

# BORCHERDS' METHOD FOR ENRIQUES SURFACES

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ABSTRACT. We classify all primitive embeddings of the lattice of numerical equivalence classes of divisors of an Enriques surface with the intersection form multiplied by 2 into an even unimodular hyperbolic lattice of rank 26. These embeddings have a property that facilitates the computation of the automorphism group of an Enriques surface by Borchers' method.

## 1. INTRODUCTION

Lattice theory is a very strong tool in the study of  $K3$  and Enriques surfaces. Let  $Z$  be a  $K3$  or an Enriques surface defined over an algebraically closed field. We denote by  $S_Z$  the lattice of numerical equivalence classes of divisors on  $Z$ . Note that  $S_Z$  is even, and if  $\text{rank } S_Z > 1$ , then  $S_Z$  is hyperbolic, that is, the signature of  $S_Z$  is  $(1, \text{rank } S_Z - 1)$ . For a positive integer  $n$  with  $n \equiv 2 \pmod{8}$ , let  $L_n$  denote an even unimodular hyperbolic lattice of rank  $n$ , which is unique up to isomorphism. *Borchers' method* [2, 3] is a procedure to calculate the automorphism group  $\text{Aut}(X)$  of a  $K3$  surface  $X$  by embedding  $S_X$  primitively into  $L_{26}$ , and applying Conway's result [6] on  $O(L_{26})$ . After the work of Kondo [13], this method has been applied to many  $K3$  surfaces, and automatized for computer calculation (see [21] and the references therein).

Let  $Y$  be an Enriques surface in characteristic  $\neq 2$  with the universal covering  $\pi: X \rightarrow Y$ . Then we have a primitive embedding  $\pi^*: S_Y(2) \hookrightarrow S_X$ , where, for a lattice  $L$ , we denote by  $L(2)$  the lattice with the same underlying  $\mathbb{Z}$ -module as  $L$  and with the symmetric bilinear form being two times that of  $L$ . Note that  $S_Y$  is isomorphic to  $L_{10}$ . Hence, to extend Borchers' method to Enriques surfaces, it is important to study the primitive embeddings of  $L_{10}(2)$  into  $L_{26}$ .

We identify  $O(L_{10}(2))$  with  $O(L_{10})$ . We say that two embeddings  $\iota$  and  $\iota'$  of  $L_{10}(2)$  into  $L_{26}$  are *equivalent up to the action of  $O(L_{10})$  and  $O(L_{26})$*  if there exist isometries  $g \in O(L_{10})$  and  $g' \in O(L_{26})$  such that, for all  $v \in L_{10}(2)$ , one has  $\iota(v)^{g'} = \iota'(v^g)$ . Our first main result is as follows.

**Theorem 1.1.** *Up to the action of  $O(L_{10})$  and  $O(L_{26})$ , there exist exactly 17 equivalence classes of primitive embeddings of  $L_{10}(2)$  into  $L_{26}$ , and they are given in Table 1.1.*

**Explanation of Table 1.1.** For a lattice  $L$ , we denote by  $A(L) := L^\vee/L$  the *discriminant group* of  $L$ , where  $L^\vee := \text{Hom}(L, \mathbb{Z})$  is the dual lattice of  $L$ , and by  $\mathcal{R}(L)$  the set of  $(-2)$ -vectors of a lattice  $L$ . Let  $R_\iota$  denote the orthogonal complement of the image of a primitive embedding  $\iota: L_{10}(2) \hookrightarrow L_{26}$  in  $L_{26}$ . Note

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No.	name	rt	m4	og
1	12A	$D_8$	1376	$2^{29} \cdot 3^7 \cdot 5^3 \cdot 7^2$
2	12B	$A_7$	1824	$2^{23} \cdot 3^6 \cdot 5^2 \cdot 7^2$
3	20A	$D_4 + D_5$	1760	$2^{25} \cdot 3^7 \cdot 5^2 \cdot 7$
4	20B	$2D_4$	1888	$2^{29} \cdot 3^4 \cdot 5 \cdot 7$
5	20C	$10A_1 + D_6$	1632	$2^{28} \cdot 3^6 \cdot 5^3 \cdot 7$
6	20D	$A_3 + A_4$	2016	$2^{16} \cdot 3^6 \cdot 5^3 \cdot 7$
7	20E	$5A_1 + A_5$	1952	$2^{20} \cdot 3^7 \cdot 5^3$
8	20F	$2A_3$	2080	$2^{23} \cdot 3^4 \cdot 5^2$
9	40A	$4A_1 + 2A_3$	2016	$2^{25} \cdot 3^5 \cdot 5$
10	40B	$8A_1 + 2D_4$	1760	$2^{30} \cdot 3^6 \cdot 5 \cdot 7$
11	40C	$6A_1 + A_3$	2080	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$
12	40D	$12A_1 + D_4$	1888	$2^{28} \cdot 3^5 \cdot 5^2$
13	40E	$2A_1 + 2A_2$	2144	$2^{16} \cdot 3^6 \cdot 5^2$
14	96A	$8A_1$	2144	$2^{28} \cdot 3^3$
15	96B	$16A_1$	2016	$2^{31} \cdot 3^5$
16	96C	$4A_1$	2208	$2^{22} \cdot 3^5$
17	infty		2272	$2^{26} \cdot 3^2 \cdot 5 \cdot 7$

TABLE 1.1. Primitive embeddings

that  $R_\iota$  is a negative-definite even lattice of rank 16 with  $A(R_\iota)$  being isomorphic to  $A(L_{10}(2)) \cong (\mathbb{Z}/2\mathbb{Z})^{10}$ . The item `rt` is the ADE-type of the negative-definite root lattice generated by  $\mathcal{R}(R_\iota)$ . For the embedding `infty`, the lattice  $R_\iota$  contains no  $(-2)$ -vectors. The item `m4` is the number of  $(-4)$ -vectors in  $R_\iota$ . The item `og` is the order of the group  $O(R_\iota)$ .

These 17 embeddings have a remarkable property, which is very useful for the calculation of the automorphism group of an Enriques surface. In order to state this property, we need to explain the notion of *tessellation by chambers*. Let  $L$  be an even hyperbolic lattice. A *positive cone*  $\mathcal{P}(L)$  is one of the two connected components of the subspace of  $L \otimes \mathbb{R}$  consisting of vectors  $x \in L \otimes \mathbb{R}$  such that  $\langle x, x \rangle > 0$ . We fix a positive cone  $\mathcal{P}(L)$  of  $L$ , and denote by  $O(L, \mathcal{P})$  the stabilizer subgroup of  $\mathcal{P}(L)$  in  $O(L)$ , which is of index 2 in  $O(L)$ . A *rational hyperplane*  $(v)^\perp$  is a subspace of  $\mathcal{P}(L)$  defined by  $\langle x, v \rangle = 0$ , where  $v \in L \otimes \mathbb{Q}$  is a vector satisfying  $\langle v, v \rangle < 0$ . Let  $\mathcal{F}$  be a locally finite family of rational hyperplanes of  $\mathcal{P}(L)$ . A closed subset  $D$  of  $\mathcal{P}(L)$  is said to be an  *$\mathcal{F}$ -chamber* if  $D$  is the closure in  $\mathcal{P}(L)$  of a connected component of the complement

$$\mathcal{P}(L) \setminus \bigcup_{H \in \mathcal{F}} H.$$

We say that a subset  $N$  of  $\mathcal{P}(L)$  has a *tessellation by  $\mathcal{F}$ -chambers* if  $N$  is a union of  $\mathcal{F}$ -chambers. For example, if  $\mathcal{F}'$  is a subfamily of  $\mathcal{F}$ , then every  $\mathcal{F}'$ -chamber has a tessellation by  $\mathcal{F}$ -chambers.

**Definition 1.2.** Note that  $\mathcal{P}(L)$  has a tessellation by  $\mathcal{F}$ -chambers. We say that this tessellation of  $\mathcal{P}(L)$  is *simple* if there exists a subgroup of  $O(L, \mathcal{P})$  that preserves the family  $\mathcal{F}$  of hyperplanes (and hence the set of  $\mathcal{F}$ -chambers) and acts on the set of  $\mathcal{F}$ -chambers transitively.

**Definition 1.3.** We say that  $\mathcal{F}$ -chambers  $D$  and  $D'$  are *isomorphic* if there exists an isometry  $g \in O(L, \mathcal{P})$  such that  $D^g = D'$ . The automorphism group of an  $\mathcal{F}$ -chamber is defined to be

$$O(L, D) := \{g \in O(L, \mathcal{P}) \mid D^g = D\}.$$

**Definition 1.4.** Let  $D$  be an  $\mathcal{F}$ -chamber, and  $\overline{D}$  the closure of  $D$  in  $L \otimes \mathbb{R}$ . We say that  $D$  is *quasi-finite* if  $\overline{D} \setminus D$  is contained in a union of at most countably many half-lines  $\mathbb{R}_{\geq 0}v_i \subset \partial \overline{\mathcal{P}}(L)$ , where  $v_i$  are non-zero vectors of  $L \otimes \mathbb{R}$  satisfying  $\langle v_i, v_i \rangle = 0$ ,  $\overline{\mathcal{P}}(L)$  is the closure of  $\mathcal{P}(L)$  in  $L \otimes \mathbb{R}$ , and  $\partial \overline{\mathcal{P}}(L) := \overline{\mathcal{P}}(L) \setminus \mathcal{P}(L)$ .

Each  $(-2)$ -vector  $r \in \mathcal{R}(L)$  defines the *reflection*  $s_r \in O(L, \mathcal{P})$  into the mirror  $(r)^\perp$ , which is defined by  $x^{s_r} := x + \langle x, r \rangle r$ . Let  $W(L)$  denote the subgroup of  $O(L, \mathcal{P})$  generated by reflections  $s_r$ , where  $r$  runs through  $\mathcal{R}(L)$ .

**Example 1.5.** We put  $\mathcal{R}(L)^\perp := \{(r)^\perp \mid r \in \mathcal{R}(L)\}$ , which is a locally finite family of rational hyperplanes. Then an  $\mathcal{R}(L)^\perp$ -chamber  $D_{\mathcal{R}}$  is a standard fundamental domain of the action on  $\mathcal{P}(L)$  of  $W(L)$ . Hence the tessellation of  $\mathcal{P}(L)$  by  $\mathcal{R}(L)^\perp$ -chambers is simple. Note that we have  $O(L, \mathcal{P}) = W(L) \rtimes O(L, D_{\mathcal{R}})$ .

**Definition 1.6.** The shape of an  $\mathcal{R}(L_n)^\perp$ -chamber was determined by Vinberg [30] for  $n = 10$  and  $18$ , and by Conway [6] for  $n = 26$ . Hence we call an  $\mathcal{R}(L_{10})^\perp$ -chamber a *Vinberg chamber*, and an  $\mathcal{R}(L_{26})^\perp$ -chamber a *Conway chamber*.

It is known that Vinberg chambers and Conway chambers are quasi-finite.

**Definition 1.7.** Let  $D$  be an  $\mathcal{F}$ -chamber. A *wall* of  $D$  is a closed subset  $w$  of  $D$  disjoint from the interior of  $D$  satisfying the following; there exists a hyperplane  $(v)^\perp \in \mathcal{F}$  such that  $w$  is equal to  $D \cap (v)^\perp$  and that  $w$  contains a non-empty open subset of  $(v)^\perp$ . We say that  $v \in L \otimes \mathbb{Q}$  *defines a wall*  $w$  of  $D$  if  $w$  is equal to  $D \cap (v)^\perp$  and  $\langle x, v \rangle \geq 0$  holds for all  $x \in D$ .

**Example 1.8.** Let  $D_{\mathcal{R}}$  be as in Example 1.5. Then the group  $W(L)$  is generated by reflections with respect to the  $(-2)$ -vectors defining walls of  $D_{\mathcal{R}}$ .

**Definition 1.9.** Let  $D$  be an  $\mathcal{F}$ -chamber, and  $w$  a wall of  $D$ . Then there exists a unique  $\mathcal{F}$ -chamber  $D'$  such that  $D \cap D' = w$ . We call  $D'$  the  $\mathcal{F}$ -chamber *adjacent to  $D$  across the wall  $w$* .

Let  $\iota: S \hookrightarrow L$  be an embedding of an even hyperbolic lattice  $S$ ,  $\mathcal{P}(S)$  the positive cone of  $S$  that is mapped to  $\mathcal{P}(L)$  by  $\iota \otimes \mathbb{R}$ , and  $\iota_{\mathcal{P}}: \mathcal{P}(S) \hookrightarrow \mathcal{P}(L)$  the induced inclusion. We put

$$\iota^* \mathcal{F} := \{\iota_{\mathcal{P}}^{-1}(H) \mid H \in \mathcal{F}, \emptyset \neq \iota_{\mathcal{P}}^{-1}(H) \subsetneq \mathcal{P}(S)\}.$$

Then  $\iota^* \mathcal{F}$  is a locally finite family of rational hyperplanes of  $\mathcal{P}(S)$ , and  $\mathcal{P}(S)$  has a tessellation by  $\iota^* \mathcal{F}$ -chambers. If all  $\mathcal{F}$ -chambers are quasi-finite, then so are all  $\iota^* \mathcal{F}$ -chambers.

In the following, we identify  $\mathcal{P}(L_{10}(2))$  with  $\mathcal{P}(L_{10})$ . If  $\iota: L_{10}(2) \hookrightarrow L_{26}$  is a primitive embedding, then  $\mathcal{P}(L_{10})$  has a tessellation by  $\iota^* \mathcal{R}(L_{26})^\perp$ -chambers. We

No.	name	walls	volindex	gD	orb	isom	NK
1	12A	12	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^2$	$2 + 2 + 2 + 2 + 4$		I
2	12B	12	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3$	$6 + 6$		II
3	20A	20	$2^8 \cdot 3^4 \cdot 5 \cdot 7$	$2^3 \cdot 3$	$4 + 4 + 6 + 6$		V
4	20B	20	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	$2^6$	$4 + 8 + 8$		III
5	20C	20	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$5 + 15$	20D	VII
6	20D	20	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$5 + 15$	20C	VII
7	20E	20	$2^7 \cdot 3^4 \cdot 5$	$2^3 \cdot 3 \cdot 5$	$10 + 10$		VI
8	20F	20	$2^9 \cdot 3^2 \cdot 5$	$2^6 \cdot 5$	20		IV
9	40A	40	$2^7 \cdot 3^2 \cdot 5$	$2^7 \cdot 3$	$12 + 12 + 16$		
10	40B	40	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$2^7 \cdot 3^2$	$16 + 24$	40C	
11	40C	40	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$2^7 \cdot 3^2$	$16 + 24$	40B	
12	40D	40	$2^5 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5$	$10 + 30$	40E	
13	40E	40	$2^5 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5$	$10 + 30$	40D	
14	96A	96	$2^5 \cdot 3^2$	$2^{13} \cdot 3$	$32 + 64$		
15	96B	96	$2^3 \cdot 3^2$	$2^{12} \cdot 3^3$	96	96C	
16	96C	96	$2^3 \cdot 3^2$	$2^{12} \cdot 3^3$	96	96B	
17	infty	$\infty$					

TABLE 1.2. Induced chambers

call an  $\iota^*\mathcal{R}(L_{26})^\perp$ -chamber an *induced chamber* associated with the embedding  $\iota$ . Note that every induced chamber is quasi-finite.

In the application of Borchers' method for the calculation of  $\text{Aut}(X)$  of a  $K3$  surface  $X$ , we embed  $S_X$  into  $L_{26}$  primitively and investigate the tessellation of  $\mathcal{P}_X$  by induced chambers. This tessellation is usually not simple, and in these cases, the computation of  $\text{Aut}(X)$  becomes very hard. See, for example, the case of the singular  $K3$  surface with transcendental lattice of discriminant 11 treated in [21], or the case of the supersingular  $K3$  surface of Artin invariant 1 in characteristic 5 studied in [10].

Our second main result is as follows.

**Theorem 1.10.** *Let  $\iota: L_{10}(2) \hookrightarrow L_{26}$  be a primitive embedding that is not of type `infty`. Then the number of walls of an induced chamber  $D$  is finite, and each wall of  $D$  is defined by a  $(-2)$ -vector of  $L_{10}$ . If  $r \in \mathcal{R}(L_{10})$  defines a wall  $w = D \cap (r)^\perp$  of  $D$ , then the reflection  $s_r$  with respect to  $r$  preserves the family of hyperplanes  $\iota^*\mathcal{R}(L_{26})^\perp$  and hence the set of induced chambers. In particular, the induced chamber adjacent to  $D$  across the wall  $w = D \cap (r)^\perp$  is equal to  $D^{s_r}$ .*

**Corollary 1.11.** *If  $\iota: L_{10}(2) \hookrightarrow L_{26}$  is not of type `infty`, then the tessellation of  $\mathcal{P}(L_{10})$  by induced chambers is simple.*

The data of the induced chambers  $D$  are given in Table 1.2. Before explaining the contents of Table 1.2, we recall two classical results about automorphism groups of Enriques surfaces. Let  $Y$  be an Enriques surface. We denote by  $\mathcal{P}_Y$  the positive

cone of  $S_Y \otimes \mathbb{R}$  containing an ample class. We then put

$$N_Y := \{x \in \mathcal{P}_Y \mid \langle x, [\Gamma] \rangle \geq 0 \text{ for all curves } \Gamma \text{ on } Y \}.$$

Then  $N_Y$  has a tessellation by Vinberg chambers, because  $N_Y$  is bounded by the hyperplanes  $([\Gamma])^\perp$  defined by the classes  $[\Gamma]$  of smooth rational curves  $\Gamma$  on  $Y$  and every smooth rational curve on  $Y$  has the self-intersection number  $-2$ .

Let  $Y$  be a complex *generic* Enriques surface. Then we have  $\mathcal{P}_Y = N_Y$ . Barth and Peters [1] showed that  $\text{Aut}(Y)$  is canonically identified with the kernel of the mod 2-reduction homomorphism  $\text{O}(S_Y, \mathcal{P}) \rightarrow \text{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2$ . Since a Vinberg chamber has no automorphism group, the group  $\text{O}(S_Y, \mathcal{P})$  is equal to the subgroup  $W(S_Y)$ . Since the mod 2-reduction homomorphism above is surjective (see [1] and Section 2.3 of this paper), there exists a union  $\mathcal{V}$  of

$$|\text{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2| = 46998591897600 = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$$

Vinberg chambers such that (i)  $\mathcal{P}(L_{10})$  is the union of  $\mathcal{V}^g$ , where  $g$  runs through  $\text{Aut}(Y)$ , and (ii) if  $g \in \text{Aut}(Y)$  is not the identity, then the interiors of  $\mathcal{V}$  and of  $\mathcal{V}^g$  are disjoint.

Kondo [12] and Nikulin [16] classified all complex Enriques surfaces with finite automorphism group. This classification was extended to odd characteristics by Martin [14]. It turns out that Enriques surfaces in characteristic  $\neq 2$  with finite automorphism group are divided into 7 classes I,  $\dots$ , VII. An Enriques surface  $Y$  with finite automorphism group has only a finite number of smooth rational curves  $\Gamma$ , and  $N_Y$  is bounded by the hyperplanes  $([\Gamma])^\perp$  defined by these curves. The configurations of these smooth rational curves are explicitly depicted in [12].

**Explanation of Table 1.2.** The item `walls` is the number of walls of an induced chamber  $D$ . Since every wall of  $D$  is defined by a  $(-2)$ -vector of  $L_{10}$ , it follows that  $D$  is a union of Vinberg chambers. The item `volindex` shows that the number of Vinberg chambers contained in  $D$  is equal to

$$|\text{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2| / \text{volindex} = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 / \text{volindex}.$$

The item `gd` is the order of the automorphism group  $\text{O}(L_{10}, D)$  of  $D$ . The item `orb` describes the orbit decomposition of the set of walls under the action of  $\text{O}(L_{10}, D)$ . The item `isom` shows that, for example, the induced chambers of the primitive embeddings 20C and 20D are isomorphic. The item `NK` shows that, for example, the induced chamber of the primitive embedding 12A is, under a suitable isomorphism  $L_{10} \cong S_Y$ , equal to  $N_Y$  of an Enriques surface  $Y$  with finite automorphism group of type I.

Since all 7 types I,  $\dots$ , VII appear in the column `NK`, our results on the induced chamber  $D$  can be applied to  $N_Y$  for an arbitrary Enriques surface  $Y$  with finite automorphism group in characteristic  $\neq 2$ .

Borchers' method has been applied to Enriques surfaces in [22] and [25] without using the facts proved in this paper. These facts actually give us a big advantage in the calculation of the automorphism group  $\text{Aut}(Y)$  of an Enriques surface  $Y$  by Borchers' method, as is exemplified in [5] and [26]. We can also enumerate all polarizations of  $Y$  with a fixed degree modulo  $\text{Aut}(Y)$  by means of the method in [23]. These applications will be treated in other papers. In [9], an interesting relation of our list with Enriques surfaces in characteristic 2 is discussed.

For the computation, the first author used a mixture of SageMath, PARI, GAP [29, 28, 27], and the second author used GAP [27]. The explicit computational data is available at the second author's webpage [24].

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**Notation.** To avoid possible confusions between  $L_{10}$  and  $L_{10}(2)$ , we put

$$\mathbf{S} := L_{10}(2).$$

We identify the underlying  $\mathbb{Z}$ -modules of  $L_{10}$  and  $\mathbf{S}$ , and choose positive cones so that  $\mathcal{P}(L_{10}) = \mathcal{P}(\mathbf{S})$ . We also have a natural identification  $O(L_{10}, \mathcal{P}) = O(\mathbf{S}, \mathcal{P})$ . We denote by  $\langle \cdot, \cdot \rangle_{\mathbf{S}}$ ,  $\langle \cdot, \cdot \rangle_{10}$  and  $\langle \cdot, \cdot \rangle_{26}$  the symmetric bilinear forms of  $\mathbf{S}$ ,  $L_{10}$ , and  $L_{26}$ , respectively.

## 2. PROOF OF THEOREMS 1.1 AND 1.10

**2.1. Discriminant form.** Let  $L$  be an even lattice. Recall that  $A(L) = L^\vee/L$  is the discriminant group of  $L$ . The quadratic form

$$q(L): A(L) \rightarrow \mathbb{Q}/2\mathbb{Z}$$

defined by  $u \bmod L \mapsto \langle u, u \rangle \bmod 2\mathbb{Z}$  for  $u \in L^\vee$  is called the *discriminant form* of  $L$ . Let  $O(q(L))$  denote the automorphism group of the finite quadratic form  $q(L)$ . Then we have a natural homomorphism

$$\eta(L): O(L) \rightarrow O(q(L)).$$

See Nikulin [15] for the basic properties of discriminant forms. Among these properties, the following is especially important for us:

**Proposition 2.1.** *Let  $M$  and  $N$  be even lattices. We consider the following sets:*

- (a) *the set  $\mathcal{L}$  of even unimodular lattices  $L$  contained in  $M^\vee \oplus N^\vee$ , containing  $M \oplus N$ , and containing each of  $M$  and  $N$  primitively, and*
- (b) *the set  $\mathcal{Q}$  of isomorphisms between the finite quadratic forms  $q(M)$  and  $-q(N)$ .*

*Let  $\phi$  be an isomorphism from  $q(M)$  to  $-q(N)$ , let  $\Gamma_\phi \subset A(M) \oplus A(N)$  denote the graph of  $\phi$ , and let  $L_\phi \subset M^\vee \oplus N^\vee$  be the pull-back of  $\Gamma_\phi$  by the natural projection  $M^\vee \oplus N^\vee \rightarrow A(M) \oplus A(N)$ . Then the mapping  $\phi \mapsto L_\phi$  gives rise to a bijection from  $\mathcal{Q}$  to  $\mathcal{L}$ . This bijection  $\mathcal{Q} \cong \mathcal{L}$  is compatible with the natural actions of  $O(M) \times O(N)$  on  $\mathcal{Q}$  and on  $\mathcal{L}$ .  $\square$*

Suppose that  $L \in \mathcal{L}$ , so that  $N$  is the orthogonal complement of the primitive sublattice  $M \subset L$ . Let  $\phi: q(M) \xrightarrow{\sim} -q(N)$  be the isomorphism corresponding to  $L$ , and  $O(\phi): O(q(M)) \xrightarrow{\sim} O(q(N))$  the induced isomorphism. We put

$$O(L, M) := \{ \tilde{g} \in O(L) \mid \tilde{g} \text{ preserves } M \},$$

and let  $\tilde{g} \mapsto \tilde{g}|_M$  and  $\tilde{g} \mapsto \tilde{g}|_N$  denote the restriction homomorphisms from  $O(L, M)$  to  $O(M)$  and  $O(N)$ , respectively. We say that  $\tilde{g} \in O(L, M)$  is a *lift* of  $g \in O(M)$  if  $\tilde{g}|_M = g$ .

**Corollary 2.2.** *Let  $g$  be an isometry of  $M$ . Then the homomorphism  $\tilde{g} \mapsto \tilde{g}|_N$  induces a bijection from the set of lifts  $\tilde{g}$  of  $g$  to the set of all isometries  $h \in O(N)$  of  $N$  such that  $\eta(M)(g) \in O(q(M))$  is mapped to  $\eta(N)(h) \in O(q(N))$  by  $O(\phi)$ .  $\square$*

**2.2. Kneser's neighbor method.** This method allows us to efficiently compute all lattices in a given genus. We review the basic idea. For proofs and a more complete treatment, see [11] and [19]. In this subsection, we assume that all lattices are positive or negative definite.

Recall that two lattices  $L$  and  $L'$  are *in the same genus* if we have isomorphisms

$$L \otimes \mathbb{Z}_p \cong L' \otimes \mathbb{Z}_p \quad \text{and} \quad L \otimes \mathbb{R} \cong L' \otimes \mathbb{R}$$

of  $\mathbb{Z}_p$ - or  $\mathbb{R}$ -valued quadratic modules for every prime  $p$ , where  $\mathbb{Z}_p$  denotes the ring of  $p$ -adic integers. Suppose that  $L$  and  $L'$  are in the same genus. Then, by the Hasse–Minkowski theorem, we have  $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$ . Thus we may and will assume that  $L \otimes \mathbb{Q} = L' \otimes \mathbb{Q}$ . Let  $p$  be an odd prime which does not divide the determinant  $\det L := |A(L)|$  of  $L$ . We say that two lattices  $L$  and  $L'$  are  *$p$ -neighbors* if

$$p = [L : L \cap L'] = [L' : L \cap L'].$$

Suppose that  $L$  and  $L'$  are  $p$ -neighbors. Then  $L \otimes \mathbb{Z}_q = L' \otimes \mathbb{Z}_q$  for all primes  $q \neq p$ . Moreover, since  $p$  does not divide  $\det L$ , both  $L \otimes \mathbb{Z}_p$  and  $L' \otimes \mathbb{Z}_p$  are unimodular  $\mathbb{Z}_p$ -lattices isomorphic over the field of  $p$ -adic rationals  $\mathbb{Q}_p$ . Thus  $L \otimes \mathbb{Z}_p$  and  $L' \otimes \mathbb{Z}_p$  are in fact isomorphic. We have proved that the  $p$ -neighbors  $L$  and  $L'$  are in the same genus.

For a given genus  $\mathcal{G}$ , we denote by  $C$  the set of isomorphism classes  $[L]$  of lattices  $L$  in this genus. Let  $p$  be an odd prime. Set

$$E := \{([L], [L']) \in C \times C \mid L \text{ and } L' \text{ are } p\text{-neighbors}\}.$$

Then  $(C, E)$  is called the  *$p$ -neighbor graph* of  $\mathcal{G}$ . Assume further that  $L \otimes \mathbb{Z}_p$  represents 0 for a lattice  $L$  in this genus. This is certainly the case if the rank of  $L$  is at least 5. In general each connected component of this graph is the union of several so called proper spinor genera. In the case relevant to us, the genus consists of a single proper spinor genus, so this does not concern us.

For given  $L$  and  $v \in L \setminus pL$  with  $\langle v, v \rangle \in p^2\mathbb{Z}_p$ , the lattice

$$L(v) := L_v + \mathbb{Z}(v/p) \quad \text{where} \quad L_v = \{x \in L \mid \langle x, v \rangle \in p\mathbb{Z}\}$$

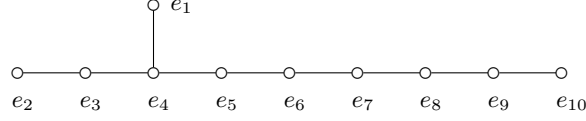
is called the  *$p$ -neighbor of  $L$  with respect to  $v$* . One can show that  $L$  and  $L(v)$  are indeed  $p$ -neighbors, that  $L_v$  depends only on  $v \bmod pL$  (as long as  $\langle v, v \rangle$  stays divisible by  $p^2$ ), and that every  $p$ -neighbor of  $L$  arises in this fashion.

Thus one can classify lattices in the genus  $\mathcal{G}$  by iteratively computing the neighbors of the lattices in  $C$  and testing for isomorphism (see [18]). One can speed this up by computing the neighbors of a given lattice only up to the action of the orthogonal group. When we are interested only in the vertices and not in the edges, we can break the computation when we have “explored” all vertices. The *mass* of the genus  $\mathcal{G}$  is defined as

$$\text{mass}(\mathcal{G}) := \sum_{[L] \in \mathcal{G}} \frac{1}{|O(L)|}.$$

It can be calculated from the invariants of  $\mathcal{G}$  alone as described in [7]. We can break the computation as soon as the sum of the reciprocals of  $|O(L)|$  reaches  $\text{mass}(\mathcal{G})$ .

This procedure is implemented for example in **Magma** [4]. In the example relevant to us, the computation with **Magma** simply exhausted all memory available. Thus we had to resort to a modified strategy: A random walk through the neighbor graph.

FIGURE 2.1. Basis of  $L_{10}$ 

**2.3. Proof of Theorem 1.1.** Let  $e_1, \dots, e_{10}$  be a basis of  $L_{10}$  consisting of  $(-2)$ -vectors that form the configuration in Figure 2.1. Then

$$(2.1) \quad V := \{x \in \mathcal{P}(L_{10}) \mid \langle x, e_i \rangle_{10} \geq 0 \text{ for } i = 1, \dots, 10\}$$

is a Vinberg chamber, and each  $V \cap (e_i)^\perp$  is a wall of  $V$  (see Vinberg [30]). Since the graph in Figure 2.1 has no non-trivial automorphisms, the group  $O(L_{10}, \mathcal{P})$  is generated by the 10 reflections  $s_1, \dots, s_{10}$  with respect to  $e_1, \dots, e_{10}$ . Recall from the paragraph Notation at the end of Introduction that we put  $\mathbf{S} := L_{10}(2)$ . In  $L_{10} \otimes \mathbb{Q} = \mathbf{S} \otimes \mathbb{Q}$ , we have  $L_{10} = L_{10}^\vee = \mathbf{S} \subset \mathbf{S}^\vee$ , and the mapping  $v \mapsto v/2$  gives an isomorphism  $L_{10} \cong \mathbf{S}^\vee$  of  $\mathbb{Z}$ -modules, which gives rise to an isomorphism

$$(L_{10}/2L_{10}, q_L) \cong q(\mathbf{S}), \quad \text{where } q_L(u \bmod 2L_{10}) := \frac{1}{2} \langle u, u \rangle_{10} \bmod 2\mathbb{Z}$$

of finite quadratic forms. Hence we see that  $|O(q(\mathbf{S}))| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$  by Proposition 1.7 of [1]. Since we have explicit generators  $s_1, \dots, s_{10}$  of  $O(L_{10}, \mathcal{P}) = O(\mathbf{S}, \mathcal{P})$ , we can confirm that  $\eta(\mathbf{S}): O(\mathbf{S}) \rightarrow O(q(\mathbf{S}))$  restricted to  $O(L_{10}, \mathcal{P})$  is surjective. Let  $\iota: \mathbf{S} \hookrightarrow L_{26}$  be a primitive embedding, and let  $R_\iota$  be the orthogonal complement of the image of  $\iota$  in  $L_{26}$ . Then  $R_\iota$  is of signature  $(0, 16)$ . By Proposition 2.1, the discriminant form  $q(R_\iota)$  is isomorphic to  $-q(\mathbf{S})$ . Since  $\eta(\mathbf{S})$  is surjective, Proposition 2.1 implies that, if a primitive embedding  $\iota': \mathbf{S} \hookrightarrow L_{26}$  satisfies  $R_{\iota'} \cong R_\iota$ , then  $\iota'$  is equivalent to  $\iota$  up to the action of  $O(\mathbf{S}) = O(L_{10})$  and  $O(L_{26})$ . Hence the proof of Theorem 1.1 is reduced to the classification of isomorphism classes of even lattices  $R$  with signature  $(0, 16)$  such that  $q(R) \cong -q(\mathbf{S})$ . Note that these conditions on signature and discriminant form determine the genus  $\mathcal{G}_R$  of  $R$ . By the mass formula [7], we see that the mass of this genus is

$$\text{mass}(\mathcal{G}_R) = 64150367/28766348771328000.$$

Let  $u: \mathbb{F}_2^2 \rightarrow \mathbb{Q}/2\mathbb{Z}$  be defined by  $u(x, y) := xy$ . Then one calculates that

$$-q(\mathbf{S}) \cong -u^{\oplus 5} = u^{\oplus 5} \cong q(R)$$

and that  $q(D_8) \cong u$  and  $q(E_8(2)) \cong u^{\oplus 4}$ , where  $D_8$  and  $E_8$  are the negative-definite root lattices of ADE-type  $D_8$  and  $E_8$ , respectively. Thus we have found a first lattice  $L = D_8 \oplus E_8(2)$  in  $\mathcal{G}_R$ . To find representatives up to isomorphism, we use a variant of Kneser's neighbor method for  $p = 3$ . Start by inserting  $L$  into a list  $C$ . Then enter the following loop. Pick a random  $L$  in  $C$  and a random  $v \in L \setminus 3L$  with  $\langle v, v \rangle$  divisible by 3, replace  $v$  by  $v + 3w$  for  $w \in L$  such that  $\langle v, v \rangle$  is divisible by 9. Calculate the 3-neighbor  $L(v)$  and check if it is isomorphic to any lattice in the list  $C$ . If not add it to  $C$ . Break the loop when the mass of the lattices in  $C$  matches  $\text{mass}(\mathcal{G}_R)$ . By this computation, it turns out that  $\mathcal{G}_R$  is constituted by 17 isomorphism classes in Table 1.1, and hence Theorems 1.1 follows.  $\square$



**2.4. Conway theory.** Let  $\mathbf{w}$  be a non-zero primitive vector of  $L_{26}$  contained in  $\partial\overline{\mathcal{P}}(L_{26})$ . Note that  $\langle \mathbf{w}, \mathbf{w} \rangle_{26} = 0$ . We put

$$[\mathbf{w}] := \mathbb{Z}\mathbf{w}, \quad [\mathbf{w}]^\perp := \{v \in L_{26} \mid \langle v, \mathbf{w} \rangle_{26} = 0\}.$$

Then  $[\mathbf{w}]^\perp/[\mathbf{w}]$  has a natural structure of an even unimodular negative-definite lattice, and hence is isomorphic to  $N(-1)$ , where  $N$  is one of the 24 Niemeier lattices (see, for example, Chapter 16 of [8]).

**Definition 2.3.** We say that  $\mathbf{w}$  is a Weyl vector if  $[\mathbf{w}]^\perp/[\mathbf{w}]$  is isomorphic to the negative-definite Leech lattice.

Since the Leech lattice is characterized as the unique Niemeier lattice with no roots, we can determine whether  $\mathbf{w}$  is a Weyl vector or not by calculating the set  $\mathcal{R}([\mathbf{w}]^\perp/[\mathbf{w}])$  of  $(-2)$ -vectors in  $[\mathbf{w}]^\perp/[\mathbf{w}]$ .

For a Weyl vector  $\mathbf{w}$ , we put

$$C(\mathbf{w}) := \{x \in \mathcal{P}(L_{26}) \mid \langle x, r \rangle_{26} \geq 0 \text{ for all } r \in \mathcal{R}(L_{26}) \text{ with } \langle r, \mathbf{w} \rangle_{26} = 1\}.$$

The following theorem is very important.

**Theorem 2.4** (Conway [6]). *The mapping  $\mathbf{w} \mapsto C(\mathbf{w})$  gives a bijection from the set of Weyl vectors to the set of Conway chambers.*  $\square$

*Remark 2.5.* Let  $\mathbf{w}$  be a Weyl vector. Since  $\mathbf{w}$  is primitive and  $L_{26}$  is unimodular, there exists a vector  $\mathbf{w}'$  such that  $\langle \mathbf{w}', \mathbf{w}' \rangle_{26} = 0$  and  $\langle \mathbf{w}, \mathbf{w}' \rangle_{26} = 1$ . Then every  $(-2)$ -vector  $r$  of  $L_{26}$  with  $\langle \mathbf{w}, r \rangle_{26} = 1$  is written as

$$\alpha_\lambda \mathbf{w} + \mathbf{w}' + \lambda, \quad \text{where } \alpha_\lambda = \frac{-\langle \lambda, \lambda \rangle_{26} - 2}{2} \text{ and } \langle \mathbf{w}, \lambda \rangle_{26} = \langle \mathbf{w}', \lambda \rangle_{26} = 0.$$

Since  $\langle \mathbf{w}, \lambda \rangle_{26} = \langle \mathbf{w}', \lambda \rangle_{26} = 0$  implies  $\langle \lambda, \lambda \rangle_{26} \leq 0$ , we see that  $a_{26} := 2\mathbf{w} + \mathbf{w}'$  is an interior point of  $C(\mathbf{w})$ .

**2.5. Proof of Theorem 1.10.** In Section 2.3, we have calculated the 17 primitive embeddings  $\iota: \mathbf{S} \hookrightarrow L_{26}$  explicitly. As was said in Notation, we identify  $\mathcal{P}(\mathbf{S})$  and  $\mathcal{P}(L_{10})$ , and denote by  $\iota_{\mathcal{P}}: \mathcal{P}(L_{10}) \hookrightarrow \mathcal{P}(L_{26})$  the induced inclusion.

Our first task is to find a Weyl vector  $\mathbf{w}$  such that  $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$  is an induced chamber, that is,  $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$  contains a non-empty open subset of  $\mathcal{P}(L_{10})$ . Recall that we have fixed a basis  $e_1, \dots, e_{10}$  of  $L_{10}$ . We put  $a_{10} := e_1^\vee + \dots + e_{10}^\vee$ , where  $e_1^\vee, \dots, e_{10}^\vee$  are the basis of  $L_{10}^\vee = L_{10}$  dual to  $e_1, \dots, e_{10}$ . Then  $a_{10}$  is an interior point of the Vinberg chamber  $V$  defined by (2.1), and we have  $\langle a_{10}, a_{10} \rangle_{10} = 1240$ . By direct calculation, we confirm the equality

$$(2.2) \quad \{r \in \mathcal{R}(L_{26}) \mid \langle r, \iota(a_{10}) \rangle_{26} = 0\} = \{r \in \mathcal{R}(L_{26}) \mid \langle r, \iota(x) \rangle_{26} = 0 \text{ for all } x \in L_{10}\},$$

which means that, if a hyperplane  $(r)^\perp$  of  $\mathcal{P}(L_{26})$  defined by  $r \in \mathcal{R}(L_{26})$  passes through  $\iota(a_{10})$ , then  $(r)^\perp$  contains the image of  $\iota_{\mathcal{P}}$ . (Note that the second set in (2.2) is identified with  $\mathcal{R}(R_i)$  by the embedding  $R_i \hookrightarrow L_{26}$ .) Therefore  $a_{10}$  is an interior point of an induced chamber  $D$ .

**Definition 2.6.** Let  $L$  be an even hyperbolic lattice. Suppose that  $v_1, v_2$  are vectors of  $\mathcal{P}(L) \cap (L \otimes \mathbb{Q})$ . We say that a  $(-2)$ -vector  $r \in \mathcal{R}(L)$  separates  $v_1$  and  $v_2$  if  $\langle r, v_1 \rangle \cdot \langle r, v_2 \rangle < 0$ . We can calculate the set of  $(-2)$ -vectors of  $L$  separating  $v_1$  and  $v_2$  by the algorithm given in Section 3.3 of [20].

We perturb  $a_{10}$  to  $a'_{10} \in \mathcal{P}(L_{10}) \cap (L_{10} \otimes \mathbb{Q})$  in a general direction so that  $a'_{10}$  is also an interior point of the same induced chamber  $D$  as  $a_{10}$ , that is, the equality (2.2) remains true with  $\iota(a_{10})$  replaced by  $\iota(a'_{10})$  and there exist no  $(-2)$ -vectors  $r$  of  $L_{26}$  separating  $\iota(a_{10})$  and  $\iota(a'_{10})$ . We choose an arbitrary Weyl vector  $\mathbf{w}_0$  of  $L_{26}$ , and calculate a vector  $a_{26} \in L_{26}$  in the interior of  $C(\mathbf{w}_0)$  by Remark 2.5. We then calculate the set  $\{\pm r_1, \dots, \pm r_N\}$  of  $(-2)$ -vectors of  $L_{26}$  separating  $\iota(a'_{10})$  and  $a_{26}$ . We sort these  $(-2)$ -vectors  $r_1, \dots, r_N$  in such a way that the line segment from  $a_{26}$  to  $\iota(a'_{10})$  intersects the hyperplanes  $(r_1)^\perp, \dots, (r_N)^\perp$  in this order. Since  $a'_{10}$  is a result of general perturbation, these  $N$  intersection points are distinct. Let  $s_\nu \in \mathcal{O}(L_{26}, \mathcal{P})$  be the reflection with respect to  $r_\nu$ . We move  $\mathbf{w}_0$  by  $s_1, \dots, s_N$  in this order and obtain a new Weyl vector  $\mathbf{w} := \mathbf{w}_0^{s_1 \cdots s_N}$ . Then  $C(\mathbf{w})$  contains  $a_{26}^{s_1 \cdots s_N}$  in its interior, and there exist no  $(-2)$ -vectors of  $L_{26}$  separating  $\iota(a'_{10})$  and  $a_{26}^{s_1 \cdots s_N}$ . Therefore  $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$  is the induced chamber  $D$  containing  $a_{10}$  in its interior.

Next we calculate the set of walls of the induced chamber  $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ . We denote by  $v \mapsto v_{\mathbf{S}}$  and  $v \mapsto v_R$  the orthogonal projections  $L_{26} \rightarrow \mathbf{S}^\vee$  and  $L_{26} \rightarrow R_l^\vee$ , and let  $\langle \cdot, \cdot \rangle_R$  denote the symmetric bilinear form of  $R_l$ . It turns out that  $\langle \mathbf{w}_{\mathbf{S}}, \mathbf{w}_{\mathbf{S}} \rangle_{\mathbf{S}} > 0$  holds except for the case where  $\iota$  is of type **inf**ty. Henceforth we assume that  $\iota$  is not of type **inf**ty. We put

$$\begin{aligned} \mathcal{R}(L_{26}, \mathbf{w}) &:= \{r \in \mathcal{R}(L_{26}) \mid \langle \mathbf{w}, r \rangle_{26} = 1\}, \\ \mathcal{R}(L_{26}, D) &:= \{r \in \mathcal{R}(L_{26}, \mathbf{w}) \mid \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0\}, \\ \mathcal{R}_D &:= \{r_{\mathbf{S}} \mid r \in \mathcal{R}(L_{26}, D)\}. \end{aligned}$$

For  $r \in \mathcal{R}(L_{26})$ , the hyperplane  $(r)^\perp$  of  $\mathcal{P}(L_{26})$  intersects the image of  $\iota_{\mathcal{P}}$  if and only if  $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0$ . Hence we have

$$D = \{x \in \mathcal{P}(L_{10}) \mid \langle r_{\mathbf{S}}, x \rangle_{\mathbf{S}} \geq 0 \text{ for all } r_{\mathbf{S}} \in \mathcal{R}_D\}.$$

The set  $\mathcal{R}_D$  can be calculated explicitly as follows. Suppose that  $r \in \mathcal{R}(L_{26}, D)$ . Then we have

$$\langle \mathbf{w}_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} + \langle \mathbf{w}_R, r_R \rangle_R = 1, \quad \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} + \langle r_R, r_R \rangle_R = -2.$$

Since  $R_l$  is negative-definite, the condition  $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0$  implies  $-2 < \langle r_R, r_R \rangle_R \leq 0$ , and if  $\langle r_R, r_R \rangle_R = 0$ , then we would have  $r_R = 0$ ,  $r = \iota(r_{\mathbf{S}})$  and hence  $r_{\mathbf{S}} \in \mathbf{S}$ , which is impossible because  $\langle r, r \rangle_{26} = -2$  whereas  $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} = 2\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{10}$  is a multiple of 4. Since  $\mathbf{S}^\vee = \frac{1}{2}\mathbf{S}$ , the discriminant form of  $\mathbf{S}$  takes values in  $\mathbb{Z}/2\mathbb{Z}$ . Naturally, the same is true for  $q(R_l) \cong -q(\mathbf{S})$ . This means that  $\langle v, v \rangle_R$  is integral for any  $v \in R_l^\vee$ . In particular, if  $v \in R_l^\vee$  satisfies  $-2 < \langle v, v \rangle_R < 0$ , then  $\langle v, v \rangle_R = -1$ . Therefore we have  $r_R \in V_R$  for all  $r \in \mathcal{R}(L_{26}, D)$ , where

$$V_R := \{v \in R_l^\vee \mid \langle v, v \rangle_R = -1\}.$$

For  $v \in V_R$ , we put  $a(v) := 1 - \langle \mathbf{w}_R, v \rangle_R$ , and let  $a(V_R)$  be the set  $\{a(v) \mid v \in V_R\}$ . For each  $a \in a(V_R)$ , we calculate

$$V_{\mathbf{S}}(a) := \{u \in \mathbf{S}^\vee \mid \langle \mathbf{w}_{\mathbf{S}}, u \rangle_{\mathbf{S}} = a, \langle u, u \rangle_{\mathbf{S}} = -1\},$$

which is finite because  $\langle \mathbf{w}_{\mathbf{S}}, \mathbf{w}_{\mathbf{S}} \rangle_{\mathbf{S}} > 0$ . We calculate the sum  $u + v \in \mathbf{S}^\vee \oplus R_l^\vee$  for all pairs  $(v, u)$  of  $v \in V_R$  and  $u \in V_{\mathbf{S}}(a(v))$ , and check whether  $u + v$  is in  $L_{26}$  or not. Thus we calculate

$$\mathcal{R}(L_{26}, D) = L_{26} \cap \{u + v \mid v \in V_R, u \in V_{\mathbf{S}}(a(v))\}$$

and the set  $\mathcal{R}_D$ . It turns out that the hyperplanes  $(r_{\mathbf{S}})^\perp$  for  $r_{\mathbf{S}} \in \mathcal{R}_D$  are distinct, and that  $\mathcal{R}_D$  spans  $L_{10} \otimes \mathbb{Q}$  over  $\mathbb{Q}$ . We will show that each  $r_{\mathbf{S}} \in \mathcal{R}_D$  defines a wall of

$D$ . We can calculate the finite group  $G$  of isometries of  $L_{10}$  that preserves the finite set  $\mathcal{R}_D \subset L_{10}$ . Recall that  $a_{10}$  is an interior point of  $D$ . We put  $\sigma_{10} := \sum_{g \in G} a_{10}^g$ , which is an interior point of  $D$  fixed by  $G$ . We put  $t := -\langle \sigma_{10}, r_{\mathbf{S}} \rangle_{\mathbf{S}} / \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}}$ , and consider the point  $\sigma'_{10} := \sigma_{10} + t \cdot r_{\mathbf{S}}$  on  $(r_{\mathbf{S}})^{\perp}$ . Then we have  $\langle \sigma'_{10}, r'_{\mathbf{S}} \rangle_{\mathbf{S}} > 0$  for all  $r'_{\mathbf{S}} \in \mathcal{R}_D \setminus \{r_{\mathbf{S}}\}$ , which means that  $\sigma'_{10}$  is an interior point of the subset  $D \cap (r_{\mathbf{S}})^{\perp}$  of  $(r_{\mathbf{S}})^{\perp}$ . Therefore  $D \cap (r_{\mathbf{S}})^{\perp}$  is a wall of  $D$ . Note that, since  $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} = -1$  for  $r_{\mathbf{S}} \in \mathcal{R}_D$  and  $2\mathbf{S}^{\vee} \subset \mathbf{S}$ , we see that  $2r_{\mathbf{S}} \in L_{10}$  and  $\langle 2r_{\mathbf{S}}, 2r_{\mathbf{S}} \rangle_{10} = -2$ . Therefore each wall of  $D$  is defined by a  $(-2)$ -vector  $2r_{\mathbf{S}}$  of  $L_{10}$ . The group  $G$  is equal to  $O(L_{10}, D)$ .

The assertions in Theorem 1.10 and Table 1.2 about the walls of an induced chamber are now proved for the induced chamber  $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$  defined by this particular Weyl vector  $\mathbf{w}$ . The data `volindex` of  $D$  in Table 1.2 is calculated by the method given in [25].

To prove that the induced tessellation of  $\mathcal{P}(L_{10})$  is simple and thus complete the proof of Theorem 1.10, it is enough to prove the following:

**Proposition 2.7.** *For each wall  $D \cap (r)^{\perp}$  of  $D$ , where  $r \in \mathcal{R}(L_{10})$ , there exists an isometry  $\tilde{g} \in O(L_{26}, \mathcal{P})$  with the following property: the isometry  $\tilde{g}$  preserves the image of  $\iota: \mathbf{S} \hookrightarrow L_{26}$  and its restriction  $\tilde{g}|_{\mathbf{S}} \in O(\mathbf{S}, \mathcal{P}) = O(L_{10}, \mathcal{P})$  to  $\mathbf{S}$  is equal to the reflection  $s_r \in O(L_{10}, \mathcal{P})$  with respect to  $r \in \mathcal{R}(L_{10})$ .*

Suppose that Proposition 2.7 is proved. Since  $\tilde{g}$  preserves  $\mathcal{R}(L_{26})$ , the isometry  $\tilde{g}|_{\mathbf{S}} = s_r$  of  $\mathbf{S}$  preserves the family  $\iota^* \mathcal{R}(L_{26})^{\perp}$  of hyperplanes and hence preserves the tessellation of  $\mathcal{P}(L_{10})$  by induced chambers. Since  $D$  and  $D^{s_r}$  have the common wall  $D \cap (r)^{\perp}$ , it follows that  $D^{s_r}$  is the induced chamber adjacent to  $D$  across the wall  $D \cap (r)^{\perp}$ . For any induced chamber  $D'$ , there exists a chain of induced chambers

$$D = D^{(0)}, D^{(1)}, \dots, D^{(m)} = D'$$

such that  $D^{(i-1)}$  and  $D^{(i)}$  are adjacent for  $i = 1, \dots, m$ . By induction on the length  $m$  of the chain, we can prove that there exists an isometry  $\tilde{g}' \in O(L_{26}, \mathcal{P})$  preserving  $\iota(\mathbf{S})$  such that the induced isometry  $\tilde{g}'|_{\mathbf{S}}$  of  $\mathbf{S}$  maps  $D$  to  $D'$ . Therefore the tessellation of  $\mathcal{P}(L_{10})$  by induced chambers is simple.

*Proof of Proposition 2.7.* The isometry  $\tilde{g}$  with the hoped-for property is explicitly given in [24] for each wall of  $D$ , and thus the proof of Theorem 1.10 is completed.  $\square$

We explain the method by which we found the isometry  $\tilde{g} \in O(L_{26}, \mathcal{P})$ . It is based on some optimistic heuristics. The isometry  $\tilde{g}$  does not necessarily have to satisfy the conditions (i) and (ii) below. Fortunately, this method worked for every wall  $w = D \cap (r)^{\perp}$  of the induced chamber  $D$ . We put

$$Q := \{v \in L_{26} \mid \langle v, x \rangle_{26} = 0 \text{ for all } x \in \iota_{\mathcal{P}}(w)\}.$$

Since  $\dim(r)^{\perp} = 9$ , the even lattice  $Q$  is negative-definite of rank  $26 - 9 = 17$  and contains  $R_{\iota}$ . Let  $\langle \cdot, \cdot \rangle_Q$  denote the symmetric bilinear form of  $Q$ . We calculate the set  $\mathcal{R}(Q)$  of  $(-2)$ -vectors of  $Q$ . The hyperplanes of  $Q \otimes \mathbb{R}$  defined by  $\langle x, r' \rangle_Q = 0$ , where  $r' \in \mathcal{R}(Q)$ , divide  $Q \otimes \mathbb{R}$  into a finite number of regions, and they correspond bijectively to the Conway chambers containing  $\iota_{\mathcal{P}}(w)$ . We put

$$(2.3) \quad \Sigma := \{r' \in \mathcal{R}(Q) \mid \langle \mathbf{w}, r' \rangle_{26} = 1\},$$

No.	name	$ V_R $	$d_{\mathbf{w}}$	$n_{\mathbf{w}}$	$a_r$	ADE-type of $\Sigma$	numb
1	12A	256	1	70	1	$2A_1 + D_8$	10
					8	$D_9$	2
2	12B	144	2	21	1	$2A_1 + A_7$	12
3	20A	160	1	22	1	$2A_1 + D_4 + D_5$	10
					4	$2D_5$	6
					5	$D_4 + D_6$	4
4	20B	128	1	14	1	$2A_1 + 2D_4$	12
					4	$D_4 + D_5$	8
5	20C	192	1	30	2	$8A_1 + A_3 + D_6$	15
					6	$10A_1 + D_7$	5
6	20D	96	2	15/2	1	$2A_1 + A_3 + A_4$	15
					3	$A_4 + D_4$	5
7	20E	112	1	10	1	$7A_1 + A_5$	10
					2	$3A_1 + A_3 + A_5$	10
8	20F	80	2	5	1	$2A_1 + 2A_3$	20
9	40A	96	1	6	1	$6A_1 + 2A_3$	12
					2	$2A_1 + 3A_3$	12
					3	$4A_1 + A_3 + D_4$	16
10	40B	160	1	16	2	$6A_1 + A_3 + 2D_4$	16
					4	$8A_1 + D_4 + D_5$	24
11	40C	80	1	4	1	$8A_1 + A_3$	16
					2	$4A_1 + 2A_3$	24
12	40D	128	1	10	2	$10A_1 + A_3 + D_4$	30
					4	$12A_1 + D_5$	10
13	40E	64	2	5/2	1	$4A_1 + 2A_2$	30
					2	$2A_2 + A_3$	10
14	96A	64	1	2	1	$10A_1$	32
					2	$6A_1 + A_3$	64
15	96B	96	1	4	2	$14A_1 + A_3$	96
16	96C	48	2	1	1	$6A_1$	96
17	infty	32	1	0			

TABLE 2.1. Walls of  $D$ 

where we regard  $\mathcal{R}(Q)$  as a subset of  $\mathcal{R}(L_{26})$  by the embedding  $Q \hookrightarrow L_{26}$ . Let  $P_w$  be an interior point of the wall  $w = D \cap (r)^\perp$  in  $(r)^\perp$ . Locally around  $P_w$ , the Conway chamber  $C(\mathbf{w})$  is defined by the inequalities  $\langle x, r' \rangle_{26} \geq 0$ , where  $r'$  runs through  $\Sigma$ . Let  $C(\mathbf{w}')$  be the Conway chamber defined locally around  $P_w$  by the opposite inequalities  $\langle x, r' \rangle_{26} \leq 0$ , where  $r'$  runs through  $\Sigma$ . Then  $C(\mathbf{w}')$  is one of the Conway chambers that induce the induced chamber  $D'$  adjacent to  $D$  across the wall  $w$ ;  $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}')) = D'$ . We search for isometries  $\tilde{g}$  of  $L_{26}$  such that

- (i) the isometry  $\tilde{g}$  maps  $\Sigma$  to  $-\Sigma$ , and
- (ii) the restriction  $\tilde{g}|_{\langle \Sigma \rangle^\perp}$  of  $\tilde{g}$  to the orthogonal complement  $\langle \Sigma \rangle^\perp$  in  $L_{26}$  of the sublattice  $\langle \Sigma \rangle$  generated by  $\Sigma$  is the identity.

If  $\tilde{g}$  satisfies (i) and (ii), then  $\tilde{g}$  fixes  $\iota_{\mathcal{P}}(P_w) \in \langle \Sigma \rangle^\perp \otimes \mathbb{R}$  and  $\tilde{g}$  maps  $C(\mathbf{w})$  to  $C(\mathbf{w}')$ . We then check the following conditions:

- (iii) the isometry  $\tilde{g}$  preserves the image of  $\iota$ , and hence its restriction  $\tilde{g}|_{\mathbf{S}}$  to  $\mathbf{S}$  maps  $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$  to the adjacent chamber  $D' = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}'))$ , and
- (iv) the restriction  $\tilde{g}|_{\mathbf{S}}$  is equal to the reflection  $s_r$ .

If we find an isometry  $\tilde{g}$  satisfying (iii) and (iv), we are done.

**Explanation of Table 2.1.** We denote by  $d_{\mathbf{w}}$  the minimal positive integer such that  $d_{\mathbf{w}}\mathbf{w}_{\mathbf{S}} \in \mathbf{S}$ , where  $\mathbf{w}_{\mathbf{S}}$  is the image of  $\mathbf{w}$  by the orthogonal projection  $L_{26} \rightarrow \mathbf{S}^{\vee}$ . We put  $n_{\mathbf{w}} := \langle \mathbf{w}_{\mathbf{S}}, \mathbf{w}_{\mathbf{S}} \rangle_{10}$ . For a  $(-2)$ -vector  $r$  defining a wall of  $D$ , we put  $a_r := \langle \mathbf{w}_{\mathbf{S}}, r \rangle_{10}$ . The root system  $\Sigma$  is defined by (2.3). The item `numb` is the number of walls with the described properties.

*Remark 2.8.* Let  $\phi: q(\mathbf{S}) \xrightarrow{\sim} -q(R_{\iota})$  be the isomorphism induced by  $L_{26} \subset \mathbf{S}^{\vee} \oplus R_{\iota}^{\vee}$ , and  $O(\phi): O(q(\mathbf{S})) \xrightarrow{\sim} O(q(R_{\iota}))$  the isomorphism induced by  $\phi$ . Proposition 2.7 can also be proved by showing that the image of  $\eta(\mathbf{S})(s_r) \in O(q(\mathbf{S}))$  by  $O(\phi)$  belongs to the image of  $\eta(R_{\iota}): O(R_{\iota}) \rightarrow O(q(R_{\iota}))$ .

**Example 2.9.** In [17], Ohashi classified all fixed-point free involutions of the Kummer surface  $X := \text{Km}(\text{Jac}(C))$  associated with the Jacobian variety of a generic genus-2 curve  $C$ , and showed that  $X$  has exactly  $6 + 15 + 10$  fixed-point free involutions modulo conjugation in  $\text{Aut}(X)$ . The automorphism group  $\text{Aut}(X)$  had been calculated by Kondo [13] by Borcherds' method. Let  $\pi: X \rightarrow Y$  be the quotient morphism by a fixed-point free involution of  $X$ . We compose the embedding  $\iota_X: S_X \hookrightarrow L_{26}$  used in [13] with the pull-back homomorphism  $\pi^*: S_Y(2) \hookrightarrow S_X$ , and obtain a primitive embedding  $\iota_Y: S_Y(2) \hookrightarrow L_{26}$ . We see that  $\iota_Y$  is of type 20E for 6 Enriques surfaces, of type 40A for 15 Enriques surfaces, and of type 40C for 10 Enriques surfaces.

See [26] for more examples, and for applications to the calculation of  $\text{Aut}(Y)$ .

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