ON KUMMER TYPE CONSTRUCTION OF SUPERSINGULAR $K3$ SURFACES IN CHARACTERISTIC 2

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Abstract. We show that every supersingular $K3$ surface in characteristic 2 with Artin invariant ≤ 2 is obtained by the Kummer type construction of Schröer.

1. Introduction

We work over an algebraically closed field k. A $K3$ surface X is called *supersingular* (in the sense of Shioda) if the rank of the Néron-Severi lattice $\mathrm{NS}(X)$ of X attains the possible maximum 22. Supersingular $K3$ surfaces exist only when $\operatorname{char} k$ is positive. The Artin invariant $\sigma(X)$ of a supersingular $K3$ surface X is defined in [3] by

$$\operatorname{disc} \mathrm{NS}(X) = -p^{2\sigma(X)},$$

where $p = \operatorname{char} k > 0$. It is known that $\sigma(X)$ is a positive integer ≤ 10.

Let A be an abelian surface, and let $\iota : A \to A$ be the involution $x \mapsto -x$. If $\operatorname{char} k \neq 2$, then the minimal resolution of the quotient surface $A/\langle \iota \rangle$ is a $K3$ surface, which is called the *Kummer surface associated with* A.

An abelian surface A in positive characteristic is called *supersingular* if A is isogenous to a product of supersingular elliptic curves. Ogus [12, 13] proved that, if $\operatorname{char} k > 2$, the supersingular $K3$ surfaces with Artin invariant ≤ 2 are exactly the Kummer surfaces associated with supersingular abelian surfaces. (See also Shioda [22].) On the other hand, Shioda [23] and Katsura [10] observed that, if $\operatorname{char} k = 2$, then the minimal resolution of the quotient of a supersingular abelian surface by the involution $x \mapsto -x$ is a rational surface.

In [17], Schröer presented a Kummer type construction of supersingular $K3$ surfaces in characteristic 2. We assume that $\operatorname{char} k = 2$ in this paragraph. Let $C \times C$ be the self-product of the rational curve C with one ordinary cusp. We put

$$\begin{array}{rcll} C & = & \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}] & \text{for the first factor, and} \\ C & = & \operatorname{Spec} k[v^2, v^3] \cup \operatorname{Spec} k[v^{-1}] & \text{for the second factor.} \end{array}$$

Let r and s be constants in k such that $(r, s) \neq (0, 0)$. Then the derivation

$$(u^{-2} + r)\frac{\partial}{\partial u} + (v^{-2} + s)\frac{\partial}{\partial v} \tag{1.1}$$

defines a global vector field δ on $C \times C$ satisfying $\delta^{[2]} = 0$. Hence δ corresponds to an action of the infinitesimal group scheme α_2 on $C \times C$. Let $X_{r,s}$ be the minimal resolution of the quotient surface $(C \times C)/\alpha_2$.

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Theorem 1.1 ([17])**.** *The surface $X_{r,s}$ is a supersingular $K3$ surface with Artin invariant*

$$\sigma(X_{r,s}) = \begin{cases} 1 & \text{if } r = 0 \text{ or } s = 0 \text{ or } r^3 = s^3, \\ 2 & \text{otherwise.} \end{cases}$$

The purpose of this paper is to prove the following:

Theorem 1.2. *Let X' be a supersingular $K3$ surface in characteristic 2 with Artin invariant ≤ 2. Then there exist constants $r, s \in k$ with $(r,s) \neq (0,0)$ such that X' is isomorphic to Schröer's Kummer surface $X_{r,s}$.*

Even though the moduli curve of marked supersingular $K3$ surfaces with Artin invariant ≤ 2 is constructed ([13, 15]), it is not separated. Hence the existence of the complete family of Schröer's Kummer surfaces of dimension 1 does not imply Theorem 1.2 immediately.

The main ingredient of the proof is the following structure theorem for Néron-Severi lattices of supersingular $K3$ surfaces due to Rudakov and Shafarevich [15]:

Theorem 1.3. *Let X and X' be supersingular $K3$ surfaces defined over the same algebraically closed field. If $\sigma(X) = \sigma(X')$, then the lattices $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic.*

Indeed, the Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular $K3$ surface X in characteristic p is p-elementary ([15, Theorem in Section 8], see also [3]). If $p = 2$, then $\mathrm{NS}(X)$ is of type I ([15, Proposition in Section 5]). Hence the classification theorem of even hyperbolic p-elementary lattices ([15, Theorem in Section 1]) implies Theorem 1.3.

The outline of the proof of Theorem 1.2 is as follows. First note that, by [17, Proposition 6.2], if $\sigma(X_{r,s}) = 2$, then Schröer's Kummer surface $X_{r,s}$ is birational to a purely inseparable double cover $Y_{r,s}$ of $\mathbb{P}^2$ defined by

$$w^2 = x(y^4 + s^2 y^2) + y(x^4 + r^2 x^2),$$

which has rational double points of type $4D_4 + 5A_1$. Let us assume, for simplicity, that the given supersingular $K3$ surface X' is of Artin invariant 2. We choose one of Schröer's Kummer surfaces X with Artin invariant 2 (for example, we put $X := X_{1,s}$ with $s \notin \mathbb{F}_4$). Using the isomorphism between $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$, we can show that X' is also birational to a double cover Y' of $\mathbb{P}^2$ with rational double points of type $4D_4 + 5A_1$. By means of the notion of *half-lines* and *splitting lines*, we can show that the covering morphism $Y' \to \mathbb{P}^2$ is purely inseparable, and then we can determine the defining equation of Y'. It turns out that the defining equation of Y' is equal to that of $Y_{t,1}$ for some non-zero constant $t \in k$. Therefore X' is isomorphic to Schröer's Kummer surface $X_{t,1}$.

A surface birational to a purely inseparable cover of $\mathbb{P}^2$ is called a *Zariski surface*, and its basic properties have been studied in [5]. In [18] and [19], we showed that every supersingular $K3$ surface in characteristic 2 is birational to a purely inseparable double cover of $\mathbb{P}^2$ with 21 ordinary nodes, and studied the Néron-Severi lattice of such a surface. Using the results obtained in [19], we have determined in [20] the moduli curve of polarized supersingular $K3$ surfaces with Artin invariant ≤ 2 and with 21 ordinary nodes. In [17], Schröer showed that, as r and s varies, his Kummer surfaces $X_{r,s}$ form a smooth family over the projective line $\operatorname{Proj} k[\sqrt{r}, \sqrt{s}]$.

It would be an interesting problem to investigate the relation between the moduli curve in [20] and Schröer's projective line.

On the other hand, in [21], we investigated supersingular $K3$ surfaces with 10 ordinary cusps. Such supersingular $K3$ surfaces exist only in characteristic 3. An example is obtained as a purely inseparable *triple* cover of $\mathbb{P}^1 \times \mathbb{P}^1$. The proof in the present article of the fact that $Y' \to \mathbb{P}^2$ is purely inseparable uses an argument developed in [21].

The plan of this paper is as follows. In §2, we collect from the lattice theory some definitions and facts that will be used in this paper. The very elementary Lemmas 2.4 and 2.5 play an important role in the proof of the fact that Y' is purely inseparable over $\mathbb{P}^2$. In §3, we review some properties of the Néron-Severi lattice of a $K3$ surface. We then introduce the notion of half-lines and splitting lines for a polarized $K3$ surface of degree 2 in §4. After investigating the purely inseparable double cover $Y_{r,s} \to \mathbb{P}^2$ birational to Schröer's Kummer surface $X_{r,s}$, we prove Theorem 1.2 in §6.

2. Preliminaries on lattices

A free $\mathbb{Z}$-module Λ of finite rank with a non-degenerate symmetric bilinear form

$$\Lambda \times \Lambda \to \mathbb{Z} \tag{2.1}$$

denoted by $(u, v) \mapsto uv$ is called a *lattice*. Let Λ be a lattice. The *dual lattice* $\Lambda^\vee$ of Λ is the $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda, \mathbb{Z})$. Then Λ is naturally embedded into $\Lambda^\vee$ as a submodule of finite index. The *discriminant group* of Λ is, by definition, the finite abelian group $\Lambda^\vee/\Lambda$. There exists a unique symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q} \tag{2.2}$$

that extends (2.1). An *overlattice* of Λ is a submodule N of $\Lambda^\vee$ containing Λ such that the bilinear form (2.2) takes values in $\mathbb{Z}$ on $N \times N$. If Λ is a sublattice of a lattice Λ' with finite index, then Λ' is embedded into $\Lambda^\vee$ in a natural way, and hence is regarded as an overlattice of Λ.

We say that Λ is *even* if $u^2 \in 2\mathbb{Z}$ holds for every $u \in \Lambda$. The signature (s_+, s_-) of a lattice Λ is the numbers of positive and negative eigenvalues of the intersection matrix of Λ. We say that Λ is *negative-definite* if $s_+ = 0$, and that Λ is *hyperbolic* if $s_+ = 1$. By abuse of language, a positive definite lattice of rank 1 is also called hyperbolic.

Let Λ be an even negative-definite lattice. A vector $r \in \Lambda$ is called a *root* if $r^2 = -2$. We denote by $\mathrm{Roots}(\Lambda)$ the set of roots in Λ. We define an equivalence relation $\sim$ on $\mathrm{Roots}(\Lambda)$ by the following: $r \sim r'$ if there exists a sequence $r_0 = r, r_1, \dots, r_{m-1}, r_m = r'$ of roots in Λ such that $r_i r_{i+1} \neq 0$ for $i = 0, \dots, m-1$. Let $R_1, \dots, R_k$ be the equivalence classes of $\sim$. We call the decomposition

$$\mathrm{Roots}(\Lambda) = R_1 \sqcup \cdots \sqcup R_k$$

the *irreducible decomposition* of $\mathrm{Roots}(\Lambda)$. Suppose that we are given a linear form

$$\alpha : \Lambda \to \mathbb{R}$$

such that $\alpha(r) \neq 0$ for any $r \in \mathrm{Roots}(\Lambda)$. We put

$$R_i^+ := \{\, r \in R_i \mid \alpha(r) > 0 \,\}. \tag{2.3}$$

A root $r \in R_i^+$ is called *decomposable* if there exist $r_1, r_2 \in R_i^+$ such that $r = r_1 + r_2$, and r is called *indecomposable* if it is not decomposable. For the proof of the following results, see [8] or [6], for example.

Proposition 2.1. *Let r be an element of R_i^+ such that $\alpha(r) > 0$. Then r can be written in a unique way as a linear combination of indecomposable elements of R_i^+. Moreover the coefficients are all non-negative integers.*

Proposition 2.2. *Let Λ_i be the sublattice of Λ generated by the roots in R_i. Then $\Lambda_1, \dots, \Lambda_k$ form an orthogonal direct sum in Λ. The indecomposable elements of R_i^+ form a basis of the lattice Λ_i, and the intersection matrix of Λ_i with respect to this basis is a Cartan matrix of type ADE multiplied by -1.*

The indecomposable elements of R_i^+ have the following characterization:

Corollary 2.3. *Let $\varepsilon_1, \dots, \varepsilon_d$ be elements of R_i^+ such that every element of R_i^+ is written as a linear combination of $\varepsilon_1, \dots, \varepsilon_d$ with non-negative integer coefficients in a unique way. Then $\{\varepsilon_1, \dots, \varepsilon_d\}$ is equal to the set of indecomposable elements of R_i^+.*

Proof. Suppose that ε_i is decomposable. There exist $r_1, r_2 \in R_i^+$ such that $\varepsilon_i = r_1 + r_2$. Since each of r_1 and r_2 is written as a linear combination of $\varepsilon_1, \dots, \varepsilon_d$ with non-negative integer coefficients, we obtain a contradiction to the uniqueness of the way to write ε_i as a linear combination of $\varepsilon_1, \dots, \varepsilon_d$ with non-negative integer coefficients. Therefore each of $\varepsilon_1, \dots, \varepsilon_d$ is indecomposable.

Suppose that $r \in R_i^+$ is indecomposable. We can write r as a linear combination of $\varepsilon_1, \dots, \varepsilon_d$ with non-negative integer coefficients. Since each ε_i is indecomposable, the uniqueness of the way to write r as a linear combination of indecomposable elements of R_i^+ with non-negative integer coefficients implies that r is equal to one of $\varepsilon_1, \dots, \varepsilon_d$. $\square$

Let τ_i be the ADE-type of the Cartan matrix of the intersection matrix of Λ_i given in Proposition 2.2. We define the *root type of* Λ to be the formal sum $\tau_1 + \cdots + \tau_k$.

We say that Λ is a *root lattice* if Λ is generated by $\mathrm{Roots}(\Lambda)$. For later use, we present properties of root lattices of type A_1 and D_4.

Let Λ be the root lattice of type A_1, and let $a \in \Lambda$ be a root, which generates Λ. We put $a^\vee := -a/2$, which generates $\Lambda^\vee$. Then the discriminant group of Λ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The proof of the following is elementary:

Lemma 2.4. *Let $v \in \Lambda^\vee$ be a vector such that $va \geq 0$. If $v \equiv 0 \bmod \Lambda$, then we have $v^2 = 0$ or $v^2 \leq -2$, and $v^2 = 0$ holds if and only if $v = 0$. If $v \equiv a^\vee \bmod \Lambda$, then we have $v^2 = -1/2$ or $v^2 \leq -9/2$, and $v^2 = -1/2$ holds if and only if $v = a^\vee$.*

Let Λ be the root lattice of type D_4 generated by the roots $d_1, \dots, d_4$ whose intersection numbers are given by the Dynkin diagram in Figure 2.1. Let $d_1^\vee, \dots, d_4^\vee$ be the basis of $\Lambda^\vee$ dual to $d_1, \dots, d_4$. We have

$$
[d_1^\vee, d_2^\vee, d_3^\vee, d_4^\vee] = [d_1, d_2, d_3, d_4] \begin{bmatrix} -1 & -1/2 & -1 & -1/2 \\ -1/2 & -1 & -1 & -1/2 \\ -1 & -1 & -2 & -1 \\ -1/2 & -1/2 & -1 & -1 \end{bmatrix}. \tag{2.4}
$$

The discriminant group of Λ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, and is generated by $d_1^\vee \bmod \Lambda$ and $d_4^\vee \bmod \Lambda$.

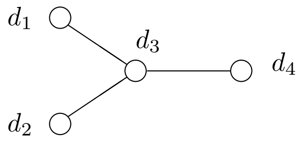

FIGURE 2.1. The Dynkin diagram of type D_4

Lemma 2.5. *Let $v \in \Lambda^\vee$ be a vector such that $vd_i \ge 0$ holds for $i = 1, \dots, 4$. If $v \equiv 0 \bmod \Lambda$, then we have $v^2 = 0$ or $v^2 \le -2$, and $v^2 = 0$ holds if and only if $v = 0$. If $v \equiv d_1^\vee \bmod \Lambda$, then we have $v^2 = -1$ or $v^2 \le -3$, and $v^2 = -1$ holds if and only if $v = d_1^\vee$.*

Proof. The first assertion is obvious. Suppose that $v \equiv d_1^\vee \bmod \Lambda$. Then we can put

$$v = d_1^\vee + x_1 d_1 + x_2 d_2 + x_3 d_3 + x_4 d_4,$$

where $x_1, \dots, x_4 \in \mathbb{Z}$. From the condition $vd_i \ge 0$ for $i = 1, \dots, 4$, we obtain the following inequalities:

$$(2.5) \quad 1 - 2x_1 + x_3 \ge 0,\ -2x_2 + x_3 \ge 0,\ x_1 + x_2 - 2x_3 + x_4 \ge 0,\ x_3 - 2x_4 \ge 0.$$

Using (2.4), we calculate

$$\begin{aligned} v^2 &= -1 - 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1 - x_1x_3 - x_2x_3 - x_3x_4) \\ &= -1 - \{(1 - 2x_1 + x_3)^2 + (-2x_2 + x_3)^2 + (x_3 - 2x_4)^2 + (x_3 - 1)^2 - 2\}/2. \end{aligned}$$

Therefore v^2 is a negative odd integer, and $v^2 = -1$ holds if and only if two of the four integers $1 - 2x_1 + x_3$, $-2x_2 + x_3$, $x_3 - 2x_4$, $x_3 - 1$ are ± 1 and the other two are 0. Combining this with the inequalities (2.5), we see that $v^2 = -1$ holds if and only if $x_1 = x_2 = x_3 = x_4 = 0$. $\square$

3. THE NÉRON-SEVERI LATTICE OF A $K3$ SURFACE

In this section, we work over an algebraically closed field of arbitrary characteristic. Let X be an (algebraic) $K3$ surface, and let $\mathrm{NS}(X)$ be the Néron-Severi lattice of X, which is an even hyperbolic lattice. For a divisor D on X, we denote by $[D] \in \mathrm{NS}(X)$ the class of D.

3.1. **The nef-cone.** We put

$$\mathrm{Nef}(X) := \{\ v \in \mathrm{NS}(X) \otimes \mathbb{R} \ \mid\ v[D] \ge 0 \text{ for any effective divisor } D \text{ on } X\ \}.$$

Let A be an ample divisor on X, and let $\mathcal{C}^+(X)$ be the connected component of

$$\{\ v \in \mathrm{NS}(X) \otimes \mathbb{R} \ \mid\ v^2 > 0\ \}$$

that contains $[A]$. For a vector $v \in \mathrm{NS}(X)$, we put

$$\langle v \rangle_{\mathbb{R}}^{\perp} := \{\ w \in \mathrm{NS}(X) \otimes \mathbb{R} \ \mid\ vw = 0\ \}.$$

Then the family of hyperplanes $\{\langle r \rangle_{\mathbb{R}}^{\perp} \mid r^2 = -2\}$ of $\mathrm{NS}(X) \otimes \mathbb{R}$ is locally finite in $\mathcal{C}^+(X)$. It is well-known and easy to prove that $[A] \notin \langle r \rangle_{\mathbb{R}}^{\perp}$ for any vector r with

$r^2 = -2$, and that $\mathrm{Nef}(X)$ is equal to the closure in $\mathrm{NS}(X) \otimes \mathbb{R}$ of the connected component of

$$\mathcal{C}^+(X) \setminus \bigcup \langle r \rangle_{\mathbb{R}}^{\perp}$$

that contains $[A]$. By the argument of Proposition 3 in [15, Section 3], we obtain the following:

Proposition 3.1. *Let X and X' be two algebraic $K3$ surfaces such that $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic. Then there exists an isomorphism $\phi : \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X')$ such that $\phi \otimes \mathbb{R}$ maps $\mathrm{Nef}(X)$ to $\mathrm{Nef}(X')$.*

3.2. **Polarizations.**

Proposition 3.2. *Let H be a divisor on an algebraic $K3$ surface X such that $[H] \in \mathrm{Nef}(X)$ and $H^2 > 0$. Then the following conditions are equivalent to each other:*

(i) *The complete linear system $|H|$ has no fixed components.*
(ii) *There exist no vectors $e \in \mathrm{NS}(X)$ such that $e[H] = 1$ and $e^2 = 0$.*

Proof. The implication (i)$\Longrightarrow$(ii) follows from the argument in the proof of (4)$\Longrightarrow$(1) in [24, Proposition 1.7]. The other implication (ii)$\Longrightarrow$(i) follows from [11, Proposition 0.1]. □

Definition 3.3. A *polarization* of an algebraic $K3$ surface X is a divisor H on X satisfying $[H] \in \mathrm{Nef}(X)$, $H^2 > 0$, and the conditions (i) and (ii) in Proposition 3.2. The positive integer H^2 is called the *degree* of the polarization H. By [11, Proposition 0.1], if H is a polarization of degree d, then $|H|$ is base-point free by Saint-Donat [16, Corollary 3.2], and we have $\dim |H| = 1 + d/2$.

A pair (X, H) of a $K3$ surface X and a polarization H of X is called a *polarized $K3$ surface.*

Combining Propositions 3.1 and 3.2, we obtain the following:

Corollary 3.4. *Let X and X' be two $K3$ surfaces such that $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic, and let H be a polarization of X. If $\phi : \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X')$ is an isomorphism such that $\phi \otimes \mathbb{R}$ maps $\mathrm{Nef}(X)$ to $\mathrm{Nef}(X')$, then $\phi([H])$ is the class of a polarization H' of X'.*

A curve C on X is called a (-2)-*curve* on X if it satisfies the following conditions that are equivalent to each other:

(i) C is a smooth rational curve,
(ii) C is reduced irreducible with negative self-intersection,
(iii) C is irreducible and $C^2 = -2$.

Let (X, H) be a polarized $K3$ surface. Then the complete linear system $|H|$ defines a morphism $\Phi_{|H|}$ from X to a projective space $\mathbb{P}^N$ $(N = 1 + H^2/2)$ that is generically finite over the image. We denote by

$$X \xrightarrow{\rho} Y \xrightarrow{\pi} \mathbb{P}^N \tag{3.1}$$

the Stein factorization of $\Phi_{|H|}$; that is, ρ is birational, Y is normal, and π is finite. The normal $K3$ surface Y has only rational double points as its singularities, and hence ρ is a contraction of an ADE-configuration of (-2)-curves. (See [1, 2].) Let $\mathcal{E}$ be the set of (-2)-curves that are contracted by ρ. The classes $[E]$ of $E \in \mathcal{E}$ are determined by the following procedure. Let $[H]^{\perp}$ be the orthogonal complement of

$[H]$ in $\mathrm{NS}(X)$. Since $\mathrm{NS}(X)$ is even hyperbolic and $[H]^2$ is positive, $[H]^\perp$ is even and negative-definite. We can therefore consider the set $\mathrm{Roots}([H]^\perp)$ of roots in $[H]^\perp$.

Lemma 3.5. *Let r be an element of $\mathrm{Roots}([H]^\perp)$. Then there exists a unique effective divisor E such that $r = [E]$ or $r = -[E]$ holds. Moreover, the integral component of E is a (-2)-curve.*

Proof. By the Riemann-Roch theorem and the Serre duality, we see that either r or $-r$ is the class of an effective divisor. Replacing r with $-r$, if necessary, we can assume that r is the class of an effective divisor E. Let $E = F + M$ be the decomposition of E into the sum of the fixed part F and the movable part M. Since $[H] \in \mathrm{Nef}(X)$, we have $HF \geq 0$ and $HM \geq 0$. Because $HE = 0$, we have $HM = 0$. Since $[H]^\perp$ is negative-definite and $M^2 \geq 0$, we obtain $M = 0$. Therefore E is unique and every irreducible component of E has negative self-intersection number. Thus the reduced part of every irreducible component of E is a (-2)-curve. □

Let $\mathrm{Roots}([H]^\perp) = R_1 \sqcup \cdots \sqcup R_k$ be the irreducible decomposition of $\mathrm{Roots}([H]^\perp)$ defined in §2. We choose an interior point a of $\mathrm{Nef}(X)$ (for example, the class of an ample divisor on X), and let $\alpha : \mathrm{NS}(X) \to \mathbb{R}$ be the linear form given by $\alpha(x) := ax$. By Lemma 3.5, we see that $\alpha(r) \neq 0$ for any $r \in \mathrm{Roots}([H]^\perp)$. We thus can define R_i^+ by (2.3), and consider the indecomposable roots of R_i^+. Note that $R_i^+ \subset R_i$ does not depend on the choice of the interior point a of $\mathrm{Nef}(X)$.

Proposition 3.6. *Let $\mathrm{Sing}(Y)$ be the set of singular points of Y. There exists a bijection from the set $\{R_1, \ldots, R_k\}$ to $\mathrm{Sing}(Y)$ with the following property. Let $P_i \in \mathrm{Sing}(Y)$ be the point corresponding to R_i. Then the classes of (-2)-curves contracted by ρ to P_i are exactly the indecomposable roots of R_i^+.*

Proof. Let r be an element of R_i^+. By Lemma 3.5 and $\alpha(r) > 0$, r is the class of a unique effective divisor of the form $a_1E_1 + \cdots + a_lE_l$, where $E_1, \ldots, E_l$ are (-2)-curves and $a_1, \ldots, a_l$ are positive integers. Since $[H] \in \mathrm{Nef}(X)$ and $r \in [H]^\perp$, we have $[E_\nu] \in [H]^\perp$ for $\nu = 1, \ldots, l$. In particular, we have $E_\nu \in \mathcal{E}$ for $\nu = 1, \ldots, l$. Let Λ_j be the sublattice of $[H]^\perp$ generated by the roots in R_j for $j = 1, \ldots, k$. Since $\Lambda_1, \ldots, \Lambda_k$ form a direct sum in $\mathrm{NS}(X)$, the uniqueness of the effective divisor representing $r \in \Lambda_i$ implies that $[E_1], \ldots, [E_l]$ are all in R_i. Since $\alpha([E_\nu]) > 0$, we have $[E_\nu] \in R_i^+$. Thus we have shown that every element of R_i^+ is written as a linear combination of the classes of (-2)-curves in R_i^+ with non-negative integer coefficients in a unique way. By Corollary 2.3, we see that r is the class of a (-2)-curve in $\mathcal{E}$ if and only if r is indecomposable in R_i^+. □

Let (X', H') be another polarized $K3$ surface. Let

$$X' \xrightarrow{\rho'} Y' \xrightarrow{\pi'} \mathbb{P}^{N'}$$

be the Stein factorization of the morphism $\Phi_{|H'|}$ defined by $|H'|$, and let $\mathcal{E}'$ be the set of (-2)-curves contracted by ρ'.

Corollary 3.7. *Suppose that there exists an isomorphism $\phi : \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X')$ such that $\phi \otimes \mathbb{R}$ maps $\mathrm{Nef}(X)$ to $\mathrm{Nef}(X')$, and that $\phi([H])$ is equal to $[H']$. Then the ADE-type of $\mathrm{Sing}(Y)$ coincides with that of $\mathrm{Sing}(Y')$. Moreover, there exist bijections*

$$\phi_{\mathcal{E}} : \mathcal{E} \xrightarrow{\sim} \mathcal{E}' \quad \textit{and} \quad \phi_{\mathrm{Sing}} : \mathrm{Sing}(Y) \xrightarrow{\sim} \mathrm{Sing}(Y')$$

such that the following diagram is commutative;

$$
\begin{array}{ccc}
\mathrm{NS}(X) & \xrightarrow{\phi} & \mathrm{NS}(X') \\
\uparrow & & \uparrow \\
\mathcal{E} & \xrightarrow{\phi_{\mathcal{E}}} & \mathcal{E}' \\
\downarrow & & \downarrow \\
\mathrm{Sing}(Y) & \xrightarrow{\phi_{\mathrm{Sing}}} & \mathrm{Sing}(Y')
\end{array}
\quad , \tag{3.2}
$$

where the up-arrows are given by $E \mapsto [E] \in \mathrm{NS}(X)$ *and* $E' \mapsto [E'] \in \mathrm{NS}(X')$*, respectively, and the down-arrows are given by* $E \mapsto \rho(E) \in \mathrm{Sing}(Y)$ *and* $E' \mapsto \rho'(E') \in \mathrm{Sing}(Y')$*, respectively.*

3.3. **Polarizations with maximal rational double points.**

Definition 3.8. We say that a polarized $K3$ surface (X, H) has *maximal rational double points* if the total Milnor number of $\mathrm{Sing}(Y)$ is equal to $\operatorname{rank} \mathrm{NS}(X) - 1$; or equivalently, the root lattice generated by $\mathrm{Roots}([H]^{\perp})$ is of finite index in $[H]^{\perp}$.

Let (X, H) be a polarized $K3$ surface with maximal rational double points. Consider the Stein factorization (3.1) of $\Phi_{|H|}$. For $P \in \mathrm{Sing}(Y)$, we denote by $\mathcal{E}_P \subset \mathcal{E}$ the set of (-2)-curves that are contracted to P by ρ, by $\Lambda_P \subset \mathrm{NS}(X)$ the sublattice generated by the classes $[E]$ of the curves $E \in \mathcal{E}_P$, and by Δ_P the discriminant group $\Lambda_P^{\vee}/\Lambda_P$ of Λ_P. We also denote by $\Lambda_H \subset \mathrm{NS}(X)$ the sublattice of rank 1 generated by $[H]$, and by Δ_H the discriminant group $\Lambda_H^{\vee}/\Lambda_H$ of Λ_H, which is a cyclic group of order equal to H^2. We then put

$$\Lambda := \Lambda_H \oplus \bigoplus_{P \in \mathrm{Sing}(Y)} \Lambda_P \quad \text{and} \quad \Delta := \Lambda^{\vee}/\Lambda.$$

We have natural decompositions

$$\Lambda^{\vee} = \Lambda_H^{\vee} \oplus \bigoplus_{P \in \mathrm{Sing}(Y)} \Lambda_P^{\vee} \qquad \text{and} \qquad \Delta = \Delta_H \oplus \bigoplus_{P \in \mathrm{Sing}(Y)} \Delta_P.$$

By the assumption, Λ is of finite index in $\mathrm{NS}(X)$, and hence $\mathrm{NS}(X)$ is an overlattice of Λ. Let v be a vector of $\mathrm{NS}(X)$. Using the direct-sum decomposition of $\Lambda^{\vee}$ and the natural embedding $\mathrm{NS}(X) \hookrightarrow \Lambda^{\vee}$, we can define the *$H$-component* $v_H \in \Lambda_H^{\vee}$ and the *P-components* $v_P \in \Lambda_P^{\vee}$ of v. We denote by $\bar{v} \in \Delta$ the class of v modulo Λ. Then the *H-component* $\bar{v}_H \in \Delta_H$ and the *P-components* $\bar{v}_P \in \Delta_P$ of $\bar{v}$ are also defined.

4. Polarizations of degree 2 in characteristic 2

From now on to the end of this paper, we assume that the base field k is of characteristic 2.

Let (X, H) be a polarized $K3$ surface of degree 2. Then the Stein factorization of $\Phi_{|H|}$ is of the form

$$X \xrightarrow{\rho} Y \xrightarrow{\pi} \mathbb{P}^2,$$

where $\pi : Y \to \mathbb{P}^2$ is a finite double cover. We have $h^0(X, \mathcal{O}_X(mH)) = m^2 + 2$ for every $m \geq 1$ by [11, Proposition 0.1]. Therefore the finite double cover $\pi : Y \to \mathbb{P}^2$ is defined by the equation

$$w^2 + w\, C(x_0, x_1, x_2) + G(x_0, x_1, x_2) = 0 \tag{4.1}$$

in the total space of the line bundle $V \to \mathbb{P}^2$ corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(3)$, where w is a fiber coordinate of V, $[x_0 : x_1 : x_2]$ is a homogeneous coordinate system of $\mathbb{P}^2$, and C and G are homogeneous polynomials of degree 3 and 6 that are regarded as sections of V and $V^{\otimes 2}$, respectively. If $C \neq 0$, then π is separable, while if $C = 0$, then π is purely inseparable.

Definition 4.1. An irreducible curve $F \subset X$ is called a *half-line of* (X, H) if $FH = 1$ holds. A line $L \subset \mathbb{P}^2$ is said to *be splitting in* (X, H) if the proper transform of L in X is non-reduced or reducible, or equivalently, if the scheme-theoretic pre-image $\pi^{-1}(L) \subset Y$ of L by π is non-reduced or reducible.

Let F be a half-line of (X, H). Then $\Phi_{|H|}$ induces an isomorphism from F to a line $L \subset \mathbb{P}^2$, and this line L is splitting in (X, H). In particular, a half-line is a (-2)-curve.

Definition 4.2. If $L \subset \mathbb{P}^2$ is a line splitting in (X, H), then the proper transform of L in X is written as $F + F'$, where F and F' are half-lines of (X, H). These half-lines are said to be *lying over* L. We say that L is *of non-reduced type* if $F = F'$, while L is *of reduced type* if $F \neq F'$.

Lemma 4.3. *Suppose that π is separable. Then the number of splitting lines of non-reduced type is at most* 3.

Proof. Let L be a splitting line of non-reduced type. We choose homogeneous coordinates $[x_0 : x_1 : x_2]$ of $\mathbb{P}^2$ such that L is defined by $x_2 = 0$. Putting $x_2 = 0$ in the defining equation (4.1), we see that the curve defined by

$$w^2 + w\,C(x_0, x_1, 0) + G(x_0, x_1, 0) = 0 \tag{4.2}$$

in the total space of the line bundle $V|_L \to L$ on L is non-reduced. Let $\gamma(w, x_0, x_1)$ be the left-hand side of (4.2). Since $\operatorname{char} k = 2$, we have $\partial\gamma/\partial w = C(x_0, x_1, 0)$. Therefore $C(x_0, x_1, 0)$ is constantly equal to zero. Thus we have shown that the defining equation of a splitting line of non-reduced type divides $C(x_0, x_1, x_2)$. Therefore, if $C \neq 0$, then the number of splitting lines of non-reduced type is at most $\deg C = 3$. □

Next we investigate the case where π is purely inseparable. In this case, π is given by the equation

$$w^2 + G(x_0, x_1, x_2) = 0. \tag{4.3}$$

Note that every splitting line is now of non-reduced type.

Remark 4.4. Let $\Gamma(x_0, x_1, x_2)$ be a homogeneous polynomial of degree 3. Then the equations $w^2 = G$ and $w^2 = G + \Gamma^2$ define surfaces isomorphic over $\mathbb{P}^2$.

We have the following relation between splitting lines and rational double points of Y. See [4] or [9] for the normal form of defining equations of rational double points in characteristic 2.

Lemma 4.5. *Let $L \subset \mathbb{P}^2$ be a line defined by $\ell(x_0, x_1, x_2) = 0$.*

(1) *The line L is splitting in (X, H) if and only if there exist homogeneous polynomials $Q(x_0, x_1, x_2)$ and $\Gamma(x_0, x_1, x_2)$ of degree* 5 *and* 3*, respectively, such that $G = \ell Q + \Gamma^2$.*

(2) *Suppose that L is splitting in (X, H), and let Q be a polynomial of degree* 5 *such that $G + \ell Q$ is a square of a cubic polynomial. We denote by $T \subset \mathbb{P}^2$ the*

quintic curve defined by $Q = 0$. Let p be a point of L, and P the point of Y such that $\pi(P) = p$. Then P is a smooth point of Y if and only if $p \notin T$, and P is an A_1-singular point of Y if and only if T intersects L transversely at p.

Proof. We can assume that $\ell = x_2$. Since the curve defined by $w^2 + G(x_0, x_1, 0) = 0$ in $V|_L$ is non-reduced, we see that $G(x_0, x_1, 0)$ is the square of a polynomial of degree 3. Hence the assertion (1) follows. Let (x, y) be an affine coordinate system of $\mathbb{P}^2$ with the origin p such that L is defined by $y = 0$. We write (4.3) as $w^2 = g(x, y)$. Let g_{ij} be the coefficient of $x^i y^j$ of g. Then P is a smooth point of Y if and only if $g_{01} \neq 0$ or $g_{10} \neq 0$, and P is an A_1-singular point of Y if and only if $g_{01} = g_{10} = 0$ and $g_{11} \neq 0$. Let $q(x, y)$ be the inhomogeneous polynomial corresponding to Q, and let q_{ij} be the coefficients of $x^i y^j$ of q. Then, up to a multiplicative constant, we have $g_{01} = q_{00}$, $g_{10} = 0$, $g_{11} = q_{10}$. Therefore the assertion (2) follows. □

Remark 4.6. The polynomials Q and Γ such that $G = \ell Q + \Gamma^2$ are not determined uniquely by G and ℓ. However, the homogeneous polynomial $Q|L$ on the line L is determined uniquely by G and ℓ.

5. Schröer's Kummer surfaces as Zariski surfaces

Let r and s be constants in k such that $r \neq 0$, $s \neq 0$ and $r^3 \neq s^3$. Then Schröer's supersingular $K3$ surface $X_{r,s}$ defined in Introduction is of Artin invariant 2. By Proposition 6.2 of [17], the quotient surface $(C \times C)/\alpha_2$ of the α_2-action on $C \times C$ defined by the vector field (1.1) contains an open subset U isomorphic to

$$\operatorname{Spec} k[a, b, c]/(\, c^2 + a(b^4 + s^2 b^2) + b(a^4 + r^2 a^2)\,).$$

The singular locus of U consists of four D_4-singular points coming from the fixed points of the α_2-action on the smooth part of $C \times C$. Let

$$\pi_{r,s} : Y_{r,s} \to \mathbb{P}^2$$

be the purely inseparable double cover defined by

$$w^2 = [x_0(x_1^4 + s^2 x_1^2 x_2^2) + x_1(x_0^4 + r^2 x_0^2 x_2^2)]x_2,$$

which is a projective completion of U. Then $Y_{r,s}$ is birational to $X_{r,s}$, and hence there exists a morphism $\rho_{r,s} : X_{r,s} \to Y_{r,s}$ that is the minimal resolution. The pull-back of a line of $\mathbb{P}^2$ by $\pi_{r,s} \circ \rho_{r,s}$ is a polarization $H_{r,s}$ of degree 2 of $X_{r,s}$. Then

$$X_{r,s} \xrightarrow{\rho_{r,s}} Y_{r,s} \xrightarrow{\pi_{r,s}} \mathbb{P}^2$$

is the Stein factorization of $\Phi_{|H_{r,s}|}$. The singular locus of $Y_{r,s}$ consists of four D_4-singular points $P(00), P(01), P(10), P(11)$ in U and five A_1-singular points $Q(0), Q(1), Q(\omega), Q(\bar{\omega}), Q(\infty)$ lying on the line defined by $x_2 = 0$. Here ω is a primitive third root of 1, and $\bar{\omega} = \omega^2$. These singular points are indexed in such a way that their images by $\pi_{r,s}$ are given in Table 5.1, where $p(\alpha\beta) := \pi_{r,s}(P(\alpha\beta))$ for $\alpha\beta = 00, 01, 10, 11$, and $q(\gamma) := \pi_{r,s}(Q(\gamma))$ for $\gamma = 0, 1, \omega, \bar{\omega}, \infty$. It is easy to see that the five lines listed below are splitting in $(X_{r,s}, H_{r,s})$:

$$\begin{aligned} &L(\infty) := \{x_2 = 0\}, \\ &L(0*) := \{x_0 = 0\}, \quad L(1*) := \{x_0 + r x_2 = 0\}, \\ &L(*0) := \{x_1 = 0\}, \quad L(*1) := \{x_1 + s x_2 = 0\}. \end{aligned}$$

$p(00)$	$=$	$[0:0:1]$	$q(0)$	$=$	$[1:0:0]$
$p(01)$	$=$	$[0:s:1]$	$q(1)$	$=$	$[1:1:0]$
$p(10)$	$=$	$[r:0:1]$	$q(\omega)$	$=$	$[1:\omega:0]$
$p(11)$	$=$	$[r:s:1]$	$q(\bar{\omega})$	$=$	$[1:\bar{\omega}:0]$
			$q(\infty)$	$=$	$[0:1:0]$

TABLE 5.1. The coordinates of the singular points of $Y_{r,s}$

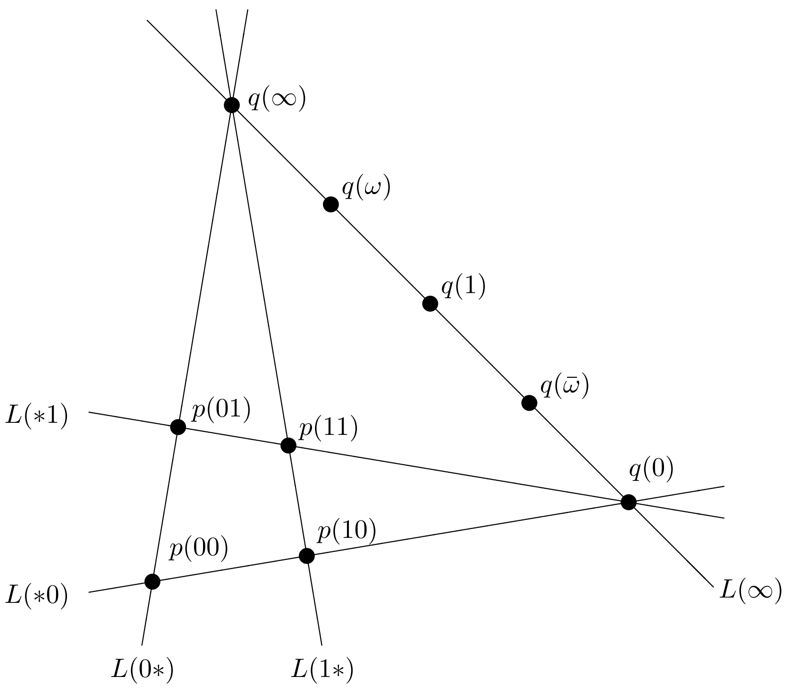

FIGURE 5.1. The configuration of splitting lines

To simplify the notation, we put

$$\mathcal{P} := \{00, 01, 10, 11\}, \quad \mathcal{Q} := \{0, 1, \omega, \bar{\omega}, \infty\}, \quad \mathcal{L} := \{\infty, 0*, 1*, *0, *1\}.$$

The configuration of the splitting lines $L(\lambda)$ $(\lambda \in \mathcal{L})$ and the points $p(\alpha\beta)$ $(\alpha\beta \in \mathcal{P})$, $q(\gamma)$ $(\gamma \in \mathcal{Q})$ are given in Figure 5.1. For a splitting line $L(\lambda)$ $(\lambda \in \mathcal{L})$, we denote by $F(\lambda)$ the half-line of $(X_{r,s}, H_{r,s})$ lying over $L(\lambda)$. By blowing up $Y_{r,s}$ at their singular points explicitly, we see that the half-lines $F(\lambda)$ and the exceptional divisors of $\rho_{r,s} : X_{r,s} \to Y_{r,s}$ intersect as in Figure 5.2. We denote the exceptional curves over the D_4-singular points $P(\alpha\beta)$ $(\alpha\beta \in \mathcal{P})$ as in Figure 5.3, and denote the exceptional curves over the A_1-singular points $Q(\gamma)$ $(\gamma \in \mathcal{Q})$ by $A(\gamma)$. The polarized $K3$ surface

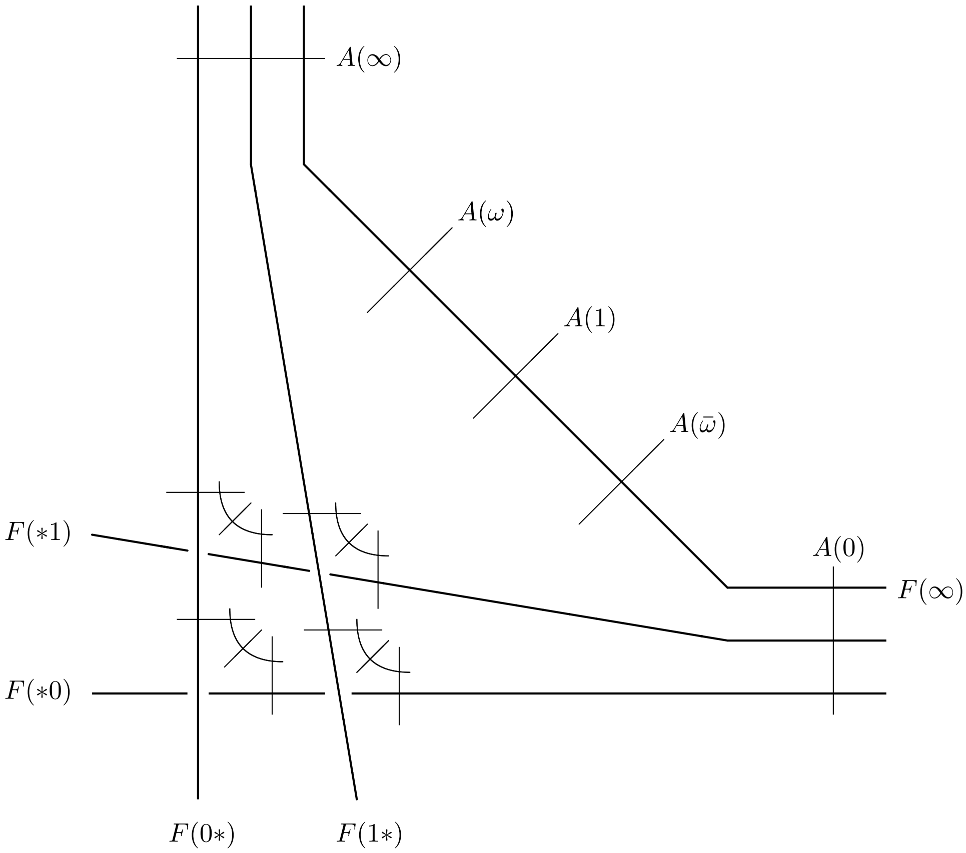

FIGURE 5.2. The configuration of half-lines and exceptional curves

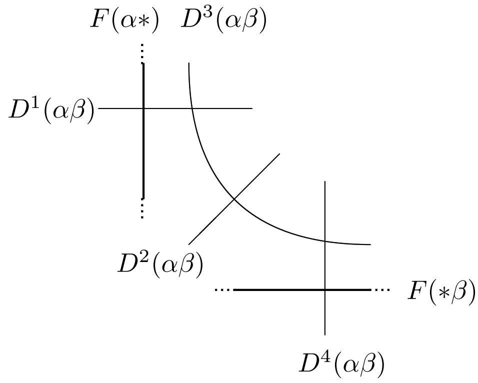

FIGURE 5.3. The exceptional curves over $P(\alpha\beta)$

$(X_{r,s}, H_{r,s})$ has maximal rational double points. Consider the sublattice

$$\Lambda_{r,s} := \Lambda_H \oplus \bigoplus_{\alpha\beta\in\mathcal{P}} \Lambda_{P(\alpha\beta)} \oplus \bigoplus_{\gamma\in\mathcal{Q}} \Lambda_{Q(\gamma)} \tag{5.1}$$

of $\mathrm{NS}(X_{r,s})$ with finite index, as in the subsection 3.3. The lattice Λ_H is of rank 1 generated by $h := [H_{r,s}]$, and $\Lambda_H^\vee$ is generated by $h^\vee := h/2$. The lattice $\Lambda_{P(\alpha\beta)}$ is of rank 4 with basis $d^i(\alpha\beta) := [D^i(\alpha\beta)]$ $(i = 1, \dots, 4)$. We denote the basis of $\Lambda_{P(\alpha\beta)}^\vee$ dual to $d^1(\alpha\beta), \dots, d^4(\alpha\beta)$ by $d^1(\alpha\beta)^\vee, \dots, d^4(\alpha\beta)^\vee$. The relation between $d^1(\alpha\beta), \dots, d^4(\alpha\beta)$ and $d^1(\alpha\beta)^\vee, \dots, d^4(\alpha\beta)^\vee$ are given by (2.4). The lattice $\Lambda_{Q(\gamma)}$ is of rank 1 generated by $a(\gamma) := [A(\gamma)]$, and $\Lambda_{Q(\gamma)}^\vee$ is generated by $a(\gamma)^\vee := -a(\gamma)/2$. From Figures 5.2 and 5.3, we see that the classes of half-lines $F(\lambda)$ $(\lambda \in \mathcal{L})$ are given as follows:

$$\begin{aligned}
[F(\infty)] &= h^\vee + a(0)^\vee + a(\omega)^\vee + a(1)^\vee + a(\bar\omega)^\vee + a(\infty)^\vee,\\
[F(0*)] &= h^\vee + d^1(00)^\vee + d^1(01)^\vee + a(\infty)^\vee,\\
[F(1*)] &= h^\vee + d^1(10)^\vee + d^1(11)^\vee + a(\infty)^\vee,\\
[F(*0)] &= h^\vee + d^4(00)^\vee + d^4(10)^\vee + a(0)^\vee,\\
[F(*1)] &= h^\vee + d^4(01)^\vee + d^4(11)^\vee + a(0)^\vee.
\end{aligned} \tag{5.2}$$

We then put

$$\Delta_{r,s} := (\Lambda_{r,s})^\vee/\Lambda_{r,s} = \Delta_H \oplus \bigoplus_{\alpha\beta\in\mathcal{P}} \Delta_{P(\alpha\beta)} \oplus \bigoplus_{\gamma\in\mathcal{Q}} \Delta_{Q(\gamma)}, \tag{5.3}$$

which is an $\mathbb{F}_2$-vector space of dimension 14. Since the discriminant of $\mathrm{NS}(X_{r,s})$ is $-2^{2\sigma(X_{r,s})} = -2^4$, we see that $\mathrm{NS}(X_{r,s})/\Lambda_{r,s} \subset \Delta_{r,s}$ is a subspace of dimension 5. It is easy to prove that the five elements

$$\overline{[F(\lambda)]} := [F(\lambda)] \bmod \Lambda_{r,s} \quad (\lambda \in \mathcal{L})$$

of $\mathrm{NS}(X_{r,s})/\Lambda_{r,s}$ are linearly independent. Therefore $\mathrm{NS}(X_{r,s})$ is generated by the classes $h = [H_{r,s}]$, $d^i(\alpha\beta) = [D^i(\alpha\beta)]$, $a(\gamma) = [A(\gamma)]$ and $[F(\lambda)]$.

Remark 5.1. Suppose that $r^3 = s^3$. Then there exists $c \in \mathbb{F}_4^\times = \{1, \omega, \bar\omega\}$ such that $s = cr$ holds. The three points $p(00), p(11)$ and $q(c)$ on $\mathbb{P}^2$ are collinear. Let M be the line passing through these points. Then M is a splitting line for $(X_{r,cr}, H_{r,cr})$. Let G be the half-line lying over M. By blowing up $Y_{r,cr}$ at the points $P(00), P(11)$ and $Q(c)$, we see that

$$[G] = h^\vee + d^2(00)^\vee + d^2(11)^\vee + a(c)^\vee.$$

Note that $d^2(\alpha\beta)^\vee \equiv d^1(\alpha\beta)^\vee + d^4(\alpha\beta)^\vee \bmod \Lambda_{P(\alpha\beta)}$ by (2.4). Hence $\overline{[G]} := [G] \bmod \Lambda_{r,cr}$ is linearly independent from the set of vectors $\overline{[F(\lambda)]}$ $(\lambda \in \mathcal{L})$ in $\Delta_{r,cr}$. In particular, the linear subspace $\mathrm{NS}(X_{r,cr})/\Lambda_{r,cr}$ of $\Delta_{r,cr}$ is of dimension 6 generated by $\overline{[F(\lambda)]}$ $(\lambda \in \mathcal{L})$ and $\overline{[G]}$, and the Artin invariant of $X_{r,cr}$ is 1.

6. Proof of main theorem

Note that supersingular $K3$ surfaces with Artin invariant 1 are isomorphic to each other [7, 13]. Therefore it is enough to prove Theorem 1.2 under the additional assumption that the Artin invariant of X' is 2.

We choose a Schröer's Kummer surface X with $\sigma(X) = 2$. To fix the ideas, we choose $s \in k \setminus \mathbb{F}_4$, and put $X := X_{1,s}$, and set

$$H := H_{1,s}, \quad Y := Y_{1,s}, \quad \rho := \rho_{1,s}, \quad \pi := \pi_{1,s}, \quad \Delta := \Delta_{1,s}, \quad \Lambda := \Delta_{1,s}.$$

Let X' be a supersingular $K3$ surface with Artin invariant 2. Theorem 1.3 implies that $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic. By Proposition 3.1, there exists an isomorphism $\phi : \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X')$ such that $\phi \otimes \mathbb{R}$ maps $\mathrm{Nef}(X)$ to $\mathrm{Nef}(X')$. We fix such an isomorphism ϕ once and for all. By Corollary 3.4, we have a polarization H' of X' with degree 2 such that $[H'] = \phi([H])$. As before, let

$$X' \xrightarrow{\rho'} Y' \xrightarrow{\pi'} \mathbb{P}^2$$

be the Stein factorization of $\Phi_{|H'|}$. By Corollary 3.7, there exist bijections $\phi_{\mathcal{E}} : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ and $\phi_{\mathrm{Sing}} : \mathrm{Sing}(Y) \xrightarrow{\sim} \mathrm{Sing}(Y')$ such that the diagram (3.2) is commutative. For $P \in \mathrm{Sing}(Y)$, we write $P' \in \mathrm{Sing}(Y')$ instead of $\phi_{\mathrm{Sing}}(P)$, and for $E \in \mathcal{E}$, we write $E' \in \mathcal{E}'$ instead of $\phi_{\mathcal{E}}(E)$. Therefore $\mathrm{Sing}(Y')$ consists of four D_4-singular points $P(\alpha\beta)'$ $(\alpha\beta \in \mathcal{P})$, and five A_1-singular points $Q(\gamma)'$ $(\gamma \in \mathcal{Q})$. For example, the (-2)-curves contracted to $P(\alpha\beta)'$ by ρ' are $D^1(\alpha\beta)'$, $D^2(\alpha\beta)'$, $D^3(\alpha\beta)'$ and $D^4(\alpha\beta)'$. We then put

$$p(\alpha\beta)' := \pi'(P(\alpha\beta)') \quad \text{and} \quad q(\gamma)' := \pi'(Q(\gamma)').$$

We also set

$$\Lambda' := \Lambda_{H'} \oplus \bigoplus_{\alpha\beta \in \mathcal{P}} \Lambda_{P(\alpha\beta)'} \oplus \bigoplus_{\gamma \in \mathcal{Q}} \Lambda_{Q(\gamma)'} \tag{6.1}$$

and

$$\Delta' := (\Lambda')^{\vee}/\Lambda' = \Delta_{H'} \oplus \bigoplus_{\alpha\beta \in \mathcal{P}} \Delta_{P(\alpha\beta)'} \oplus \bigoplus_{\gamma \in \mathcal{Q}} \Delta_{Q(\gamma)'} \tag{6.2}$$

as (5.1) and (5.3). Note that ϕ induces isomorphisms

$$\phi_{\Lambda} : \Lambda \xrightarrow{\sim} \Lambda' \quad \text{and} \quad \phi_{\Delta} : \Delta \xrightarrow{\sim} \Delta'$$

that are compatible with the direct-sum decompositions (5.1), (6.1), and (5.3), (6.2).

Let $L \subset \mathbb{P}^2$ be a line splitting in (X, H), and let F be the half-line of (X, H) lying over L. We can define a line $L' \subset \mathbb{P}^2$ splitting in (X', H') and a half-line F' of (X', H') lying over L' as follows.

Claim 6.1. There exists a unique effective divisor D' that represents $\phi([F])$.

Proof. Since $\phi([F])^2 = -2$ and $\phi([F])[H'] = 1$, there exists an effective divisor D' that represents $\phi([F])$. Let $D' = \Gamma' + M'$ be the decomposition of D' into the sum of the fixed part Γ' and the movable part M'. Suppose that $M' \neq 0$. If $M'H' = 0$, then $M'^{\,2} < 0$ because $[H']^{\perp}$ is negative-definite. Therefore we have $M'H' > 0$. Since $\Gamma'H' \geq 0$, we have $M'H' = 1$, which implies that $\Phi_{|H'|}$ induces an isomorphism from M' to a line on $\mathbb{P}^2$. Hence M' is a smooth rational curve, which is a contradiction. □

Since $D'H' = 1$, there exists a unique irreducible component F' of D' such that $F'H' = 1$. Then F' is a half-line of (X', H'). We define $L' \subset \mathbb{P}^2$ to be the image of F' by $\rho' \circ \pi'$.

Claim 6.2. Let F'' be a half-line for (X', H') lying over L'. Then $\overline{[F'']} = \overline{[F']}$ holds in Δ', where $\overline{[F'']} = [F''] \mod \Lambda'$ and $\overline{[F']} = [F'] \mod \Lambda'$.

Proof. The case where $F' = F''$ is obvious. Suppose that $F' \neq F''$. Then $F' + F''$ is the total transform of L' in X' minus a linear combination of curves in $\mathcal{E}'$, and hence $[F'] + [F''] \in \Lambda'$. Because Δ' is a 2-elementary abelian group, we obtain $\overline{[F'']} = \overline{[F']}$. □

Claim 6.3. We have $\phi_\Delta(\overline{[F]}) = \overline{[F']}$.

Proof. Since $\phi([F]) = [D']$, we have $\phi_\Delta(\overline{[F]}) = \overline{[D']}$. Since $D' - F'$ is effective and $(D' - F')H' = 0$, each irreducible component of $D' - F'$ is contracted to a point by ρ'. Therefore we have $[D'] - [F'] \in \Lambda'$, and hence $\overline{[D']} = \overline{[F']}$. □

Now we have half-lines $F(\lambda)'$ and splitting lines $L(\lambda)'$ of (X', H') for each $\lambda \in \mathcal{L}$. By Claim 6.3, the elements $\overline{[F(\lambda)']}$ of Δ' are distinct to each other. Hence, by Claim 6.2, the lines $L(\lambda)'$ are distinct to each other.

Claim 6.4. Let P be a point of $\mathrm{Sing}(Y)$. If $\pi(P) \in L(\lambda)$, then $\pi'(P') \in L(\lambda)'$.

Proof. If $\pi(P) \in L(\lambda)$, then the P-component of $\overline{[F(\lambda)]} \in \Delta$ is not zero by (5.2). Hence the P'-component of $\overline{[F(\lambda)']} \in \Delta'$ is not zero by Claim 6.3. Consequently, there exists $E' \in \mathcal{E}'_{P'}$ such that $F(\lambda)'E' \neq 0$. Therefore the image $L(\lambda)'$ of $F(\lambda)'$ passes through $\pi'(P') \in \mathbb{P}^2$. □

Claim 6.5. The splitting line $L(\lambda)'$ is of non-reduced type for any $\lambda \in \mathcal{L}$.

Proof. Let G' be an arbitrary half-line of (X', H') lying over $L(\lambda)'$. Then the class $g' := [G'] \in \mathrm{NS}(X')$ satisfies the following:

(i) $(g')^2 = -2$,
(ii) $g'[H'] = 1$, and
(iii) for every $E' \in \mathcal{E}'$, we have $g'[E'] \geq 0$.

Suppose that $L(\lambda)'$ is of reduced type. Then there exists a half-line F''lying over $L(\lambda)'$ that is distinct from $F(\lambda)'$. Since $[F''][F(\lambda)'] \geq 0$, we have $[F''] \neq [F(\lambda)']$. By Claim 6.2, we have $\overline{[F(\lambda)']} = \overline{[F'']}$ in Δ'. Consequently, it is enough to show that there exists only one class g' in $\mathrm{NS}(X')$ satisfying (i), (ii), (iii) above and

(iv) $\overline{(g')} = \overline{[F(\lambda)']} = \phi_\Delta(\overline{[F(\lambda)]})$,

where the second equality follows from Claim 6.3. We denote by $g'_{H'}$ and $g'_{P'}$ the H'- and P'-components of g', respectively, where $P' \in \mathrm{Sing}(S')$. By (ii), we have $g'_{H'} = [H']/2$. Combining this with (i), we have

$$\sum_{\alpha\beta \in \mathcal{P}} (g'_{P(\alpha\beta)'})^2 + \sum_{\gamma \in \mathcal{Q}} (g'_{Q(\gamma)'})^2 = -5/2. \tag{6.3}$$

The case where $\lambda = \infty$. By (iii), (iv), (5.2) and Lemmas 2.4, 2.5, we have

$$(g'_{P(\alpha\beta)'})^2 = 0 \text{ or } \leq -2 \quad \text{and} \quad (g'_{Q(\gamma)'})^2 = -1/2 \text{ or } \leq -9/2.$$

Combining this with (6.3), we have

$$(g'_{P(\alpha\beta)'})^2 = 0 \quad \text{and} \quad (g'_{Q(\gamma)'})^2 = -1/2.$$

By (iii) and Lemmas 2.4, 2.5 again, we have

$$g'_{P(\alpha\beta)'} = 0 \quad \text{and} \quad g'_{Q(\gamma)'} = -[A(\gamma)']/2.$$

Thus the uniqueness of g' is proved.

The case where $\lambda = 0*$. By (iii), (iv), (5.2) and Lemmas 2.4, 2.5, we have

$$\begin{array}{rll}
(g'_{P(\alpha\beta)'})^2 & = -1 \text{ or } \le -3 & \text{if } \alpha\beta = 00 \text{ or } 01, \\
(g'_{P(\alpha\beta)'})^2 & = 0 \text{ or } \le -2 & \text{if } \alpha\beta = 10 \text{ or } 11, \\
(g'_{Q(\gamma)'})^2 & = -1/2 \text{ or } \le -9/2 & \text{if } \gamma = \infty, \\
(g'_{Q(\gamma)'})^2 & = 0 \text{ or } \le -2 & \text{if } \gamma \neq \infty.
\end{array}$$

Combining this with (6.3), we have

$$\begin{aligned}
&(g'_{P(00)'})^2 = (g'_{P(01)'})^2 = -1, \quad (g'_{P(10)'})^2 = (g'_{P(11)'})^2 = 0, \\
&(g'_{Q(\infty)'})^2 = -1/2, \quad (g'_{Q(\gamma)'})^2 = 0 \quad \text{for } \gamma \neq \infty.
\end{aligned}$$

By (iii) and Lemmas 2.4, 2.5 again, we have

$$\begin{aligned}
&g'_{P(00)'} = \delta^1(00), \quad g'_{P(01)'} = \delta^1(01), \quad g'_{P(10)'} = g'_{P(11)'} = 0, \\
&g'_{Q(\infty)'} = -[A(\infty)']/2, \quad g'_{Q(\gamma)'} = 0 \quad \text{for } \gamma \neq \infty,
\end{aligned}$$

where

$$\delta^1(\alpha\beta) = -[D^1(\alpha\beta)'] - [D^2(\alpha\beta)']/2 - [D^3(\alpha\beta)'] - [D^4(\alpha\beta)']/2.$$

(See (2.4).) Thus the uniqueness of g' is proved.

The other cases $\lambda = 1*, *0, *1$ can be treated in the same way. □

We have five distinct splitting lines $L(\lambda)'$ ($\lambda \in \mathcal{L}$) for (X', H'), which are of non-reduced type by Claim 6.5. By Lemma 4.3, we see that $\pi' : Y' \to \mathbb{P}^2$ is purely inseparable. By Claim 6.4, the configuration of the lines $L(\lambda)'$ and the points $p(\alpha\beta)', q(\gamma)'$ are exactly the same as the configuration depicted in Figure 5.1 with superscript prime ($'$) being put to everything.

There exists a homogeneous coordinate system $[x : y : z]$ of $\mathbb{P}^2$ such that

$$\begin{aligned}
&q(\infty)' = [0 : 1 : 0], \quad q(1)' = [1 : 1 : 0], \quad q(0)' = [1 : 0 : 0], \\
&p(00)' = [0 : 0 : 1], \quad p(10)' = [1 : 0 : 1].
\end{aligned}$$

We put

$$p(01)' = [0 : t : 1],$$

where t is a non-zero constant. Then we have $p(11)' = [1 : t : 1]$ by Figure$'$ 5.1. Let

$$w^2 = G(x, y, z)$$

be the defining equation of Y', where G is a homogeneous polynomial of degree 6, and let G_{lmn} ($l + m + n = 6$) be the coefficient of $x^l y^m z^n$ in G. By Remark 4.4, we can assume

$$G_{lmn} = 0 \quad \text{if} \quad l \equiv m \equiv n \equiv 0 \mod 2.$$

Because $L(0*)' = \{x = 0\}$ is splitting, Lemma 4.5(1) implies

$$G_{015} = G_{033} = G_{051} = 0.$$

Because $L(*0)' = \{y = 0\}$ is splitting, Lemma 4.5(1) implies

$$G_{105} = G_{303} = G_{501} = 0.$$

Because $L(\infty)' = \{z = 0\}$ is splitting, Lemma 4.5(1) implies

$$G_{150} = G_{330} = G_{510} = 0.$$

Therefore we have

$$G(x, y, z) = xyz\, C(x, y, z),$$

where C is a homogeneous polynomial of degree 3. By Lemma 4.5(2), the line $L(0*)' = \{x = 0\}$ and the quintic curve defined by $yz\, C(x, y, z) = 0$ intersect transversely at $q(\infty)'$ and with multiplicity ≥ 2 at $p(00)'$ and $p(01)'$. Therefore there exists a constant A such that $yzC(0, y, z) = Ay^2z(y + tz)^2$. In particular, we obtain

$$G_{132} = G_{114} = 0 \quad \text{and} \quad G_{123} = t^2G_{141}.$$

By Lemma 4.5(2), the line $L(*0)' = \{y = 0\}$ and the curve $xz\, C(x, y, z) = 0$ intersect transversely at $q(0)'$ and with multiplicity ≥ 2 at $p(00)'$ and $p(10)'$. Therefore there exists a constant B such that $xzC(x, 0, z) = Bx^2z(x + z)^2$. In particular, we obtain

$$G_{312} = G_{114} = 0 \quad \text{and} \quad G_{213} = G_{411}.$$

By Lemma 4.5(2), the line $L(\infty)' = \{z = 0\}$ and the curve $xy\, C(x, y, z) = 0$ intersect transversely at the five points $q(\gamma)'$ ($\gamma \in \mathcal{Q}$). In particular, the curve $xy\, C(x, y, z) = 0$ passes through $q(1)'$, and hence we obtain

$$G_{141} + G_{231} + G_{321} + G_{411} = 0.$$

Combining these, we see that Y' is defined by

$$w^2 = xyz(t^2ayz^2 + dxz^2 + ay^3 + bxy^2 + cx^2y + dx^3),$$

where a, b, c, d are constants such that $a + b + c + d = 0$. Because $L(1*)' = \{x = z\}$ is splitting, the polynomial $yz^2(t^2yz^2 + ay^3 + bzy^2 + cz^2y)$ of y and z is a square of a cubic polynomial. Therefore $b = 0$. Because $L(*1)' = \{y = tz\}$ is splitting, the polynomial $txz^2(dxz^2 + bt^2xz^2 + ctx^2z + dx^3)$ of x and z is a square of a cubic polynomial. Therefore $c = 0$. Because $a + b + c + d = 0$, we have $a = d$. Therefore Y' is defined by

$$w^2 = xyz(t^2yz^2 + xz^2 + y^3 + x^3).$$

Hence Y' is isomorphic to Schröer's normal $K3$ surface $Y_{t,1}$, and hence X' is isomorphic to Schröer's Kummer surface $X_{t,1}$.

Remark 6.6. In [14], it is shown that every supersingular $K3$ surface in characteristic 5 with Artin invariant ≤ 3 is obtained as a double cover of the projective plane with the branch curve defined by $y^5 - f(x) = 0$, where $f(x)$ is a polynomial of degree 6, and hence it is unirational.

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