpha')$ are distinct inner walls, then $w(\alpha) \cap w(\alpha')$ is an 8-dimensional inner face. For each $\beta \in B$, there exist exactly three elements $\alpha, \alpha', \alpha'' \in A$ that contain β, and the three 8-dimensional inner faces $w(\alpha) \cap w(\alpha')$, $w(\alpha') \cap w(\alpha'')$ and $w(\alpha'') \cap w(\alpha)$ form an $\mathrm{aut}(Y)$-equivalence class. The face $w(\alpha) \cap w(\alpha')$ corresponds to the relation

$$\bar{g}(\alpha)\,\bar{g}(\alpha'')\,\bar{g}(\alpha')\,\bar{g}(\alpha)\,\bar{g}(\alpha'')\,\bar{g}(\alpha') \;=\; \mathrm{id}.$$

For each $i \in \{1, \dots, 5\}$, there exist exactly three pairs $\{\alpha_\nu, \alpha'_\nu\}$ $(\nu = 1, 2, 3)$ of elements of A such that $\alpha_\nu \cap \alpha'_\nu = \{i\}$. For each pair $\{\alpha_\nu, \alpha'_\nu\}$, the 8-dimensional inner face $w(\alpha_\nu) \cap w(\alpha'_\nu)$ forms an $\mathrm{aut}(Y)$-equivalence class consisting of only one element. The face $w(\alpha_\nu) \cap w(\alpha'_\nu)$ corresponds to the relation

$$\bar{g}(\alpha_\nu)\,\bar{g}(\alpha'_\nu)\,\bar{g}(\alpha_\nu)\,\bar{g}(\alpha'_\nu) \;=\; \mathrm{id}.$$

Thus Proposition 7.16 is proved. □

7.4. **Proof of Theorem 1.1.** In [20], Enriques surfaces Z with automorphisms that act on S_Z trivially are classified. (See also [14] and [19].) It follows that the action of $\mathrm{Aut}(Y)$ on S_Y is faithful. Then Theorem 1.1 follows immediately from Corollary 7.10 and Proposition 7.16.

7.5. **Elliptic fibrations of Y.** We prove Theorem 1.5. The $657 = 57 + 600$ one-dimensional faces of D_Y are divided into $44 = 21 + 23$ $\mathrm{aut}(Y)$-equivalence classes. Among them, there exist exactly 57 ideal faces, and they are divided into 21 $\mathrm{aut}(Y)$-equivalence classes.

Let $\phi\colon Y \to \mathbb{P}^1$ be an elliptic fibration, and let $2f_\phi \in S_Y$ denote the class of a fiber of ϕ. There exists an isometry $g \in \mathrm{aut}(Y)$ that maps f_ϕ in an ideal face of D_Y. Conversely, let F be an ideal face of D_Y, and let $f \in S_Y$ be the primitive vector such that $F = \mathbb{R}_{\geq 0} f$. Since f is nef and satisfies $\langle f, f\rangle_Y = 0$, there exists an elliptic fibration $\phi\colon Y \to \mathbb{P}^1$ such that $f = f_\phi$. Therefore there exists a bijection between the set of elliptic fibrations modulo the action of $\mathrm{Aut}(Y)$ and the set of $\mathrm{aut}(Y)$-equivalence classes of ideal faces of D_Y.

For each ideal face $F = \mathbb{R}_{\geq 0} f$ with $f \in S_Y$ primitive, the ADE-type of reducible fibers of the corresponding elliptic fibration can be calculated from f, the ample class h_Y, and the sets $\mathcal{R}_d$ calculated in Proposition 7.12 by the method described in Section 5.3. Thus we obtain Table 1.1 and hence Theorem 1.5 is proved.

7.6. RDP-**configurations on Y.** We prove Theorem 1.6. Let $\psi\colon Y \to \overline{Y}$ be a birational morphism to a surface $\overline{Y}$ that has only rational double points as its singularities, and let $h_\psi \in S_Y$ be the pullback of the class of a hyperplane section of $\overline{Y}$. Composing ψ with an automorphism of Y, we assume that $h_\psi \in D_Y$. We see that the set

$$\mathcal{R}(\psi) := \{\, [C] \in S_Y \;\mid\; C \text{ is a smooth rational curve on } Y \text{ contracted by } \psi \,\}$$

can be calculated from the face F of D_Y that contains h_ψ in its interior. Indeed, since $\langle h_\psi, h_\psi\rangle_Y > 0$, the face F is not an ideal face, and hence we can calculate the set $\mathcal{G}(F)$ defined in Proposition 7.15. Then $\mathcal{R}(\psi)$ is equal to

$$\left\{ \;\bar{u}_\alpha^{\bar{g}} \;\middle|\; \begin{array}{l} \bar{g} \text{ is an element of } \mathcal{G}(F) \text{ such that the wall} \\ (D_Y \cap (\bar{u}_\alpha)^\perp)^{\bar{g}} \text{ of } D_Y^{\bar{g}} \text{ contains } F. \end{array} \right\}$$

Therefore we write $\mathcal{R}(\psi)$ as $\mathcal{R}(F)$. Conversely, let F be a non-ideal face of D_Y, and let h_F be an element of $F \cap S_Y$ that is not contained in any wall of F. Multiplying h_F by a positive integer if necessary, we can assume that the line bundle $L_F \to Y$ corresponding to h_F is globally generated and defines a morphism $\Phi_{|L_F|}\colon Y \to \mathbb{P}^m$. Let

$$Y \;\xrightarrow{\psi}\; \overline{Y} \;\longrightarrow\; \mathbb{P}^m$$

be the Stein factorization of $\Phi_{|L_F|}$. Then we have $\mathcal{R}(\psi) = \mathcal{R}(F)$.

We calculate $\mathcal{R}(F)$ for all non-ideal faces of D_Y. For two non-ideal faces F and F', we put $F \leq F'$ if F is a face of F' and $\mathcal{R}(F) = \mathcal{R}(F')$ holds. In the previous version of this paper, we look at the maximal faces with respect to this partial ordering, and divide them into $\mathrm{aut}(Y)$-equivalence classes. In order to obtain $\mathrm{Aut}(Y)$-equivalence classes of RDP-configurations, however, we have to devide these maximal faces by a coarser equivalence relation. For this new equivalence relation, see [31].

7.7. **Vinberg chambers in D_Y.** In this subsection, we identify S_Y with L_{10} by the isometry $L_{10} \xrightarrow{\sim} S_Y$ given by (6.5). In particular, the chamber D_Y is contained in $\mathcal{P}_{10}$. Note that the primitive defining vectors $\bar{u}_\alpha$ and $\bar{v}_\alpha$ of walls of D_Y are roots (see (7.1)), and hence D_Y is tessellated by Vinberg chambers.

Proof of Theorem 1.8. Let V_0 denote the Vinberg chamber D_{10} defined in Section 3.1. We put $\varepsilon_i(V_0) := e_i$ for $i = 1, \dots, 10$, and let $m_i(V_0)$ denote the wall $V_0 \cap (\varepsilon_i(V_0))^\perp$ of V_0. Then, for each $i = 1, \dots, 9$, there exists a unique non-ideal one-dimensional face $F_i(V_0)$ of V_0 that is not contained in the wall $m_i(V_0)$. We denote by $f_i(V_0) \in L_{10}$ the primitive vector such that $F_i(V_0) = \mathbb{R}_{\ge 0} f_i(V_0)$. Then we have $f_1(V_0) = h_Y$ under the identification $L_{10} = S_Y$. (In fact, we have chosen the isomorphism (6.5) in such a way that $f_1(V_0) = h_Y$ holds.)

Let V be an arbitrary Vinberg chamber in $\mathcal{P}_{10}$. Since the automorphism group $\mathrm{aut}(V_0)$ of V_0 is trivial, there exists a unique isometry $\bar{g}(V) \in \mathrm{O}^+(L_{10})$ that maps V_0 to V. We put

$$\varepsilon_i(V) := \varepsilon_i(V_0)^{\bar{g}(V)}, \ \ m_i(V) := m_i(V_0)^{\bar{g}(V)}, \ \ f_i(V) := f_i(V_0)^{\bar{g}(V)}.$$

We say that a primitive vector v of L_{10} is an *f_1-vector* if $v = f_1(V)$ for some Vinberg chamber V. Let $v = f_1(V)$ be an f_1-vector. We put

$$S(v) := \{ \ V' \ \mid \ V' \text{ is a Vinberg chamber such that } v = f_1(V') \ \}.$$

Since the defining roots $\varepsilon_2(V), \dots, \varepsilon_{10}(V)$ of the walls $m_2(V), \dots, m_{10}(V)$ of V containing $f_1(V)$ form a Dynkin diagram of type A_9, the cardinality of $S(v)$ is equal to $|\mathfrak{S}_{10}|$. We then put

$$\Sigma(v) := \bigcup_{V' \in S(v)} V',$$

and call it a Σ-*chamber with the center* v. It is obvious that $\mathcal{P}_{10}$ is tessellated by Σ-chambers. The defining roots $\varepsilon_2(V), \varepsilon_3(V), \varepsilon_5(V), \dots, \varepsilon_{10}(V)$ of the walls of V that contain $f_1(V)$ and are perpendicular to the wall $m_1(V)$ opposite to $f_1(V)$ form a Dynkin diagram of type $A_2 + A_6$. Hence there exist exactly $|\mathfrak{S}_3 \times \mathfrak{S}_7|$ Vinberg chambers V' in $S(v)$ such that $m_1(V)$ and $m_1(V')$ are supported on the same hyperplane. Hence the number of walls of the chamber $\Sigma(v)$ is

$$\frac{|\mathfrak{S}_{10}|}{|\mathfrak{S}_3 \times \mathfrak{S}_7|} = \frac{10!}{3! \times 7!} = 120.$$

In particular, we see that the number of Σ-chambers that are adjacent to the Σ-chamber $\Sigma(v)$ is 120. Moreover, we can calculate the list $\{v_1, \dots, v_{120}\}$ of centers of these adjacent Σ-chambers.

Let v be an f_1-vector. If v belongs to the interior of D_Y, then all 10! Vinberg chambers contained in $\Sigma(v)$ are contained in D_Y. If v does not belong to D_Y, then none of Vinberg chambers in $\Sigma(v)$ are contained in D_Y. Suppose that v is located on the boundary of D_Y. We calculate the Dynkin diagram $\mathit{\Delta}$ formed by the roots

$$\{\bar{u}_\alpha \,|\, \langle \bar{u}_\alpha, v\rangle_Y = 0\} \ \cup \ \{\bar{v}_\alpha \,|\, \langle \bar{v}_\alpha, v\rangle_Y = 0\}$$

that define walls of D_Y containing v. This Dynkin diagram Δ is a sub-diagram of the Dynkin diagram of type A_9. Let $W(\Delta) \subset \mathfrak{S}_{10}$ denote the corresponding subgroup. Then, among the 10! Vinberg chambers in $\Sigma(v)$, exactly $|\mathfrak{S}_{10}|/|W(\Delta)|$ Vinberg chambers are contained in D_Y.

Starting from the f_1-vector h_Y, we cover D_Y by Σ-chambers, and count the number of Vinberg chambers in D_Y. By this method, Theorem 1.8 is proved. $\square$

Recall that Z_{gen} is a generic Enriques surface. The natural representation $\mathrm{Aut}(Z_{\mathrm{gen}}) \to \mathrm{O}^+(L_{10})$ is injective by [2]. Let $\mathrm{aut}(Z_{\mathrm{gen}})$ denote the image of this homomorphism. Let $\mathrm{aut}'(Y)$ denote the subgroup of $\mathrm{O}^+(L_{10})$ generated by $\mathrm{aut}(Y)$ and the ten reflections $\bar{\sigma}_\alpha$ with respect to the roots $\bar{u}_\alpha \in S_Y = L_{10}$.

Theorem 7.17. (1) *The group* $\mathrm{aut}'(Y)$ *contains* $\mathrm{aut}(Z_{\mathrm{gen}})$ *as a normal subgroup, and* $\mathrm{aut}'(Y)/\mathrm{aut}(Z_{\mathrm{gen}})$ *is isomorphic to the Weyl group* $W(E_6)$ *of type* E_6. (2) *The induced chamber* D_Y *is a fundamental domain of the action of* $\mathrm{aut}'(Y)$ *on* $\mathcal{P}_Y$.

Proof. Recall that the natural homomorphism $\rho\colon \mathrm{O}^+(L_{10}) \to \mathrm{O}(q_{L_{10}(2)}) \cong \mathrm{GO}^+_{10}(2)$ is surjective. By [2], we know that $\mathrm{aut}(Z_{\mathrm{gen}})$ is equal to $\mathrm{Ker}\,\rho$, and hence

$$[\,\mathrm{O}^+(L_{10}) : \mathrm{aut}(Z_{\mathrm{gen}})\,] = 46998591897600 = 51840 \cdot 906608640.$$

By the brute force method using [10], we see that the order of the subgroup of $\mathrm{GO}^+_{10}(2)$ generated by $10+10$ elements $\rho(\bar{\sigma}_\alpha)$ and $\rho(\bar{g}_\alpha)$ is 51840. Hence we have

$$[\,\mathrm{aut}'(Y) : \mathrm{aut}'(Y) \cap \mathrm{aut}(Z_{\mathrm{gen}})\,] = 51840.$$

On the other hand, by Propositions 7.4, 7.8, and Theorem 1.8, we have

$$[\,\mathrm{O}^+(L_{10}) : \mathrm{aut}'(Y)\,] = \frac{906608640}{|\,\mathrm{aut}'(Y) \cap \mathrm{aut}(D_Y)\,|}.$$

Hence we obtain $|\,\mathrm{aut}'(Y) \cap \mathrm{aut}(D_Y)\,| = 1$, which implies the assertion (2). Moreover, we have $\mathrm{aut}(Z_{\mathrm{gen}}) \lhd \mathrm{aut}'(Y)$ and $|\mathrm{aut}'(Y)/\mathrm{aut}(Z_{\mathrm{gen}})| = 51840$.

In the following, we denote by $D(L)$ the discriminant group $L^\vee/L$ of an even lattice L, by $q(L)$ the discriminant form of L, and by $\eta(L)\colon \mathrm{O}(L) \to \mathrm{O}(q(L))$ the natural homomorphism. We have $|D(S_X)| = 2^4 \cdot 3$, $|D(S_X^+)| = 2^{10}$, and by Proposition 7.11, $|D(S_X^-)| = 2^6 \cdot 3$. By [22], the even overlattice S_X of $S_X^+ \oplus S_X^-$ defines an isotropic subgroup

$$H_S := S_X/(S_X^+ \oplus S_X^-) \subset D(S_X^+) \oplus D(S_X^-)$$

such that $H_S \cap D(S_X^+) = 0$, $H_S \cap D(S_X^-) = 0$, and $|H_S^\perp/H_S| = 2^4 \cdot 3$. Hence we have $|H_S| = 2^6$. Let $(D(S_X^-)_2, q(S_X^-)_2)$ denote the 2-part of

$$(D(S_X^-), q(S_X^-)) \cong (D(E_6(2)), q(E_6(2))),$$

which can be regarded as a quadratic form of Witt defect 1 on $\mathbb{F}_2^6$. The automorphism group $\mathrm{GO}_6^-(2)$ of this quadratic form is isomorphic to $W(E_6)$. (See page 26 of [1].) Since $|H_S| = |D(S_X^-)_2| = 2^6$, the second projection induces an isomorphism $\gamma_H\colon H_S \xrightarrow{\sim} D(S_X^-)_2$, and the composite of γ_H^{-1} and the first projection defines an embedding

$$\gamma\,:\,(D(E_6(2)), q(E_6(2))) \cong (D(S_X^-)_2, q(S_X^-)_2) \;\hookrightarrow\; (D(S_X^+), q(S_X^+)).$$

Note that the image H of γ is equal to the image of the natural homomorphism $S_X \to S_X^{+\vee} \to D(S_X^+)$. Let q_H denote the restriction of $q(S_X^+)$ to H, and $q_{H^\perp}$ the restriction to the orthogonal complement $H^\perp$ of H in $(D(S_X^+), q(S_X^+))$. Since q_H

is isomorphic to $-q(S_X^-)_2$, we have $\mathrm{O}(q_H) \cong \mathrm{GO}_6^-(2) \cong W(E_6)$. We consider the homomorphism

$$\rho' : \mathrm{aut}'(Y) \hookrightarrow \mathrm{O}^+(S_Y) \cong \mathrm{O}^+(S_X^+) \xrightarrow{\eta(S_X^+)} \mathrm{O}(q(S_X^+)).$$

Since the homomorphism $\mathrm{O}^+(S_Y) \cong \mathrm{O}^+(S_X^+) \to \mathrm{O}(q(S_X^+))$ is identified with ρ, the homomorphism ρ' embeds $\mathrm{aut}'(Y)/\mathrm{aut}(Z_{\mathrm{gen}})$ into $\mathrm{O}(q(S_X^+))$. Every element of the image of

$$\mathrm{aut}'(Y) \hookrightarrow \mathrm{O}^+(S_Y) \cong \mathrm{O}^+(S_X^+)$$

lifts to an element of $\mathrm{O}(S_X)$. Indeed, $\bar{g}_\alpha \in \mathrm{aut}'(Y)$ lifts to $g_\alpha \in \mathrm{O}(S_X)$, and $\bar{\sigma}_\alpha \in \mathrm{aut}'(Y)$ lifts to $\sigma_\alpha\sigma_{\bar{\alpha}} \in \mathrm{O}(S_X)$ (see the proof of Proposition 7.8). Hence every element in the image of ρ' preserves the factors H and $H^\perp$ of $D(S_X^+) = H \oplus H^\perp$. By direct computation, we see that $\rho'(\bar{g}_\alpha)$ and $\rho'(\bar{\sigma}_\alpha)$ act on $H^\perp$ trivially. Therefore $\mathrm{aut}'(Y)/\mathrm{aut}(Z_{\mathrm{gen}})$ can be regarded as a subgroup of $\mathrm{O}(q_H)$. Comparing the order, we obtain $\mathrm{aut}'(Y)/\mathrm{aut}(Z_{\mathrm{gen}}) \cong \mathrm{O}(q_H) \cong W(E_6)$. □

Remark 7.18. Note that the lift $\sigma_\alpha\sigma_{\bar{\alpha}} \in \mathrm{O}^+(S_X)$ of $\bar{\sigma}_\alpha \in \mathrm{aut}'(Y)$ satisfies the period condition $\eta_{S_X}(\sigma_\alpha\sigma_{\bar{\alpha}}) \in \{\pm 1\}$ (see Proposition 6.1). By the specialization of Z_{gen} to Y, the period condition is weakened and $\mathrm{aut}(Z_{\mathrm{gen}})$ becomes the larger group $\mathrm{aut}'(Y)$ with $10 + 10$ generators $\bar{g}_\alpha$ and $\bar{\sigma}_\alpha$. The presence of smooth rational curves on Y, however, prevents the 10 generators $\bar{\sigma}_\alpha$ from entering into $\mathrm{aut}(Y)$.

8. Entropy

Recently, many works have appeared on the distribution of entropies $\log \lambda(g)$ of automorphisms g of $K3$ or Enriques surfaces, where $\lambda(g)$ is the spectral radius of the action of g on the Néron-Severi lattice of the surface. In particular, the problem to determine the minimum of the positive entropies in a certain class of automorphisms has been studied, for example, in [18], [24] and [8] . In this section, we report the result of a computational experiment on the entropies of automorphisms of Z_{gen}.

By the result of [2] and Theorem 7.17 above, we have the following equalities:

$$\begin{aligned} \mathrm{aut}(Z_{\mathrm{gen}}) &= \mathrm{Ker}(\rho \colon \mathrm{O}^+(L_{10}) \twoheadrightarrow \mathrm{GO}_{10}^+(2)) \\ &= \mathrm{Ker}(\rho|_{\mathrm{aut}'(Y)} \colon \mathrm{aut}'(Y) \twoheadrightarrow \mathrm{GO}_6^-(2)). \end{aligned}$$

Since we know finite sets of generators for $\mathrm{O}^+(L_{10})$ and for $\mathrm{aut}'(Y)$, we can obtain a finite set of generators of $\mathrm{aut}(Z_{\mathrm{gen}})$ by the Reidemeister-Schreier method (see Chapter 2 of [17]) from each of these descriptions of $\mathrm{aut}(Z_{\mathrm{gen}})$. Since $|\mathrm{GO}_{10}^+(2)|$ is very large, however, making use of the first equality is not practical. On the other hand, since $|\mathrm{GO}_6^-(2)|$ is much smaller compared with $|\mathrm{GO}_{10}^+(2)|$, we have managed to obtain a finite set of generators of $\mathrm{aut}(Z_{\mathrm{gen}})$ in a reasonable computation time by means of the second equality.

Using this generating set, we search for elements $g \in \mathrm{aut}(Z_{\mathrm{gen}})$ with small $\lambda(g)$ for each degree $d = 2, 4, \dots, 10$ of the minimal polynomial $s_{\lambda(g)}$ of the Salem number $\lambda(g)$. Below is the list of the smallest values of $\lambda(g)$ among the ones we found by an extensive random search of elements of $\mathrm{aut}(Z_{\mathrm{gen}})$. See [32] for the matrices g

with these spectral radii.

d	$s_{\lambda(g)}$	$\lambda(g)$
2	$t^2 - 14\,t + \dots$	$13.9282\dots$
4	$t^4 - 16\,t^3 + 14\,t^2 - \dots$	$15.1450\dots$
6	$t^6 - 38\,t^5 - 49\,t^4 - 84\,t^3 - \dots$	$39.3019\dots$
8	$t^8 - 68\,t^7 + 68\,t^6 - 188\,t^5 + 118\,t^4 - \dots$	$67.0269\dots$
10	$t^{10} - 138\,t^9 - 19\,t^8 - 248\,t^7 + 18\,t^6 - 252\,t^5 + \dots$	$138.1505\dots$

Remark 8.1. The famous Lehmer's number $\lambda_{\mathrm{Leh}} = 1.17628\dots$ is the spectral radius of a Coxeter element c of $\mathrm{O}^+(L_{10}) = W(L_{10})$. The order of $\rho(c) \in \mathrm{GO}^+_{10}(2)$ is 31, and the spectral radius of $c^{31} \in \mathrm{aut}(Z_{\mathrm{gen}})$ is equal to $\lambda_{\mathrm{Leh}}^{31} = 153.4056\dots$.

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