A NOTE ON CONFIGURATIONS OF (-2)-VECTORS ON ENRIQUES SURFACES

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1. Introduction

This note is a supplement of the joint paper [3] with S. Brandhorst.

It was established by Nikulin [7], Kondo [5], and Martin [6] that Enriques surfaces in characteristic $\neq 2$ with finite automorphism group are divided into seven classes $\mathrm{I}, \mathrm{II}, \dots, \mathrm{VII}$. The configurations of smooth rational curves on these Enriques surfaces are depicted in Kondo [5] by beautiful but complicated graphs.

A lattice of rank n is *hyperbolic* if the signature is $(1, n-1)$. A *positive cone* of a hyperbolic lattice L is a connected component of $\{\, x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0 \,\}$. For a positive integer n with $n \equiv 2 \bmod 8$, let L_n denote the even unimodular hyperbolic lattice of rank n, which is unique up to isomorphism. Borcherds' method [1, 2] is a method to calculate the automorphism group of an even hyperbolic lattice S by embedding S into L_{26} primitively and using the tessellation of a positive cone of L_{26} by Conway chambers. (See Chapter 27 of [4]. See [3] for the definition of Conway chambers.) This method has been applied to lattices S_X of numerical equivalence classes of divisors of various $K3$ surfaces X, and the automorphism group of these $K3$ surfaces are calculated.

The lattice S_Y of numerical equivalence classes of divisors of an Enriques surface Y is isomorphic to L_{10}. The universal covering $X \to Y$ of Y by a $K3$ surface X induces a primitive embedding $S_Y(2) \hookrightarrow S_X$, where $S_Y(2)$ is the lattice obtained from S_Y by multiplying the intersection form $\langle\ ,\ \rangle$ by 2. If S_X is embedded primitively into L_{26} in Borcherds' method, then $S_Y(2)$ is also embedded primitively into L_{26}. In [3], hoping to apply Borcherds' method to Enriques surfaces systematically, we have classified all primitive embeddings of $L_{10}(2)$ into L_{26}. It turns out that there exist exactly 17 primitive embeddings

`12A, 12B, 20A, ..., 20F, 40A, ..., 40E, 96A, 96B, 96C, infty`

up to the action of the orthogonal groups of L_{10} and L_{26}. Let $\mathcal{P}_{10}$ be a positive cone of L_{10}. For each of these primitive embeddings except for the type `infty`, we obtain a finite polyhedral cone in $\mathcal{P}_{10}$ bounded by hyperplanes perpendicular to (-2)-vectors in L_{10} such that $\mathcal{P}_{10}$ is tessellated by the image of reflections of this finite polyhedral cone with respect to the walls. The set of walls of this finite polyhedral cone defines a configuration of (-2)-vectors of L_{10}. The 7 configurations $\mathrm{I}, \mathrm{II}, \dots, \mathrm{VII}$ of Nikulin-Kondo appear among these 16 configurations.

In this note, we give a combinatorial description for each of these configurations. The result includes new descriptions of the Nikulin-Kondo configurations, which we hope are handier than the picturesque graphs of [5] in some situations.

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An explicit computational data is available at [10]. We used `GAP` [11] for the calculation.

Conventions. (1) A configuration is a pair (Γ, μ) of a finite set Γ and a mapping $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ such that $\mu(x, y) = \mu(y, x)$ for all $x, y \in \Gamma$. In this note, we always assume that

$$\mu(x, x) = -2 \quad \text{for all} \quad x \in \Gamma. \tag{1.1}$$

The automorphism group of a configuration (Γ, μ) is the group of permutations of Γ that preserve μ. The size of a configuration (Γ, μ) is $|\Gamma|$.

(2) The cyclic group of order n is denoted by C_n. The symmetric group of degree n is denoted by $\mathfrak{S}_n$, and the alternating group of degree n is denoted by $\mathfrak{A}_n$. Let I_n denote the identity matrix of size n. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the square matrix of size n whose components are all 1 and all 0, respectively.

2. Combinatorial descriptions

2.1. `12A`. The configuration of type `12A` is the configuration of Nikulin-Kondo type I (Fig. 1.4 of [5]). The automorphism group is isomorphic to $C_2 \times C_2$.

2.2. `12B`. The configuration of type `12B` is the configuration of Nikulin-Kondo type II (Fig. 2.4 of [5]). The automorphism group is isomorphic to $C_2 \times \mathfrak{S}_4$.

2.3. `20A`. The configuration of type `20A` is isomorphic to the configuration of Nikulin-Kondo type V (Fig. 5.5 of [5]).

Let A be the set $\{1, 2, 3, 4\}$, and B the set of subsets $\{i, j\}$ of A with size 2. Let A_1 and A_2 be two copies of A with the natural bijection to A denoted by $a \mapsto \bar{a}$. Let B_1 and B_2 be two copies of B with the natural bijection to B denoted by $b \mapsto \bar{b}$. We then put

$$\Gamma := A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2,$$

and define a symmetric function $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $a, a' \in A_1$ with $a \neq a'$. Then $\mu(a, a') = 0$.
- Suppose that $a \in A_1$ and $a' \in A_2$. Then
$$\mu(a, a') = \begin{cases} 2 & \text{if } \bar{a} = \bar{a}', \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $a, a' \in A_2$ with $a \neq a'$. Then $\mu(a, a') = 2$.
- Suppose that $a \in A_1$ and $b \in B_1$. Then $\mu(a, b) = 0$.
- Suppose that $a \in A_1$ and $b \in B_2$. Then
$$\mu(a, b) = \begin{cases} 1 & \text{if } \bar{a} \in \bar{b}, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $a \in A_2$ and $b \in B_1$. Then
$$\mu(a, b) = \begin{cases} 2 & \text{if } \bar{a} \in \bar{b}, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $a \in A_2$ and $b \in B_2$. Then $\mu(a, b) = 0$.
- Suppose that $b, b' \in B_1$ with $b \neq b'$. Then
$$\mu(a, b) = \begin{cases} 2 & \text{if } \bar{b} \cap \bar{b}' = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

- Suppose that $b \in B_1$ and $b' \in B_2$. Then

$$\mu(a,b) = \begin{cases} 2 & \text{if } \bar{b} \cap \bar{b}' = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $b, b' \in B_2$ with $b \neq b'$. Then $\mu(b,b') = 0$.

Then (Γ, μ) defines the configuration of type `20A`.

Remark 2.1. The automorphism group of (Γ, μ) is isomorphic to $\mathfrak{S}_4$, acting naturally on A.

2.4. **`20B`.** The configuration of type `20B` is isomorphic to the configuration of Nikulin-Kondo type III (Fig. 3.5 of [5]).

We put $P := \{1, 2, 3, 4\}$. Let Q_1 and Q_2 be quadrangles. For $i = 1, 2$, let VQ_i be the set of vertices of Q_i, and let EQ_i be the set of edges of Q_i. Let $EQ_i = \{a_i, a'_i\} \cup \{b_i, b'_i\}$ be the decomposition such that a_i and a'_i (resp. b_i and b'_i) have no common vertex. We then put

$$\Gamma := P \sqcup VQ_1 \sqcup VQ_2 \sqcup EQ_1 \sqcup EQ_2,$$

and define a symmetric function $\mu \colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $p_1, p_2 \in P$ with $p_1 \neq p_2$. Then $\mu(p_1, p_2) = 0$.
- Suppose that $p \in P$ and $v \in VQ_1 \sqcup VQ_2$. Then $\mu(p, v) = 0$.
- Suppose that $p \in P$ and $e_1 \in EQ_1$. Then

$$\mu(p, e_1) = \begin{cases} 1 & \text{if } (\, p \in \{1,2\} \text{ and } e_1 \in \{a_1, a'_1\}\,) \text{ or } (\, p \in \{3,4\} \text{ and } e_1 \in \{b_1, b'_1\}\,), \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $p \in P$ and $e_2 \in EQ_2$. Then

$$\mu(p, e_2) = \begin{cases} 1 & \text{if } (\, p \in \{1,3\} \text{ and } e_2 \in \{a_2, a'_2\}\,) \text{ or } (\, p \in \{2,4\} \text{ and } e_2 \in \{b_2, b'_2\}\,), \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $v_1, v_2 \in VQ_1 \sqcup VQ_2$ with $v_1 \neq v_2$. Then

$$\mu(v_1, v_2) = \begin{cases} 0 & \text{if } v_1 \text{ and } v_2 \text{ are the end-points of an edge,} \\ 2 & \text{otherwise.} \end{cases}$$

- Suppose that $v \in VQ_1 \sqcup VQ_2$ and $e \in EQ_1 \sqcup EQ_2$. Then

$$\mu(v, e) = \begin{cases} 2 & \text{if } v \text{ is an end-point of } e, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $e_1, e_2 \in EQ_1 \sqcup EQ_2$ with $e_1 \neq e_2$. Then $\mu(e_1, e_2) = 0$.

Then (Γ, μ) defines the configuration of type `20B`.

Remark 2.2. The automorphism group of (Γ, μ) is the group of the automorphism of the disjoint union $Q_1 \sqcup Q_2$ of two quadrangles, that is, $D_8^2 \rtimes C_2$.

2.5. **`20C` and `20D`.** The configurations of type `20C` and of type `20D` are isomorphic, and they are isomorphic to the configuration of Nikulin-Kondo type VII (Fig. 7.7 of [5]).

Let A be $\{1,\dots,5\}$, and let B be the set of non-ordered pairs $\{(ij),(kl)\}$ of disjoint subsets $(ij)=\{i,j\}$ and $(kl)=\{k,l\}$ of A with size 2. For $b=\{(ij),(kl)\}\in B$, let $\bar{b}\in A$ denote the unique element of A that is not contained in $(ij)\cup(kl)$. We then put

$$\Gamma := A \sqcup B,$$

and define a symmetric function $\mu\colon \Gamma\times\Gamma\to\mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $a,a'\in A$ with $a\ne a'$. Then we have $\mu(a,a')=2$.
- Suppose that $a\in A$ and $b\in B$. Then we have

$$\mu(a,b) := \begin{cases} 2 & \text{if } a=\bar{b}, \\ 0 & \text{otherwise.}\end{cases}$$

- Suppose that $b,b'\in B$ with $b\ne b'$. Then we have

$$\mu(b,b') := \begin{cases} 1 & \text{if } b\cap b'\ne\emptyset, \\ 0 & \text{otherwise.}\end{cases}$$

Then (Γ,μ) defines the configurations of type `20C` and type `20D`.

Remark 2.3. The automorphism group of (Γ,μ) is isomorphic to $\mathfrak{S}_5$.

2.6. **`20E`.** The configuration of type `20E` is isomorphic to the configuration of Nikulin-Kondo type VI (Fig. 6.4 of [5]). The description below of this configuration was obtained in [9].

Let A be the set of subsets of $\{1,\dots,5\}$ with size 3. Let A_1 and A_2 be two copies of A with the natural bijection to A denoted by $a\mapsto\bar{a}$. We then put

$$\Gamma := A_1\sqcup A_2,$$

and define a symmetric function $\mu\colon \Gamma\times\Gamma\to\mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $a,a'\in A_1$ with $a\ne a'$. Then

$$\mu(a,a') = \begin{cases} 1 & \text{if } |a\cap a'|=1, \\ 0 & \text{otherwise.}\end{cases}$$

- Suppose that $a,a'\in A_2$ with $a\ne a'$. Then

$$\mu(a,a') = \begin{cases} 1 & \text{if } |a\cap a'|=2, \\ 0 & \text{otherwise.}\end{cases}$$

- Suppose that $a\in A_1$ and $a'\in A_2$. Then

$$\mu(a,a') = \begin{cases} 2 & \text{if } \bar{a}=\bar{a}', \\ 0 & \text{otherwise.}\end{cases}$$

Then (Γ,μ) defines the configuration of type `20E`.

Remark 2.4. The sub-configuration $(A_1,\mu|A_1)$ is isomorphic to the Petersen graph, and the sub-configuration $(A_2,\mu|A_2)$ is isomorphic to the complement of the Petersen graph. The automorphism group of (Γ,μ) is equal to the automorphism group of the Petersen graph, which is isomorphic to $\mathfrak{S}_5$.

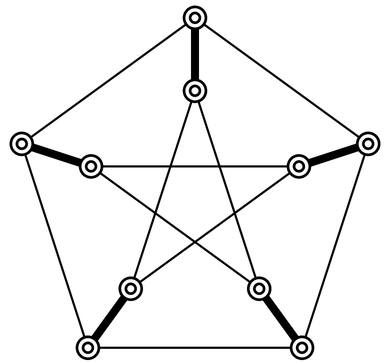

FIGURE 2.1. Graph for Nikulin-Kondo type VI

2.7. **20F.** The configuration of type 20F is isomorphic to the configuration of Nikulin-Kondo type IV (Fig. 4.4 of [5]). The description below of this configuration was obtained in [8].

Let $\overline{\Gamma}$ be the set of vertices of the Petersen graph P, and let Γ be the set with 20 vertices with a map $\rho\colon \Gamma \to \overline{\Gamma}$ such that $|\rho^{-1}(\bar{v})| = 2$ for every $\bar{v} \in \overline{\Gamma}$. We fix a numbering v_1, v_2 of the elements in each fiber $\rho^{-1}(\bar{v}) = \{v_1, v_2\}$ of ρ. We then define a symmetric function $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- We have $\mu(v, v') = 0$ if $\rho(v) = \rho(v')$.
- We have $\mu(v, v') = 0$ if $\rho(v)$ and $\rho(v')$ are not connected in P.
- We have $\mu(v, v') = 1$ if $\rho(v)$ and $\rho(v')$ are connected by a thin line in Figure 2.1.
- Suppose that $\bar{v}$ and $\bar{v}'$ are connected by a thick line in Figure 2.1. Let $\rho^{-1}(\bar{v}) = \{v_1, v_2\}$ and $\rho^{-1}(\bar{v}') = \{v'_1, v'_2\}$ be the fibers with the fixed numberings. Then
$$\mu(v_i, v'_j) = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then the isomorphism class of the configuration (Γ, μ) does not depend on the choice of numberings of two elements in fibers of ρ, and (Γ, μ) defines the configuration of type 20F.

Remark 2.5. The group $\mathrm{Aut}(\Gamma, \mu)$ is of order 640. The action of $\mathrm{Aut}(\Gamma, \mu)$ on Γ preserves the fibers of $\rho\colon \Gamma \to \overline{\Gamma}$, and we have a natural homomorphism from $\mathrm{Aut}(\Gamma, \mu)$ to the automorphism group $\mathrm{Aut}(P)$ of the Petersen graph, which is isomorphic to $\mathfrak{S}_5$. Thus we obtain an exact sequence

$$0 \longrightarrow C_2^5 \longrightarrow \mathrm{Aut}(\Gamma, \mu) \longrightarrow G_{20} \longrightarrow 1, \tag{2.1}$$

where G_{20} is the subgroup of $\mathrm{Aut}(P) \cong \mathfrak{S}_5$ consisting of elements that preserve the thick edges in Figure 2.1. As a subgroup of $\mathfrak{S}_5$, the group G_{20} is conjugate to the subgroup generated by (12345) and (2354).

2.8. **40A.** Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two copies of the cubes $I^3 \subset \mathbb{R}^3$, where $I \subset \mathbb{R}$ is the unit interval. Let ε be $+$ or $-$. A vertex of $\mathcal{C}_\varepsilon$ is written as $((a_x, a_y, a_z), \varepsilon)$, where $a_x, a_y, a_z \in \{0, 1\}$, and a face of $\mathcal{C}_\varepsilon$ is written as $(w = a, \varepsilon)$, where $w \in \{x, y, z\}$ and $a \in \{0, 1\}$. Let V be the set of vertices of $\mathcal{C}_\pm$, and let F be the set of faces of $\mathcal{C}_\pm$.

Let P be the set of pairs of a face $f_+ = (w = a_+)$ of $\mathcal{C}_+$ and a face $f_- = (w = a_-)$ of $\mathcal{C}_-$ that are parallel. Each element of P is written as $(w = a_+, w = a_-)$, where $w \in \{x, y, z\}$ and $a_\pm \in \{0, 1\}$. We have $|V| = 16$, $|F| = 12$, $|P| = 12$. We put

$$\Gamma := V \sqcup F \sqcup P,$$

and define a symmetric function $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $v_1, v_2 \in V$ with $v_1 \neq v_2$. Then

$$\mu(v_1, v_2) = \begin{cases} 0 & \text{if } v_1v_2 \text{ is an edge of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\ 4 & \text{if } v_1v_2 \text{ is a diagonal of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\ 2 & \text{otherwise.} \end{cases}$$

- Suppose that $v \in V$ and $f \in F$. Then

$$\mu(v, f) = \begin{cases} 2 & \text{if } v \in f, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $v \in V$ and $p = (f_+, f_-) \in P$. Then

$$\mu(v, p) = \begin{cases} 2 & \text{if } v \in f_+ \cup f_-, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $f_1, f_2 \in F$ with $f_1 \neq f_2$. Let f_i be $(w_i = a_i, \varepsilon_i)$, where $w_i \in \{x, y, z\}$, $a_i \in \{0, 1\}$, and $\varepsilon_i \in \{+, -\}$. Then

$$\mu(f_1, f_2) = \begin{cases} 1 & \text{if } \varepsilon_1 \neq \varepsilon_2 \text{ and } w_1 \neq w_2, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $f = (w = a, \varepsilon) \in F$ and $p = (f'_+, f'_-) \in P$. Let $\bar{f}$ be the unique face of $\mathcal{C}_\varepsilon$ that is disjoint from f. Then

$$\mu(f, p) = \begin{cases} 2 & \text{if } \bar{f} = f'_+ \text{ or } \bar{f} = f'_-, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $p_1, p_2 \in P$ with $p_1 \neq p_2$. Let $\mathrm{faces}(p_i)$ denote the set of 2 faces contained in p_i, and let $\mathrm{verts}(p_i)$ denote the set of 8 vertices contained in the two faces of p_i.

$$\mu(p_1, p_2) = \begin{cases} 2 & \text{if } \mathrm{verts}(p_1) \cap \mathrm{verts}(p_2) = \emptyset, \\ 0 & \text{if } \mathrm{faces}(p_1) \cap \mathrm{faces}(p_2) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Then (Γ, μ) defines the configuration of type `40A`.

Remark 2.6. The automorphism group $\mathrm{Aut}(\Gamma, \mu)$ is of order 768, and V, F, P are the orbits of the action on Γ. Let V_+ and V_- be the set of vertices of $\mathcal{C}_+$ and $\mathcal{C}_-$, regarded as graphs with edges being the edges of the cubes. The automorphism group of the graph V_+ is of order 48. The stabilizer subgroup $\mathrm{Stab}(V_+)$ of V_+ in $\mathrm{Aut}(\Gamma, \mu)$ is of index 2, the natural homomorphism $\mathrm{Stab}(V_+) \to \mathrm{Aut}(V_+)$ is surjective, and its kernel is isomorphic to C_2^3 acting on V_- as $((a_x, a_y, a_z), -) \mapsto ((\pm a_x, \pm a_y, \pm a_z), -)$.

2.9. `40B` **and** `40C`. The configurations of type `40B` and of `40C` are isomorphic.

We put $F := \{1,2,3,4\}$. Let P be the set $F \times F$ with the projections $\mathrm{pr}_1 \colon P \to F$ and $\mathrm{pr}_2 \colon P \to F$. Let B be the set of bijections $f \colon F \to F$. We put

$$\Gamma := P \sqcup B,$$

and define a symmetric function $\mu \colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $p, p' \in P$ with $p \neq p'$. Then
$$\mu(p,p') = \begin{cases} 1 & \text{if } \mathrm{pr}_1(p) = \mathrm{pr}_1(p') \text{ or } \mathrm{pr}_2(p) = \mathrm{pr}_2(p'), \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $p \in P$ and $f \in B$. Then
$$\mu(p,f) = \begin{cases} 2 & \text{if } f(\mathrm{pr}_1(p)) = \mathrm{pr}_2(p), \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $f, f' \in B$ with $f \neq f'$. Then $\gamma := f f'^{-1}$ is a permutation of F. Let $\tau(\gamma)$ denote the lengths of cycles in the cycle decomposition of $\gamma \in \mathfrak{S}_4$. Then
$$\mu(f,f') = \begin{cases} 2 & \text{if } \tau(\gamma) = 4, \\ 2 & \text{if } \tau(\gamma) = 2+2, \\ 1 & \text{if } \tau(\gamma) = 3+1, \\ 0 & \text{if } \tau(\gamma) = 2+1+1. \end{cases}$$

Then (Γ, μ) defines the configurations of type `40B` and `40C`.

Remark 2.7. The group $\mathrm{Aut}(\Gamma,\mu)$ is isomorphic to $(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes C_2$, which acts on P in the natural way.

2.10. `40D` **and** `40E`. The configurations of type `40D` and of `40E` are isomorphic.

A subset $(ij) := \{i,j\}$ of size 2 of $\{1,\dots,6\}$ is called a *duad*, and a subset $(ijk) := \{i,j,k\}$ of size 3 of $\{1,\dots,6\}$ is called a *trio*. A *syntheme* is a non-ordered set $(ij)(kl)(mn) := \{(ij),(kl),(mn)\}$ of 3 duads such that $\{i,j,k,l,m,n\} = \{1,\dots,6\}$. A *double trio* is a non-ordered pair $(ijk)(lmn) := \{(ijk),(lmn)\}$ of complementary trios. Let D, S and T be the set of duads, synthemes, and double trios, respectively. We have $|D| = 15$, $|S| = 15$, and $|T| = 10$. We then put

$$\Gamma := D \sqcup S \sqcup T,$$

and define a symmetric function $\mu \colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $\delta_1, \delta_2 \in D$ with $\delta_1 \neq \delta_2$. Then
$$\mu(\delta_1,\delta_2) = \begin{cases} 1 & \text{if } |\delta_1 \cap \delta_2| = 1, \\ 0 & \text{if } |\delta_1 \cap \delta_2| = 0. \end{cases}$$
- Suppose that $\delta \in D$ ans $\sigma \in S$. Then
$$\mu(\delta,\sigma) = \begin{cases} 2 & \text{if } \delta \in \sigma, \\ 0 & \text{if } \delta \notin \sigma. \end{cases}$$
- Suppose that $\delta \in D$ ans $\tau = \{t_1, t_2\} \in T$, where t_1 and t_2 are trios. Then
$$\mu(\delta,\tau) = \begin{cases} 2 & \text{if } \delta \subset t_1 \text{ or } \delta \subset t_2, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $\sigma_1, \sigma_2 \in S$ with $\sigma_1 \neq \sigma_2$. Then
$$\mu(\sigma_1, \sigma_2) = \begin{cases} 1 & \text{if } \sigma_1 \cap \sigma_2 = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $\sigma \in S$ and $\tau \in T$. Then
$$\mu(\sigma, \tau) = \begin{cases} 2 & \text{if } |\delta \cap t| = 1 \text{ for any duad } \delta \in \sigma \text{ and any trio } t \in \tau, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $\tau_1, \tau_2 \in T$ with $\tau_1 \neq \tau_2$. Then $\mu(\tau_1, \tau_2) = 2$.

Then (Γ, μ) defines the configurations of type `40D` and `40E`.

Remark 2.8. By construction, the symmetric group $\mathfrak{S}_6$ acts on (Γ, μ), and D, S, T are the orbits. The full automorphism group of the configuration (Γ, μ) is isomorphic to the automorphism group $\mathrm{Aut}(\mathfrak{A}_6)$ of the alternating group $\mathfrak{A}_6$. The group $\mathrm{Aut}(\mathfrak{A}_6)$ contains $\mathfrak{A}_6$ as a normal subgroup of index 4 such that $\mathrm{Aut}(\mathfrak{A}_6)/\mathfrak{A}_6$ is isomorphic to C_2^2, and contains $\mathfrak{S}_6$, $\mathrm{PGL}_2(9)$ and M_{10} as subgroups of index 2. (See, for example, Section 1.5, Chapter 10 of [4].) We can construct $\mathrm{Aut}(\mathfrak{A}_6)$ from $\mathfrak{S}_6$ by adding an automorphism θ that induces the non-trivial outer automorphism of $\mathfrak{S}_6$. Correspondingly, the action of $\mathrm{Aut}(\mathfrak{A}_6)$ on (Γ, μ) fuses the duads D and the synthemes S, and decomposes Γ into two orbits $D \sqcup S$ and T.

2.11. **`96A`.** Recall that $\mathbf{0}_n$ is the $n \times n$ zero matrix, and $\mathbf{1}_n$ is the $n \times n$ matrix with all components 1. We consider the matrix
$$\Sigma_{16} := \begin{bmatrix} -2I_4 & \mathbf{1}_4 & 2I_4 & \mathbf{0}_4 \\ \mathbf{1}_4 & -2I_4 & \mathbf{0}_4 & 2I_4 \\ 2I_4 & \mathbf{0}_4 & -2I_4 & \mathbf{1}_4 \\ \mathbf{0}_4 & 2I_4 & \mathbf{1}_4 & -2I_4 \end{bmatrix}.$$
We put
$$d := \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, \quad t_+ := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad t_- := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$
and
$$D_8 := \begin{bmatrix} d & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & d & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & d & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & d \end{bmatrix}, \quad T_8 := \begin{bmatrix} t_+ & t_- & t_- & t_- \\ t_- & t_+ & t_- & t_- \\ t_- & t_- & t_+ & t_- \\ t_- & t_- & t_- & t_+ \end{bmatrix}.$$
We then consider the matrix
$$\Sigma_{32} := \begin{bmatrix} D_8 & T_8 & \mathbf{1}_8 & \mathbf{0}_8 \\ T_8 & D_8 & \mathbf{0}_8 & \mathbf{1}_8 \\ \mathbf{1}_8 & \mathbf{0}_8 & D_8 & T_8 \\ \mathbf{0}_8 & \mathbf{1}_8 & T_8 & D_8 \end{bmatrix}.$$

For $k = 16$ and $k = 32$, let (Γ_k, μ_k) be the configuration of size k with the symmetric bilinear form $\mu_k \colon \Gamma_k \times \Gamma_k \to \mathbb{Z}$ given by the matrix Σ_k defined above. Then there exist exactly 64 sub-configurations $(\mathit{\Gamma}', \mu_{32}|\mathit{\Gamma}')$ of (Γ_{32}, μ_{32}) with $\mathit{\Gamma}' \subset \Gamma_{32}$ that are isomorphic to (Γ_{16}, μ_{16}). We denote by Γ_{64} the set of sub-configurations of (Γ_{32}, μ_{32}) isomorphic to (Γ_{16}, μ_{16}), and define $\mu_{64}\colon \Gamma_{64} \times \Gamma_{64} \to \mathbb{Z}$ by

$$\mu_{64}(\mathit{\Gamma}', \mathit{\Gamma}'') := \begin{cases} 6 & \text{if } |\mathit{\Gamma}' \cap \mathit{\Gamma}''| = 0, \\ 4 & \text{if } |\mathit{\Gamma}' \cap \mathit{\Gamma}''| = 4, \\ 2 & \text{if } |\mathit{\Gamma}' \cap \mathit{\Gamma}''| = 8, \\ 0 & \text{if } |\mathit{\Gamma}' \cap \mathit{\Gamma}''| = 12, \\ -2 & \text{if } |\mathit{\Gamma}' \cap \mathit{\Gamma}''| = 16. \end{cases}$$

We then put

$$\Gamma := \Gamma_{32} \sqcup \Gamma_{64},$$

and define a symmetric function $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $v, v' \in \Gamma_{32}$. Then $\mu(v, v') := \mu_{32}(v, v')$.
- Suppose that $\mathit{\Gamma}', \mathit{\Gamma}'' \in \Gamma_{64}$. Then $\mu(\mathit{\Gamma}', \mathit{\Gamma}'') := \mu_{64}(\mathit{\Gamma}', \mathit{\Gamma}'')$.
- Suppose that $v \in \Gamma_{32}$ and $\mathit{\Gamma}' \in \Gamma_{64}$. Then

$$\mu(v, \mathit{\Gamma}') := \begin{cases} 2 & \text{if } v \in \mathit{\Gamma}', \\ 0 & \text{otherwise.} \end{cases}$$

Then (Γ, μ) defines the configuration of type `96A`.

Remark 2.9. The order of the automorphism group of (Γ, μ) is 147456. The natural homomorphism $\mathrm{Aut}(\Gamma_{32}, \mu_{32}) \to \mathrm{Aut}(\Gamma, \mu)$ is an isomorphism. The set Γ_{32} is regarded as the indexes $\{1, \dots, 32\}$ of row vectors of the matrix Σ_{32}. We have a decomposition

$$\Gamma_{32} = o_1 \sqcup \cdots \sqcup o_4, \quad o_i := \{8(i-1)+1, \dots, 8(i-1)+8\}.$$

The action of $\mathrm{Aut}(\Gamma_{32}, \mu_{32})$ on Γ_{32} preserves this decomposition, and hence we have a homomorphism

$$\pi\colon\ \mathrm{Aut}(\Gamma_{32}, \mu_{32}) \to \mathfrak{S}_4$$

to the permutation group of $o_1, \dots, o_4$. The image is isomorphic to C_2^2. Each o_i is equipped with a structure of the configuration by $\mu_{32}|o_i\colon o_i \times o_i \to \mathbb{Z}$, or equivalently, by the matrix D_8. The automorphism group $\mathrm{Aut}(o_i)$ of this configuration $(o_i, \mu_{32}|o_i)$ is isomorphic to $C_2^4 \rtimes \mathfrak{S}_4$. Let G_{192} denote the subgroup $\mathrm{Aut}(o_i) \cap \mathfrak{A}_8$ of $\mathrm{Aut}(o_i)$, where the intersection is taken in the full permutation group $\mathfrak{S}_8$ of o_i. Then the natural homomorphism

$$\mathrm{Ker}\,\pi \to \mathrm{Aut}(o_1) \times \mathrm{Aut}(o_3)$$

is injective, and the image is equal to $G_{192} \times G_{192}$. Thus we have an exact sequence

$$1 \longrightarrow G_{192} \times G_{192} \longrightarrow \mathrm{Aut}(\Gamma, \mu) \longrightarrow C_2^2 \longrightarrow 0.$$

2.12. **`96B` and `96C`.** The configurations of type `96B` and of `96C` are isomorphic.

We put

$$m := \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}, \quad t_+ := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad t_- := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

We then define an 8×8 matrix D by

$$D := \begin{bmatrix} m & t_+ & t_+ & t_+ \\ t_+ & m & t_+ & t_+ \\ t_+ & t_+ & m & t_+ \\ t_+ & t_+ & t_+ & m \end{bmatrix},$$

$$S_1 := \begin{bmatrix} - & - & + & + \\ - & - & + & + \\ + & + & - & - \\ + & + & - & - \end{bmatrix}, \quad S_2 := \begin{bmatrix} - & - & + & + \\ + & + & - & - \\ - & - & + & + \\ + & + & - & - \end{bmatrix}, \quad S_3 := \begin{bmatrix} - & - & + & + \\ + & + & - & - \\ + & + & - & - \\ - & - & + & + \end{bmatrix},$$

$$S_4 := \begin{bmatrix} - & + & - & + \\ - & + & - & + \\ + & - & + & - \\ + & - & + & - \end{bmatrix}, \quad S_5 := \begin{bmatrix} - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{bmatrix}, \quad S_6 := \begin{bmatrix} - & + & - & + \\ + & - & + & - \\ + & - & + & - \\ - & + & - & + \end{bmatrix},$$

$$S_7 := \begin{bmatrix} - & + & + & - \\ - & + & + & - \\ + & - & - & + \\ + & - & - & + \end{bmatrix}, \quad S_8 := \begin{bmatrix} - & + & + & - \\ + & - & - & + \\ - & + & + & - \\ + & - & - & + \end{bmatrix}, \quad S_9 := \begin{bmatrix} - & + & + & - \\ + & - & - & + \\ + & - & - & + \\ - & + & + & - \end{bmatrix},$$

$$S_{10} := \begin{bmatrix} + & - & - & + \\ - & + & + & - \\ - & + & + & - \\ + & - & - & + \end{bmatrix}, \quad \cdots\cdots \quad S_{18} := \begin{bmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{bmatrix}.$$

TABLE 2.1. Eighteen matrices $S_1, \dots, S_{18}$

and a 24×24 matrix T by

$$T := \begin{bmatrix} D & \mathbf{1}_8 & \mathbf{1}_8 \\ \mathbf{1}_8 & D & \mathbf{1}_8 \\ \mathbf{1}_8 & \mathbf{1}_8 & D \end{bmatrix}. \tag{2.2}$$

Let $\mathcal{S}$ be the set of 18 square matrices $S_1, \dots, S_{18}$ of size 4 with components in $\{+, -\}$ obtained from S_1 in Table 2.1 by permuting rows and columns. For a 3×3 matrix

$$\nu := \begin{bmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ i_{31} & i_{32} & i_{33} \end{bmatrix}$$

with components $i_{\alpha\beta}$ in $\{1, \dots, 18\}$, let $S[\nu]$ denote the 24×24 matrix obtained from ν by first replacing each $i_{\alpha\beta}$ with the member $S_{i_{\alpha\beta}}$ of $\mathcal{S}$ indexed by $i_{\alpha\beta}$ and then replacing $+$ with t_+ and $-$ with t_-. We put

$$\nu_1 := \begin{bmatrix} 9 & 8 & 7 \\ 6 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}, \quad \nu_2 := \begin{bmatrix} 5 & 9 & 7 \\ 9 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad \nu_3 := \begin{bmatrix} 4 & 8 & 9 \\ 9 & 5 & 4 \\ 2 & 1 & 3 \end{bmatrix},$$

$$\nu_4 := \begin{bmatrix} 8 & 9 & 1 \\ 6 & 1 & 5 \\ 1 & 5 & 9 \end{bmatrix}, \quad \nu_5 := \begin{bmatrix} 8 & 1 & 9 \\ 1 & 9 & 5 \\ 6 & 5 & 1 \end{bmatrix}, \quad \nu_6 := \begin{bmatrix} 9 & 8 & 1 \\ 1 & 5 & 9 \\ 6 & 1 & 5 \end{bmatrix}.$$

Then the 96×96 *symmetric* matrix

$$\begin{bmatrix} T & S[\nu_1] & S[\nu_2] & S[\nu_3] \\ & T & S[\nu_4] & S[\nu_5] \\ & & T & S[\nu_6] \\ & & & T \end{bmatrix} \tag{2.3}$$

defines the configurations of type `96B` and `96C`.

Remark 2.10. The group $\mathfrak{S}_4$ acts on $\mathcal{S}$ as $S \mapsto \sigma S$ for $S \in \mathcal{S}$ and $\sigma \in \mathfrak{S}_4$, where σS is obtained from S by permuting *rows* of S by σ. Let G_{row} be the subgroup of the full permutation group $\mathfrak{S}(\mathcal{S})$ of $\mathcal{S}$ generated by the action of $\mathfrak{S}_4$ on rows and the flipping $+ \leftrightarrow -$. Then $|G_{\mathrm{row}}| = 48$, and $\mathcal{S}$ is decomposed by G_{row} into 3 orbits, each of which is of size 6. Similarly, we define G_{col} to be the subgroup of $\mathfrak{S}(\mathcal{S})$ generated by the action of $\mathfrak{S}_4$ on *columns* and the flipping. Then $|G_{\mathrm{col}}| = 48$ and $\mathcal{S}$ is decomposed by G_{col} into 3 orbits of size 6. The intersection of any orbit of G_{row} and any orbit of G_{col} consists of two matrices that are interchanged by the flipping.

Let $\mathcal{M}$ be the set of 3×3 matrices with components in the set $\{1, \dots, 18\}$ of indexes of $\mathcal{S}$. The groups G_{row} and G_{col} act on $\{1, \dots, 18\}$ as described in the previous paragraph. Let $\mathcal{G}$ be the subgroup of the full permutation group of $\mathcal{M}$ generated by the following permutations:

- the permutations of 3 rows,
- choosing a row and making an element of G_{row} act on the 3 components of the row,
- the permutations of 3 columns, and
- choosing a column and making an element of G_{col} act on the 3 components of the column.

Then we confirm that there exists one and only one orbit O of the action of $\mathcal{G}$ on $\mathcal{M}$ with the following property: for every $\nu \in O$, each row of ν consists of 3 distinct elements, and each column of ν consists of 3 distinct elements. We have $|O| = 23887872$.

The 6 matrices $\nu_1, \dots, \nu_6$ above belong to this orbit O. We tried to characterize the 6-tuple $\nu_1, \dots, \nu_6$ of elements of O combinatorially, but we could not find a nice description.

Remark 2.11. The automorphism group of (Γ, μ) is of order 221184. The set Γ is decomposed into 48 pairs $\{v, v'\}$ with $\mu(v, v') = 4$. Let P_{48} be the set of these pairs. The kernel of the natural homomorphism

$$\pi\colon\ \mathrm{Aut}(\Gamma, \mu) \to \mathfrak{S}(P_{48})$$

is isomorphic to C_2. The set P_{48} is decomposed into the disjoint union of 4 subsets $t_1, \dots, t_4$ of size 12, each of which corresponds to the diagonal block T of the matrix (2.3). The natural homomorphism

$$\rho\colon\ \mathrm{Im}\,\pi \to \mathfrak{S}_4$$

is surjective. Hence $\mathrm{Ker}\,\rho$ is of order 4608. The kernel of the natural homomorphism

$$\sigma\colon\ \mathrm{Ker}\,\rho \to \mathfrak{S}(t_1)$$

is isomorphic to C_2^2, and hence Im σ is of order 1152. The set t_1 is then decomposed into the disjoint union of 3 subsets $d_1, \dots, d_3$ of size 4, each of which corresponds to the diagonal block D of the matrix (2.2). The natural homomorphism

$$\tau\colon\ \mathrm{Im}\,\sigma \to \mathfrak{S}_3$$

is surjective. Hence $\mathrm{Ker}\,\tau$ is of order 192, which is isomorphic to $C_2^6\colon C_3$.

References

[1] Richard Borcherds. Automorphism groups of Lorentzian lattices. *J. Algebra*, 111(1):133–153, 1987.

[2] Richard E. Borcherds. Coxeter groups, Lorentzian lattices, and $K3$ surfaces. *Internat. Math. Res. Notices*, 1998(19):1011–1031, 1998.

[3] Simon Brandhorst and Ichiro Shimada. Borcherds' method for Enriques surfaces, 2019. Preprint, arXiv:1903.01087.

[4] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, third edition, 1999.

[5] Shigeyuki Kondo. Enriques surfaces with finite automorphism groups. *Japan. J. Math. (N.S.)*, 12(2):191–282, 1986.

[6] Gebhard Martin. Enriques surfaces with finite automorphism group in positive characteristic. Preprint, arXiv:1703.08419, 2017.

[7] V. V. Nikulin. Description of automorphism groups of Enriques surfaces. *Dokl. Akad. Nauk SSSR*, 277(6):1324–1327, 1984. Soviet Math. Dokl. 30 (1984), No.1 282–285.

[8] Ichiro Shimada. The elliptic modular surface of level 4 and its reduction modulo 3, 2018. Preprint. arXiv:1806.05787.

[9] Ichiro Shimada. On an Enriques surface associated with a quartic Hessian surface. *Canad. J. Math.*, 71(1):213–246, 2019.

[10] Ichiro Shimada. A note on configurations of (-2)-vectors on Enriques surfaces: computational data. http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html, 2019.

[11] The GAP Group. *GAP - Groups, Algorithms, and Programming*. Version 4.8.6; 2016 (http://www.gap-system.org).

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