RECONFIRMATION OF THE ENUMERATION OF HOLES OF THE LEECH LATTICE

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This note is a detailed version of Remark 2.10 of the author's preprint [3]. Therefore we use the notation of [3] freely. In the following, TABLE means Table 25.1 of [1, Chapter 25] calculated by Borcherds, Conway, and Queen. In TABLE, the equivalence classes of holes of the Leech lattice Λ are enumerated. The purpose of this note is to explain a method to reconfirm the correctness of TABLE.

The fact that there exist at least $23 + 284$ equivalence classes of holes can be established by giving explicitly the set $P_{\mathbf{c}}$ of vertices of the polytope $\overline{P}_{\mathbf{c}}$ for a representative $\mathbf{c}$ of each equivalence class $[\mathbf{c}]$. See Remark 3.1 of [3] and the computational data given in the author's web page [4].

In order to see that there exist no other equivalence classes, Borcherds, Conway, and Queen used the volume formula

$$\sum_{[\mathbf{c}]} \frac{\mathrm{vol}(\overline{P}_{\mathbf{c}})}{|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|} = \frac{1}{|\mathrm{Co}_0|}. \tag{0.1}$$

The volume $\mathrm{vol}(\overline{P}_{\mathbf{c}})$ of $\overline{P}_{\mathbf{c}}$ can be easily calculated from the set $P_{\mathbf{c}}$ of vertices, and the result coincides with the values given in the third column of TABLE. The equality (0.1) holds when $|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|$ is replaced by the value $g = g(\mathbf{c})$ given in the second column of TABLE and the summation is taken over the set of the equivalence classes of holes listed in TABLE. Therefore, in order to show the completeness of TABLE, it is enough to prove the inequality

$$|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)| \;\le\; g(\mathbf{c}) \tag{0.2}$$

for each hole $\mathbf{c}$ that appears in TABLE. The groups $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ for deep holes are studied in detail in [1, Chapters 23 and 24]. Hence we will prove the inequality (0.2) for shallow holes $\mathbf{c}$.

Let $\mathbf{c}$ be a shallow hole that appears in TABLE. Then $\overline{P}_{\mathbf{c}}$ is a 24-dimensional simplex, and $P_{\mathbf{c}}$ consists of 25 points of Λ. Recall that $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is the group of permutations g of $P_{\mathbf{c}}$ such that $\|p^g - q^g\| = \|p - q\|$ holds for any $p, q \in P_{\mathbf{c}}$. Each permutation $g \in \mathrm{Aut}(\overline{P}_{\mathbf{c}})$ induces an affine isometry $g_\Lambda \colon \Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda \otimes \mathbb{Q}$, and we have

$$g \in \mathrm{Aut}(P_{\mathbf{c}}, \Lambda) \iff g_\Lambda \text{ preserves } \Lambda \subset \Lambda \otimes \mathbb{Q}. \tag{0.3}$$

When $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is not very large, we can make the list of elements of $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by the criterion (0.3). We can also use the following trick to reduce the amount of the computation.

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no.	type	α	β	ν	$\lvert\mathrm{Aut}(\overline{P}_{\mathbf{c}})\rvert$	$g(\mathbf{c})$
293	$a_5a_2^{10}$	a_5	a_2	10	$2\cdot 2^{10}\cdot 10!$	720
299	$d_4a_1^{21}$	d_4	a_1	21	$6\cdot 21!$	120960
303	$a_3a_2^{11}$	a_3	a_2	11	$2\cdot 2^{11}\cdot 11!$	7920
304	$a_3a_1^{22}$	a_3	a_1	22	$2\cdot 22!$	887040
305	$a_1a_2^{12}$	a_1	a_2	12	$2^{12}\cdot 12!$	190080
306	$a_2a_1^{23}$	a_2	a_1	23	$2\cdot 23!$	10200960
307	a_1^{25}	a_1	a_1	24	$25!$	244823040

TABLE 0.1. Shallow holes with large $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$

$$
\begin{array}{rcrcl}
|M_{21}| & = & 20160 & = & g(\mathbf{c}_{299})/6 \\
|M_{22}| & = & 443520 & = & g(\mathbf{c}_{304})/2 \\
|M_{23}| & = & 10200960 & = & g(\mathbf{c}_{306}) \\
|M_{24}| & = & 244823040 & = & g(\mathbf{c}_{307}) \\
|M_{11}| & = & 7920 & = & g(\mathbf{c}_{303}) \\
|M_{12}| & = & 95040 & = & g(\mathbf{c}_{305})/2
\end{array}
$$

TABLE 0.2. Orders of Mathieu groups

Example 0.1. Consider the shallow hole $\mathbf{c}_{297}$ of type $d_4^4a_1^9$. We have $|\mathrm{Aut}(\overline{P}_{\mathbf{c}})| = 6^4\cdot 4!\cdot 9! = 11287019520$. We choose two vertices v_1 and v_2 that correspond to nodes of two a_1 in $d_4^4a_1^9$, and consider the subgroup $\mathrm{Stab}(v_1, v_2)$ of $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ consisting of permutations that fix each of v_1 and v_2. Then the index of $\mathrm{Stab}(v_1, v_2)$ in $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is at most 72. We see by the criterion (0.3) that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)\cap\mathrm{Stab}(v_1, v_2)$ is of order 6, and hence $|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|$ is at most $72 \times 6 = 432 = g(\mathbf{c}_{297})$. In fact, $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to $(((C_3 \times C_3) : Q_8) : C_3) : C_2$, where C_n is the cyclic group of order n and Q_8 is the quaternion group.

This brute-force method works for shallow holes except for the seven cases listed in Table 0.1.

0.1. **Golay codes and Mathieu groups.** The values $g(\mathbf{c})$ in Table 0.1 suggest that the groups $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ are related to Mathieu groups. (See Table 0.2.) For each shallow hole $\mathbf{c}$ in Table 0.1, we construct a code that is related to a Golay code, and clarify the relation between $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ and the corresponding Mathieu group.

Remark 0.2. In Remarks (ii) of [1, Chapter 25], it is stated that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to the Mathieu group M_{24} for the shallow hole $\mathbf{c}_{307}$ of type a_1^{25}.

We fix notions and notation about codes, and recall the definitions of Golay codes and Mathieu groups. Let $\mathbb{F}$ be either $\mathbb{F}_2$ or $\mathbb{F}_3$, and let l be a positive integer. A *code of length* l over $\mathbb{F}$ is a linear subspace of $\mathbb{F}^l$. Let C be a code of length l. When $\mathbb{F} = \mathbb{F}_2$, we say that C is *binary*, and when $\mathbb{F} = \mathbb{F}_3$, we say that C is *ternary*. When $\dim C = d$, we say that C is an (l, d)-code. Each element of C is called a *codeword*. The *weight* $\mathrm{wt}(x)$ of a codeword $x = (x_1, \dots, x_l)$ is defined to be the cardinality of $\{i \mid x_i \neq 0\}$. The *minimal weight* of C is the minimum of

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&1&1&0&0&0&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&1&0&1&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&1&0&1&1\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&1&1&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1&0&1
\end{bmatrix}$$

TABLE 0.3. A basis of $\mathcal{C}_{24}$

$$\begin{bmatrix}
1&0&0&0&0&0&0&1&1&1&1&1\\
0&1&0&0&0&0&2&0&1&2&2&1\\
0&0&1&0&0&0&2&1&0&1&2&2\\
0&0&0&1&0&0&2&2&1&0&1&2\\
0&0&0&0&1&0&2&2&2&1&0&1\\
0&0&0&0&0&1&2&1&2&2&1&0
\end{bmatrix}$$

TABLE 0.4. A basis of $\mathcal{C}_{12}$

$\{\mathrm{wt}(x) \mid x \in C \setminus \{0\}\}$. The *weight distribution of a code* C is the expression

$$0^1\, w_1^{n_1}\, w_2^{n_2}\, \dots\, w_m^{n_m}$$

that indicates that C contains exactly n_i codewords of weight w_i for $i = 1, \dots, m$, where $0, w_1, \dots, w_m$ are distinct weights, and that $|C| = 1 + n_1 + \cdots + n_m$ holds.

For a linear subspace V of $\mathbb{F}^l$, the intersection $C \cap V$ is also a code of length l. For a positive integer $k < l$, let $\mathrm{pr}_k \colon \mathbb{F}^l \to \mathbb{F}^k$ denote the projection

$$(x_1, \dots, x_l) \mapsto (x_1, \dots, x_k).$$

Then $\mathrm{pr}_k(C)$ is a code of length k.

Let $\mathcal{G}_l$ denote the subgroup of $GL_l(\mathbb{F})$ consisting of monomial transformations, that is, $\mathcal{G}_l$ is the group of linear automorphisms of $\mathbb{F}^l$ generated by permutations of coordinates and multiplications by a non-zero scalar on one coordinate. When $\mathbb{F} = \mathbb{F}_2$, we have $\mathcal{G}_l \cong \mathfrak{S}_l$, and when $\mathbb{F} = \mathbb{F}_3$, we have $\mathcal{G}_l \cong \{\pm 1\}^l \rtimes \mathfrak{S}_l$. The automorphism group of a code C of length l is defined to be

$$\mathrm{Aut}(C) := \{\ g \in \mathcal{G}_l \ \mid\ C^g = C\ \}.$$

Two codes C and C' of length l are said to be *equivalent* if there exists a monomial transformation $g \in \mathcal{G}_l$ such that $C' = C^g$. The weight distribution and the isomorphism class of the automorphism group depend only on the equivalence class of codes.

The *binary Golay code* $\mathcal{C}_{24}$ is the binary $(24, 12)$-code generated by the row vectors of the matrix in Table 0.3. The *ternary Golay code* $\mathcal{C}_{12}$ is the ternary $(12, 6)$-code generated by the row vectors of the matrix in Table 0.4. We have the following theorem, which will be used frequently in the next section.

Theorem 0.3 (Pless [2]). (1) *Let C be a binary $(24, 12)$-code. Then the following conditions are equivalent:*

- *C is equivalent to the binary Golay code $\mathcal{C}_{24}$,*
- *the minimal weight of C is 8, and*
- *the weight distribution of C is $0^1\, 8^{759}\, 12^{2576}\, 16^{759}\, 24^1$.*

(2) *Let C be a ternary $(12, 6)$-code. Then the following conditions are equivalent:*

- *C is equivalent to the ternary Golay code $\mathcal{C}_{12}$,*
- *the minimal weight of C is 6, and*
- *the weight distribution of C is $0^1\, 6^{264}\, 9^{440}\, 12^{24}$.*

Let $\mathbb{F}$ be $\mathbb{F}_2$. The automorphism group of $\mathcal{C}_{24}$ is the Mathieu group M_{24}. As a subgroup of the full symmetric group $\mathfrak{S}_{24}$ of the set $\{x_1, \dots, x_{24}\}$ of coordinate positions of $\mathbb{F}_2^{24}$, the Mathieu group M_{24} is 5-transitive. For a positive integer $k < 24$, let $\mathfrak{S}_k$ denote the subgroup of $\mathfrak{S}_{24}$ consisting of permutations that fix each of $x_{k+1}, \dots, x_{24}$. For $k = 21, 22, 23$, we define the Mathieu group M_k by

$$M_k := M_{24} \cap \mathfrak{S}_k.$$

Let $\mathbb{F}$ be $\mathbb{F}_3$. We have a natural homomorphism from $\mathcal{G}_{12}$ to the full symmetric group $\mathfrak{S}_{12}$ of the set $\{x_1, \dots, x_{12}\}$ of coordinate positions of $\mathbb{F}_3^{12}$. The image of $\mathrm{Aut}(\mathcal{C}_{12})$ by this homomorphism is the Mathieu group M_{12}. The kernel of the projection $\mathrm{Aut}(\mathcal{C}_{12}) \to M_{12}$ is of order 2 and is generated by the scalar multiplication by -1. The action of M_{12} on $\{x_1, \dots, x_{12}\}$ is 5-transitive. The stabilizer subgroup of x_{12} in M_{12} is the Mathieu group M_{11}.

0.2. **Construction of a code.** Let $[\mathbf{c}]$ be one of the equivalence classes listed in Table 0.1. The hole type $\tau(\mathbf{c})$ is of the form $\alpha\beta^\nu$, where α, β, and ν are given in Table 0.1. We put

$$\begin{aligned} &p = 2, \quad \mathbb{F} = \mathbb{F}_2, \quad \text{when } \beta = a_1, \quad \text{and} \\ &p = 3, \quad \mathbb{F} = \mathbb{F}_3, \quad \text{when } \beta = a_2. \end{aligned}$$

We consider the case $\mathbf{c} \neq \mathbf{c}_{307}$. (The case $\mathbf{c} = \mathbf{c}_{307}$ will be treated in Section 0.4.) We decompose $P_\mathbf{c}$ to the disjoint union of A and B, where the vertices in A correspond to the nodes of α and the vertices in B correspond to the nodes of β^ν. Since $\alpha \neq \beta$, we have a direct product decomposition

$$\mathrm{Aut}(\overline{P}_\mathbf{c}) = \mathrm{Aut}(A) \times \mathrm{Aut}(B),$$

where $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are the groups of symmetries of the Coxeter–Dynkin diagrams α of A and β^ν of B, respectively. Since $\mathrm{Aut}(A)$ is very small, we can easily calculate $\mathrm{Aut}(A) \cap \mathrm{Aut}(P_\mathbf{c}, \Lambda)$ by the criterion (0.3). It turns out that, in all cases, the group $\mathrm{Aut}(A) \cap \mathrm{Aut}(P_\mathbf{c}, \Lambda)$ is trivial. Therefore the second projection $\mathrm{Aut}(\overline{P}_\mathbf{c}) \to \mathrm{Aut}(B)$ embeds $\mathrm{Aut}(P_\mathbf{c}, \Lambda)$ into $\mathrm{Aut}(B)$. We denote by

$$\mathrm{Aut}_B(P_\mathbf{c}, \Lambda) \subset \mathrm{Aut}(B)$$

the image of $\mathrm{Aut}(P_\mathbf{c}, \Lambda)$. For the proof of the inequality (0.2), it is enough to show that the order of $\mathrm{Aut}_B(P_\mathbf{c}, \Lambda)$ is at most $g(\mathbf{c})$.

Let $\langle A \rangle$ and $\langle B \rangle$ denote the minimal affine subspaces of $\Lambda_\mathbb{R}$ that contain A and B, respectively. We have

$$\dim\langle A \rangle = |A| - 1, \quad \dim\langle B \rangle = |B| - 1, \quad \dim\langle A \rangle + \dim\langle B \rangle = 23, \quad \langle A \rangle \cap \langle B \rangle = \emptyset.$$

Let $\Lambda_{\mathbb{R}}/\langle A\rangle$ be the quotient of $\Lambda_{\mathbb{R}}$ by the equivalence relation

$$x \sim y \iff a + x - y \in \langle A\rangle \text{ for one (and hence all) } a \in \langle A\rangle,$$

that is, we have $x \sim y$ if and only if $x - y$ is parallel to $\langle A\rangle$. We denote by

$$\rho :\ \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}/\langle A\rangle$$

the quotient map. Then $\Lambda_{\mathbb{R}}/\langle A\rangle$ has a natural structure of the linear space of dimension $|B|$ over $\mathbb{R}$ with $\rho(\langle A\rangle)$ being the origin, and

$$L := \rho(\Lambda)$$

is a discrete $\mathbb{Z}$-submodule of $\Lambda_{\mathbb{R}}/\langle A\rangle$ with full rank. Let M denote the $\mathbb{Z}$-submodule of $\Lambda_{\mathbb{R}}/\langle A\rangle$ generated by $\rho(B)$. Then M is also a discrete $\mathbb{Z}$-submodule with full rank, and is equipped with a canonical basis $\{\rho(b)\,|\,b \in B\}$. It is obvious that M is contained in L. Therefore we have

$$M \subset L \subset M \otimes \mathbb{Q}.$$

Note that $\operatorname{Aut}(B)$ acts on M naturally, and that each element of the subgroup $\operatorname{Aut}_B(P_{\mathbf{c}}, \Lambda)$ of $\operatorname{Aut}(B)$ preserves $L \subset M \otimes \mathbb{Q}$.

Let n denote the least positive integer such that $nL \subset M$. Then we have a submodule nL/nM of $M/nM = (\mathbb{Z}/n\mathbb{Z})^B$. It turns out that n is divisible by p. We define a submodule F of M/nM as follows.

- When $\beta = a_1$, we put $\tilde{b} := (n/2)\, b$, and

$$F := \bigoplus_{b\in B} (\mathbb{Z}/n\mathbb{Z})\, \tilde{b}.$$

- Suppose that $\beta = a_2$. We label the elements of B as $b_1, b_1', \dots, b_\nu, b_\nu'$ in such a way that the nodes corresponding to b_i and b_i' are connected in the Coxeter–Dynkin diagram a_2^ν. We then put $\tilde{b}_i := (n/3)\, b_i + (2n/3)\, b_i'$, and

$$F := \bigoplus_{i=1}^{\nu} (\mathbb{Z}/n\mathbb{Z})\, \tilde{b}_i.$$

  Note that F does not change even if we interchange b_i and b_i', because we have $(n/3)\,(b_i + 2b_i') = -(n/3)\,(2b_i + b_i')$ in M/nM.

Then we have $F = \mathbb{F}^\nu$. We define a code Γ of length ν over $\mathbb{F}$ by

$$\Gamma := (nL/nM) \cap F.$$

The group $\operatorname{Aut}(B)$ acts on F, and is identified with the group $\mathcal{G}_\nu$ of monomial transformations of $\mathbb{F}^\nu$. (When $\beta = \alpha_2$, the transposition of b_i and b_i' corresponds to the multiplication by -1 on the ith coordinate of $\mathbb{F}^\nu$.) Under this identification, we have

$$\operatorname{Aut}_B(P_{\mathbf{c}}, \Lambda) \subset \operatorname{Aut}(\Gamma).$$

In the next section, we describe this code Γ explicitly, and derive an upper bound of $|\operatorname{Aut}(P_{\mathbf{c}}, \Lambda)| = |\operatorname{Aut}_B(P_{\mathbf{c}}, \Lambda)|$ from $\operatorname{Aut}(\Gamma)$.

0.3. **Description of the code Γ.**

0.3.1. *The shallow hole* $\mathbf{c}_{293}$ *of type* $a_5 a_2^{10}$. In this case, we have $n = 15$. The ternary code Γ is a $(10, 5)$-code with weight distribution

$$0^1\, 4^{30}\, 6^{60}\, 7^{120}\, 9^{20}\, 10^{12}.$$

It turns out that Γ is equivalent to the code $\mathrm{pr}_{10}(\mathcal{C}_{12} \cap V)$, where V is the linear subspace of $\mathbb{F}_3^{12}$ defined by $x_{11} + x_{12} = 0$. We can calculate its automorphism group directly, and see that $\mathrm{Aut}(\Gamma)$ is of order 1440. Hence $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is contained in the group $\mathrm{Aut}(A) \times \mathrm{Aut}(\Gamma)$ of order 2880. We calculate $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by applying the criterion (0.3) to these 2880 elements. Then we see that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to the symmetric group of degree 6, and hence its order is $g(\mathbf{c}_{293}) = 720$.

0.3.2. *The shallow hole* $\mathbf{c}_{299}$ *of type* $d_4 a_1^{21}$. In this case, we have $n = 14$. The binary code Γ is a $(21, 11)$-code with weight distribution

$$0^1\, 6^{168}\, 8^{210}\, 10^{1008}\, 12^{280}\, 14^{360}\, 16^{21}.$$

We construct a linear embedding

$$\iota\colon \Gamma \hookrightarrow \mathbb{F}_2^{24}$$

such that $\mathrm{pr}_{21} \circ \iota$ is the identity map of Γ, and that every codeword of the image $\Gamma' := \iota(\Gamma)$ is of weight 0, 8, 12, or 16. Let $\beta_1, \dots, \beta_{11}$ be a basis of Γ. We define $\beta_i' \in \mathbb{F}_2^{24}$ as follows. When the weight of β_i is 6, 10, or 14, we put

$$\beta_i' := (\,\beta_i \,|\, 0, 1, 1\,), \quad \text{or} \quad \beta_i' := (\beta_i \,|\, 1, 0, 1\,), \quad \text{or} \quad \beta_i' := (\,\beta_i \,|\, 1, 1, 0\,). \tag{0.4}$$

When the weight of β_i is 8, 12, or 16, we put

$$\beta_i' := (\,\beta_i \,|\, 0, 0, 0\,).$$

We search for a combination of choices in (0.4) such that every element of the linear subspace of $\mathbb{F}_2^{24}$ generated by $\beta_1', \dots, \beta_{11}'$ has weight 0, 8, 12, or 16. If $\beta_1', \dots, \beta_{11}'$ satisfy this condition, then the linear embedding $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ defined by $\beta_i \mapsto \beta_i'$ satisfies the properties required for ι. By this method, we find exactly six such embeddings. We fix one of them. The weight distribution of Γ' is

$$0^1\, 8^{378}\, 12^{1288}\, 16^{381}.$$

Then the code $\tilde{\Gamma}$ generated by Γ' and the vector $\varepsilon := (1, 1, \dots, 1) \in \mathbb{F}_2^{24}$ of weight 24 is equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{21}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{22} + x_{23} + x_{24} = 0$.

Let $\mathfrak{S}_3'$ be the full symmetric group of the coordinate positions $\{x_{22}, x_{23}, x_{24}\}$. We have $\mathfrak{S}_{21} \times \mathfrak{S}_3' \subset \mathfrak{S}_{24}$. We will construct an injective homomorphism

$$\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\tilde{\Gamma}) \cap (\mathfrak{S}_{21} \times \mathfrak{S}_3').$$

Since $\mathrm{Aut}(\tilde{\Gamma}) \cap \mathfrak{S}_{21}$ is isomorphic to M_{21}, the order of $\mathrm{Aut}(\tilde{\Gamma}) \cap (\mathfrak{S}_{21} \times \mathfrak{S}_3')$ is at most $6 \times |M_{21}| = g(\mathbf{c}_{299})$. Since $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda) \subset \mathrm{Aut}(\Gamma)$, the existence of such an injective homomorphism will imply the desired inequality $|\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)| \le g(\mathbf{c}_{299})$.

Let $\mathrm{pr}_3'\colon \mathbb{F}_2^{24} \to \mathbb{F}_2^3$ denote the projection $(x_1, \dots, x_{24}) \mapsto (x_{22}, x_{23}, x_{24})$. Then $T := \mathrm{pr}_3'(\Gamma')$ is defined in $\mathbb{F}_2^3$ by $x_{22} + x_{23} + x_{24} = 0$, and hence we have a natural identification

$$GL(T) = \mathfrak{S}_3'. \tag{0.5}$$

Let $g \in \mathfrak{S}_{21}$ be an automorphism of Γ. Then, via $\iota\colon \Gamma \cong \Gamma'$, the automorphism g induces a linear automorphism g' of the linear space Γ'. Since the linear subspace $\iota^{-1}(\mathrm{Ker}\ \mathrm{pr}_3'|_{\Gamma'})$ of Γ consists exactly of codewords of weight 0, 8, 12, and 16, it is preserved by g, and hence g' induces a linear automorphism of T. By (0.5), there exists a unique permutation $g'' \in \mathfrak{S}'_3$ such that $(g, g'') \in \mathfrak{S}_{21} \times \mathfrak{S}'_3$ preserves Γ'. Since (g, g'') preserves $\varepsilon = (1, 1, \ldots, 1)$, this pair (g, g'') is in fact an automorphism of $\tilde{\Gamma}$.

0.3.3. *The shallow hole* $\mathbf{c}_{303}$ *of type* $a_3 a_2^{11}$. In this case, we have $n = 18$. The ternary code Γ is an $(11, 5)$-code with weight distribution

$$0^1\, 6^{132}\, 9^{110}.$$

Let $\Gamma \hookrightarrow \mathbb{F}_3^{12}$ be the linear embedding given by $x \mapsto (\, x \,|\, 0 \,)$, and let Γ' denote its image. We put

$$Y := \{\ y \in \mathbb{F}_3^{11} \ \mid\ \mathrm{wt}(y) = 11, \text{ and } \mathrm{wt}(x + y) \equiv 2 \bmod 3 \text{ for all } x \in \Gamma\ \}.$$

Then Y consists of 24 vectors. We choose an element $y_0 \in Y$, and let $\tilde{\Gamma}_1$ (resp. $\tilde{\Gamma}_2$) be the code of length 12 generated by Γ' and $(\, y_0 \,|\, 1 \,)$ (resp. $(\, y_0 \,|\, 2 \,)$). Then both of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are equivalent to $\mathcal{C}_{12}$. This means that Γ is equivalent to $\mathrm{pr}_{11}(\mathcal{C}_{12} \cap V)$, where V is the linear subspace of $\mathbb{F}_3^{12}$ defined by $x_{12} = 0$. Moreover, the two codes $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are distinct, and for each $y \in Y$, one and only one of the following holds:

$$(\ (\, y \,|\, 1 \,) \in \tilde{\Gamma}_1 \ \text{ and }\ (\, y \,|\, 2 \,) \in \tilde{\Gamma}_2\ ) \quad \text{or} \quad (\ (\, y \,|\, 1 \,) \in \tilde{\Gamma}_2 \ \text{ and }\ (\, y \,|\, 2 \,) \in \tilde{\Gamma}_1\ ).$$

Let $g \in \mathcal{G}_{11}$ be an automorphism of Γ. Since g preserves Y, one and only one of $(\, g \,|\, 1 \,) \in \mathcal{G}_{12}$ or $(\, g \,|\, -1 \,) \in \mathcal{G}_{12}$ is an automorphism of $\tilde{\Gamma}_1$. Hence $|\mathrm{Aut}(\Gamma)|$ is bounded by the order of $2.M_{11}$.

On the other hand, let $f_A \in \mathrm{Aut}(A)$ be the non-trivial element of $\mathrm{Aut}(A) \cong \mathbb{Z}/2\mathbb{Z}$, and let f_B be the element of $\mathrm{Aut}(B)$ which corresponds to the scalar multiplication by -1, that is, f_B is the product of transpositions of b_i and b'_i for $i = 1, \ldots, 11$. Note that f_B belongs to $\mathrm{Aut}(\Gamma)$. By the criterion (0.3), we see that neither f_B nor $f_A f_B$ is in $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$. Hence $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)$ is a proper subgroup of $\mathrm{Aut}(\Gamma)$. In particular, its order is at most $|M_{11}| = 7920 = g(\mathbf{c}_{303})$.

0.3.4. *The shallow hole* $\mathbf{c}_{304}$ *of type* $a_3 a_1^{22}$. In this case, we have $n = 16$. The binary code Γ is a $(22, 11)$-code with weight distribution

$$0^1\, 6^{77}\, 8^{330}\, 10^{616}\, 12^{616}\, 14^{330}\, 16^{77}\, 22^1.$$

Let $\beta_1, \ldots, \beta_{11}$ be a basis of Γ. We define $\beta'_i \in \mathbb{F}_2^{24}$ by

$$\beta'_i := \begin{cases} (\, \beta_i \,|\, 0, 0 \,) & \text{if } \mathrm{wt}(\beta_i) \text{ is 8, 12, or 16,} \\ (\, \beta_i \,|\, 1, 1 \,) & \text{if } \mathrm{wt}(\beta_i) \text{ is 6, 10, 14, or 22.} \end{cases}$$

Then the image Γ' of the linear embedding $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ defined by $\beta_i \mapsto \beta'_i$ is a binary $(24, 11)$-code with weight distribution

$$0^1\, 8^{407}\, 12^{1232}\, 16^{407}\, 24^1.$$

We enumerate the set

$$Y := \{\ y \in \mathbb{F}_2^{22} \ \mid\ \mathrm{wt}(y) = 7, \text{ and } \mathrm{wt}(x + y) \equiv 3 \bmod 4 \text{ for all } x \in \Gamma\ \}.$$

Then Y consists of 352 vectors. We choose $y_0 \in Y$, and define the code $\tilde{\Gamma}_{01}$ (resp. $\tilde{\Gamma}_{10}$) to be the code of length 24 generated by Γ' and $(\, y_0 \,|\, 0, 1 \,)$ (resp. $(\, y_0 \,|\, 1, 0 \,)$). Then both of $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$ are equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{22}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{23} + x_{24} = 0$. Moreover, the two codes $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$ are distinct, and for each $y \in Y$, one and only one of the following holds:

$$\Big((y \,|\, 0,1) \in \tilde{\Gamma}_{01} \text{ and } (y \,|\, 1,0) \in \tilde{\Gamma}_{10} \Big) \quad \text{or} \quad \Big((y \,|\, 0,1) \in \tilde{\Gamma}_{10} \text{ and } (y \,|\, 1,0) \in \tilde{\Gamma}_{01} \Big).$$

Let $\sigma \in \mathfrak{S}_{24}$ denote the transposition of x_{23} and x_{24}, and let $\mathfrak{S}_2'$ be the subgroup $\{\mathrm{id}, \sigma\}$ of $\mathfrak{S}_{24}$. We have $\mathfrak{S}_{22} \times \mathfrak{S}_2' \subset \mathfrak{S}_{24}$. Since $\mathrm{Aut}(\tilde{\Gamma}_{01}) \cap \mathfrak{S}_{22}$ is isomorphic to M_{22} and $2 \times |M_{22}| = g(\mathbf{c}_{304})$, it is enough to construct an injective homomorphism

$$\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\tilde{\Gamma}_{01}) \cap (\mathfrak{S}_{22} \times \mathfrak{S}_2').$$

Note that σ interchanges $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$. Let $g \in \mathfrak{S}_{22}$ be an automorphism of Γ. Since g preserves Y, one and only one of $(g, \mathrm{id}) \in \mathfrak{S}_{22} \times \mathfrak{S}_2'$ or $(g, \sigma) \in \mathfrak{S}_{22} \times \mathfrak{S}_2'$ induces an isomorphism of $\tilde{\Gamma}_{01}$. Hence the mapping

$$g \mapsto \begin{cases} (g, \mathrm{id}) & \text{if } (g, \mathrm{id}) \text{ maps } \tilde{\Gamma}_{01} \text{ to } \tilde{\Gamma}_{01}, \\ (g, \sigma) & \text{if } (g, \mathrm{id}) \text{ maps } \tilde{\Gamma}_{01} \text{ to } \tilde{\Gamma}_{10}, \end{cases}$$

gives the desired injective homomorphism.

0.3.5. *The shallow hole* $\mathbf{c}_{305}$ *of type* $a_1 a_2^{12}$. In this case, we have $n = 21$. The ternary code Γ is a $(12, 6)$-code of minimal weigh 6, and hence is equivalent to $\mathcal{C}_{12}$. Therefore $|\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)|$ is at most $|2.M_{12}| = 2 \times 95040 = g(\mathbf{c}_{305})$.

0.3.6. *The shallow hole* $\mathbf{c}_{306}$ *of type* $a_2 a_1^{23}$. In this case, we have $n = 18$. The binary code Γ is a $(23, 11)$-code with weight distribution

$$0^1\, 8^{506}\, 12^{1288}\, 16^{253}.$$

Let $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ be the linear embedding given by $x \mapsto (\, x \,|\, 0\,)$. Then the code $\tilde{\Gamma}$ in $\mathbb{F}_2^{24}$ generated by the image of this embedding and the vector $\varepsilon = (1, 1, \dots, 1) \in \mathbb{F}_2^{24}$ is equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{23}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{24} = 0$. Hence we obtain an injective homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\tilde{\Gamma}) \cap \mathfrak{S}_{23} \cong M_{23}$.

0.4. **The shallow hole $\mathbf{c}_{307}$ of type a_1^{25}.** Let $\mathbf{c}$ be a shallow hole with $\tau(\mathbf{c}) = a_1^{25}$. Let $v_0, \dots, v_{24}$ be the vertices of $\overline{P}_{\mathbf{c}}$, and let c_i be the circumcenter of the 23-dimensional face of $\overline{P}_{\mathbf{c}}$ that does not contain v_i. Then there exists a unique vertex v_k such that $m(c_k) = 12$ and $m(c_j) = 24$ for $j \neq k$, where $m\colon \Lambda \otimes \mathbb{Q} \to \mathbb{Z}_{>0}$ is defined in Section 3 of [3]. We put $A := \{v_k\}$ and $B := P_{\mathbf{c}} \setminus A$. Then $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is contained in $\mathrm{Aut}(B) \subset \mathrm{Aut}(\overline{P}_{\mathbf{c}})$. We construct a code Γ of length 24 by the method described in Section 0.2. In this case, the quotient map $\rho\colon \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}/\langle A \rangle$ is just the translation $x \mapsto x - v_k$, and M is the sublattice of Λ generated by $v_j - v_k$ ($j \neq k$). We have $n = 10$, and the binary code $\Gamma := (10\Lambda \cap 5M)/10M$ of length 24 is equivalent to $\mathcal{C}_{24}$. Hence $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is embedded into M_{24}.

References

[1] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, New York, third edition, 1999.

[2] Vera Pless. On the uniqueness of the Golay codes. *J. Combinatorial Theory*, 5:215–228, 1968.

[3] Ichiro Shimada. Holes of the Leech lattice and the projective models of $K3$ surfaces. Preprint, 2015. arXiv:1502.02099.

[4] Ichiro Shimada. The list of holes of the Leech lattice, 2016. http://www.math.sci.hiroshima-u.ac.jp/~shimada/Leech.html.

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