ON A SMOOTH QUARTIC SURFACE CONTAINING 56 LINES WHICH IS ISOMORPHIC AS A $K3$ SURFACE TO THE FERMAT QUARTIC

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Abstract. We give a defining equation of a complex smooth quartic surface containing 56 lines, and investigate its reductions to positive characteristics. This surface is isomorphic to the complex Fermat quartic surface, which contains only 48 lines. We give the isomorphism explicitly.

1. Introduction

The complex Fermat quartic surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

is a very interesting surface, because it lies at the intersection point of the two important classes of algebraic varieties; $K3$ surfaces and Fermat varieties. In this paper, we show that the complex $K3$ surface underlying the Fermat quartic surface has another smooth quartic surface $X_{56} \subset \mathbb{P}^3$ as a projective model. While the Fermat quartic surface contains only 48 lines, our new quartic surface X_{56} contains 56 lines, and hence these two surfaces are not projectively isomorphic. We present an explicit defining equation of X_{56}, and describe the isomorphism between these two surfaces. It turns out that the isomorphism $X_{48} \xrightarrow{\sim} X_{56}$, the lines on X_{48} and X_{56}, and the automorphisms of X_{48} and X_{56} are all defined over the 8th cyclotomic field $\mathbb{Q}(\zeta)$, where

$$\zeta := \exp(2\pi\sqrt{-1}/8).$$

We then study the reductions of X_{56} at primes of $\mathbb{Z}[\zeta]$.

In the following, we denote the complex Fermat quartic surface by X_{48}, and the complex $K3$ surface underlying X_{48} by X.

The existence of a complex smooth quartic surface containing 56 lines that is isomorphic to X_{48} has been implicitly shown in the paper [1] by Degtyarev, Itenberg and Sertöz. This paper is one of the several works [1, 16, 17] on the number of lines lying on a smooth quartic surface that have been done after the seminal work [15], in which Rams and Schütt revised and corrected B. Segre's classical work [21]. Degtyarev, Itenberg and Sertöz [1] proved the following theorem.

Theorem 1.1. *The number of lines lying on a complex smooth quartic surface is either in* $\{64, 60, 56, 54\}$ *or* ≤ 52.

The maximum number 64 of lines lying on a complex smooth quartic surface is attained by the Schur quartic. Schütt [20] has discovered the defining equations of complex smooth quartic surfaces containing 60 lines. On the other hand, in [1],

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the transcendental lattices of complex smooth quartic surfaces containing 56 lines are calculated. One of these surfaces, which we denote by X_{56}, has the oriented transcendental lattice isomorphic to that of X_{48}, and hence they are isomorphic over the complex number field by the result of [27] on the classification of complex $K3$ surfaces with Picard number 20. However, the defining equation of X_{56} and the description of the isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ have been unknown.

Our main results are as follows. Let $\mathbf{P}^3$ and $\mathbb{P}^3$ be the projective spaces with homogeneous coordinates $(x_1 : x_2 : x_3 : x_4)$ and $(y_1 : y_2 : y_3 : y_4)$, respectively. Let S_X denote the Néron-Severi lattice of X_{48}, and let $h_{48} \in S_X$ be the class of a hyperplane section of $X_{48} \subset \mathbf{P}^3$.

Theorem 1.2. *If $h \in S_X$ is a very ample class such that $\langle h, h\rangle = 4$ and $\langle h, h_{48}\rangle = 6$, then the quartic surface model X_h of X corresponding to h contains exactly 56 lines. The number of very ample classes $h \in S_X$ satisfying $\langle h, h\rangle = 4$ and $\langle h, h_{48}\rangle = 6$ is 384.*

Our quartic surface $X_{56} \subset \mathbb{P}^3$ corresponds to one of the 384 very ample classes in Theorem 1.2. For a prime P of $\mathbb{Z}[\zeta]$, let κ_P denote the residue field at P, and $\bar{\kappa}_P$ an algebraic closure of κ_P.

Theorem 1.3. (1) *We put $A := -1 - 2\zeta - 2\zeta^3$, $B := 3 + A$, and*

$$\begin{aligned}\Psi \quad := \quad & y_1^3 y_2 + y_1 y_2^3 + y_3^3 y_4 + y_3 y_4^3 \\ & + (y_1 y_4 + y_2 y_3)(A(y_1 y_3 + y_2 y_4) + B(y_1 y_2 - y_3 y_4)).\end{aligned}$$

Let X_{56} denote the surface in $\mathbb{P}^3$ defined over $\mathbb{Q}(\zeta)$ by the equation $\Psi = 0$. Then the complex surface $X_{56} \otimes \mathbb{C}$ is smooth, and contains exactly 56 lines, each of which is defined over $\mathbb{Q}(\zeta)$. The projective automorphism group of $X_{56} \otimes \mathbb{C}$ is of order 64. Moreover, if P is a prime of $\mathbb{Z}[\zeta]$ over a prime integer $p > 3$, then the surface $X_{56} \otimes \bar{\kappa}_P$ is also smooth, contains exactly 56 lines, each of which is defined over the finite field κ_P.

(2) *There exists an isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ defined over $\mathbb{Q}(\zeta)$ such that the class $h_{56} \in S_X$ of the pull-back of a hyperplane section of X_{56} satisfies $\langle h_{48}, h_{56}\rangle = 6$. This isomorphism induces an isomorphism from $X_{48} \otimes \bar{\kappa}_P$ to $X_{56} \otimes \bar{\kappa}_P$, if P is a prime of $\mathbb{Z}[\zeta]$ over a prime integer $p > 3$.*

Remark 1.4. Since $A = -1 - 2\sqrt{-2}$, the surface X_{56} is in fact defined over $\mathbb{Q}(\sqrt{-2})$.

The precise description and a geometric characterization of the class $h_{56} \in S_X$ are given in Section 4. An explicit description of the isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ is given in Table 4.1, where the pull-back of the rational functions y_i/y_j on X_{56} are the rational functions f_i/f_j on X_{48}.

In the study of reductions of X_{56}, we have to calculate Gröbner bases of ideals in polynomial rings with coefficients in $\mathbb{Z}[\zeta]$ over the residue field κ_P at infinitely many primes P of $\mathbb{Z}[\zeta]$. A simple computational trick for this task will be given in Section 5. For the actual computation, we used GAP [5]. Computational data is available from the author's webpage [26].

The Néron-Severi lattices of the Fermat quartic surfaces in characteristic 0 and in characteristic 3 were studied by Mizukami and Inose in 1970's. In particular, they proved that these Néron-Severi lattices are generated by the classes of lines. This fact is crucial for our construction of X_{56}. See [10] and Section 6.1 of [19].

Note that we have the following classical theorem due to Matsumura and Monsky [9, Theorem 2].

Theorem 1.5. *If two smooth hypersurfaces of degree $d \geq 3$ in $\mathbb{P}^n$ with $n \geq 3$ are isomorphic as abstract varieties but not projectively equivalent, then we have $(d, n) = (4, 3)$.*

Recently, Oguiso [13] informed us of his method of constructing pairs of complex smooth quartic surfaces that are isomorphic as $K3$ surfaces but are not projectively isomorphic. His result shows in particular that the graph of the isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ is a complete intersection of 4 hypersurfaces of bi-degree $(1, 1)$ in $\mathbf{P}^3 \times \mathbb{P}^3$.

After the first version of this paper is submitted, Degtyarev [2] has proved that, up to projective equivalence, the $K3$ surface underlying X_{48} has exactly three smooth quartic surface models; X_{48}, X_{56}, and its complex conjugate $\overline{X_{56}}$.

The plan of this paper is as follows. In Section 2, we review the theory of lattices, fix notation, and present two algorithms that are used throughout this paper. In Section 3, we describe the Néron-Severi lattice S_X of X_{48} by means of the 48 lines on it. In Section 4, we study very ample line bundles on X_{48} that give rise to an isomorphism to X_{56}, and show how to obtain the defining equation of X_{56}. We also compute the projective automorphism group of X_{56}. In Section 5, we investigate the reductions of X_{56} at primes of $\mathbb{Z}[\zeta]$.

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2. Preliminaries on lattices

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}$. The *orthogonal group* $\mathrm{O}(L)$ of L acts on L from the *right*. The *dual lattice* $L^\vee$ of L is a submodule of $L \otimes \mathbb{Q}$ consisting of vectors $v \in L \otimes \mathbb{Q}$ such that $\langle v, x \rangle \in \mathbb{Z}$ holds for any $x \in L$. The *discriminant group* $\mathrm{disc}(L)$ of L is defined to be $L^\vee/L$. A lattice L is *unimodular* if $\mathrm{disc}(L)$ is trivial. A lattice L is *even* if $\langle v, v \rangle \in 2\mathbb{Z}$ holds for any $v \in L$. Let L be an even lattice. The $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ that extends $\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}$ defines a finite quadratic form $q_L\colon \mathrm{disc}(L) \to \mathbb{Q}/2\mathbb{Z}$, which is called the *discriminant form* of L. Let $\mathrm{O}(q_L)$ denote the automorphism group of the finite quadratic form q_L, which acts on $\mathrm{disc}(L)$ from the right. Then we have a natural homomorphism

$$\eta_L\colon \mathrm{O}(L) \to \mathrm{O}(q_L).$$

See [11] for applications of the theory of discriminant forms.

A lattice L of rank $n > 1$ is *hyperbolic* if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(1, n-1)$. We have the following algorithms. See [25] for details.

Algorithm 2.1. Let M be a free $\mathbb{Z}$-module of finite rank $n > 1$ with a $\mathbb{Q}$-valued non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon M \times M \to \mathbb{Q}$ such that $M \otimes \mathbb{R}$ is of signature $(1, n-1)$. (For example, M is a hyperbolic lattice or the dual lattice of a hyperbolic lattice.) Let $h \in M$ be a vector such that $\langle h, h \rangle > 0$. Then, for given rational numbers a and b, we can make the list of all vectors x of M that satisfy $\langle h, x \rangle = a$ and $\langle x, x \rangle = b$. ∎

Algorithm 2.2. Let L be a hyperbolic lattice, and let h, h' be vectors of L that satisfy $\langle h, h \rangle > 0$, $\langle h', h' \rangle > 0$ and $\langle h, h' \rangle > 0$. Then, for a negative integer d, we can make the list of all vectors x of L that satisfy $\langle h, x \rangle > 0$, $\langle h', x \rangle < 0$ and $\langle x, x \rangle = d$. ∎

Remark 2.3. These algorithms are based on an algorithm of positive-definite quadratic forms described in Section 3.1 of [25]. This algorithm can be made much faster by means of the lattice reduction basis [8]. See Section 2.7 of [3].

Let L be an even hyperbolic lattice, and let $\mathcal{P}(L)$ be one of the two connected components of $\{x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$, which we call a *positive cone* of L. A vector $r \in L$ is called a (-2)*-vector* if $\langle r, r\rangle = -2$ holds. For a (-2)-vector r, we have a *reflection* $s_r \colon x \mapsto x + \langle x, r\rangle\, r$ in the hyperplane

$$(r)^{\perp} := \{\ x \in L \otimes \mathbb{R}\ \mid\ \langle r, x\rangle = 0\ \}.$$

Each $s_r \in \mathrm{O}(L)$ acts on $\mathcal{P}(L)$. The reflections s_r with respect to all (-2)-vectors r generate a subgroup $W(L)$ of $\mathrm{O}(L)$. The closure in $\mathcal{P}(L)$ of a connected component of

$$\mathcal{P}(L)\ \setminus\ \bigcup\,(r)^{\perp}$$

is called a *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}(L)$. Let $\mathcal{N}$ be a standard fundamental domain, and let h be an element of $\mathcal{N} \cap L$ such that $h \notin (r)^{\perp}$ for any (-2)-vectors r. Then Algorithm 2.2 applied to $d = -2$ provides us with a method to determine whether a given vector $h' \in \mathcal{P}(L) \cap L$ is contained in $\mathcal{N}$ or not.

The *Néron-Severi lattice* S_Y of an algebraic $K3$ surface Y is the $\mathbb{Z}$-module of numerical equivalence classes of divisors on Y with the intersection pairing. The lattice S_Y is even, and if its rank is > 1, it is hyperbolic. The class of a curve C on Y is denoted by $[C] \in S_Y$. Suppose that $\operatorname{rank} S_X > 1$. It is well-known that the *nef cone*

$$\{\ x \in S_Y \otimes \mathbb{R}\ \mid\ \langle x, [C]\rangle \geq 0 \text{ for any curve } C \text{ on } Y\ \}$$

is a standard fundamental domain of the action of $W(S_X)$. When Y is defined over $\mathbb{C}$, the second cohomology group $H^2(Y, \mathbb{Z})$ of a complex $K3$ surface Y with the cup product is an even, unimodular lattice of signature $(3, 19)$ containing S_Y as a primitive sublattice.

3. The Fermat quartic surface X_{48}

The complex Fermat quartic surface is denoted by $X_{48} \subset \mathbf{P}^3$. The complex surface underlying X_{48} is simply denoted by X. We describe the Néron-Severi lattice S_X of X in terms of the lines on X_{48}. In particular, we study the condition for an isometry $g \in \mathrm{O}(S_X)$ of S_X to extend to a Hodge isometry of $H^2(X, \mathbb{Z})$.

The *transcendental lattice* T_X of X is defined to be the orthogonal complement of S_X in $H^2(X, \mathbb{Z})$. It is known [10], [27] that S_X is of rank 20, and T_X is isomorphic to the positive-definite lattice

$$T := \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \tag{3.1}$$

of rank 2. By [11], we have an anti-isometry $q_{S_X} \xrightarrow{\sim} -q_{T_X}$ of discriminant forms, and hence $|\operatorname{disc}(S_X)| = 64$.

The surface X_{48} contains exactly 48 lines. These lines are labelled by the tags $[i, [\mu, \nu]]$ in the following way, where i is an integer satisfying $1 < i \leq 4$, and μ and ν are positive odd integers ≤ 7. Let j and k be integers such that $j < k$ and $\{1, i, j, k\} = \{1, 2, 3, 4\}$. Then the line on X_{48} labelled by $[i, [\mu, \nu]]$ is defined by

$$x_1 + \zeta^{\mu} x_i = 0, \quad x_j + \zeta^{\nu} x_k = 0.$$

$$
\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2
\end{bmatrix}
$$

TABLE 3.1. Gram matrix G_{S_X} of S_X

All lines on X_{48} are obtained in this way. Following [22], we call a point $Q \in X_{48}$ a *τ-point* if the intersection $X_{48} \cap T_Q(X_{48})$ of X_{48} and the tangent plane $T_Q(X_{48}) \subset \mathbf{P}^3$ to X_{48} at Q consists of four lines passing though Q. There exist exactly 24 τ-points, and each line on X_{48} contains exactly two τ-points. If three distinct lines on X_{48} have a common point Q, then Q is a τ-point, and hence these three lines are coplanar. The converse is also true; if three distinct lines on X_{48} are coplanar, then they have a common point, which is a τ-point.

We choose the 20 lines $l_1, \dots, l_{20}$ labelled by the following tags respectively:

$$
\begin{aligned}
&[2,[1,1]],\ [2,[1,3]],\ [2,[1,5]],\ [2,[1,7]],\\
&[2,[3,1]],\ [2,[3,3]],\ [2,[3,5]],\ [2,[5,1]],\ [2,[5,3]], [2,[5,5]],\ [3,[1,1]],\ [3,[1,3]],\\
&[3,[1,5]],\ [3,[3,1]],\ [3,[3,3]],\ [3,[3,5]],\ [4,[1,1]],\ [4,[1,3]], [4,[1,5]],\ [4,[3,1]].
\end{aligned}
$$

Their intersection matrix G_{S_X} is given in Table 3.1. Since the determinant of this matrix is $-64 = -|\operatorname{disc}(S_X)|$, we see that the classes of these 20 lines form a basis of S_X, and the matrix G_{S_X} is the Gram matrix of S_X with respect to this basis. From now on, every vector of S_X is written as a *row* vector with respect to this basis. Since the four lines $l_1, \dots, l_4$ are on the plane $x_1 + \zeta x_2 = 0$, the class h_{48} of the hyperplane section of $X_{48} \hookrightarrow \mathbf{P}^3$ is given by

$$h_{48} = [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0].$$

By Riemann-Roch theorem, the set of classes $[\ell]$ of lines ℓ on X_{48} is equal to

$$\mathcal{F}_{48} := \{\ r \in S_X \ \mid\ \langle r, r\rangle = -2,\ \ \langle r, h_{48}\rangle = 1\ \}.$$

This set can be calculated by Algorithm 2.1. The class of each line is also computed from the intersection numbers with $l_1, \dots, l_{20}$.

We have a basis of $S_X^\vee$ dual to the fixed basis $[l_1], \dots, [l_{20}]$ of S_X. To distinguish the vector representation with respect to the non-dual basis $[l_1], \dots, [l_{20}]$ of S_X and that with respect to the dual basis of $S_X^\vee$, we put a superscript $^\vee$ on the dual representation. Thus we have a relation

$$x^\vee = x\,\mathrm{G}_{S_X}$$

between the non-dual vector representation $x \in \mathbb{Q}^{20}$ of an element $v \in S_X \otimes \mathbb{Q}$ and the dual representation $x^\vee$ of v.

Consider the following vectors of $S_X^\vee$:

$$\begin{aligned} s_1 &:= [3, 1, 2, 2, 1, 3, 2, 2, 2, 2, 2, 3, 1, 2, 1, 2, 2, 1, 3, 1]^\vee, \\ s_2 &:= [1, 3, 1, 1, 1, 1, 3, 2, 1, 0, 1, 1, 2, 2, 3, -1, 1, 2, 0, 2]^\vee. \end{aligned}$$

Then the elements

$$\sigma_1 := s_1 \bmod S_X,\ \ \sigma_2 := s_2 \bmod S_X$$

of $\mathrm{disc}(S_X)$ form a basis of $\mathrm{disc}(S_X) \cong (\mathbb{Z}/8\mathbb{Z})^2$, under which the discriminant form q_{S_X} is given by the matrix

$$\frac{1}{8}\begin{bmatrix} 11 & 5 \\ 5 & 14 \end{bmatrix},$$

where the diagonal components are in $\mathbb{Q}/2\mathbb{Z}$ and the off-diagonal components are in $\mathbb{Q}/\mathbb{Z}$. Let P be the 20×2 matrix

$$\begin{bmatrix} 7 & 2 & 5 & 6 & 0 & 6 & 6 & 7 & 2 & 7 & 6 & 4 & 6 & 2 & 4 & 2 & 4 & 0 & 4 & 0 \\ 0 & 5 & 3 & 2 & 7 & 6 & 3 & 1 & 7 & 6 & 0 & 6 & 2 & 0 & 2 & 6 & 4 & 4 & 4 & 4 \end{bmatrix}^T$$

with components in $\mathbb{Z}/8\mathbb{Z}$. Then the quotient homomorphism $S_X^\vee \to \mathrm{disc}(S_X)$ is given by $x^\vee \mapsto x^\vee P$ with respect to the basis σ_1, σ_2 of $\mathrm{disc}(S_X)$. When we are given an element g of $\mathrm{O}(S_X)$ as a 20×20 matrix R_g with respect to the basis $[l_1], \dots, [l_{20}]$ of S_X, the automorphism $\eta_{S_X}(g)$ of q_{S_X} induced by g is represented with respect to the basis σ_1, σ_2 by the 2×2 matrix

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \mathrm{G}_{S_X}^{-1} R_g \mathrm{G}_{S_X} P \tag{3.2}$$

with components in $\mathbb{Z}/8\mathbb{Z}$.

Let t_1 and t_2 be a basis of the lattice T under which the Gram matrix is given in (3.1), and let $\mathbb{C}\omega$ be a totally isotropic subspace of $T \otimes \mathbb{C}$. We have

$$T \otimes \mathbb{C} = \mathbb{C}\omega \oplus \mathbb{C}\bar{\omega}.$$

Let $t_1^\vee, t_2^\vee$ be the basis of $T^\vee$ dual to the basis t_1, t_2 of T. Then the elements $\tau_1 := t_1^\vee \bmod T$ and $\tau_2 := t_2^\vee \bmod T$ form a basis of $\mathrm{disc}(T) \cong (\mathbb{Z}/8\mathbb{Z})^2$, under which q_T is given by the matrix

$$\frac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As above, we can calculate the natural homomorphism $\eta_T\colon \mathrm{O}(T) \to \mathrm{O}(q_T)$ explicitly. It is easy to see that $\mathrm{O}(T)$ is of order 8, $\mathrm{O}(q_T)$ is of order 16, and η_T is injective. Moreover, we see that the group

$$\tilde{\Gamma}_T := \{\ g \in \mathrm{O}(T)\ \mid\ \mathbb{C}\omega^g = \mathbb{C}\omega\ \}$$

is of order 4. Since T is isomorphic to the transcendental lattice T_X, there exists an isomorphism $q_{S_X} \cong -q_T$ by [11]. In fact, since $\mathrm{O}(q_T)$ is of order 16, there exist exactly 16 isomorphisms from q_{S_X} to $-q_T$. For an isomorphism $\varphi\colon q_{S_X} \xrightarrow{\sim} -q_T$, let $\varphi_*\colon \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_T)$ be the induced isomorphism. It turns out that the subgroup

$$\Gamma_{S_X} := \{\ \gamma \in \mathrm{O}(q_{S_X})\ \mid\ \varphi_*(\gamma) \in \eta_T(\tilde{\Gamma}_T)\ \}$$

of $\mathrm{O}(q_{S_X})$ does *not* depend on the choice of φ; we have

$$\Gamma_{S_X} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 5 \\ 6 & 3 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}). \tag{3.3}$$

Note that an isometry $\tilde{g}$ of the lattice $H^2(X,\mathbb{Z})$ is a Hodge isometry if and only if $\tilde{g}$ preserves T_X and its orientation. If $\tilde{g}$ preserves T_X, then the orientation of T_X is preserved if and only if $\tilde{g}|_{T_X}$ belongs to $\tilde{\Gamma}_T$ under an (and hence any) isometry $T \cong T_X$. Hence, by [11], we see that an isometry $g \in \mathrm{O}(S_X)$ extends to a Hodge isometry of $H^2(X,\mathbb{Z})$ if and only if

$$\eta_{S_X}(g) \in \Gamma_{S_X}. \tag{3.4}$$

This condition can be checked computationally using (3.2) and (3.3).

We let the automorphism group $\mathrm{Aut}(X)$ act on X from the left, and act on S_X from the right by the pull-back. The following facts can be checked by direct computation by means of the data we have prepared so far. We consider the subgroup

$$\mathrm{Aut}(X_{48}) := \{\ \gamma \in \mathrm{PGL}_4(\mathbb{C})\ \mid\ \gamma(X_{48}) = X_{48}\ \}$$

on $\mathrm{Aut}(X)$, which is known to be of order 1536 and generated by the permutations of coordinates $x_1, \dots, x_4$ and the scalar-multiplications by ζ^2 of coordinates. We denote by

$$G_{48} := \mathrm{Im}(\mathrm{Aut}(X_{48}) \to \mathrm{O}(S_X))$$

the image of $\mathrm{Aut}(X_{48})$ in $\mathrm{O}(S_X)$ by the natural representation, which is injective. Since the set $\mathcal{F}_{48}$ of classes of lines on X_{48} spans S_X, the stabilizer subgroup

$$\tilde{G}_{48} := \{\ g \in \mathrm{O}(S_X)\ \mid\ h_{48}^g = h_{48}\ \}$$

of h_{48} is isomorphic to the group of permutations of $\mathcal{F}_{48}$ that preserve the intersection numbers. The mapping

$$g \mapsto ([l_1]^g, \dots, [l_{20}]^g)$$

gives a bijection from $\tilde{G}_{48}$ to the set of ordered lists $([l'_1], \dots, [l'_{20}])$ of elements of $\mathcal{F}_{48}$ that satisfy

$$\langle [l'_i], [l'_j] \rangle = \langle [l_i], [l_j] \rangle \ \text{ for all } i, j = 1, \dots, 20.$$

We calculate the set of all these $([l'_1], \dots, [l'_{20}])$ by the standard backtrack program (see [6] for the meaning of the backtrack program), and calculate $\tilde{G}_{48}$ as a list of elements of $\mathrm{O}(S_X)$. Since each line on X_{48} is defined over $\mathbb{Q}(\zeta)$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ also acts on $\mathcal{F}_{48}$ preserving the intersection numbers, and hence acts on S_X. It turns out that $\tilde{G}_{48}$ is of order 6144 and is generated by G_{48}

$$A'(o_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A'(o_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A'(o_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A'(o_4) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad A'(o_5) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A'(o_6) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(o_7) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A'(o_8) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

TABLE 3.2. Matrices $A'(o_i)$

and $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. By Torelli theorem [14], the subgroup G_{48} of $\tilde{G}_{48}$ consists of elements $g \in \tilde{G}_{48}$ that satisfy the period-preserving condition (3.4).

We review the result of B. Segre [22] on the set of pairs of lines on X_{48}. The group $\mathrm{Aut}(X_{48})$ acts on the 48 lines transitively. Let $\mathcal{P}_i$ be the set of pairs of intersecting lines on X_{48}, and let $\mathcal{P}_d$ be the set of pairs of disjoint lines on X_{48}. The orbit decompositions $\mathcal{P}_i = o_1 \sqcup o_2 \sqcup o_3$ and $\mathcal{P}_d = o_4 \sqcup \cdots \sqcup o_8$ of these sets by the action of $\mathrm{Aut}(X_{48})$ are as follows:

$$\begin{array}{llllll}
\{ [2,[1,1]],\ [2,[1,5]] \} & \in & o_1 & \subset & \mathcal{P}_i, & |o_1| = 48, \\
\{ [2,[1,1]],\ [2,[1,3]] \} & \in & o_2 & \subset & \mathcal{P}_i, & |o_2| = 96, \\
\{ [2,[1,1]],\ [3,[1,1]] \} & \in & o_3 & \subset & \mathcal{P}_i, & |o_3| = 192, \\
\hline
\{ [2,[1,1]],\ [2,[5,5]] \} & \in & o_4 & \subset & \mathcal{P}_d, & |o_4| = 24, \\
\{ [2,[1,1]],\ [2,[3,3]] \} & \in & o_5 & \subset & \mathcal{P}_d, & |o_5| = 96, \\
\{ [2,[1,1]],\ [2,[3,5]] \} & \in & o_6 & \subset & \mathcal{P}_d, & |o_6| = 96, \\
\{ [2,[1,1]],\ [3,[1,5]] \} & \in & o_7 & \subset & \mathcal{P}_d, & |o_7| = 192, \\
\{ [2,[1,1]],\ [3,[1,3]] \} & \in & o_8 & \subset & \mathcal{P}_d, & |o_8| = 384.
\end{array}$$

For each orbit o_i, we define an 8×8 matrix $A(o_i) = (a_{jk})$ as follows. Let $\{\ell, \ell'\}$ be a pair in o_i. We put

$$a_{jk} := \text{the number of lines } \ell'' \text{ such that } \{\ell, \ell''\} \in o_j \text{ and } \{\ell', \ell''\} \in o_k.$$

The 3×3 upper-left part $A'(o_i) = (a_{jk})_{1 \le j,k \le 3}$ of each of these matrices are given in Table 3.2.

Remark 3.1. Let $\{\ell, \ell'\}$ be a pair of intersecting lines. Then ℓ and ℓ' intersect at a τ-point if and only if $\{\ell, \ell'\}$ belongs to o_1 or to o_2.

4. THE QUARTIC SURFACE X_{56}

For $v \in S_X$, let $\mathcal{L}_v \to X$ be a line bundle whose class is v. We say that $h \in S_X$ is a *polarization of degree* 4 if $\langle h, h \rangle = 4$ and the complete linear system $|\mathcal{L}_h|$ is fixed-component free. By [18], if h is a polarization of degree 4, then $|\mathcal{L}_h|$ is base-point free and defines a morphism $\Phi_h \colon X \to \mathbb{P}^3$. We say that a polarization h of degree 4 is *very ample* if Φ_h is an embedding.

Theorem 4.1. *A class $h \in S_X$ with $\langle h, h \rangle = 4$ is a very ample polarization of degree* 4 *if and only if the following hold:*

(a) $\langle h, h_{48}\rangle > 0$,
(b) $\{ r \in S_X \mid \langle r,r\rangle = -2,\ \langle r, h_{48}\rangle > 0,\ \langle r,h\rangle < 0 \}$ *is empty,*
(c) $\{ e \in S_X \mid \langle e,e\rangle = 0,\ \langle e,h\rangle = 1 \}$ *is empty,*
(d) $\{ e \in S_X \mid \langle e,e\rangle = 0,\ \langle e,h\rangle = 2 \}$ *is empty, and*
(e) $\{ r \in S_X \mid \langle r,r\rangle = -2,\ \langle r,h\rangle = 0 \}$ *is empty.*

If $h \in S_X$ is a very ample polarization of degree 4, then the set $\mathcal{F}_h$ of classes of lines contained in the image X_h of $\Phi_h\colon X \to \mathbb{P}^3$ is equal to

$$\{ r \in S_X \mid \langle r,r\rangle = -2,\ \langle r,h\rangle = 1 \}.$$

Proof. The condition (a) is equivalent to the condition that h is in the positive cone $\mathcal{P}(S_X)$ of $S_X \otimes \mathbb{R}$ containing h_{48}. Suppose that (a) holds. Since the nef-cone of X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}(S_X)$, the condition (b) is equivalent to the condition that h is nef. Suppose that (a) and (b) hold. By Proposition 0.1 of [12], the condition (c) is equivalent to the condition that $|\mathcal{L}_h|$ is fixed-component free, and hence defines a morphism $\Phi_h\colon X \to \mathbb{P}^3$. Suppose that (a)–(c) hold. By [18], the condition (d) is equivalent to the condition that Φ_h is not hyperelliptic, that is, Φ_h is generically injective. Suppose that (a)–(d) hold. The condition (e) is equivalent to the condition that Φ_h does not contract any (-2)-curves, that is, the image X_h of Φ_h is smooth. The second assertion is obvious. □

Note that the conditions (a)–(e) can be checked by means of Algorithms 2.1 and 2.2, and that the set $\mathcal{F}_h$ can be calculated by Algorithm 2.1.

We say that $h \in S_X$ is an *X_{56}-polarization* if h is a very ample polarization of degree 4 such that $X_h \subset \mathbb{P}^3$ contains exactly 56 lines. The *relative degree* of a very ample polarization h of degree 4 is defined to be $\langle h, h_{48}\rangle$.

Using Algorithm 2.1, we calculate the set

$$\mathcal{H}_d := \{ v \in S_X \mid \langle v, h_{48}\rangle = d,\ \ \langle v,v\rangle = 4 \}$$

for $d = 1, \dots, 6$. Note that G_{48} acts on each $\mathcal{H}_d$. We have $\mathcal{H}_d = \emptyset$ for $d < 4$, and

$$\mathcal{H}_4 = \{h_{48}\},\quad |\mathcal{H}_5| = 48,\quad |\mathcal{H}_6| = 48264.$$

The action of G_{48} on $\mathcal{H}_5$ is transitive, and no vectors in $\mathcal{H}_5$ are nef. The action of G_{48} decomposes $\mathcal{H}_6$ into 60 orbits. Among the vectors in $\mathcal{H}_6$,

- 792 vectors in 5 orbits are not nef,
- 792 vectors in 5 orbits are nef, fixed-component free, but define hyperelliptic morphism,
- 46296 vectors in 48 orbits are nef, fixed-component free, define non-hyperelliptic morphism, but the images are singular, and
- the remaining 384 vectors in 2 orbits are very ample, and the images contain exactly 56 lines. The larger group $\tilde{G}_{48}$ acts on these 384 vectors transitively.

Thus we obtain the following theorem.

Theorem 4.2. (1) *If $h \in S_X$ is a very ample polarization of degree 4 with relative degree 6, then h is an X_{56}-polarization.* (2) *There exist exactly 384 X_{56}-polarizations of relative degree 6. Up to the action of $\mathrm{Aut}(X_{48})$ and $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, there exists only one X_{56}-polarization of relative degree 6.*

We describe X_{56}-polarizations of relative degree 6 geometrically.

Definition 4.3. An ordered list $(\ell_1, \ell_2, m_1, m_2, m_3, m_4, n)$ of seven lines on X_{48} is called an X_{56}-*configuration* if the following conditions are satisfied:

- $\{\ell_1, \ell_2\} \in o_4$,
- $\{\ell_1, m_1\} \in o_1$, $\{\ell_2, m_1\} \in o_1$, and $\{\ell_1, m_i\} \in o_3$, $\{\ell_2, m_i\} \in o_3$ for $i = 2, 3, 4$,
- $\{m_1, m_k\} \in o_7$ for $k = 2, 3, 4$,
- $\{m_2, m_3\} \in o_5$, $\{m_2, m_4\} \in o_8$, $\{m_3, m_4\} \in o_8$, and
- $\{\ell_1, n\} \in o_8$, $\{\ell_2, n\} \in o_8$, $\{m_1, n\} \in o_8$, $\{m_2, n\} \in o_2$, $\{m_3, n\} \in o_2$, $\{m_4, n\} \in o_7$.

We make the list of X_{56}-configurations. It turns out that the number of X_{56}-configurations is 6144. Comparing this list with the list of X_{56}-polarizations of relative degree 6, we obtain the following theorem.

Theorem 4.4. *If $(\ell_1, \ell_2, m_1, \dots, m_4, n)$ is an X_{56}-configuration, then the vector*

$$h := 3h_{48} - ([\ell_1] + [\ell_2] + [m_1] + \cdots + [m_4]) \tag{4.1}$$

is an X_{56}-polarization of relative degree 6.

Consider the seven lines $\ell_1, \ell_2, m_1, \dots, m_4, n$ labelled by the tags

$$[2,[1,1]],\ [2,[5,5]],\ [2,[1,5]],\ [3,[1,1]],\ [3,[3,3]],\ [4,[1,7]],\ [3,[1,3]], \tag{4.2}$$

respectively. These lines form an X_{56}-configuration, and the corresponding X_{56}-polarization h_{56} is given by

$$h_{56} = [1, 2, 1, 2, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, -1, 0, 1, 1, 1, 0].$$

Theorem 4.5. *Let $\Phi_{56} \colon X_{48} \hookrightarrow \mathbb{P}^3$ be the embedding induced by h_{56}, and X_{56} the image of Φ_{56}. With a suitable choice of the homogeneous coordinates of $\mathbb{P}^3$, the surface X_{56} is defined by the equation $\Psi = 0$, where Ψ is given in Theorem 1.3.*

Proof. Let Γ_d be the space of all homogeneous polynomials of degree d in the variables x_1, x_2, x_3, x_4. We have a natural identification

$$\Gamma_3 = H^0(X_{48}, \mathcal{L}_{3h_{48}}).$$

Since $\mathcal{L}_{h_{56}}$ is isomorphic to $\mathcal{L}_{3h_{48}}(-\ell_1 - \ell_2 - m_1 - \cdots - m_4)$ as an invertible sheaf, the space $H^0(X_{48}, \mathcal{L}_{h_{56}})$ is identified with the space of homogeneous polynomials of degree 3 in x_1, x_2, x_3, x_4 that vanish along each of the lines $\ell_1, \ell_2, m_1, \dots, m_4$. Since we have explicit defining equations (4.2) of these lines, we can confirm that $H^0(X_{48}, \mathcal{L}_{h_{56}})$ is of dimension 4, and calculate a basis f_1, f_2, f_3, f_4 of $H^0(X_{48}, \mathcal{L}_{h_{56}})$ by elementary linear algebra. We fix a basis of $H^0(X_{48}, \mathcal{L}_{h_{56}})$ as in Table 4.1. Let $\bar{\Gamma}_{12}$ denote the space of all homogeneous polynomials of degree 12 in x_1, x_2, x_3, x_4 such that the degree with respect to x_1 is ≤ 3. For $g \in \Gamma_{12}$, let $\rho(g)$ denote the remainder on the division by $x_1^4 + x_2^4 + x_3^4 + x_4^4$ under the lex monomial ordering $x_1 > x_2 > x_3 > x_4$. Then we obtain a surjective homomorphism

$$\rho \colon \Gamma_{12} \to \bar{\Gamma}_{12}.$$

Therefore we have a natural identification

$$\bar{\Gamma}_{12} = H^0(X_{48}, \mathcal{L}_{12h_{48}}).$$

Let Σ_4 be the linear space of all homogeneous polynomials of degree 4 in the variables y_1, y_2, y_3, y_4. The substitution $y_i \mapsto f_i$ for $i = 1, \dots, 4$ gives rise to a linear homomorphism $\sigma \colon \Sigma_4 \to \Gamma_{12}$. The linear homomorphism $\rho \circ \sigma$ is represented by a 290×35 matrix. The kernel of $\rho \circ \sigma \colon \Sigma_4 \to \bar{\Gamma}_{12}$ is of dimension 1 and is generated by the polynomial $\Psi \in \Sigma_4$. □

$$
\begin{aligned}
f_1 \;=\;& \left(1+\zeta-\zeta^3\right) {x_1}^3+\left(\zeta+\zeta^2+\zeta^3\right) {x_1}^2 x_3+(1+\zeta) {x_1}^2 x_4+\left(-\zeta-\zeta^2-\zeta^3\right) x_1 {x_2}^2+ \\
& (-1-\zeta) x_1 x_2 x_3+\left(\zeta+\zeta^2\right) x_1 x_2 x_4-x_1 {x_3}^2+\left(\zeta+\zeta^2\right) x_1 x_3 x_4-\zeta^3 x_1 {x_4}^2+ \\
& \left(1-\zeta^2-\zeta^3\right) {x_2}^2 x_3+\left(-\zeta-\zeta^2\right) x_2 {x_3}^2+\left(\zeta^2+\zeta^3\right) x_2 x_3 x_4+\zeta^2 {x_3}^3+x_3 {x_4}^2 \\
f_2 \;=\;& {x_1}^3-\zeta^2 {x_1}^2 x_3+\left(-1+\zeta^3\right) {x_1}^2 x_4-\zeta^2 x_1 {x_2}^2+\left(1-\zeta^3\right) x_1 x_2 x_3+(-1-\zeta) x_1 x_2 x_4+ \\
& \left(1+\zeta-\zeta^3\right) x_1 {x_3}^2+\left(-\zeta^2-\zeta^3\right) x_1 x_3 x_4+\left(-1-\zeta-\zeta^2\right) x_1 {x_4}^2+\zeta {x_2}^2 x_3+ \\
& \left(\zeta^2+\zeta^3\right) x_2 {x_3}^2+\left(1-\zeta^3\right) x_2 x_3 x_4+\left(\zeta+\zeta^2+\zeta^3\right) {x_3}^3+\left(1+\zeta-\zeta^3\right) x_3 {x_4}^2 \\
f_3 \;=\;& \left(1+\zeta+\zeta^2\right) {x_1}^2 x_2+\left(\zeta+\zeta^2+\zeta^3\right) {x_1}^2 x_4+(-1-\zeta) x_1 x_2 x_3+\left(\zeta+\zeta^2\right) x_1 x_2 x_4+ \\
& \left(-\zeta-\zeta^2\right) x_1 x_3 x_4+\left(\zeta^2+\zeta^3\right) x_1 {x_4}^2+\left(1-\zeta^2-\zeta^3\right) {x_2}^3+\left(-\zeta-\zeta^2\right) {x_2}^2 x_3+ \\
& \left(1+\zeta+\zeta^2\right) {x_2}^2 x_4+\zeta^2 x_2 {x_3}^2+\left(-\zeta^2-\zeta^3\right) x_2 x_3 x_4+\zeta^3 x_2 {x_4}^2+\zeta^3 {x_3}^2 x_4+\zeta {x_4}^3 \\
f_4 \;=\;& -\zeta {x_1}^2 x_2+{x_1}^2 x_4+\left(-1+\zeta^3\right) x_1 x_2 x_3+(1+\zeta) x_1 x_2 x_4+\left(-\zeta^2-\zeta^3\right) x_1 x_3 x_4+ \\
& \left(-1+\zeta^3\right) x_1 {x_4}^2+\zeta^3 {x_2}^3+(-1-\zeta) {x_2}^2 x_3+\zeta {x_2}^2 x_4+\left(-1-\zeta+\zeta^3\right) x_2 {x_3}^2+ \\
& \left(1-\zeta^3\right) x_2 x_3 x_4+\left(-1+\zeta^2+\zeta^3\right) x_2 {x_4}^2+\left(1-\zeta^2-\zeta^3\right) {x_3}^2 x_4+\left(-1-\zeta-\zeta^2\right) {x_4}^3
\end{aligned}
$$

TABLE 4.1. Basis of $H^0(X_{48}, \mathcal{L}_{h_{56}})$

We study the projective geometry of the surface X_{56} more closely. The set

$$\mathcal{F}_{56} := \{\ r \in S_X \ \mid\ \langle r, r\rangle = -2,\ \ \langle r, h_{56}\rangle = 1\ \}$$

of classes of lines on X_{56} can be easily calculated by Algorithm 2.1. It turns out that $\mathcal{F}_{56}$ spans S_X, and hence the stabilizer subgroup

$$\tilde{G}_{56} := \{\ g \in \mathrm{O}(S_X)\ \mid\ h_{56}^g = h_{56}\ \}$$

of h_{56} in $\mathrm{O}(S_X)$ is naturally isomorphic to the group of permutations of $\mathcal{F}_{56}$ that preserve the intersection numbers. We fix a list of vectors $[\lambda_1], \dots, [\lambda_{20}]$ of $\mathcal{F}_{56}$ that form a basis of S_X, and calculate $\tilde{G}_{56}$ by the standard backtrack program in the same way as the calculation of $\tilde{G}_{48}$. It turns out that $\tilde{G}_{56}$ is of order 128. We put

$$\mathrm{Aut}(X_{56}) := \{\ \gamma \in \mathrm{PGL}_4(\mathbb{C})\ \mid\ \gamma(X_{56}) = X_{56}\ \},$$

and consider its image

$$G_{56} := \mathrm{Im}(\mathrm{Aut}(X_{56}) \to \mathrm{O}(S_X))$$

by the natural representation. Then G_{56} is a subgroup of $\tilde{G}_{56}$ consisting of elements that satisfy the period-preserving condition (3.4). It turns out that G_{56} is isomorphic to the group

$$(((\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2\mathbb{Z}$$

of order 64. The action of $\mathrm{Aut}(X_{56})$ decomposes $\mathcal{F}_{56}$ into three orbits O_8, O_{16}, O_{32} of size 8, 16, and 32, respectively. We have

$$h_{56} = \frac{1}{8} \sum_{r \in O_{32}} r = \frac{1}{8} \left(2 \sum_{r \in O_8} r + \sum_{r \in O_{16}} r \right). \tag{4.3}$$

The intersection $\mathcal{F}_{48} \cap \mathcal{F}_{56}$ consists of 30 classes. (In fact, in searching for a basis of $H^0(X_{48}, \mathcal{L}_{h_{56}})$ that gives a simple defining equation of X_{56}, we have used these

30 lines as a clue.) Since we know the defining equations of the 48 lines on X_{48} and the morphism $\Phi_{56}\colon X_{48} \overset{\sim}{\to} X_{56}$ explicitly, we can easily compute the defining equations of these 30 lines.

We say that a finite set $\{\mu_1, \dots, \mu_N\}$ of lines in $\mathbb{P}^3$ *has a unique common intersecting line* if there exists a unique line in $\mathbb{P}^3$ that intersects all of $\mu_1, \dots, \mu_N$. If we know the defining equations of $\mu_1, \dots, \mu_N$, then we can determine whether $\{\mu_1, \dots, \mu_N\}$ has a unique common intersecting line, and if it has, we can calculate the defining equation of the common intersecting line. Suppose that we know the defining equations of lines $\lambda'_1, \dots, \lambda'_N$ on X_{56}, but we do not know the defining equation of a line λ'_{N+1} on X_{56}. Since we have the set $\mathcal{F}_{56}$ of classes of lines on X_{56}, we can make the subset $\{\lambda'_{i_1}, \dots, \lambda'_{i_k}\}$ of $\{\lambda'_1, \dots, \lambda'_N\}$ consisting of lines that intersect λ'. If $\{\lambda'_{i_1}, \dots, \lambda'_{i_k}\}$ has a unique common intersecting line, then we can calculate the defining equation of λ'_{N+1}. Starting from the 30 lines, we can calculate the defining equations of the remaining 26 lines on X_{56}. Since we know the permutation action of G_{56} on $\mathcal{F}_{56}$, we can calculate each element of $\mathrm{Aut}(X_{56})$. By these calculations, we obtain the following theorem.

Theorem 4.6. *The subgroup* $\mathrm{Aut}(X_{56})$ *of* $\mathrm{PGL}_4(\mathbb{C})$ *is generated by the following elements of order* 4.

$$\gamma_1 = \begin{bmatrix} 1 & \zeta^2 & -\zeta+\zeta^2-\zeta^3 & 1-\zeta+\zeta^3 \\ -1+\zeta-\zeta^3 & -\zeta+\zeta^2-\zeta^3 & -\zeta^2 & 1 \\ -1 & \zeta^2 & -\zeta+\zeta^2-\zeta^3 & -1+\zeta-\zeta^3 \\ -1+\zeta-\zeta^3 & \zeta-\zeta^2+\zeta^3 & \zeta^2 & 1 \end{bmatrix},$$

$$\gamma_2 = \begin{bmatrix} 1 & \zeta^2 & \zeta+\zeta^2-\zeta^3 & 1-\zeta-\zeta^3 \\ \zeta^2 & 1 & 1-\zeta-\zeta^3 & \zeta+\zeta^2-\zeta^3 \\ \zeta+\zeta^2-\zeta^3 & 1-\zeta-\zeta^3 & -1 & -\zeta^2 \\ 1-\zeta-\zeta^3 & \zeta+\zeta^2-\zeta^3 & -\zeta^2 & -1 \end{bmatrix}.$$

The lines on X_{56} *are obtained from the following lines* $\lambda^{(8)}, \lambda^{(16)}, \lambda^{(32)}$ *by the action of* $\mathrm{Aut}(X_{56})$*, where the size of the orbit of* $\lambda^{(n)}$ *are* n.

$$\begin{aligned} \lambda^{(8)} &: \ y_1+\zeta^2 y_4 = y_2-\zeta^2 y_3 = 0, \\ \lambda^{(16)} &: \ y_1+(-\zeta+\zeta^2-\zeta^3)\, y_4 = y_2+(-\zeta+\zeta^2-\zeta^3)\, y_3 = 0, \\ \lambda^{(32)} &: \ y_1 = 3y_2+(-1-\zeta-\zeta^3)y_3+(-\zeta+\zeta^2+\zeta^3)\, y_4 = 0. \end{aligned}$$

Corollary 4.7. *Every line* λ *on* X_{56} *is defined by an equation* $M_\lambda\, y = 0$*, where* M_λ *is a* 2×4 *matrix in the row-reduced echelon form with components in* $\mathbb{Z}[\zeta, 1/3]$*, and* $y = [y_1, y_2, y_3, y_4]^T$.

5. Reductions of X_{56} at primes of $\mathbb{Z}[\zeta]$

5.1. **Buchberger algorithm and the reduction.** We need a slight enhancement of the Buchberger algorithm to calculate Gröbner bases over κ_P at all but finitely many primes P of $\mathbb{Z}[\zeta]$ simultaneously. This method must have been used by many people without fanfare, but we cannot find any appropriate references.

We fix a monomial ordering on the set of monomials of variables $z_1, \dots, z_n$. Let F be a field. We use the notation in Chapter 2 of [4]. In particular, for a non-zero polynomial $f \in F[z_1, \dots, z_n]$, let $\mathrm{LC}(f) \in F$ denote the leading coefficient, and for $f, g \in F[z_1, \dots, z_n]$ and a finite subset $H \subset F[z_1, \dots, z_n]$, let $S(f, g)$ be the S-polynomial of f and g, and $\overline{f}^H$ the remainder on the division of f by H.

Suppose that F is a number field, and let R be the integer ring of F. For a prime P of R, let R_P denote the localization of R at P, $R_P^\times$ the group of units of R_P, and κ_P the residue field of R at P. For a polynomial $f \in R_P[z_1, \dots, z_n]$, let $f \bmod P \in \kappa_P[z_1, \dots, z_n]$ denote the reduction of f at P, and for a subset H of $R_P[z_1, \dots, z_n]$, let $H \bmod P$ denote the set of reductions at P of polynomials in H. The following lemma follows immediately from the definition of the S-polynomial and the division algorithm.

Lemma 5.1. *Let f and g be polynomials in $R_P[z_1, \dots, z_n]$, and H a finite subset of $R_P[z_1, \dots, z_n]$. We have $S(f, g) \in F[z_1, \dots, z_n]$ and $\overline{f}^H \in F[z_1, \dots, z_n]$.*

(1) *Suppose that both of $\mathrm{LC}(f)$ and $\mathrm{LC}(g)$ belong to $R_P^\times$. Then $S(f, g)$ belongs to $R_P[z_1, \dots, z_n]$, and $S(f, g) \bmod P$ is equal to the S-polynomial of the polynomials $f \bmod P$ and $g \bmod P$ in $\kappa_P[z_1, \dots, z_n]$.*

(2) *Suppose that $\mathrm{LC}(h) \in R_P^\times$ for any $h \in H$. Then $\overline{f}^H$ belongs to $R_P[z_1, \dots, z_n]$, and $\overline{f}^H \bmod P$ is equal to the remainder on the division of $f \bmod P \in \kappa_P[z_1, \dots, z_n]$ by the subset $H \bmod P$ of $\kappa_P[z_1, \dots, z_n]$.*

Suppose that a finite set $\{f_1, \dots, f_s\}$ of non-zero polynomials in $R[z_1, \dots, z_n]$ is given. Let I_F be the ideal of $F[z_1, \dots, z_n]$ generated by $\{f_1, \dots, f_s\}$. For a prime P of R, let I_P be the ideal of $\kappa_P[z_1, \dots, z_n]$ generated by $\{f_1, \dots, f_s\} \bmod P$. A Gröbner basis G of I_F is calculated by the Buchberger algorithm. We initialize $G := \{f_1, \dots, f_s\}$. If $\overline{S(f_i, f_j)}^G$ is non-zero for a pair of $f_i, f_j \in G$, we add $f_t := \overline{S(f_i, f_j)}^G$ to G. We continue this process until no new non-zero polynomials $\overline{S(f_i, f_j)}^G$ are obtained.

We introduce a variable set C in the Buchberger algorithm. We initialize

$$C := \{ \mathrm{LC}(f_i) \mid i = 1, \dots, s \},$$

and, whenever a new non-zero polynomial $f_t = \overline{S(f_i, f_j)}^G$ is added to G, we add $\mathrm{LC}(f_t)$ to C. From Lemma 5.1, we obtain the following proposition.

Proposition 5.2. *Let P be a prime of R. Suppose that, when the algorithm terminates, we have $C \subset R_P^\times$. Then we have $G \subset R_P[z_1, \dots, z_n]$, and $G \bmod P$ is a Gröbner basis of the ideal I_P of $\kappa_P[z_1, \dots, z_n]$.*

Since C is a finite set, we can calculate a finite set S of prime integers such that $G \bmod P$ is a Gröbner basis of I_P for any prime P over $p \notin S$. More precisely, for $\alpha \in F$, let $d(\alpha)$ denote the least positive integer such that $d(\alpha)\alpha \in R$, and let $n(\alpha) \in \mathbb{Z}$ be the norm of $d(\alpha)\alpha \in R$ over $\mathbb{Z}$. Let $\tilde{C}$ denote the subset

$$\{d(\alpha) \mid \alpha \in C\} \cup \{n(\alpha) \mid \alpha \in C\} \tag{5.1}$$

of $\mathbb{Z} \setminus \{0\}$. For a finite set T of non-zero integers, let $\mathcal{P}(T)$ denote the set of prime integers that divide at least one element of T. Then $C \subset R_P^\times$ holds for any prime P over $p \notin \mathcal{P}(\tilde{C})$.

In fact, this naive method often fails to work in practice, because some elements of $\tilde{C}$ can be so large that we cannot calculate their prime factors. (For example, in the proof of Theorem 5.6 below, this method led us to a factorization of a composite integer $> 10^{80}$, which was impossible.) To overcome this difficulty, we use the following trick. Let $T_1, \dots, T_N$ be finite sets of non-zero integers. Then we have

$$\mathcal{P}(T_1) \cap \cdots \cap \mathcal{P}(T_N) = \mathcal{P}(\mathrm{gcds}(T_1, \dots, T_N)),$$

where

$$\mathrm{gcds}(T_1, \dots, T_N) := \{ \gcd(t_1, \dots, t_N) \mid t_1 \in T_1, \dots, t_N \in T_N \}.$$

Since the calculation of the greatest common divisor of large integers is much easier than the calculation of prime factors of these integers, we often manage to calculate $\mathcal{P}(T_1) \cap \cdots \cap \mathcal{P}(T_N)$ even when the calculation of $\mathcal{P}(T_i)$ is intractable.

For example, suppose that I_F contains 1, and let us calculate a finite set S of prime integers such that $1 \in I_P$ holds for any prime P of R over $p \notin S$. We carry out the Buchberger algorithm several times under various choices of monomial ordering, and obtain the sets $\tilde{C}_1, \dots, \tilde{C}_N$ of non-zero integers for these choices. Note that, if $p \notin \mathcal{P}(\tilde{C}_\nu)$ for at least one ν, then $i \in I_P$ for any prime P of R over p. Hence the intersection S of these $\mathcal{P}(\tilde{C}_i)$ has the desired property.

By means of this method, we write the following algorithms.

Algorithm 5.3. Let V be a subscheme of $\mathbb{P}^3$ defined by a homogeneous polynomial $\psi \in R[z_1, \dots, z_4]$ such that $V \otimes F$ is a smooth surface. Then we can make a finite set S of prime integers such that $V \otimes \kappa_P$ is a smooth surface for any prime P of R over $p \notin S$. Executing the Buchberger algorithm in the field κ_P for the primes P over $p \in S$, we can make the complete set of primes P such that $V \otimes \kappa_P$ is not a smooth surface. ∎

Algorithm 5.4. We say that a finite set $\{\mu_1, \dots, \mu_N\}$ of lines in $\mathbb{P}^3$ defined over a field *has no common intersecting lines* if there exist no lines in $\mathbb{P}^3$ that intersect all of $\mu_1, \dots, \mu_N$. Let $\{\mu_1, \dots, \mu_N\}$ be a set of subschemes of $\mathbb{P}^3$ defined over R such that $\{\mu_1 \otimes F, \dots, \mu_N \otimes F\}$ is a set of distinct N lines with no common intersecting lines. We can make a complete set B of primes of R such that $\{\mu_1 \otimes \kappa_P, \dots, \mu_N \otimes \kappa_P\}$ is a set of distinct N lines that has no common intersecting lines for any $P \notin B$. ∎

5.2. **Reductions of X_{56}.** Let $\mathcal{X}_{56}$ be the projective scheme over $\mathbb{Z}[\zeta]$ defined by the homogeneous equation $\Psi = 0$ in $\mathbb{P}^3$. The generic fiber $\mathcal{X}_{56} \otimes \mathbb{Q}(\zeta)$ is the surface X_{56}. For a prime P of $\mathbb{Z}[\zeta]$, let $\bar{\kappa}_P$ denote an algebraic closure of κ_P. We define $X_{56}(P)$ to be the pullback of $\mathcal{X}_{56}$ by $\mathbb{Z}[\zeta] \to \bar{\kappa}_P$.

Recall that $A = -1 - 2\zeta - 2\zeta^3$ and $B = 3 + A$. There exists only one prime P_2 of $\mathbb{Z}[\zeta]$ over 2, and we have $A \bmod P_2 = 1$. There exist exactly two primes P_3 and P_3' of $\mathbb{Z}[\zeta]$ over 3, for which we have $A \bmod P_3 = 0$ and $A \bmod P_3' = 1$. It is easy to see that $X_{56}(P_2)$ and $X_{56}(P_3')$ are singular at the point $(1 : 0 : \sqrt{-1} : 0)$.

Proposition 5.5. *The surface $X_{56}(P_3)$ is projectively isomorphic over $\kappa_{P_3} \cong \mathbb{F}_9$ to the Fermat quartic surface in characteristic* 3. *In particular, $X_{56}(P_3)$ is smooth, and contains* 112 *lines, each of which is defined over κ_{P_3}.*

Proof. The surface $X_{56}(P_3)$ is defined by $y_1^3 y_2 + y_1 y_2^3 + y_3^3 y_4 + y_3 y_4^3 = 0$, which is a non-degenerate Hermitian form in 4 variables over $\mathbb{F}_9$. Hence $X_{56}(P_3)$ is projectively isomorphic to the Fermat quartic surface in characteristic 3 over $\mathbb{F}_9$ by [23]. For the number of lines on this surface, see [23] or [7]. □

By Corollary 4.7, the lines on X_{56} reduce to lines on $X_{56}(P)$ for any prime P over $p > 3$.

Theorem 5.6. *Suppose that P is a prime of $\mathbb{Z}[\zeta]$ over a prime integer $p > 3$. Then $X_{56}(P)$ is smooth, and contains exactly 56 lines, each of which is obtained by the reduction of a line on X_{56} at P. Moreover, the isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ given in Table 4.1 reduces at P to an isomorphism $X_{48} \otimes \bar{\kappa}_P \xrightarrow{\sim} X_{56}(P)$.*

Proof. The smoothness of $X_{56}(P)$ can be proved by Algorithm 5.3. We show that $X_{56}(P)$ contains exactly 56 lines, and that they are obtained by the reduction of lines on X_{56}. The fact that the 56 lines on X_{56} reduce to distinct 56 lines on $X_{56}(P)$ keeping the intersection numbers can be easily proved. We will show that there exist no other lines on $X_{56}(P)$. Let $S_{X_{56}(P)}$ denote the Néron-Severi lattice of $X_{56}(P)$. Recall that the set $\mathcal{F}_{56}$ of classes of the 56 lines on X_{56} spans S_X. Hence the reduction of lines on X_{56} induces a natural embedding

$$S_X \hookrightarrow S_{X_{56}(P)}$$

of lattices. From now on, we regard S_X as a sublattice of $S_{X_{56}(P)}$ by this embedding. In particular, $\mathcal{F}_{56}$ is a subset of the set of classes of lines on $X_{56}(P)$. It is enough to show that, if λ is a line on $X_{56}(P)$, then its class $[\lambda] \in S_{X_{56}(P)}$ is in $\mathcal{F}_{56}$. Let Q_P denote the orthogonal complement of S_X in $S_{X_{56}(P)}$. Then Q_P is either of rank 0 or negative-definite of rank 2. (See [24] for the problem when Q_P is of rank 2.) We have

$$S_X \oplus Q_P \quad \subset \quad S_{X_{56}(P)} \quad \subset \quad S_X^\vee \oplus Q_P^\vee.$$

We denote the projections by

$$\mathrm{pr}_S \colon S_{X_{56}(P)} \to S_X^\vee, \qquad \mathrm{pr}_Q \colon S_{X_{56}(P)} \to Q_P^\vee.$$

Let $h_{56}(P) \in S_{X_{56}(P)}$ denote the class of a hyperplane section of $X_{56}(P) \subset \mathbb{P}^3$. Since $\mathcal{F}_{56}$ is perpendicular to $Q_P^\vee$, we have $\langle \mathrm{pr}_S(h_{56}(P)), r\rangle = \langle h_{56}(P), r\rangle = 1$ for any $r \in \mathcal{F}_{56}$. The class $h_{56} \in S_X$ is characterized in $S_X \otimes \mathbb{Q}$ by the property that $\langle h_{56}, r\rangle = 1$ holds for any $r \in \mathcal{F}_{56}$. Hence we have $\mathrm{pr}_S(h_{56}(P)) = h_{56}$. Since $\langle h_{56}(P), h_{56}(P)\rangle = \langle h_{56}, h_{56}\rangle = 4$ and Q_P is either 0 or negative-definite, we obtain $\mathrm{pr}_Q(h_{56}(P)) = 0$, and therefore we have

$$h_{56}(P) = h_{56}.$$

Suppose that there exists a line ν on $X_{56}(P)$ such that $[\nu] \notin \mathcal{F}_{56}$. Since $h_{56}(P) = h_{56}$ and $h_{56} \perp Q_P$, we have $\langle \mathrm{pr}_S([\nu]), h_{56}\rangle = 1$. Since $\langle \mathrm{pr}_Q([\nu]), \mathrm{pr}_Q([\nu])\rangle \le 0$, we have $\langle \mathrm{pr}_S([\nu]), \mathrm{pr}_S([\nu])\rangle \ge -2$. For any $[\lambda] \in \mathcal{F}_{56}$, we have $\langle [\nu], [\lambda]\rangle \in \{0, 1\}$. Since $Q_P \perp \mathcal{F}_{56}$, we have $\langle \mathrm{pr}_S([\nu]), [\lambda]\rangle \in \{0, 1\}$ for any $[\lambda] \in \mathcal{F}_{56}$. Therefore $\mathrm{pr}_S([\nu])$ belongs to

$$\mathcal{F}'_{56} := \{\ r' \in S_X^\vee \ \mid\ \langle r', h_{56}\rangle = 1,\ \langle r', r'\rangle \ge -2,\ \langle r', [\lambda]\rangle \in \{0,1\} \text{ for any } [\lambda] \in \mathcal{F}_{56}\ \}.$$

We calculate $\mathcal{F}'_{56}$ by Algorithm 2.1. It turns out that $\mathcal{F}'_{56}$ consists of 56 vectors. For each $r' \in \mathcal{F}'_{56}$, we calculate the set $\Lambda(r') = \{\lambda'_1, \dots, \lambda'_k\}$ of lines on X_{56} such that

$$\{[\lambda'_1], \dots, [\lambda'_k]\} = \{\ [\lambda] \in \mathcal{F}_{56}\ \mid\ \langle [\lambda], r'\rangle = 1\ \}.$$

Then ν is a common intersecting line of the set

$$\Lambda(\mathrm{pr}_S([\nu])) \otimes \kappa_P = \{\lambda'_1 \otimes \kappa_P, \dots, \lambda'_k \otimes \kappa_P\}$$

of lines over κ_P. On the other hand, since we know the defining equations over $\mathbb{Z}[\zeta, 1/3]$ of lines $\lambda'_1, \dots, \lambda'_k$, we can see by Algorithm 5.4 that $\Lambda(r') \otimes \kappa_P$ has no common intersecting lines for any $r' \in \mathcal{F}'_{56}$ if P is over $p > 3$. Thus we obtain a contradiction. (See Remark 5.7 for what happens when $p = 3$.)

Next, we investigate the reduction at P of the isomorphism $\Phi_{56}\colon X_{48} \xrightarrow{\sim} X_{56}$ given in Table 4.1. The equality (4.3) implies that, as an invertible sheaf, the line bundle $\mathcal{L}_{h_{56}}^{\otimes 8}$ is isomorphic to $\mathcal{O}(D)$, where D is a linear combination of lines on X_{56}. Hence the embedding $S_X \hookrightarrow S_{X_{56}(P)}$ induced by the reduction of lines on X_{56} maps the class h_{56} of $\mathcal{L}_{h_{56}}$ to the class of the line bundle $\mathcal{L}_{h_{56}} \otimes \bar{\kappa}_P$. Since $h_{56} = h_{56}(P)$, the line bundle $\mathcal{L}_{h_{56}} \otimes \bar{\kappa}_P$ is the very ample line bundle associated with the embedding $X_{56}(P) \hookrightarrow \mathbb{P}^3$. We confirm that

$$f \bmod P := (f_1 \bmod P, \dots, f_4 \bmod P)$$

are linearly independent over κ_P. Hence $f \bmod P$ form a basis of the space of the global sections of $\mathcal{L}_{h_{56}} \otimes \bar{\kappa}_P$. □

Remark 5.7. We investigate the lines on $X_{56}(P_3)$, where P_3 is the prime of $\mathbb{Z}[\zeta]$ in Proposition 5.5. For each line λ on X_{56}, we have an invertible 2×2 matrix U_λ with components in $\mathbb{Q}(\zeta)$ such that $U_\lambda M_\lambda$ has components in the localization $\mathbb{Z}[\zeta]_{P_3}$, and that $\overline{M}'_\lambda := U_\lambda M_\lambda \bmod P_3$ is a matrix of row-reduced echelon form of rank 2. Hence the equation $\overline{M}'_\lambda\, y = 0$ defines a line on $X_{56}(P_3)$ defined over κ_{P_3}. Using these equations, we can make the reduction $\lambda \mapsto \lambda \otimes \kappa_{P_3}$ of lines on X_{56} to lines on $X_{56}(P_3)$. Since this reduction keeps the intersection numbers, it induces an embedding $S_X \hookrightarrow S_{X_{56}(P_3)}$. For each $r' \in \mathcal{F}'_{56}$, the set $\Lambda(r') \otimes \kappa_{P_3}$ of lines over κ_{P_3} has a unique common intersecting line, and the common intersecting line is contained in $X_{56}(P_3)$. Thus we obtain the $112 = |\mathcal{F}_{56}| + |\mathcal{F}'_{56}|$ lines on $X_{56}(P_3)$.

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