SINGULARITIES OF DUAL VARIETIES IN CHARACTERISTIC 3

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Abstract. We investigate singularities of a general plane section of the dual variety of a smooth projective variety, or more generally, the discriminant variety associated with a linear system of divisors on a smooth projective variety. We show that, in characteristic 3, singular points of E_6-type take the place of ordinary cusps in characteristic 0.

1. Introduction

We work over an algebraically closed field k.

Let X be a smooth projective variety of dimension $n > 0$, and let $\mathcal{L}$ be a line bundle on X. We consider the m-dimensional linear system $|M|$ of divisors on X corresponding to a linear subspace M of $H^0(X, \mathcal{L})$ with dimension $m + 1 > 1$. The *discriminant variety* of $|M|$ is the locus of all points $t \in \mathbb{P}_*(M)$ such that the corresponding divisor $D_t \in |M|$ is singular ([2, Section 2]). When the linear system $|M|$ embeds X into a projective space $\mathbb{P}^m$, then the parameter space $\mathbb{P}_*(M)$ of the linear system $|M|$ is identified with the dual projective space $(\mathbb{P}^m)^\vee$ of $\mathbb{P}^m$, and the discriminant variety of $|M|$ is called the *dual variety* of $X \subset \mathbb{P}^m$.

Since the paper of Wallace [24], it has been noticed that the geometry of dual varieties in positive characteristics is quite different from that in characteristic 0. For example, the reflexivity property does not hold in general in positive characteristics. See [17] and [8] for the definition and detailed accounts of the reflexivity. Many papers have been written about this failure of the reflexivity property in positive characteristics. For example, see [6, 7, 9, 12, 11, 13, 19].

However, if the linear system $|M|$ is sufficiently ample, then the peculiarity about the reflexivity in positive characteristics vanishes except for the case when $\operatorname{char} k$ is 2 and $\dim X$ is odd. Namely we have the following theorem ([14, Théorème 2.5], [8, Theorem (5.4)]):

Theorem 1.1. *Suppose that* $\operatorname{char} k \neq 2$ *or* $\dim X$ *is even. Let* $\mathcal{A}$ *be a very ample line bundle of* X*, and let* X *be embedded in* $\mathbb{P}^m$ *by the complete linear system* $|\mathcal{A}^{\otimes d}|$ *with* $d \geq 2$*. Then the dual variety of* $X \subset \mathbb{P}^m$ *is a hypersurface of* $(\mathbb{P}^m)^\vee$*, and* $X \subset \mathbb{P}^m$ *is reflexive.*

In this paper, we show that the singularity of the dual variety has a peculiar feature in characteristic 3 that does not vanish however ample the linear system may be.

We assume that $|M|$ is sufficiently ample. By cutting the dual variety by a general plane in $\mathbb{P}_*(M) = (\mathbb{P}^m)^\vee$, we obtain a singular plane curve. If $\operatorname{char} k > 3$

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or $\operatorname{char} k = 0$, the plane curve has only ordinary cusps as its unibranched singular points. We show that, if $\operatorname{char} k = 3$, the plane curve has E_6-singular points as its unibranched singular points.

In fact, we prove our results in the more general setting of discriminant varieties associated with (not necessarily very ample) linear systems. Here in Introduction, however, we state our results in the case of dual varieties.

We assume that the base field k is of characteristic $\neq 2$. Let $X \subset \mathbb{P}^m$ be a smooth projective variety of dimension $n > 0$. We assume that X is not contained in any hyperplane of $\mathbb{P}^m$, so that the dual projective space

$$\mathbf{P} := (\mathbb{P}^m)^\vee$$

of $\mathbb{P}^m$ is regarded as the parameter space $\mathbb{P}_*(M)$ of the linear system $|M|$ of hyperplane sections on X, where M is a linear subspace of $H^0(X, \mathcal{O}_X(1))$. We use the same letter to denote a point $H \in \mathbf{P}$ and the corresponding hyperplane $H \subset \mathbb{P}^m$. We denote by $\mathcal{D} \subset X \times \mathbf{P}$ the universal family of hyperplane sections. The support of $\mathcal{D}$ is equal to the closed subset

$$\{ \ (p, H) \in X \times \mathbf{P} \ \mid \ p \in H \ \}$$

of $X \times \mathbf{P}$. It is easy to see that $\mathcal{D}$ is smooth of dimension $n + m - 1$. Let $\mathcal{C}$ be the critical locus of the second projection $\mathcal{D} \to \mathbf{P}$ with the canonical scheme structure (Definition 2.15). Then $\mathcal{C}$ is smooth, irreducible and of dimension $m - 1$. In fact, if $\mathcal{N}$ is the conormal sheaf of $X \subset \mathbb{P}^m$, then $\mathcal{C}$ is isomorphic to $\mathbb{P}^*(\mathcal{N})$ ([14, Remarque 3.1.5]). The support of $\mathcal{C}$ is equal to the set

$$\{ \ (p, H) \in \mathcal{D} \ \mid \ \text{the divisor } H \cap X \text{ of } X \text{ is singular at } p \ \}.$$

The image of $\mathcal{C}$ by the projection to $\mathbf{P}$ is called the *dual variety* of $X \subset \mathbb{P}^m$, or the *discriminant variety* of the linear system $|M|$ on X.

We will study the singularity of the dual variety by investigating the critical locus $\mathcal{E}$ of the second projection $\mathcal{C} \to \mathbf{P}$. The codimension of $\mathcal{E}$ in $\mathcal{C}$ is ≤ 1. If the codimension is 0, then either the dual variety is not a hypersurface of $\mathbf{P}$, or $\mathcal{C}$ is inseparable over the dual variety. By [14, Proposition 3.3] or Proposition 3.14 of this paper, the complement $\mathcal{C} \setminus \mathcal{E}$ is set-theoretically equal to

$$\{ \ (p, H) \in \mathcal{C} \ \mid \ \text{the Hessian of the singularity of } H \cap X \text{ at } p \text{ is non-degenerate} \ \}.$$

We equip the critical locus $\mathcal{E}$ with the canonical scheme structure by Definition 2.15, and put

$$\mathcal{E}^{\mathrm{sm}} := \{ \ (p, H) \in \mathcal{E} \ \mid \ \mathcal{E} \text{ is smooth of dimension } m - 2 \text{ at } (p, H) \ \},$$

which is a Zariski open (possibly empty) subset of $\mathcal{E}$. Note that, if $\mathcal{E}^{\mathrm{sm}}$ is non-empty, then $\mathcal{E}$ is of codimension 1 in $\mathcal{C}$, and hence the dual variety is a hypersurface in $\mathbf{P}$. Moreover, if $\mathcal{E}^{\mathrm{sm}}$ is non-empty, then the generalized Monge-Segre-Wallace criterion ([16, Theorem (4.4)] or [17, Theorem (4)]) implies that $X \subset \mathbb{P}^m$ is reflexive.

We put

$$\mathcal{E}^{A_2} := \{ \ (p, H) \in \mathcal{E} \ \mid \ \text{the singularity of } H \cap X \text{ at } p \text{ is of type } A_2 \ \}.$$

See Definition 2.13 for the definition of the hypersurface singularity of type A_2.

We will show that $\mathcal{E}$ is irreducible and the loci $\mathcal{E}^{\mathrm{sm}}$ and $\mathcal{E}^{A_2}$ are dense in $\mathcal{E}$ if $|M|$ is sufficiently ample (Proposition 4.9).

Let $P = (p, H)$ be a closed point of $\mathcal{E}$, and let $\Lambda \subset \mathbf{P}$ be a general plane passing through $H \in \mathbf{P}$. We denote by C_Λ the pull-back of Λ by the projection $\mathcal{C} \to \mathbf{P}$. Our main goal is to investigate the singularity of the morphism $C_\Lambda \to \Lambda$ at $P \in C_\Lambda$.

Theorem 1.2. *Suppose that* $\operatorname{char} k > 3$ *or* $\operatorname{char} k = 0$. *Then the following two conditions are equivalent:*

(i) $P \in \mathcal{E}^{A_2}$,
(ii) $P \in \mathcal{E}^{\mathrm{sm}}$, *and the projection* $\mathcal{E} \to \mathbf{P}$ *induces a surjective homomorphism*

$$(\mathcal{O}_{\mathbf{P},H})^\wedge \twoheadrightarrow (\mathcal{O}_{\mathcal{E},P})^\wedge$$

on the completions of the local rings.

Moreover, if these conditions are satisfied, then C_Λ *is smooth of dimension* 1 *at* P, *and the morphism* $C_\Lambda \to \Lambda$ *has a critical point of* A_2*-type at* P *(Definition 2.1).*

This result seems to be classically known. See Proposition 4.4 and Theorem 5.2 (1) of this paper for the proof.

Now we assume that k is of characteristic 3. Then $P \in \mathcal{E}^{A_2}$ does not necessarily imply $P \in \mathcal{E}^{\mathrm{sm}}$. Our main results are as follows.

(**I**) The projection $\mathcal{E}^{\mathrm{sm}} \to \mathbf{P}$ factors as

$$\mathcal{E}^{\mathrm{sm}} \xrightarrow{q} (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}} \xrightarrow{\tau} \mathbf{P},$$

where $q : \mathcal{E}^{\mathrm{sm}} \to (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}}$ is the quotient morphism by an integrable subbundle $\mathcal{K}$ of the tangent vector bundle $T(\mathcal{E}^{\mathrm{sm}})$ of $\mathcal{E}^{\mathrm{sm}}$ with rank 1 (Definition 2.18). In particular, q is a purely inseparable finite morphism of degree 3.

(**II**) Suppose that $P = (p, H)$ is a point of $\mathcal{E}^{\mathrm{sm}} \cap \mathcal{E}^{A_2}$. Then the morphism $\tau : (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}} \to \mathbf{P}$ induces a surjective homomorphism

$$(\mathcal{O}_{\mathbf{P},H})^\wedge \twoheadrightarrow (\mathcal{O}_{(\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}},q(P)})^\wedge.$$

Moreover, the scheme C_Λ is smooth of dimension 1 at P, and the morphism $C_\Lambda \to \Lambda$ has a critical point of E_6-type at P (Definition 2.3).

In the case where $(n, m) = (1, 2)$, the locus $\mathcal{E}^{\mathrm{sm}}$ is always empty. In this case, we have the following result. Let $X \subset \mathbb{P}^2$ be a smooth projective plane curve. The first projection $\mathcal{C} \to X$ is then an isomorphism with the inverse morphism given by $p \mapsto (p, T_p(X))$, where $T_p(X) \subset \mathbb{P}^2$ is the tangent line to X at p. The projection $\mathcal{C} \to \mathbf{P} = (\mathbb{P}^2)^\vee$ is therefore identified with the Gauss map

$$\gamma_X : X \to \mathbf{P}$$

that maps $p \in X$ to $T_p(X) \in \mathbf{P}$. The image of γ_X is the dual curve $X^\vee$ of X. A point $P = (p, T_p(X))$ of $\mathcal{C}$ is a point of $\mathcal{E}$ if and only if $T_p(X)$ is a flex tangent line to X at p, and P is a point of $\mathcal{E}^{A_2}$ if and only if $T_p(X)$ is an ordinary flex tangent line to X at p.

(**III**) Suppose that γ_X induces a separable morphism from X to $X^\vee$. Then $\mathcal{E}$ is of dimension 0. Let $P = (p, T_p(X))$ be a point of $\mathcal{E}$. Then the length of $\mathcal{O}_{\mathcal{E},P}$ is divisible by 3. Suppose that p is an ordinary flex point of X. Then γ_X is formally isomorphic at p to the morphism

$$T_l : t \mapsto (u, v) = (t^{3l+1}, t^3 + t^{3l+2})$$

from $\operatorname{Spec} k[[t]]$ to $\operatorname{Spec} k[[u,v]]$, where $l := \operatorname{length} \mathcal{O}_{\mathcal{E},P}/3$. Hence the singular point $T_p(X)$ of $X^\vee$ is formally isomorphic to the plane curve singularity defined by

$$x^{3l+1} + y^3 + x^{2l}y^2 = 0.$$

Suppose that all flex points of $X \subset \mathbb{P}^2$ are ordinary. Let t_l be the number of critical points of T_l-type in the morphism γ_X. Then we have

$$\sum l t_l = d - 2 + 2g, \tag{1.1}$$

where d is the degree of $X \subset \mathbb{P}^2$ and g is the genus of X.

Remark 1.3. The critical point of T_1-type is a critical point of E_6-type.

Remark 1.4. By the Monge-Segre-Wallace criterion, the condition that X be separable over $X^\vee$ by γ_X is equivalent to the condition that the plane curve $X \subset \mathbb{P}^2$ is reflexive. See [7, 9, 11, 19] for the properties of non-reflexive curves.

Remark 1.5. If $\operatorname{char} k > 3$ or $\operatorname{char} k = 0$, and if the dual curve $X^\vee$ has only ordinary nodes and ordinary cusps as its singularities, then the number of the ordinary cusps is equal to $3(d - 2 + 2g)$.

The simplest example of the result (III) is as follows. Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. We fix a flex point $O \in E$, and regard E as an elliptic curve with the origin O. Since $\operatorname{char}(k) \neq 2$, the dual curve $E^\vee$ is of degree 6, and the Gauss map γ_E induces a birational morphism from E to $E^\vee$. The singular points of $E^\vee$ are in one-to-one correspondence with the flex points of E via γ_E. On the other hand, the flex points of E are in one-to-one correspondence with the 3-torsion subgroup $E[3]$ of the elliptic curve E. We have

$$E[3] \;\cong\; \begin{cases} \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} & \text{if } \operatorname{char}(k) \neq 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } \operatorname{char}(k) = 3 \text{ and } E \text{ is not supersingular,} \\ 0 & \text{if } \operatorname{char}(k) = 3 \text{ and } E \text{ is supersingular.} \end{cases}$$

Then the critical locus of $\gamma_E : E \to \mathbf{P}$ consists of

$$\begin{cases} 9 \text{ points of } A_2\text{-type} & \text{if } \operatorname{char}(k) \neq 3, \\ 3 \text{ points of } E_6\text{-type} & \text{if } \operatorname{char}(k) = 3 \text{ and } E \text{ is not supersingular,} \\ 1 \text{ point of } T_3\text{-type} & \text{if } \operatorname{char}(k) = 3 \text{ and } E \text{ is supersingular.} \end{cases}$$

The plan of this paper is as follows. In §2, we fix some notions and notation. In §3, we define the schemes $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{E}$ in the setting of discriminant varieties, and study their properties. The results in this section are valid in any characteristics including the case where $\operatorname{char} k = 2$. In §4, we assume that $\operatorname{char} k \neq 2$, and study the scheme $\mathcal{E}$ more closely. Then we show that, in characteristic 3, the projection from $\mathcal{E}^{\mathrm{sm}}$ to $\mathbf{P}$ factors through the quotient morphism by an integrable tangent vector bundle of rank 1 (Theorem 4.5). In §5, we prove a normal form theorem (Theorem 5.2) on the critical points of the morphism $C_\Lambda \to \Lambda$ under the assumption that $\operatorname{char} k \neq 2$, and prove the result (II) above. In §6, we treat the case where $\operatorname{char} k = 3$ and $(n,m) = (1,2)$, and prove the result (III) above, except for the formula (1.1). In §7, we calculate the degree of $\mathcal{E}$ with respect to $\mathcal{O}_{\mathbf{P}}(1)$, count the number of the unibranched singular points on C_Λ, and prove (1.1).

In the paper [22], we will study the singularity of discriminant varieties in characteristic 2 in the case where $\dim X$ is even.

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Notation and Terminology.

(1) Throughout this paper, we work over an algebraically closed field k. A *variety* is a reduced irreducible quasi-projective scheme over k. A *point* means a closed point unless otherwise stated.
(2) Let X be a variety, and P a point of X. We denote by $T_P(X)$ the Zariski tangent space to X at P. When X is smooth, we denote by $T(X)$ the tangent bundle of X.
(3) Let $f : X \to Y$ be a morphism from a smooth variety X to a smooth variety Y, and let P be a point of X. Then f is said to be a *closed immersion formally at* P if the differential homomorphism $d_P f : T_P(X) \to T_{f(P)}(Y)$ of f at P is injective, or equivalently, the induced homomorphism $(\mathcal{O}_{Y,f(P)})^\wedge \to (\mathcal{O}_{X,P})^\wedge$ from the formal completion $(\mathcal{O}_{Y,f(P)})^\wedge$ of $\mathcal{O}_{Y,f(P)}$ to the formal completion $(\mathcal{O}_{X,P})^\wedge$ of $\mathcal{O}_{X,P}$ is surjective.

2. Definitions

2.1. **Curve singularities.** Let $\varphi : C \to S$ be a morphism from a smooth curve C to a smooth surface S. Let P be a point of C, t a formal parameter of $(\mathcal{O}_{C,P})^\wedge$, and (u, v) a formal parameter system of $(\mathcal{O}_{S,\varphi(P)})^\wedge$. We have a local homomorphism

$$\varphi^* : (\mathcal{O}_{S,\varphi(P)})^\wedge = k[[u, v]] \to (\mathcal{O}_{C,P})^\wedge = k[[t]].$$

Definition 2.1. We say that φ has a *critical point of A_2-type* at P if

$$\begin{aligned} \varphi^* u &= a\,t^2 + b\,t^3 + (\text{terms of degree} \geq 4) \quad \text{and} \\ \varphi^* v &= c\,t^2 + d\,t^3 + (\text{terms of degree} \geq 4) \end{aligned}$$

with $ad - bc \neq 0$ hold.

Remark 2.2. If φ has a critical point of A_2-type at P, then it is possible to choose t and (u, v) in such a way that

$$\varphi^* u = t^2 \quad \text{and} \quad \varphi^* v = t^3.$$

The image of the germ (C, P) by φ is then defined by $u^3 - v^2 = 0$. This holds even when $\operatorname{char} k$ is 2.

Definition 2.3. We say that φ has a *critical point of E_6-type* at P if

$$\begin{aligned} \varphi^* u &= a\,t^3 + b\,t^4 + (\text{terms of degree} \geq 5) \quad \text{and} \\ \varphi^* v &= c\,t^3 + d\,t^4 + (\text{terms of degree} \geq 5) \end{aligned}$$

with $ad - bc \neq 0$ hold.

Remark 2.4. Suppose that φ has a critical point of E_6-type at P. If $\operatorname{char} k$ is not 2 nor 3, then, under suitable choice of t and (u, v), we have

$$\varphi^* u = t^3 \quad \text{and} \quad \varphi^* v = t^4,$$

and the image of the germ (C, P) is given by $u^4 - v^3 = 0$. If $\operatorname{char} k = 3$, then, under suitable choice of t and (u, v), we have either

$$(\varphi^* u = t^3,\ \ \varphi^* v = t^4) \quad \text{or} \quad (\varphi^* u = t^3 + t^5,\ \ \varphi^* v = t^4).$$

In the former case, the image of the germ (C, P) is given by $u^4 - v^3 = 0$, while in the latter case, the image is formally isomorphic to the germ of a plane curve singularity defined by

$$x^4 + y^3 + x^2 y^2 = 0.$$

In the notation of Artin [1] and Greuel-Kröning [4], they are denoted by E_6^0 and E_6^1, respectively. See Remark 2.7 and Propositions 6.2 and 6.3.

From now until the end of this subsection, we assume that $\operatorname{char} k = 3$. For $F \in (\mathcal{O}_{S,\varphi(P)})^\wedge$, we denote by $F_{[t,\nu]}$ the coefficient of t^ν in the formal power series $\varphi^* F$ of t.

Definition 2.5. Let l be a positive integer. We say that φ has a *critical point of T_l-type* at P if the following conditions are satisfied:

$$\begin{array}{lcl} u_{[t,\nu]} \neq 0 & \Longrightarrow & \nu > 3l \ \text{ or } \ 3|\nu, \\ v_{[t,\nu]} \neq 0 & \Longrightarrow & \nu > 3l \ \text{ or } \ 3|\nu, \end{array} \qquad \text{and} \tag{2.1}$$

$$\begin{vmatrix} u_{[t,3]} & u_{[t,3l+1]} \\ v_{[t,3]} & v_{[t,3l+1]} \end{vmatrix} \neq 0, \qquad \begin{vmatrix} u_{[t,3l+1]} & u_{[t,3l+2]} \\ v_{[t,3l+1]} & v_{[t,3l+2]} \end{vmatrix} \neq 0. \tag{2.2}$$

Remark 2.6. Note that the conditions (2.1) and (2.2) do not depend on the choice of the formal parameters t and (u, v). Indeed, suppose that (u, v) satisfies (2.1). If

$$u' = \sum \alpha_{ij} u^i v^j \quad \text{and} \quad v' = \sum \beta_{ij} u^i v^j$$

form another formal parameter system of $(\mathcal{O}_{S,\varphi(P)})^\wedge$, then (u', v') also satisfies (2.1), and

$$\begin{bmatrix} u'_{[t,3]} & u'_{[t,3l+1]} & u'_{[t,3l+2]} \\ v'_{[t,3]} & v'_{[t,3l+1]} & v'_{[t,3l+2]} \end{bmatrix} = \begin{bmatrix} \alpha_{10} & \alpha_{01} \\ \beta_{10} & \beta_{01} \end{bmatrix} \begin{bmatrix} u_{[t,3]} & u_{[t,3l+1]} & u_{[t,3l+2]} \\ v_{[t,3]} & v_{[t,3l+1]} & v_{[t,3l+2]} \end{bmatrix}$$

holds. If s is another formal parameter of $(\mathcal{O}_{C,P})^\wedge$ that relates to t by

$$t = \sum \gamma_i s^i,$$

then $u_{[s,\nu]}$ and $v_{[s,\nu]}$ satisfy (2.1), and we have

$$\begin{bmatrix} u_{[s,3]} & u_{[s,3l+1]} & u_{[s,3l+2]} \\ v_{[s,3]} & v_{[s,3l+1]} & v_{[s,3l+2]} \end{bmatrix} = \begin{bmatrix} u_{[t,3]} & u_{[t,3l+1]} & u_{[t,3l+2]} \\ v_{[t,3]} & v_{[t,3l+1]} & v_{[t,3l+2]} \end{bmatrix} \begin{bmatrix} \gamma_1^3 & 0 & 0 \\ 0 & \gamma_1^{3l+1} & 0 \\ 0 & 0 & \gamma_1^{3l+2} \end{bmatrix}.$$

Remark 2.7. The critical point of T_1-type is just the critical point of E_6^1-type.

Remark 2.8. In §6, we will show that, if φ has a critical point of T_l-type at P, then, by choosing appropriate formal parameters t and (u, v), we have

$$\varphi^* u = t^{3l+1} \quad \text{and} \quad \varphi^* v = t^3 + t^{3l+2},$$

and the image of the germ (C, P) by φ is formally isomorphic to the germ of a plane curve singularity defined by

$$x^{3l+1} + y^3 + x^{2l} y^2 = 0.$$

2.2. **Hypersurface singularities.** Let X be a smooth variety of dimension n, and let $D \subset X$ be an effective divisor of X that is passing through a point $P \in X$ and is singular at P. Let $(x_1, \dots, x_n)$ be a formal parameter system of X at P, and let $f = 0$ be the local defining equation of D at P. The symmetric bilinear form

$$H_{f,P} \,:\, T_P(X) \times T_P(X) \,\to\, k$$

defined by

$$H_{f,P}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(P)$$

does not depend on the choice of the formal parameter system $(x_1, \dots, x_n)$, and does not depend on the choice of f except for multiplicative constants. We call $H_{f,P}$ the *Hessian* of D at P.

Definition 2.9. We say that the singularity of D at P is *non-degenerate* if $H_{f,P}$ is non-degenerate.

From now on to the end of this subsection, we assume that $\operatorname{char} k$ is *not* 2.

Definition 2.10. A formal parameter system $(x_1, \dots, x_n)$ of X at P is called *admissible with respect to* f if

$$f = x_1^2 + \cdots + x_r^2 + (\text{terms of degree} \geq 3)$$

holds in $(\mathcal{O}_{X,P})^\wedge = k[[x_1, \dots, x_n]]$, where r is the rank of $H_{f,P}$.

Remark 2.11. Since $\operatorname{char} k$ is not 2, any formal parameter system at P can be turned into an admissible formal parameter system with respect to f by means of a *linear* transformation of parameters.

Proposition 2.12. *Suppose that the Hessian of D at P is of rank $n-1$. Then the following two conditions are equivalent.*

(i) *There exist a local defining equation $f = 0$ of D at P and a formal parameter system $(x_1, \dots, x_n)$ admissible with respect to f such that the coefficient of x_n^3 in f is non-zero.*

(ii) *For any local defining equation $f = 0$ of D at P and for every formal parameter system $(x_1, \dots, x_n)$ admissible with respect to f, the coefficient of x_n^3 in f is non-zero.*

Proof. Let $f = 0$ and $g = 0$ be local defining equations of D at P. Suppose that $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are formal parameter systems of X at P admissible with respect to f and g, respectively. Let T be the $n \times n$-matrix whose (i,j)-component is

$$\frac{\partial y_i}{\partial x_j}(P).$$

Since the rank of the Hessian of D at P is $n-1$, we have

$${}^tT \left[\begin{array}{c|c} I_{n-1} & \mathbf{0} \\ \hline {}^t\mathbf{0} & 0 \end{array}\right] T = c \left[\begin{array}{c|c} I_{n-1} & \mathbf{0} \\ \hline {}^t\mathbf{0} & 0 \end{array}\right],$$

where c is a non-zero constant. Therefore we have

$$\frac{\partial y_i}{\partial x_n}(P) \neq 0 \quad \Longleftrightarrow \quad i = n. \tag{2.3}$$

There exists a formal parameter series $u(x_1, \dots, x_n)$ with $u(0, \dots, 0) \neq 0$ such that

$$f(x_1, \dots, x_n) = u(x_1, \dots, x_n) g(y_1, \dots, y_n)$$

holds. Expanding $u(x_1, \dots, x_n) g(y_1, \dots, y_n)$ in the formal power series of $(x_1, \dots, x_n)$ using (2.3), we see that the coefficient of x_n^3 in f is equal to

$$u(0, \dots, 0) \left(\frac{\partial y_n}{\partial x_n}(P) \right)^3$$

times the coefficient of y_n^3 in g. □

Definition 2.13. We say that the singularity of D at P is of *type* A_2 if the Hessian of D at P is of rank $n-1$, and the conditions (i) and (ii) in Proposition 2.12 above are satisfied.

2.3. **Degeneracy subschemes.**

Definition 2.14. Let X be a variety, and let E and F be vector bundles on X with rank e and f, respectively. We put $r := \min(e, f)$. For a homomorphism $\sigma : E \to F$, we denote by $\mathbf{D}(\sigma)$ the closed subscheme of X defined locally on X by all r-minors of the $f \times e$-matrix expressing σ, and call $\mathbf{D}(\sigma)$ the *degeneracy subscheme* of σ.

For $P \in X$, let $\mathfrak{m}_P$ denote the maximal ideal of $\mathcal{O}_P := \mathcal{O}_{X,P}$, and let

$$\sigma_P := \sigma \otimes \mathcal{O}_P/\mathfrak{m}_P \;:\; E \otimes \mathcal{O}_P/\mathfrak{m}_P \;\to\; F \otimes \mathcal{O}_P/\mathfrak{m}_P$$

be the linear homomorphism induced from σ on the fibers over P. The support of $\mathbf{D}(\sigma)$ is equal to

$$\{\, P \in X \;\mid\; \text{the rank of } \sigma_P \text{ is} < r \,\}.$$

Definition 2.15. Let $\phi : X \to Y$ be a morphism from a smooth variety X to a smooth variety Y. The *critical subscheme* of ϕ is the degeneracy subscheme of the homomorphism

$$d\phi \;:\; T(X) \;\to\; \phi^* T(Y),$$

and is denoted by $\mathbf{Cr}(\phi)$.

Suppose that $\dim X \leq \dim Y$. Then a point $P \in X$ is in the support of $\mathbf{Cr}(\phi)$ if and only if ϕ fails to be a closed immersion formally at P. (See Notation and Terminology (3).)

2.4. **The quotient morphism by an integrable subbundle.** In this subsection, we assume that $\operatorname{char} k = p > 0$. Let X be a smooth variety, and let $\mathcal{N}$ be a subbundle of $T(X)$.

Definition 2.16. We say that $\mathcal{N}$ is *integrable* if $\mathcal{N}$ is closed under the pth power operation $D \mapsto D^p$ and the bracket product

$$(D, D') \mapsto [D, D'] := DD' - D'D$$

of derivations.

Proposition 2.17 ([21] Théorème 2)**.** *Let X be a smooth variety, and $\mathcal{N}$ an integrable subbundle of $T(X)$. Then there exists a unique morphism $q : X \to X^{\mathcal{N}}$ with the following properties;*

(i) *q induces a homeomorphism on the underlying topological spaces,*
(ii) *q is a radical covering of height 1, and*

(iii) *the kernel of* $dq : T(X) \to q^* T(X^{\mathcal{N}})$ *coincides with* $\mathcal{N}$.

Moreover the variety $X^{\mathcal{N}}$ *is smooth, and* q *is a purely inseparable finite morphism of degree* p^r*, where* r *is the rank of* $\mathcal{N}$.

Indeed, the scheme structure of $X^{\mathcal{N}}$ is given on the topological space X^{sp} underlying X by putting

$$\Gamma(U, \mathcal{O}_{X^{\mathcal{N}}}) := \Gamma(U, \mathcal{O}_X)^{\Gamma(U,\mathcal{N})}$$

for each affine Zariski open subset U of X^{sp}, where $\Gamma(U, \mathcal{N})$ is considered as a module of derivations on $\Gamma(U, \mathcal{O}_X)$, and $\Gamma(U, \mathcal{O}_X)^{\Gamma(U,\mathcal{N})}$ is the sub-algebra of $\Gamma(U, \mathcal{O}_X)$ consisting of all the elements that are annihilated by every derivation in $\Gamma(U, \mathcal{N})$. The inclusions

$$\Gamma(U, \mathcal{O}_{X^{\mathcal{N}}}) \hookrightarrow \Gamma(U, \mathcal{O}_X)$$

together with the identity map on X^{sp} yield the radical covering $q : X \to X^{\mathcal{N}}$. See [21] for more detail.

Definition 2.18. Let X be a smooth variety, and $\mathcal{N}$ an integrable subbundle of $T(X)$. The morphism $q : X \to X^{\mathcal{N}}$ is called the *quotient morphism by* $\mathcal{N}$.

Remark 2.19. Let $q : X \to X^{\mathcal{N}}$ be as in Definition 2.18. Suppose that $\mathcal{N}$ is of rank r. Let P be a point of X. Then there exists a local parameter system $(x_1, \dots, x_n)$ of X at P such that

$$(x_1^p, \dots, x_r^p, x_{r+1}, \dots, x_n)$$

is a local parameter system of $X^{\mathcal{N}}$ at $q(P)$. See [21, Proposition 6]. In particular, $(\mathcal{O}_{X,P})^{\wedge}$ is a free module of rank p^r over $(\mathcal{O}_{X^{\mathcal{N}},q(P)})^{\wedge}$, and hence $(\mathcal{O}_{X,P})^{\wedge}$ is faithfully flat over $(\mathcal{O}_{X^{\mathcal{N}},q(P)})^{\wedge}$.

Remark 2.20. Let $f : X \to Y$ be a morphism from a smooth variety X to a smooth variety Y. Suppose that the kernel $\mathcal{K}$ of the homomorphism $df : T(X) \to f^* T(Y)$ is a subbundle of $T(X)$. (This assumption is always satisfied if we replace X with a Zariski open dense subset of X.) Then $\mathcal{K}$ is integrable, and the morphism $f : X \to Y$ factors canonically as

$$X \xrightarrow{q} X^{\mathcal{K}} \longrightarrow Y,$$

where $q : X \to X^{\mathcal{K}}$ is the quotient morphism by $\mathcal{K}$.

3. The discriminant variety of a linear system

We make no assumptions on the characteristic of the base field k in this section.

Let $\overline{X}$ be a projective variety of dimension $n > 0$. Let $\mathcal{L} \to \overline{X}$ be a line bundle on $\overline{X}$, and M a linear subspace of $H^0(\overline{X}, \mathcal{L})$ with dimension $m + 1 \geq 2$. We denote by

$$\mathbf{P} := \mathbb{P}_*(M)$$

the projective space of one-dimensional linear subspaces of M, which is the parameter space of the linear system $|M|$. We put

$$X := \overline{X} \setminus (\mathrm{Sing}(\overline{X}) \cup \mathrm{Bs}(|M|)),$$

where $\mathrm{Sing}(\overline{X})$ is the singular locus of $\overline{X}$ and $\mathrm{Bs}(|M|)$ is the base locus of the linear system $|M|$. We denote by

$$\Psi : X \to \mathbf{P}^{\vee}$$

the morphism induced by the linear system $|M|$. Let

$$\mathrm{pr}_1 : X \times \mathbf{P} \to X \quad \text{and} \quad \mathrm{pr}_2 : X \times \mathbf{P} \to \mathbf{P}$$

be the projections. For a non-zero element f of M, we denote by $[f]$ the point of $\mathbf{P}$ corresponding to f, and by $\overline{D}_{[f]} \in |M|$ the divisor of $\overline{X}$ defined by $f = 0$. We then put

$$D_{[f]} := \overline{D}_{[f]} \cap X.$$

In the vector bundle $M \otimes_k \mathcal{O}_{\mathbf{P}}$ on $\mathbf{P}$, there exists a tautological subbundle $\mathcal{S} \hookrightarrow M \otimes_k \mathcal{O}_{\mathbf{P}}$ of rank 1, which is isomorphic to $\mathcal{O}_{\mathbf{P}}(-1)$. Hence we have a canonical section

$$\mathcal{O}_{\mathbf{P}} \longrightarrow M \otimes_k \mathcal{O}_{\mathbf{P}}(1) \tag{3.1}$$

of $M \otimes_k \mathcal{O}_{\mathbf{P}}(1)$. On the other hand, the inclusion $M \hookrightarrow H^0(X, \mathcal{L})$ induces a natural homomorphism

$$M \otimes_k \mathcal{O}_X \longrightarrow \mathcal{L}. \tag{3.2}$$

We put

$$\widetilde{\mathcal{L}} := \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}}(1).$$

Composing the pull-backs of (3.1) and (3.2) to $X \times \mathbf{P}$, we obtain a section

$$\mathcal{O}_{X \times \mathbf{P}} \longrightarrow \widetilde{\mathcal{L}}. \tag{3.3}$$

Definition 3.1. We fix a non-zero element

$$\sigma \in H^0(X \times \mathbf{P}, \widetilde{\mathcal{L}})$$

corresponding to (3.3), which is unique up to multiplicative constants. We denote by $\mathcal{D}$ the subscheme of $X \times \mathbf{P}$ defined by $\sigma = 0$, and by

$$p_1 : \mathcal{D} \to X \quad \text{and} \quad p_2 : \mathcal{D} \to \mathbf{P}$$

the projections.

It is easy to see that the support of $\mathcal{D}$ coincides with the set

$$\{\ (p, [f]) \in X \times \mathbf{P} \ \mid\ p \in D_{[f]}\ \}.$$

Proposition 3.2. *The scheme $\mathcal{D}$ is smooth.*

Proof. Since the linear system $|M|$ has no base points on X, the first projection $p_1 : \mathcal{D} \to X$ is a smooth morphism with fibers being hyperplanes of $\mathbf{P}$. Since X is smooth, so is $\mathcal{D}$. □

Definition 3.3. Let $\mathcal{C}$ denote the critical subscheme $\mathbf{Cr}(p_2)$ of $p_2 : \mathcal{D} \to \mathbf{P}$.

Let U be a Zariski open subset of $X \times \mathbf{P}$. Assume that there exists a trivialization

$$\tau \ : \ \widetilde{\mathcal{L}} \,|\, U \xrightarrow{\sim} \mathcal{O}_{X \times \mathbf{P}} \,|\, U$$

of the line bundle $\widetilde{\mathcal{L}}$ over U. Let Θ be a section of $T(X \times \mathbf{P})$ over U, which is regarded as a derivation on $\Gamma(U, \mathcal{O}_{X \times \mathbf{P}})$. Since $\mathcal{D}$ is defined by $\sigma = 0$, the element

$$\tau^{-1}(\Theta(\tau(\sigma))) \,|\, \mathcal{D} \ \in \ \Gamma(U \cap \mathcal{D}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{D}})$$

does not depend on the choice of the trivialization τ. Hence we denote it by $(\Theta\sigma) \,|\, \mathcal{D}$. It is obvious that, if two sections Θ and Θ' of $T(X \times \mathbf{P})$ over U are mapped to the same element in $\Gamma(U\cap\mathcal{D}, T(X\times\mathbf{P})\otimes\mathcal{O}_{\mathcal{D}})$, then we have $(\Theta\sigma)\,|\,\mathcal{D}=(\Theta'\sigma)\,|\,\mathcal{D}$. Therefore we have a natural homomorphism

$$d\sigma\;:\;T(X\times\mathbf{P})\otimes\mathcal{O}_{\mathcal{D}}\;\to\;\widetilde{\mathcal{L}}\otimes\mathcal{O}_{\mathcal{D}}$$

of vector bundles on $\mathcal{D}$ defined by

$$\Theta\,|\,\mathcal{D}\;\;\mapsto\;\;(\Theta\sigma)\,|\,\mathcal{D}.$$

We then denote by

$$d\sigma_X\;:\;p_1^*T(X)\;\to\;\widetilde{\mathcal{L}}\otimes\mathcal{O}_{\mathcal{D}}$$

the restriction of $d\sigma$ to the direct factor $p_1^*T(X)$ of

$$T(X\times\mathbf{P})\otimes\mathcal{O}_{\mathcal{D}}=p_1^*T(X)\,\oplus\,p_2^*T(\mathbf{P}).$$

Proposition 3.4. (1) *The critical subscheme $\mathcal{C}$ of $p_2:\mathcal{D}\to\mathbf{P}$ coincides with the degeneracy subscheme $\mathbf{D}(d\sigma_X)$ of $d\sigma_X$.*

(2) *A point $(p,[f])$ of $\mathcal{D}$ is contained in $\mathcal{C}$ if and only if the divisor $D_{[f]}$ of X is singular at $p\in X$.*

Construction 3.5. In order to prove Proposition 3.4, we introduce a formal parameter system of $\mathcal{D}$ at a point $P=(p,[f])\in\mathcal{D}$. We choose a formal parameter system $(x_1,\dots,x_n)$ of X at $p\in X$. Since the linear system $|M|$ has no base points on X, we can choose a global section β of $\mathcal{L}$ such that $\beta(p)\neq 0$. Then we can choose a basis $(\boldsymbol{b}_0,\dots,\boldsymbol{b}_m)$ of M in such a way that

$$\boldsymbol{b}_0=f,\qquad \boldsymbol{b}_m=\beta,$$

and that the functions

$$\phi_i:=\boldsymbol{b}_i/\beta\qquad(i=0,\dots,m-1)$$

on X defined locally at p satisfy

$$\phi_0(p)=\cdots=\phi_{m-1}(p)=0.$$

Let $(y_1,\dots,y_m)$ be the affine coordinate system of $\mathbf{P}$ such that a point $(c_1,\dots,c_m)$ corresponds to the one-dimensional linear subspace of M spanned by

$$\boldsymbol{b}_0+c_1\boldsymbol{b}_1+\cdots+c_m\boldsymbol{b}_m\quad\in\quad M.$$

Then $[f]=[\boldsymbol{b}_0]\in\mathbf{P}$ is the origin $(0,\dots,0)$.

We will regard $\phi_0,\ \dots,\ \phi_{m-1}$ as formal power series of $(x_1,\dots,x_n)$ so that we will write $\phi_i(0)$ instead of $\phi_i(p)$, for example. We put

$$\Phi:=\phi_0+y_1\phi_1+\cdots+y_{m-1}\phi_{m-1}+y_m.$$

Then we have

$$\sigma=c\,\Phi\beta\quad\text{for some } c\in k^\times \tag{3.4}$$

in $\widetilde{\mathcal{L}}\otimes_{\mathcal{O}_P}(\mathcal{O}_P)^\wedge$, where $\mathcal{O}_P$ is the local ring $\mathcal{O}_{X\times\mathbf{P},P}$. Hence $\mathcal{D}$ is given by $\Phi=0$ locally at P. Since

$$\frac{\partial\Phi}{\partial y_m}(0,0)=1,$$

we see that

$$(\xi,\eta)=(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_{m-1}):=(p_1^*x_1,\dots,p_1^*x_n,p_2^*y_1,\dots,p_2^*y_{m-1})$$

is a formal parameter system of $\mathcal{D}$ at P.

Proof of Proposition 3.4. Let $P = (p, [f])$ be a point of $\mathcal{D}$. We use the formal parameter system (ξ, η) of $\mathcal{D}$ at P and the affine coordinate system $(y_1, \dots, y_m)$ of $\mathbf{P}$ with the origin $[f]$ given in Construction 3.5. We write the pull-back $p_2^* y_m$ of y_m to $\mathcal{D}$ as a formal power series of (ξ, η):

$$p_2^* y_m = g_m(\xi, \eta) \quad \text{in} \quad (\mathcal{O}_{\mathcal{D},P})^\wedge = k[[\xi, \eta]].$$

Then the Jacobian matrix of $p_2 : \mathcal{D} \to \mathbf{P}$ is as follows:

$$\left[\begin{array}{ccc|c} & 0 & & I_{m-1} \\ \hline \dfrac{\partial g_m}{\partial \xi_1} & \cdots & \dfrac{\partial g_m}{\partial \xi_n} & * \end{array}\right]$$

because $p_2^* y_i = \eta_i$ for $i = 1, \dots, m-1$ and $p_2^* y_m = g_m(\xi, \eta)$. Hence the degenerate subscheme $\mathcal{C}$ of $p_2 : \mathcal{D} \to \mathbf{P}$ is defined locally at P by the ideal

$$\left\langle \frac{\partial g_m}{\partial \xi_1}, \dots, \frac{\partial g_m}{\partial \xi_n} \right\rangle \subset (\mathcal{O}_{\mathcal{D},P})^\wedge = k[[\xi, \eta]]. \tag{3.5}$$

On the other hand, by (3.4), the degeneracy subscheme of $d\sigma_X : p_1^* T(X) \to \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{D}}$ is defined locally at P by the ideal

$$\left\langle \frac{\partial \Phi}{\partial x_1}\bigg|_{\mathcal{D}}, \dots, \frac{\partial \Phi}{\partial x_n}\bigg|_{\mathcal{D}} \right\rangle \subset (\mathcal{O}_{\mathcal{D},P})^\wedge. \tag{3.6}$$

By the definition of g_m, we have

$$\Phi(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_{m-1}, g_m(\xi, \eta)) \equiv 0.$$

Applying $\partial/\partial\xi_i$ to this identity, we obtain

$$\frac{\partial \Phi}{\partial x_i}\bigg|_{\mathcal{D}} + \frac{\partial \Phi}{\partial y_m}\bigg|_{\mathcal{D}} \cdot \frac{\partial g_m}{\partial \xi_i} \equiv 0.$$

Because $\partial\Phi/\partial y_m \equiv 1$, the ideals (3.5) and (3.6) coincide in $(\mathcal{O}_{\mathcal{D},P})^\wedge$. Therefore the assertion (1) is proved. Because

$$\frac{\partial \Phi}{\partial x_i}(0, 0) = \frac{\partial \phi_0}{\partial x_i}(0),$$

the origin $P \in \mathcal{D}$ is contained in the subscheme $\mathcal{C}$ of $\mathcal{D}$ defined by the ideal (3.6) if and only if we have

$$\frac{\partial \phi_0}{\partial x_1}(0) = \cdots = \frac{\partial \phi_0}{\partial x_n}(0) = 0;$$

that is, the divisor $D_{[f]} = \{\phi_0 = 0\}$ is singular at p. Thus the assertion (2) is also proved. □

Corollary 3.6. *The subscheme $\mathcal{C}$ of $X \times \mathbf{P}$ is defined by*

$$\Phi = \frac{\partial \Phi}{\partial x_1} = \cdots = \frac{\partial \Phi}{\partial x_n} = 0$$

locally at a point $P = (p, [f])$ of $\mathcal{D}$, where Φ is the function on $X \times \mathbf{P}$ defined locally at P given in Construction 3.5.

Note that the expected dimension of $\mathcal{C}$ is $m - 1$.

Proposition 3.7. *The subscheme $\mathcal{C}$ is smooth of dimension $m-1$ at a point $P = (p, [f])$ of $\mathcal{C}$ if one of the following holds;*

(i) *the singularity of $D_{[f]}$ at p is non-degenerate, or*

(ii) *the morphism $\Psi : X \to \mathbf{P}^\vee$ induced by the linear system $|M|$ is a closed immersion formally at p.*

Proof. We use the formal parameter system $(x_1, \dots, x_n, y_1, \dots, y_m)$ of $X \times \mathbf{P}$ at P given in Construction 3.5. By Corollary 3.6, the subscheme $\mathcal{C}$ is smooth of dimension $m-1$ at the origin P if and only if the $(n+m) \times (n+1)$-matrix

$$J := \left[\begin{array}{c|c}
\dfrac{\partial \phi_0}{\partial x_1}(0) & \\
\vdots & \dfrac{\partial^2 \phi_0}{\partial x_i \partial x_j}(0) \quad {\scriptstyle (i,j=1,\dots,n)} \\
\dfrac{\partial \phi_0}{\partial x_n}(0) & \\
\hline
0 & \\
\vdots & \dfrac{\partial \phi_i}{\partial x_j}(0) \quad \left({\scriptstyle i\,=\,1,\,\dots,\,m-1, \atop j\,=\,1,\,\dots,\,n}\right) \\
0 & \\
\hline
1 & 0 \qquad \cdots \qquad 0
\end{array}\right]
\begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\\a\end{array}}\right\} m \end{array}$$

is of rank $n+1$. Here we have used the following equalities:

$$\frac{\partial \Phi}{\partial x_i}(0,0) = \frac{\partial \phi_0}{\partial x_i}(0), \qquad \frac{\partial \Phi}{\partial y_j}(0,0) = \begin{cases} \phi_j(0) = 0 & \text{if } j < m, \\ 1 & \text{if } j = m, \end{cases}$$

and

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_i}\right)(0,0) = \frac{\partial^2 \phi_0}{\partial x_j \partial x_i}(0), \qquad \frac{\partial}{\partial y_j}\left(\frac{\partial \Phi}{\partial x_i}\right)(0,0) = \begin{cases} \dfrac{\partial \phi_j}{\partial x_i}(0) & \text{if } j < m, \\ 0 & \text{if } j = m. \end{cases}$$

Suppose that the condition (i) holds. Then the Hessian matrix

$$\left(\frac{\partial^2 \phi_0}{\partial x_i \partial x_j}(0)\right)$$

of $D_{[f]}$ at p is non-degenerate, and hence the matrix J is of rank $n+1$. Suppose that the condition (ii) holds. Then there exist n divisors $D_1, \dots, D_n \in |M|$ that pass through p, are smooth at p, and intersect transversely at p. The local defining equations of these D_i at P are linear combinations of $\phi_1, \dots, \phi_{m-1}$, because the divisor $D_{[f]} = \{\phi_0 = 0\}$ is singular at p and the divisor corresponding to $\boldsymbol{b}_m$ does not pass through p. Hence the $(m-1) \times n$-matrix

$$\left(\frac{\partial \phi_i}{\partial x_j}(0)\right)_{i=1,\dots,m-1,\ j=1,\dots,n}$$

is of rank n, and thus J is of rank $n+1$. $\square$

Assumption 3.8. From now on until the end of the paper, we assume that $m > n$, and that the locus

$$X^\circ := \{\, p \in X \mid \text{the morphism } \Psi : X \to \mathbf{P}^\vee \text{ is a closed immersion formally at } p \,\}$$

is dense in X.

Note that if $\overline{X}$ is smooth and the linear system $|M|$ is very ample, then X° coincides with $\overline{X}$.

Definition 3.9. We put

$$\mathcal{C}^\circ := \mathcal{C} \cap (X^\circ \times \mathbf{P}),$$

and denote by

$$\pi_1 : \mathcal{C}^\circ \to X^\circ \quad \text{and} \quad \pi_2 : \mathcal{C}^\circ \to \mathbf{P}$$

the projections.

Proposition 3.10. *The scheme $\mathcal{C}^\circ$ is a smooth irreducible closed subscheme of $X^\circ \times \mathbf{P}$ with dimension $m-1$.*

Proof. The fact that $\mathcal{C}^\circ$ is smooth of dimension $m-1$ follows from Proposition 3.7 and the definition of X°. We will prove the irreducibility of $\mathcal{C}^\circ$. For each point $p \in X^\circ$, there exists a unique n-dimensional linear subspace $L_p \subset \mathbf{P}^\vee$ passing through $\Psi(p)$ such that the image of the injective homomorphism $d_p\Psi : T_p(X^\circ) \to T_{\Psi(p)}(\mathbf{P}^\vee)$ coincides with $T_{\Psi(p)}(L_p) \subset T_{\Psi(p)}(\mathbf{P}^\vee)$. The fiber of $\pi_1 : \mathcal{C}^\circ \to X^\circ$ over p coincides with the linear subspace

$$\{ \, H \in \mathbf{P} \mid L_p \subset H \, \}$$

of $\mathbf{P}$. Hence $\mathcal{C}^\circ$ is irreducible. □

Remark 3.11. The above proof of Proposition 3.10 shows that, if $m = n+1$, then $\pi_1 : \mathcal{C}^\circ \to X^\circ$ is an isomorphism with the inverse morphism given by $p \mapsto (p, L_p)$. In this case, the morphism $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is identified with the Gauss map $X^\circ \to \mathbf{P}$ of the morphism $\Psi : X^\circ \to \mathbf{P}^\vee$.

Definition 3.12. Let $\mathcal{E}$ denote the critical subscheme $\mathbf{Cr}(\pi_2)$ of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$.

Definition 3.13. We will construct the *universal Hessian*

$$\mathcal{H} \,:\, \pi_1^* \, T(X^\circ) \otimes_{\mathcal{O}_{\mathcal{C}^\circ}} \pi_1^* \, T(X^\circ) \,\to\, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{C}^\circ}$$

on $\mathcal{C}^\circ$. Let U be a Zariski open subset of X°. Making U smaller if necessary, we may assume that there exist regular functions $(u_1, \dots, u_n)$ on U that form a coordinate system on U, and that there exists a trivialization $\mathcal{L} \,|\, U \cong \mathcal{O}_U$ of $\mathcal{L}$ over U. Let V be a Zariski open subset of $\mathbf{P}$ over which the line bundle $\mathcal{O}_{\mathbf{P}}(1)$ is trivialized. Let $\Phi_{U\times V}$ denote the regular function on $U \times V$ obtained from the fixed global section σ of $\widetilde{\mathcal{L}}$ via a trivialization $\tau : \widetilde{\mathcal{L}} \,|\, (U \times V) \cong \mathcal{O}_{U\times V}$. We define $\mathcal{H}$ on $\mathcal{C}^\circ \cap (U \times V)$ by

$$\mathcal{H}\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) := \tau^{-1}\left(\frac{\partial^2 \Phi_{U\times V}}{\partial u_i \partial u_j}\right).$$

It is easy to see that this definition does not depend on the choice of the coordinate system $(u_1, \dots, u_n)$ on U and the trivializations of the line bundles, because the functions $\Phi_{U\times V}$ and $\partial\Phi_{U\times V}/\partial u_1, \dots, \partial\Phi_{U\times V}/\partial u_n$ are constantly equal to zero on $\mathcal{C}^\circ \cap (U \times V)$ by Corollary 3.6. Therefore we can define $\mathcal{H}$ globally on $\mathcal{C}^\circ$. We denote by

$$\mathcal{H}^{\sim} \,:\, \pi_1^* \, T(X^\circ) \,\to\, \widetilde{\mathcal{L}} \otimes \pi_1^* \, T(X^\circ)^\vee$$

the homomorphism induced from $\mathcal{H}$.

The following proposition is a scheme-theoretic refinement of [14, Proposition 3.3]. See also the Hessian criterion of Hefez and Kleiman ([17, Theorem (12)], [8, Theorem 3.2]).

Proposition 3.14. *The critical subscheme $\mathcal{E}$ of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ coincides with the degeneracy subscheme $\mathbf{D}(\mathcal{H}\tilde{})$ of $\mathcal{H}\tilde{}$.*

Construction 3.15. In order to prove Proposition 3.14, we introduce a formal parameter system of $\mathcal{C}^\circ$ at a point $P = (p, [f]) \in \mathcal{C}^\circ$. We use the same notation as in Construction 3.5. Since $p \in X^\circ$, we can assume that the vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ among the basis $\boldsymbol{b}_0, \dots, \boldsymbol{b}_m$ of M define divisors that pass through p, are smooth at p, and intersect transversely at p. Then we can take $(\phi_1, \dots, \phi_n)$ as the formal parameter system $(x_1, \dots, x_n)$ of X° at p; that is, we have

$$\phi_1 = x_1, \ \dots, \ \phi_n = x_n,$$

and hence we have

$$\Phi = \phi_0 + y_1 x_1 + \dots + y_n x_n + y_{n+1}\phi_{n+1} + \dots + y_{m-1}\phi_{m-1} + y_m.$$

By a further linear transformation of the basis $\boldsymbol{b}_0, \dots, \boldsymbol{b}_m$, we can also assume that

$$\frac{\partial \phi_i}{\partial x_j}(0) = 0 \qquad \text{for} \quad i = n+1, \dots, m-1 \quad \text{and} \quad j = 1, \dots, n$$

hold; that is, the functions $\phi_{n+1}, \dots, \phi_{m-1}$ have no linear terms as formal power series of $x_1, \dots, x_n$. By Corollary 3.6, the local defining equations of $\mathcal{C}^\circ$ in $X^\circ \times \mathbf{P}$ at $P = (p, [f])$ are as follows.

$$\begin{array}{llllllll}
\phi_0 & + y_1 x_1 & + \cdots & + y_n x_n & + y_{n+1}\phi_{n+1} & + \cdots & + y_{m-1}\phi_{m-1} & + y_m = 0, \\
\dfrac{\partial \phi_0}{\partial x_1} & + y_1 & & & + y_{n+1}\dfrac{\partial \phi_{n+1}}{\partial x_1} & + \cdots & + y_{m-1}\dfrac{\partial \phi_{m-1}}{\partial x_1} & = 0, \\
 & & \cdots & & & & & \\
 & & \cdots & & & & & \\
\dfrac{\partial \phi_0}{\partial x_n} & & & + y_n & + y_{n+1}\dfrac{\partial \phi_{n+1}}{\partial x_n} & + \cdots & + y_{m-1}\dfrac{\partial \phi_{m-1}}{\partial x_n} & = 0.
\end{array}$$

We see that

$$(u, v) = (u_1, \dots, u_n, v_{n+1}, \dots, v_{m-1}) := (\pi_1^* x_1, \dots, \pi_1^* x_n, \pi_2^* y_{n+1}, \dots, \pi_2^* y_{m-1})$$

is a formal parameter system of $\mathcal{C}^\circ$ at $P = (p, [f])$.

Proof of Proposition 3.14. Let $P = (p, [f])$ be a point of $\mathcal{C}^\circ$. We use the formal parameter system (u, v) of $\mathcal{C}^\circ$ at P and the affine coordinate system $(y_1, \dots, y_m)$ of $\mathbf{P}$ with the origin $[f]$ given in Construction 3.15. We put

$$\gamma_j := \pi_2^* y_j \qquad (j = 1, \dots, m).$$

Since $\gamma_j = v_j$ for $j = n+1, \dots, m-1$, the Jacobian matrix of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is of the form

$$\left[\begin{array}{c|c} \dfrac{\partial \gamma_i}{\partial u_j}\ {\scriptstyle (i,j=1,\dots,n)} & * \\ \hline 0 & I_{m-n-1} \\ \hline \dfrac{\partial \gamma_m}{\partial u_1} \quad \dots \quad \dfrac{\partial \gamma_m}{\partial u_n} & * \end{array}\right].$$

Hence the defining ideal of the critical subscheme $\mathcal{E}$ of π_2 at P is generated by all n-minors of the $(n+1) \times n$ matrix

$$\left[\begin{array}{c} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \\ \hline \boldsymbol{a}_m \end{array}\right] := \left[\begin{array}{ccc} \dfrac{\partial \gamma_1}{\partial u_1} & \dots & \dfrac{\partial \gamma_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \gamma_n}{\partial u_1} & \dots & \dfrac{\partial \gamma_n}{\partial u_n} \\ \hline \dfrac{\partial \gamma_m}{\partial u_1} & \dots & \dfrac{\partial \gamma_m}{\partial u_n} \end{array}\right].$$

Since $\Phi \,|\, \mathcal{C}^\circ \equiv 0$, we have

$$\tilde{\phi}_0 + \gamma_1 u_1 + \dots + \gamma_n u_n + v_{n+1}\tilde{\phi}_{n+1} + \dots + v_{m-1}\tilde{\phi}_{m-1} + \gamma_m \equiv 0, \tag{3.7}$$

where

$$\tilde{\phi}_i := \phi_i(u_1, \dots, u_n) = \pi_1^* \phi_i.$$

Applying $\partial/\partial u_i$ to (3.7), we obtain

$$\frac{\partial \tilde{\phi}_0}{\partial u_i} + \gamma_i + \sum_{\nu=1}^{n} \frac{\partial \gamma_\nu}{\partial u_i} u_\nu + \sum_{\mu=n+1}^{m-1} v_\mu \frac{\partial \tilde{\phi}_\mu}{\partial u_i} + \frac{\partial \gamma_m}{\partial u_i} \equiv 0. \tag{3.8}$$

Since $(\partial \Phi/\partial x_i) \,|\, \mathcal{C}^\circ \equiv 0$ for $i = 1, \dots, n$, we have

$$\frac{\partial \tilde{\phi}_0}{\partial u_i} + \gamma_i + \sum_{\mu=n+1}^{m-1} v_\mu \frac{\partial \tilde{\phi}_\mu}{\partial u_i} \equiv 0, \tag{3.9}$$

because $(\partial \phi_j/\partial x_i) \,|\, \mathcal{C}^\circ = \partial \tilde{\phi}_j/\partial u_i$. Combining the identities (3.8) and (3.9), we obtain

$$\frac{\partial \gamma_m}{\partial u_i} \equiv -\sum_{\nu=1}^{n} \frac{\partial \gamma_\nu}{\partial u_i} u_\nu \qquad (i = 1, \dots, n).$$

Thus we have

$$\boldsymbol{a}_m = -\sum_{\nu=1}^{n} u_\nu \boldsymbol{a}_\nu.$$

Therefore the defining ideal of $\mathcal{E}$ at P is generated by

$$\det A := \det \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}$$

in $(\mathcal{O}_{\mathcal{C}^\circ, P})^\wedge$. On the other hand, we have

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\bigg|_{\mathcal{C}^\circ} \equiv \frac{\partial^2 \tilde{\phi}_0}{\partial u_i \partial u_j} + \sum_{\mu=n+1}^{m-1} v_\mu \frac{\partial^2 \tilde{\phi}_\mu}{\partial u_i \partial u_j}. \tag{3.10}$$

Applying $\partial/\partial u_j$ to (3.9), we obtain

$$\frac{\partial^2 \tilde{\phi}_0}{\partial u_i \partial u_j} + \frac{\partial \gamma_i}{\partial u_j} + \sum_{\mu=n+1}^{m-1} v_\mu \frac{\partial^2 \tilde{\phi}_\mu}{\partial u_i \partial u_j} \equiv 0. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\bigg|_{\mathcal{C}^\circ} \equiv -\frac{\partial \gamma_i}{\partial u_j}. \tag{3.12}$$

We denote by

$$S := \begin{bmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_n \end{bmatrix} = \left(\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\bigg|_{\mathcal{C}^\circ} \right)$$

the $n \times n$ matrix representing the universal Hessian $\mathcal{H}$ locally at P. From (3.12), we obtain

$$\boldsymbol{s}_i = -\boldsymbol{a}_i \qquad (i = 1, \dots, n).$$

Hence $\det A$ and $\det S$ generate the same ideal in $(\mathcal{O}_{\mathcal{C}^\circ, P})^\wedge$. Therefore $\mathcal{E}$ coincides with $\mathbf{D}(\mathcal{H}\tilde{\ })$ locally at P. □

Corollary 3.16 ([14] Proposition 3.3)**.** *The morphism $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is a closed immersion formally at a point $(p, [f]) \in \mathcal{C}^\circ$ if and only if the singularity of the divisor $D_{[f]}$ of X° at $p \in X^\circ$ is non-degenerate.*

Corollary 3.17. *The subscheme $\mathcal{E}$ of $X^\circ \times \mathbf{P}$ is defined by*

$$\Phi = \frac{\partial \Phi}{\partial x_1} = \cdots = \frac{\partial \Phi}{\partial x_n} = \det \left(\frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) = 0$$

locally at a point $P = (p, [f])$ of $\mathcal{C}^\circ$, where Φ is the function on $X^\circ \times \mathbf{P}$ defined locally at P given in Construction 3.15.

Remark 3.18. By Corollaries 3.6 and 3.17, the scheme $\mathcal{E}$ is of codimension ≤ 1 in $\mathcal{C}^\circ$. It was observed by Wallace [24] that, in positive characteristics, $\mathcal{E}$ and $\mathcal{C}^\circ$ may coincide. For example, let $\overline{X}$ be the Fermat hypersurface

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_{n+1}^{q+1} = 0$$

of degree $q+1$ in $\mathbb{P}^{n+1}$, where $q = l^\nu$ is a power of the characteristic $l > 0$ of k, and let M be the complete linear system $|\mathcal{O}_{\overline{X}}(1)|$. Then, at every point p of $\overline{X}$, the divisor $T_p(\overline{X}) \cap \overline{X}$ of $\overline{X}$ has a *degenerate* singular point at p, and hence $\mathcal{E} = \mathcal{C}^\circ$ holds. In this case, the morphism $\mathcal{C}^\circ \to \mathbf{P}$ is purely inseparable of degree q^n onto its image. See [14, Example 3.4] or [23] for the details.

4. The scheme $\mathcal{E}$

In this section, we assume that $\operatorname{char} k$ is *not* 2.

Construction 4.1. Let $P = (p, [f])$ be a point of $\mathcal{E}$, and let r be the rank of the Hessian of $D_{[f]}$ at p. By Corollary 3.16, we have $r < n$. We choose a formal parameter system $(x_1, \dots, x_n, y_1, \dots, y_m)$ of $X^\circ \times \mathbf{P}$ at P given in Construction 3.15. Since $\operatorname{char} k \neq 2$, we can assume that the functions

$$\phi_1 = x_1, \ \ \dots, \ \ \phi_n = x_n$$

form an admissible formal parameter system with respect to ϕ_0 at $p \in X^\circ$ by a linear transformation of the basis $\boldsymbol{b}_0, \dots, \boldsymbol{b}_m$ of M. (See Remark 2.11). Thus we have

$$\phi_0 = x_1^2 + \cdots + x_r^2 + (\text{terms of degree} \geq 3) \quad \text{in} \quad (\mathcal{O}_{X^\circ, p})^\wedge = k[[x_1, \dots, x_n]].$$

Definition 4.2. Let

$$\varpi_1 : \mathcal{E} \to X^\circ \quad \text{and} \quad \varpi_2 : \mathcal{E} \to \mathbf{P}$$

be the projections. We put

$$\mathcal{E}^{\mathrm{sm}} := \{ \ P \in \mathcal{E} \ \mid \ \mathcal{E} \text{ is smooth of dimension } m-2 \text{ at } P \ \},$$

which is a Zariski open subset of $\mathcal{E}$, and let

$$\varpi_1^{\mathrm{sm}} : \mathcal{E}^{\mathrm{sm}} \to X^\circ \quad \text{and} \quad \varpi_2^{\mathrm{sm}} : \mathcal{E}^{\mathrm{sm}} \to \mathbf{P}$$

be the restrictions of ϖ_1 and ϖ_2 to $\mathcal{E}^{\mathrm{sm}}$. Note that, if $\mathcal{E}^{\mathrm{sm}}$ is non-empty, then the image of the projection $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is a hypersurface.

We also put

$$\mathcal{E}^{A_2} := \{ \ (p, [f]) \in \mathcal{E} \ \mid \ \text{the singularity of the divisor } D_{[f]} \text{ at } p \text{ is of type } A_2 \ \}.$$

In the following, Proposition 4.3 concerns with both the cases of characteristic 3 and characteristic $\neq 3$, Proposition 4.4 treats the case where $\operatorname{char} k \neq 3$, and Theorem 4.5 is a result in characteristic 3.

Proposition 4.3. *If $P = (p, [f])$ is a point of $\mathcal{E}^{\mathrm{sm}}$, then the rank of the Hessian $H_{\phi_0, p}$ of the divisor $D_{[f]}$ at p is $n-1$.*

Conversely, let $P = (p, [f])$ be a point of $\mathcal{E}$, and suppose that the rank of $H_{\phi_0, p}$ is $n-1$. Let $(x_1, \dots, x_n, y_1, \dots, y_m)$ be the formal parameter system of $X^\circ \times \mathbf{P}$ at P given in Construction 4.1. Let a_i $(i = 1, \dots, n)$ be the coefficient of $x_i x_n^2$ in ϕ_0, and let b_j $(j = n+1, \dots, m-1)$ be the coefficient of x_n^2 in ϕ_j. Then $P \in \mathcal{E}^{\mathrm{sm}}$ holds if and only if at least one of

$$a_1, \dots, a_{n-1}, 3a_n, b_{n+1}, \dots, b_{m-1}$$

is not zero.

Proposition 4.4. *Suppose that* $\operatorname{char} k \neq 3$. *Then we have*

$$\mathcal{E}^{A_2} = \mathcal{E}^{\mathrm{sm}} \setminus \mathbf{Cr}(d\varpi_2^{\mathrm{sm}}).$$

Theorem 4.5. *Suppose that* $\operatorname{char} k = 3$. *We denote by $\mathcal{K}$ the kernel of the homomorphism*

$$d\varpi_2^{\mathrm{sm}} \ : \ T(\mathcal{E}^{\mathrm{sm}}) \ \to \ \varpi_2^{\mathrm{sm}*} \, T(\mathbf{P}).$$

Then $\mathcal{K}$ is an integrable subbundle of $T(\mathcal{E}^{\mathrm{sm}})$ with rank 1. Let

$$\mathcal{E}^{\mathrm{sm}} \quad \xrightarrow{q} \quad (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}} \quad \xrightarrow{\tau} \quad \mathbf{P}$$

be the canonical factorization of ϖ_2^{sm}, where q is the quotient morphism by $\mathcal{K}$. Then we have

$$q(\mathcal{E}^{A_2}\cap\mathcal{E}^{\mathrm{sm}}) \;\subset\; (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}}\setminus \mathbf{Cr}(\tau).$$

Proof of Propositions 4.3, 4.4 and Theorem 4.5. Let $P=(p,[f])$ be a point of $\mathcal{E}$, and let r be the rank of the Hessian $H_{\phi_0,p}$ of $D_{[f]}$ at p. We use the formal parameter system $(x_1,\dots,x_n,y_1,\dots,y_m)$ of $X^\circ\times\mathbf{P}$ at P given in Construction 4.1. For a formal power series F of $(x_1,\dots,x_n,y_1,\dots,y_m)$, we denote by $F^{[1]}$ the homogeneous part of degree 1 of F. Then we have

$$\begin{aligned}
\Phi^{[1]} &= y_m,\\
\left(\frac{\partial\Phi}{\partial x_i}\right)^{[1]} &= 2x_i+y_i \qquad (i=1,\dots,r),\\
\left(\frac{\partial\Phi}{\partial x_i}\right)^{[1]} &= y_i \qquad (i=r+1,\dots,n),
\end{aligned}$$

and

$$\begin{aligned}
\left(\det\left(\frac{\partial^2\Phi}{\partial x_i\partial x_j}\right)\right)^{[1]} &= \begin{cases} 0 & \text{if } r<n-1,\\ \left(\dfrac{\partial^2\Phi}{\partial x_n^2}\right)^{[1]} & \text{if } r=n-1,\end{cases}\\
&= \begin{cases} 0 & \text{if } r<n-1,\\ \begin{array}{l}2(a_1x_1+\cdots+a_{n-1}x_{n-1}+3a_nx_n+\\ \quad +b_{n+1}y_{n+1}+\cdots+b_{m-1}y_{m-1})\end{array} & \text{if } r=n-1.\end{cases}
\end{aligned}$$

By Corollary 3.17, the Zariski tangent space $T_P(\mathcal{E})$ to $\mathcal{E}$ at P is identified with the linear space defined by these $n+2$ linear forms in the $(n+m)$-dimensional linear space with coordinates $(x_1,\dots,x_n,y_1,\dots,y_m)$. Hence Proposition 4.3 is proved.

If $\operatorname{char}k\neq 3$ and $P\in\mathcal{E}^{A_2}$, then $P\in\mathcal{E}^{\mathrm{sm}}$ because $3a_n\neq 0$. Suppose that $P\in\mathcal{E}^{\mathrm{sm}}$. The kernel of the linear homomorphism

$$d_P\varpi_2^{\mathrm{sm}}\;:\;T_P(\mathcal{E}^{\mathrm{sm}})\;\to\;T_{[f]}(\mathbf{P})$$

is identified with the intersection of the linear space defined by the $n+2$ linear forms above and the linear space defined by

$$y_1=\cdots=y_m=0.$$

Hence $\operatorname{Ker}(d_P\varpi_2^{\mathrm{sm}})$ is of dimension 0 if and only if $3a_n\neq 0$. Thus Proposition 4.4 is proved.

We now assume that $\operatorname{char}k=3$. Suppose that $P=(p,[f])\in\mathcal{E}^{\mathrm{sm}}$. The kernel of the linear homomorphism $d_P\varpi_2^{\mathrm{sm}}$ is of dimension 1 and is generated by

$$\left(\frac{\partial}{\partial x_n}\right)_P \;\in\; T_P(\mathcal{E}^{\mathrm{sm}}).$$

Since this holds at every point P of $\mathcal{E}^{\mathrm{sm}}$, we see that the sub-sheaf $\mathcal{K}=\operatorname{Ker}(d\varpi_2^{\mathrm{sm}})$ of $T(\mathcal{E}^{\mathrm{sm}})$ is a subbundle of rank 1. The integrability of $\mathcal{K}$ follows trivially from the definition. From now on, we further assume that $P\in\mathcal{E}^{A_2}$; that is, $a_n\neq 0$. The fiber

$$Z:=(\varpi_2^{\mathrm{sm}})^{-1}(\,[f]\,)$$

of ϖ_2^{sm} passing through P is defined by

$$\phi_0 = \frac{\partial \phi_0}{\partial x_1} = \cdots = \frac{\partial \phi_0}{\partial x_n} = \det\left(\frac{\partial^2 \phi_0}{\partial x_i \partial x_j}\right) = 0$$

in $X^\circ \times \{[f]\} \cong X^\circ$ locally at P. We will calculate $\dim_k \mathcal{O}_{Z,P}$. Since ϖ_2^{sm} factors through the radical covering $q : \mathcal{E}^{\mathrm{sm}} \to (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}}$ of degree 3, we have

$$\dim_k \mathcal{O}_{Z,P} \geq 3.$$

We put

$$\xi_i := x_i \,|\, Z \quad (i = 1, \dots, n-1) \quad \text{and} \quad t := x_n \,|\, Z.$$

Using the identity $\partial\phi_0/\partial x_1 = \cdots = \partial\phi_0/\partial x_{n-1} = 0$ on Z and Lemma 4.6 below, we can write ξ_i in formal power series of t as follows:

$$\xi_i = a_i t^2 + (\text{terms of degree} \geq 3) \quad (i = 1, \dots, n-1).$$

Making substitutions $x_i = \xi_i$ for $i = 1, \dots, n-1$ and $x_n = t$ in ϕ_0, we obtain a formal power series

$$\phi_0 \,|\, Z = a_n t^3 + (\text{terms of degree} \geq 4).$$

Since $a_n \neq 0$, we obtain $\dim_k \mathcal{O}_{Z,P} \leq 3$. Therefore $\dim_k \mathcal{O}_{Z,P} = 3$ holds. We put

$$A := (\mathcal{O}_{\mathcal{E}^{\mathrm{sm}},P})^\wedge, \quad B := (\mathcal{O}_{(\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}},q(P)})^\wedge, \quad C := (\mathcal{O}_{\mathbf{P},[f]})^\wedge,$$

and let $\mathfrak{m}_A$, $\mathfrak{m}_B$, $\mathfrak{m}_C$ be their maximal ideals, respectively. From $\dim_k \mathcal{O}_{Z,P} = 3$ and Remark 2.19, we have

$$\dim_k(A/\mathfrak{m}_C A) = 3 = \dim_k(A/\mathfrak{m}_B A).$$

Since $\mathfrak{m}_C B \subseteq \mathfrak{m}_B$, we obtain

$$\mathfrak{m}_B A = \mathfrak{m}_C A.$$

Since A is faithfully flat over B, we obtain $\mathfrak{m}_B = \mathfrak{m}_C B$, which implies that $C \to B$ is surjective. Hence τ is a closed immersion formally at $q(P)$. Thus Theorem 4.5 is proved. □

Lemma 4.6. *Let $F_1(u,t)$, ..., $F_N(u,t)$ be formal power series of variables $(u,t) = (u_1, \dots, u_N, t)$ such that $F_1(0,0) = \cdots = F_N(0,0) = 0$ and $\det J \neq 0$, where*

$$J := \begin{bmatrix} \dfrac{\partial F_1}{\partial u_1}(0,0) & \dots & \dfrac{\partial F_1}{\partial u_N}(0,0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_N}{\partial u_1}(0,0) & \dots & \dfrac{\partial F_N}{\partial u_N}(0,0) \end{bmatrix}.$$

We put

$$\mu := \min\{\, \mathrm{ord}_{t=0}(F_i(0,t)) \;|\; i = 1, \dots, N \,\},$$

and let α_i be the coefficient of t^μ in $F_i(0,t)$. We put

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} := -J^{-1} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}.$$

Then we can solve the equation

$$F_1(u,t) = \cdots = F_N(u,t) = 0$$

with indeterminates $u_1, \dots, u_N$ in $k[[t]]$ as follows:

$$u_i = \beta_i t^\mu + (\text{terms of degree} > \mu) \quad (i = 1, \dots, N).$$

Proof. Obvious. $\square$

The following Corollary of Proposition 4.3 plays a crucial role in the proof of Theorem 5.2.

Corollary 4.7. *Suppose that* $\operatorname{char} k = 3$. *If $P \in \mathcal{E}^{\mathrm{sm}}$, then at least one of*

$$a_1, \dots, a_{n-1}, b_{n+1}, \dots b_{m-1}$$

is not zero. In particular, if $(n, m) = (1, 2)$, then $\mathcal{E}^{\mathrm{sm}} = \emptyset$.

Remark 4.8. Suppose that the Hessian $H_{\phi_0,p}$ of $D_{[f]}$ at p is of rank $n-1$. Then the condition that at least one of $a_1, \dots, a_{n-1}, 3a_n$ be non-zero is independent of the choice of the admissible formal parameter system $(x_1, \dots, x_n)$ of X at p with respect to ϕ_0. The condition that at least one of $b_{n+1}, \dots, b_{m-1}$ be non-zero is equivalent to the condition that there exists a divisor $D \in \mathbf{P}$ passing through p and having a non-degenerate singular point at p.

Next we will give a sufficient condition for $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\mathrm{sm}}$ to be dense in $\mathcal{E}$.

Proposition 4.9. *For $p \in X^\circ$, let $\mathfrak{m}_p \subset \mathcal{O}_p$ denote the maximal ideal of the local ring $\mathcal{O}_p := \mathcal{O}_{X^\circ,p}$, and let $\mathcal{L}_p$ denote the $\mathcal{O}_p$-module $\mathcal{L} \otimes \mathcal{O}_p$. Suppose that the evaluation homomorphism*

$$v_p \,:\, M \,\to\, \mathcal{L}_p/\mathfrak{m}_p^4\mathcal{L}_p \,\cong\, \mathcal{O}_p/\mathfrak{m}_p^4$$

is surjective at every point p of X°. Then $\mathcal{E}$ is irreducible, and $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\mathrm{sm}}$ are dense in $\mathcal{E}$.

Proof. The space $\mathfrak{m}_p^2/\mathfrak{m}_p^3$ is regarded as the space of symmetric bilinear forms on the Zariski tangent space $T_p(X^\circ) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$. The determinant of the symmetric matrix cuts out the irreducible subscheme D of degenerate symmetric bilinear forms in $\mathfrak{m}_p^2/\mathfrak{m}_p^3$. By Proposition 3.14, there exists a closed variety $\widetilde{D} \subset \mathfrak{m}_p^2/\mathfrak{m}_p^4 \subset \mathcal{O}_p/\mathfrak{m}_p^4$, which is a cone over $D \subset \mathfrak{m}_p^2/\mathfrak{m}_p^3$ in the subspace $\mathfrak{m}_p^2/\mathfrak{m}_p^4$ of $\mathcal{O}_p/\mathfrak{m}_p^4$ and is invariant under the multiplications by elements of $k^\times$, such that

$$\varpi_1^{-1}(p) = \mathbb{P}_*(v_p^{-1}(\widetilde{D})).$$

By the definition of hypersurface singularities of type A_2, and by Proposition 4.3, there exist Zariski open dense subsets $\widetilde{D}^{A_2}$ and $\widetilde{D}^{\mathrm{sm}}$ of $\widetilde{D}$, which are invariant under the multiplications by elements of $k^\times$, such that

$$\varpi_1^{-1}(p) \cap \mathcal{E}^{A_2} = \mathbb{P}_*(v_p^{-1}(\widetilde{D}^{A_2})) \quad \text{and} \quad \varpi_1^{-1}(p) \cap \mathcal{E}^{\mathrm{sm}} = \mathbb{P}_*(v_p^{-1}(\widetilde{D}^{\mathrm{sm}})).$$

Therefore, if v_p is surjective at every point $p \in X^\circ$, then $\mathcal{E}$ is irreducible, and $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\mathrm{sm}}$ are dense in $\mathcal{E}$. $\square$

Corollary 4.10. *Let $\mathcal{A}$ be a very ample line bundle on a smooth projective variety $\overline{X}$. If $\mathcal{L} = \mathcal{A}^{\otimes 3}$ and $M = H^0(\overline{X}, \mathcal{L})$, then $\mathcal{E}$ is irreducible, and $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\mathrm{sm}}$ are dense in $\mathcal{E}$.*

5. A general plane section of the discriminant hypersurface

In this section, we still assume that $\operatorname{char} k$ is *not* 2.

Definition 5.1. Let $P = (p, [f])$ be a point of $\mathcal{E}^{\mathrm{sm}}$, and let $\Lambda \subset \mathbf{P}$ be a general plane passing through the point $\pi_2(P) = [f]$ of $\mathbf{P}$. We denote by

$$\pi_\Lambda \,:\, C_\Lambda \,\to\, \Lambda$$

the restriction of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ to

$$C_\Lambda := \pi_2^{-1}(\Lambda) \quad \subset \quad \mathcal{C}^\circ.$$

Note that, if $\mathcal{E}^{\mathrm{sm}}$ is not empty, then the image of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is a hypersurface, and hence $\pi_2(\mathcal{C}^\circ) \cap \Lambda$ is a projective plane curve.

Theorem 5.2. *Let $P = (p, [f])$ be a point of $\mathcal{E}^{\mathrm{sm}} \cap \mathcal{E}^{A_2}$, and let Λ be a general plane in $\mathbf{P}$ passing through $[f]$. Then C_Λ is smooth of dimension 1 at $P \in C_\Lambda$.*

(1) *Suppose that $\operatorname{char} k \neq 3$. Then the morphism $\pi_\Lambda : C_\Lambda \to \Lambda$ has a critical point of A_2-type at P.*

(2) *Suppose that $\operatorname{char} k = 3$. Then the morphism $\pi_\Lambda : C_\Lambda \to \Lambda$ has a critical point of E_6-type at P.*

Proof. We use the formal parameter system

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_m)$$

of $X^\circ \times \mathbf{P}$ at $P = (p, [f]) \in \mathcal{E}^{\mathrm{sm}}$ given in Construction 4.1. Since $\Lambda \subset \mathbf{P}$ is a general plane passing through the origin $[f]$, we can take

$$u := y_n \,|\, \Lambda \quad \text{and} \quad v := y_m \,|\, \Lambda$$

as affine coordinates of Λ with the origin $[f]$. The linear embedding $\Lambda \hookrightarrow \mathbf{P}$ is given by

$$y_n = u, \qquad y_m = v, \qquad y_i = \alpha_i u + \beta_i v \quad (i \neq n, m), \tag{5.1}$$

where α_i and β_i $(i \neq n, m)$ are general elements of k. For a formal power series $F = F(x, y)$ of (x, y), we denote by F_Λ the formal power series of

$$(x, u, v) = (x_1, \dots, x_n, u, v)$$

obtained by making the substitutions (5.1) in F. In other words, we put

$$F_\Lambda(x, u, v) := F \,|\, (X^\circ \times \Lambda).$$

For simplicity, we put

$$\Phi_i := \frac{\partial \Phi}{\partial x_i}.$$

Then C_Λ is defined in $X^\circ \times \Lambda$ by the equations

$$\Phi_\Lambda = \Phi_{1,\Lambda} = \cdots = \Phi_{n,\Lambda} = 0$$

locally at P. The linear parts $\Phi_\Lambda^{[1]}, \Phi_{1,\Lambda}^{[1]}, \dots, \Phi_{n,\Lambda}^{[1]}$ of these formal power series are given as follows:

$$\begin{array}{rcll} \Phi_\Lambda^{[1]} & = & v, & \\ \Phi_{i,\Lambda}^{[1]} & = & 2x_i + \alpha_i u + \beta_i v & \qquad (i < n), \\ \Phi_{n,\Lambda}^{[1]} & = & u. & \end{array}$$

Therefore C_Λ is smooth of dimension 1 at P, and the variable

$$t := x_n \,|\, C_\Lambda$$

is a formal parameter of C_Λ at P. Hence we can write the functions $u \,|\, C_\Lambda$, $v \,|\, C_\Lambda$ and $x_i \,|\, C_\Lambda$ $(i < n)$ on C_Λ as formal power series of t with no constant terms:

$$\begin{array}{rclcl} u \,|\, C_\Lambda & = & U(t) & = & \sum_{\nu=1}^{\infty} U_\nu \, t^\nu, \\ v \,|\, C_\Lambda & = & V(t) & = & \sum_{\nu=1}^{\infty} V_\nu \, t^\nu, \\ x_i \,|\, C_\Lambda & = & X_i(t) & = & \sum_{\nu=1}^{\infty} X_{i,\nu} \, t^\nu \quad (i < n). \end{array}$$

In order to prove the assertions (1) and (2), it is enough to calculate the coefficients U_ν and V_ν up to $\nu = 3$ and up to $\nu = 4$, respectively.

The coefficients are calculated by the following algorithm. Let (S) be a set of substitutions of the form

$$(S) \left\{ \begin{array}{rcl} u & = & P_u(t), \\ v & = & P_v(t), \\ x_i & = & P_{x_i}(t) \qquad (i < n), \end{array} \right.$$

where P_u, P_v and P_{x_i} are polynomials in t with coefficients in k and without constant terms. For a formal power series F of (x, y), we denote by $s(F, S)$ the formal power series of t obtained from $F_\Lambda = F_\Lambda(x, u, v)$ by making the substitutions (S) and $x_n = t$:

$$s(F, S) := F_\Lambda(P_{x_1}(t), \ldots, P_{x_{n-1}}(t), t, P_u(t), P_v(t)).$$

We also denote by $c(F, S, l)$ the coefficient of t^l in $s(F, S)$.

The $(l+1)$-st step of the algorithm. Suppose that we have calculated the coefficients U_ν, V_ν and $X_{i,\nu}$ for $\nu \le l$ in such a way that, by making the substitutions

$$(S_l) \left\{ \begin{array}{rcl} u & = & P_u^{[l]}(t) = \sum_{\nu=1}^{l} U_\nu \, t^\nu, \\ v & = & P_v^{[l]}(t) = \sum_{\nu=1}^{l} V_\nu \, t^\nu, \\ x_i & = & P_{x_i}^{[l]}(t) = \sum_{\nu=1}^{l} X_{i,\nu} \, t^\nu \qquad (i < n) \end{array} \right.$$

and $x_n = t$ to the formal power series $\Phi_\Lambda, \Phi_{1,\Lambda}, \ldots, \Phi_{n,\Lambda}$ defining C_Λ in $X^\circ \times \Lambda$, we obtain

$$c(\Phi, S_l, \lambda) = c(\Phi_1, S_l, \lambda) = \cdots = c(\Phi_n, S_l, \lambda) = 0$$

for $\lambda \le l$. We then put

$$(S_{l+1}) \left\{ \begin{array}{rcl} u & = & P_u^{[l]}(t) + U_{l+1} \, t^{l+1}, \\ v & = & P_v^{[l]}(t) + V_{l+1} \, t^{l+1}, \\ x_i & = & P_{x_i}^{[l]} + X_{i,l+1} \, t^{l+1} \qquad (i < n), \end{array} \right.$$

and solve the equations

$$c(\Phi, S_{l+1}, l+1) = c(\Phi_1, S_{l+1}, l+1) = \cdots = c(\Phi_n, S_{l+1}, l+1) = 0$$

with indeterminates being the new coefficients U_{l+1}, V_{l+1} and $X_{i,l+1}$ $(i < n)$.

A monomial M of $x = (x_1, \ldots, x_n)$ is said to be of degree $[\lambda, \mu]$ if M is of degree λ in $(x_1, \ldots, x_{n-1})$ and of degree μ in x_n. For a formal power series F of x, we denote by $F^{[\lambda,\mu]}$ the homogeneous part of degree $[\lambda, \mu]$. Let M be a monomial of (x, y), or of (x, u, v). We say that M is of degree $[\lambda, \mu, \nu]$ if M is of degree $[\lambda, \mu]$ in

x, and is of degree ν in $y = (y_1, \dots, y_m)$ or in (u, v), respectively. Let F be a formal power series of (x, y), or of (x, u, v). We denote by $F^{[\lambda,\mu,\nu]}$ the homogeneous part of F with degree $[\lambda, \mu, \nu]$. Since the embedding $\Lambda \hookrightarrow \mathbf{P}$ is linear, we obviously have

$$(F^{[\lambda,\mu,\nu]})_\Lambda = (F_\Lambda)^{[\lambda,\mu,\nu]}$$

for a formal power series F of (x, y). If the substitutions

$$(S) \begin{cases} u &= P_u(t), \\ v &= P_v(t), \\ x_i &= P_{x_i}(t) \qquad (i < n) \end{cases}$$

satisfy

$$\operatorname{ord}_{t=0} P_u(t) \geq A, \quad \operatorname{ord}_{t=0} P_v(t) \geq A, \quad \text{and} \quad \operatorname{ord}_{t=0} P_{x_i}(t) \geq B \quad (i < n),$$

then we have

$$c(F, S, l) = \sum_{B\lambda+\mu+A\nu \leq l} c(F^{[\lambda,\mu,\nu]}, S, l).$$

Recall that

$$\Phi = \phi_0 + y_1 x_1 + \cdots + y_n x_n + y_{n+1}\phi_{n+1} + \cdots + y_{m-1}\phi_{m-1} + y_m,$$

where $\phi_0, \phi_{n+1}, \dots, \phi_{m-1}$ are formal power series of $x = (x_1, \dots, x_n)$ such that

$$\begin{aligned} &\phi_0^{[0,0]} = \phi_{n+1}^{[0,0]} = \cdots = \phi_{m-1}^{[0,0]} = 0, \\ &\phi_0^{[0,1]} = \phi_{n+1}^{[0,1]} = \cdots = \phi_{m-1}^{[0,1]} = \phi_0^{[1,0]} = \phi_{n+1}^{[1,0]} = \cdots = \phi_{m-1}^{[1,0]} = 0, \\ &\phi_0^{[2,0]} = x_1^2 + \cdots + x_{n-1}^2, \quad \phi_0^{[1,1]} = \phi_0^{[0,2]} = 0. \end{aligned}$$

Recall also that $a_1, \dots, a_n, b_{n+1}, \dots, b_{m-1}$ are defined in Proposition 4.3 by

$$\phi_0^{[1,2]} = (a_1 x_1 + \cdots + a_{n-1} x_{n-1}) x_n^2, \quad \phi_0^{[0,3]} = a_n x_n^3,$$

and

$$\phi_j^{[0,2]} = b_j x_n^2 \quad (j = n+1, \dots, m-1).$$

By the assumption $P \in \mathcal{E}^{A_2}$, we have

$$a_n \neq 0.$$

We define $e_1, \dots, e_n$ and $f_1, \dots, f_n$ by

$$\begin{aligned} \phi_0^{[1,3]} &= (e_1 x_1 + \cdots + e_{n-1} x_{n-1})\, x_n^3, \quad \phi_0^{[0,4]} = e_n x_n^4, \\ \phi_0^{[1,4]} &= (f_1 x_1 + \cdots + f_{n-1} x_{n-1})\, x_n^4, \quad \phi_0^{[0,5]} = f_n x_n^5. \end{aligned}$$

We also define homogeneous polynomials $A_i(x_1, \dots, x_{n-1})$ $(i < n)$ of degree 1 and $B(x_1, \dots, x_{n-1})$ of degree 2 by

$$A_i := \frac{1}{x_n} \frac{\partial \phi_0^{[2,1]}}{\partial x_i}, \qquad B := \frac{\partial \phi_0^{[2,1]}}{\partial x_n}.$$

Then we obtain Table 5.1.

Step 1. We put

$$(S_1) \begin{cases} u &= U_1\, t, \\ v &= V_1\, t, \\ x_i &= X_{i,1}\, t \qquad (i < n). \end{cases}$$

$[\lambda,\mu,\nu]$	$F=\Phi$	$F=\Phi_i \ \ (i<n)$	$F=\Phi_n$
$[0,0,0]$	0	0	0
$[0,0,1]$	y_m	y_i	y_n
$[0,1,0]$	0	0	0
$[0,1,1]$	$y_n x_n$	——	$2\left(\sum_{j=n+1}^{m-1} b_j y_j\right) x_n$
$[0,2,0]$	0	$a_i x_n^2$	$3\,a_n x_n^2$
$[0,3,0]$	$a_n x_n^3$	$e_i x_n^3$	$4\,e_n x_n^3$
$[0,4,0]$	$e_n x_n^4$	$f_i x_n^4$	$5\,f_n x_n^4$
$[1,0,0]$	0	$2x_i$	0
$[1,1,0]$	0	$A_i(x_0,\dots,x_{n-1})\,x_n$	$2\left(\sum_{i=1}^{n-1} a_i x_i\right) x_n$
$[1,2,0]$	$\left(\sum_{i=1}^{n-1} a_i x_i\right) x_n^2$	——	0 if $\operatorname{char} k=3$
$[2,0,0]$	$\sum_{i=1}^{n-1} x_i^2$	——	$B(x_0,\dots,x_{n-1})$
if $\nu>1$	0	0	0

TABLE 5.1. $F^{[\lambda,\mu,\nu]}$ for $F=\Phi$, Φ_i $(i<n)$ and Φ_n

Then we have

$$c(F,S_1,1)=\sum_{\lambda+\mu+\nu\le 1} c(F^{[\lambda,\mu,\nu]},S_1,1)$$

for any formal power series F of (x,y). Therefore we obtain equations

$$V_1=0, \qquad 2\,X_{i,1}+\alpha_i U_1+\beta_i V_1=0 \quad (i<n), \qquad U_1=0.$$

Hence we get

$$U_1=V_1=X_{i,1}=0 \quad (i<n).$$

Step 2. We put

$$(S_2)\begin{cases} u &=& U_2\,t^2, \\ v &=& V_2\,t^2, \\ x_i &=& X_{i,2}\,t^2 \qquad (i<n). \end{cases}$$

Then we have

$$c(F,S_2,2)=\sum_{2\lambda+\mu+2\nu\le 2} c(F^{[\lambda,\mu,\nu]},S_2,2).$$

Therefore we obtain equations

$$\begin{aligned} &V_2=0,\\ &\alpha_i U_2+\beta_i V_2+a_i+2\,X_{i,2}=0 \quad (i<n),\\ &U_2+3\,a_n=0. \end{aligned}$$

Hence we get

$$U_2=-3\,a_n, \qquad V_2=0, \qquad X_{i,2}=(3a_n\alpha_i-a_i)/2 \quad (i<n).$$

Step 3. We put

$$(S_3)\begin{cases} u &= U_2\,t^2 + U_3\,t^3, \\ v &= V_3\,t^3, \\ x_i &= X_{i,2}\,t^2 \,+\, X_{i,3}\,t^3 \qquad (i < n). \end{cases}$$

Then we have

$$c(F, S_3, 3) = \sum_{2\lambda+\mu+2\nu\le 3} c(F^{[\lambda,\mu,\nu]}, S_3, 3).$$

Putting $F = \Phi$ in this formula, we obtain an equation

$$V_3 + U_2 + a_n = 0.$$

Hence we get

$$V_3 = 2a_n.$$

Therefore we have

$$\begin{array}{rcrcl} u\,|\,C_\Lambda &=& -3\,a_n t^2 &+& (\text{terms of degree} \ge 3), \\ v\,|\,C_\Lambda &=& 2\,a_n t^3 &+& (\text{terms of degree} \ge 4). \end{array}$$

Thus the assertion (1) in $\operatorname{char} k \neq 3$ is proved.

From now on, we assume $\operatorname{char} k = 3$. Then we have

$$U_2 = 3\,a_n = 0, \qquad X_{i,2} = a_i \;\; (i < n),$$

and the substitutions (S_3) become as follows:

$$(S_3)\begin{cases} u &= U_3\,t^3, \\ v &= V_3\,t^3, \\ x_i &= X_{i,2}\,t^2 \,+\, X_{i,3}\,t^3 \qquad (i < n). \end{cases}$$

Therefore we have

$$c(F, S_3, 3) = \sum_{2\lambda+\mu+3\nu\le 3} c(F^{[\lambda,\mu,\nu]}, S_3, 3).$$

Hence we get equations

$$V_3 + a_n = 0,$$

$$\alpha_i U_3 + \beta_i V_3 + e_i + 2\,X_{i,3} + A_i(X_{1,2}, \dots, X_{n-1,2}) = 0 \quad (i < n),$$

$$U_3 + e_n + 2\,\Big(\sum_{i=1}^{n-1} a_i X_{i,2}\Big) = 0.$$

Thus we obtain

$$U_3 = 2\,e_n + \sum_{i=1}^{n-1} a_i^2, \qquad V_3 = 2\,a_n,$$

and

$$X_{i,3} = \alpha_i U_3 + \beta_i V_3 + \Xi_i = \alpha_i(2\,e_n + \sum_{i=1}^{n-1} a_i^2) + 2\,\beta_i a_n + \Xi_i \qquad (i < n),$$

where $\Xi_1, \dots, \Xi_{n-1}$ do not depend on the parameters α_j nor β_j $(j \neq n, m)$.

Step 4. We put

$$(S_4)\begin{cases} u &= U_3\,t^3 + U_4\,t^4, \\ v &= V_3\,t^3 + V_4\,t^4, \\ x_i &= X_{i,2}\,t^2 + X_{i,3}\,t^3 + X_{i,4}\,t^4 \qquad (i < n). \end{cases}$$

We have

$$c(F, S_4, 4) = \sum_{2\lambda+\mu+3\nu\le 4} c(F^{[\lambda,\mu,\nu]}, S_4, 4).$$

Putting $F = \Phi$ and $F = \Phi_n$ into this formula, we obtain equations

$$V_4 + U_3 + e_n + \sum_{i=1}^{n-1} a_i X_{i,2} + \sum_{i=1}^{n-1} X_{i,2}^2 = 0, \quad \text{and}$$

$$U_4 + 2\sum_{j=n+1}^{m-1} b_j(\alpha_j U_3 + \beta_j V_3) + 2f_n + 2\sum_{i=1}^{n-1} a_i X_{i,3} + B(X_{1,2}, \dots, X_{n-1,2}) = 0.$$

From the first equation, we obtain

$$V_4 = -U_3 - e_n - 2\sum_{i=1}^{n-1} a_i^2 = 0.$$

Since $V_3 = 2a_n \neq 0$, the critical point P of π_Λ is of E_6-type if and only if $U_4 \neq 0$. From the second equation, we obtain

$$U_4 = U_3\Big(\sum_{i=1}^{n-1} a_i\alpha_i + \sum_{j=n+1}^{m-1} b_j\alpha_j\Big) + V_3\Big(\sum_{i=1}^{n-1} a_i\beta_i + \sum_{j=n+1}^{m-1} b_j\beta_j\Big) + \Upsilon,$$

where Υ does not depend on the parameters α_j nor β_j $(j \neq n, m)$. From Corollary 4.7 and the assumption $P \in \mathcal{E}^{\mathrm{sm}}$, at least one of $a_1, \dots, a_{n-1}$, $b_{n+1}, \dots, b_{m-1}$ is not zero. Since $V_3 = 2a_n \neq 0$, by choosing $\beta_1, \dots, \beta_{n-1}, \beta_{n+1}, \dots, \beta_{m-1}$ general enough, we have $U_4 \neq 0$. □

6. The dual curve of a plane curve in characteristic 3

Throughout this section, we suppose that $\operatorname{char} k = 3$ and $(n, m) = (1, 2)$.

Recall that, in the case $(n, m) = (1, 2)$, the projection $\pi_1 : \mathcal{C}^\circ \to X^\circ$ is an isomorphism, and $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is identified with the Gauss map (Remark 3.11).

Theorem 6.1. (1) *The critical subscheme $\mathcal{E}$ of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is of dimension 0 if and only if π_2 is separable onto its image.*

(2) *Suppose that π_2 is separable onto its image. Then, at every point P of $\mathcal{E}$, the length of $\mathcal{O}_{\mathcal{E},P}$ is a multiple of 3. Let $P = (p, [f])$ be a point of $\mathcal{E}^{A_2}$. Then π_2 has a critical point of T_l-type at P, where $l := \operatorname{length} \mathcal{O}_{\mathcal{E},P}/3$.*

Proof. If π_2 is inseparable onto its image, then the generic point of $\mathcal{C}^\circ$ is contained in $\mathcal{E}$, and hence $\dim \mathcal{E} = \dim \mathcal{C}^\circ = 1$. Conversely, suppose that π_2 is separable onto its image. Let $P = (p, [f])$ be a point of $\mathcal{E}$. We use the formal parameters (x_1, y_1, y_2) of $X^\circ \times \mathbf{P}$ given in Construction 4.1. We put

$$\phi_0 = c_3\,x_1^3 + c_4\,x_1^4 + \cdots = \sum_{\nu=1}^{\infty} c_{3\nu}\,x_1^{3\nu} + \sum_{\nu=1}^{\infty} c_{3\nu+1}\,x_1^{3\nu+1} + \sum_{\nu=1}^{\infty} c_{3\nu+2}\,x_1^{3\nu+2}.$$

Then $\mathcal{C}^\circ$ is defined locally at P by the equations

$$\phi_0 + y_1 x_1 + y_2 = 0 \quad \text{and} \quad \phi_0' + y_1 = 0.$$

Therefore

$$t := x_1 \,|\, \mathcal{C}^\circ$$

is a formal parameter of $\mathcal{C}^\circ$ at P, and $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ is given by

$$\begin{array}{lrcl} \text{(6.1)} & \pi_2^* y_1 &=& -\phi_0' \,|\, \mathcal{C}^\circ \quad = \; -\sum c_{3\nu+1}\, t^{3\nu} \;+\; \sum c_{3\nu+2}\, t^{3\nu+1}, \\ & \pi_2^* y_2 &=& (\phi_0' x_1 - \phi_0) \,|\, \mathcal{C}^\circ \quad = \; -\sum c_{3\nu}\, t^{3\nu} \;+\; \sum c_{3\nu+2}\, t^{3\nu+2}. \end{array}$$

Since π_2 is separable, there exists a positive integer ν such that $c_{3\nu+2} \neq 0$. By Corollary 3.17, the scheme $\mathcal{E}$ is defined on $\mathcal{C}^\circ$ by

$$\left.\frac{\partial^2 \Phi}{\partial x_1^2}\right|_{\mathcal{C}^\circ} = \phi_0'' \,|\, \mathcal{C}^\circ = -\sum_{\nu=1}^{\infty} c_{3\nu+2}\, t^{3\nu} = 0.$$

Therefore $\dim_P \mathcal{E}$ is 0, and the length of $\mathcal{O}_{\mathcal{E},P}$ is equal to $3l$, where

$$l := \min\{\, \nu \;\mid\; c_{3\nu+2} \neq 0 \,\}.$$

If $P \in \mathcal{E}^{A_2}$, then $c_3 \neq 0$. Therefore, from (6.1), we see that π_2 has a critical point of T_l-type at P. □

In the rest of this section, we will investigate normal forms of a critical point of T_l-type. Let $\varphi : C \to S$ be a morphism given in §2.1.

Proposition 6.2. *Suppose that φ has a critical point of T_l-type at $P \in C$. Then there exist a formal parameter t of $(\mathcal{O}_{C,P})^\wedge$ and a formal parameter system (u, v) of $(\mathcal{O}_{S,\varphi(P)})^\wedge$ such that φ is given by*

$$\varphi^* u = t^{3l+1} \quad \textit{and} \quad \varphi^* v = t^3 + t^{3l+2}.$$

Proof. Let t and (u, v) be arbitrary formal parameters of $(\mathcal{O}_{C,P})^\wedge$ and $(\mathcal{O}_{S,\varphi(P)})^\wedge$, respectively. For $F \in (\mathcal{O}_{S,\varphi(P)})^\wedge$, we denote by $F_{[t,\nu]}$ the coefficient of t^ν in $\varphi^* F \in (\mathcal{O}_{C,P})^\wedge = k[[t]]$. For $A, B \in (\mathcal{O}_{C,P})^\wedge$, we write $A = B + [\geq N]$ if $A - B$ is contained in the Nth power of the maximal ideal of $(\mathcal{O}_{C,P})^\wedge$. By the definition of the critical point of T_l-type, we have

$$\begin{aligned} \varphi^* u &= u_{[t,3]} t^3 + u_{[t,6]} t^6 + \cdots + u_{[t,3l]} t^{3l} + u_{[t,3l+1]} t^{3l+1} + u_{[t,3l+2]} t^{3l+2} + [\geq 3l+3], \\ \varphi^* v &= v_{[t,3]} t^3 + v_{[t,6]} t^6 + \cdots + v_{[t,3l]} t^{3l} + v_{[t,3l+1]} t^{3l+1} + v_{[t,3l+2]} t^{3l+2} + [\geq 3l+3], \end{aligned}$$

and the coefficients $u_{[t,\nu]}$ and $v_{[t,\nu]}$ satisfy (2.2). Since $(u_{[t,3]}, v_{[t,3]}) \neq (0,0)$, we can assume that

$$\text{(6.2)} \qquad u_{[t,3]} = 0 \quad \text{and} \quad v_{[t,3]} = 1$$

by a linear transformation of (u, v). If $r \geq 2$, then we have

$$(v^r)_{[t,\nu]} \neq 0 \;\text{ and }\; \nu \not\equiv 0 \mod 3 \quad \Longrightarrow \quad \nu \geq 3l+4.$$

Therefore, replacing u with

$$u - c_2 v^2 - \cdots - c_l v^l$$

with appropriate coefficients $c_2, \ldots, c_l$, we can assume that

$$\varphi^* u = u_{[t,3l+1]} t^{3l+1} + u_{[t,3l+2]} t^{3l+2} + [\geq 3l+3].$$

By (6.2) and the condition (2.2), we have $u_{[t,3l+1]} \neq 0$. Therefore there exists a formal parameter s of $(\mathcal{O}_{C,P})^\wedge$ such that

$$\varphi^* u = s^{3l+1}.$$

By $u_{[s,3]} = 0$ and the condition (2.2), we can assume

$$v_{[s,3]} = 1 \quad \text{and} \quad v_{[s,3l+1]} = 0$$

by a linear transformation of (u, v). If $r \geq 2$, then we have

$$(v^r)_{[s,\nu]} \neq 0 \ \text{ and } \ \nu \not\equiv 0 \mod 3 \quad \Longrightarrow \quad \nu \geq 3l + 5.$$

Therefore, replacing v with

$$v - d_2 v^2 - \cdots - d_l v^l$$

with appropriate coefficients $d_2, \ldots, d_l$, we can assume that

$$\varphi^* v = s^3 + v_{[s,3l+2]} s^{3l+2} + [\geq 3l + 3].$$

By the condition (2.2) again, we have $v_{[s,3l+2]} \neq 0$. Replacing (u, v, s) with $(\alpha u, \beta v, \gamma s)$ with appropriate $\alpha, \beta, \gamma \in k^\times$, and denoting s by t, we obtain

$$\begin{array}{rcll} \varphi^* u &=& t^{3l+1}, & \text{and} \\ \varphi^* v &=& t^3 + t^{3l+2} + [\geq 3l + 3]. & \end{array}$$

We put

$$T := \{ \ 3a + (3l + 1)b \ \mid \ a, b \in \mathbb{Z}_{\geq 0} \ \},$$

and fix functions

$$m_1 : T \to \mathbb{Z}_{\geq 0} \quad \text{and} \quad m_2 : T \to \mathbb{Z}_{\geq 0}$$

such that

$$3m_1(\nu) + (3l + 1)m_2(\nu) = \nu$$

holds for every $\nu \in T$. It is easy to see that a non-negative integer ν is in T if and only if

$$\begin{array}{rl} & (\nu \leq 3l \ \text{ and } \ \nu \equiv 0 \mod 3) \\ \text{or} & (3l < \nu \leq 6l + 1 \ \text{ and } \ \nu \not\equiv 2 \mod 3) \\ \text{or} & (6l + 1 < \nu) \end{array}$$

holds. Therefore, replacing v with

$$v - \sum_{\nu \geq 3l+3, \ \nu \in T} e_\nu u^{m_2(\nu)} v^{m_1(\nu)}$$

with coefficients e_ν chosen appropriately, we obtain

$$\begin{array}{rcll} \varphi^* u &=& t^{3l+1}, & \text{and} \\ \varphi^* v &=& t^3 + t^{3l+2} + \displaystyle\sum_{\mu=1}^{l-1} A_\mu t^{3l+3\mu+2} & \end{array}$$

with $A_1, \ldots, A_{l-1} \in k$. If the coefficients A_μ are all zero, then the proof is finished. Assume that $A_\mu \neq 0$ for some $\mu < l$, and put

$$m := \min\{ \ \mu \ \mid \ A_\mu \neq 0 \ \}.$$

We put

$$u' := u - A_m u v^m.$$

Then we have

$$\varphi^* u' = t^{3l+1} - A_m t^{3l+3m+1} + [\geq 6l + 3m].$$

There exists a formal parameter s of $(\mathcal{O}_{C,P})^\wedge$ such that

$$\varphi^* u' = s^{3l+1}.$$

Then we have

$$s = t - A_m t^{3m+1} + [\geq 3m + 2],$$

and therefore

$$t = s + A_m s^{3m+1} + [\geq 3m + 2].$$

Let R_r $(r \geq 3m+1)$ be the coefficients in

$$t^3 = s^3 + \sum_{r \geq 3m+1} R_r s^{3r}.$$

Because $3l + 2 \equiv -1 \mod 3$, we have

$$t^{3l+2} + A_m t^{3l+3m+2} + [\geq 3l + 3m + 3] = s^{3l+2} + [\geq 3l + 3m + 3].$$

Therefore we obtain

$$\varphi^* v = s^3 + \sum_{r=3m+1}^{l+m} R_r s^{3r} + s^{3l+2} + [\geq 3l + 3m + 3].$$

If $r \geq 3m+1$, then we have

$$(v^r)_{[s,\nu]} \neq 0 \;\text{ and }\; \nu \not\equiv 0 \mod 3 \quad \Longrightarrow \quad \nu \geq 3(r-1) + 3l + 2 \geq 3l + 3m + 3.$$

Therefore, replacing v with

$$v - \sum_{r=3m+1}^{l+m} R'_r v^r$$

with appropriate coefficients R'_ν, we can assume that

$$\varphi^* v = s^3 + s^{3l+2} + [\geq 3l + 3m + 3].$$

Replacing v with

$$v - \sum_{\nu \geq 3l+3m+3,\ \nu \in T} f_\nu u^{m_2(\nu)} v^{m_1(\nu)}$$

with appropriate coefficients f_ν and denoting u' by u and s by t, we get

$$\begin{aligned} \varphi^* u &= t^{3l+1} \qquad \text{and} \\ \varphi^* v &= t^3 + t^{3l+2} + \sum_{\mu=m+1}^{l-1} A'_\mu t^{3l+3\mu+2} \end{aligned}$$

with new coefficients $A'_{m+1}, \ldots, A'_{l-1}$. Thus we have

$$\min\{\, \mu \mid A'_\mu \neq 0 \,\} \;>\; m = \min\{\, \mu \mid A_\mu \neq 0 \,\}.$$

Therefore, after repeating this process finitely often, we obtain formal power series with the desired properties. □

Proposition 6.3. *Suppose that φ has a critical point of T_l-type at $P \in C$. Then the image of the germ (C, P) by φ is formally isomorphic to the germ of a plane curve singularity defined by*

$$x^{3l+1} + y^3 + x^{2l} y^2 = 0. \qquad (6.3)$$

Proof. Let $C_l \subset \mathbb{A}^2$ be the affine curve defined by the equation (6.3), and let

$$\nu : \widetilde{C_l} \to C_l$$

be the normalization in a neighborhood of $O := (0,0)$. Let $P \in \widetilde{C_l}$ be a point such that $\nu(P) = O$. It is enough to show that $\nu^{-1}(O)$ consists of a single point P (that is, C_l is locally irreducible at O), and that the composite of ν and the inclusion $C_l \hookrightarrow \mathbb{A}^2$ has a critical point of T_l-type at P.

We denote by $D_{m,n}$ the affine curve defined by

$$x^{m+1} + y^3 + x^n y^2 = 0.$$

We have $C_l = D_{3l,2l}$. Let $\beta : (\mathbb{A}^2)^\sim \to \mathbb{A}^2$ be the blowing-up at O. The proper transform of $D_{m,n}$ $(m \geq 3, n \geq 2)$ by β is isomorphic to $D_{m-3,n-1}$, and the proper birational morphism

$$\psi_{m,n} := \beta \,|\, D_{m-3,n-1}$$

is given by $(x, y) \mapsto (x, xy)$. We also have

$$\psi_{m,n}^{-1}(O) = \{O\}.$$

Since

$$D_{0,l} \quad : \quad x + y^3 + x^l y^2 = 0$$

is smooth at O, the curve $D_{3l,2l} = C_l$ is locally irreducible at O, and the composite

$$\nu \quad : \quad D_{0,l} \xrightarrow{\psi_{3,l+1}} D_{3,l+1} \xrightarrow{\psi_{6,l+2}} \cdots \xrightarrow{\psi_{3l,2l}} D_{3l,2l} = C_l$$

is the normalization of C_l in a neighborhood of O. We put

$$t := y \,|\, D_{0,l},$$

which is a formal parameter of $D_{0,l}$ at O. Then

$$x \,|\, D_{0,l} = -t^3 - (-1)^l\, t^{3l+2} + (\text{terms of degree} \geq 3l+3).$$

Since

$$\begin{array}{lcll} \nu^*(x \,|\, C_l) = x \,|\, D_{0,l} & = & -t^3 - (-1)^l\, t^{3l+2} + (\text{terms of degree} \geq 3l+3) & \text{and} \\ \nu^*(y \,|\, C_l) = (x^l y) \,|\, D_{0,l} & = & (-1)^l\, t^{3l+1} + (\text{terms of degree} \geq 3l+3), & \end{array}$$

we see that the composite of $\nu : D_{0,l} \to C_l$ and the inclusion $C_l \hookrightarrow \mathbb{A}^2$ has a critical point of T_l-type at $O \in D_{0,l}$. $\square$

7. The degree of $\mathcal{E}$

For a smooth projective variety V, we denote by $A_k(V) = A^{\dim V - k}(V)$ the abelian group of rational equivalence classes of k-cycles of V, and by $A_*(V)$ the Chow group of V. For a closed subscheme W of V, let $[W] \in A_*(V)$ be the class of W. We denote by

$$\int_V \, : \, A_0(V) \to \mathbb{Z}$$

the degree map $\sum_P n_P[P] \mapsto \sum_P n_P$.

In this section, we assume the following:

$$\overline{X} = X = X^\circ; \tag{7.1}$$

that is, $\overline{X}$ is smooth, the linear system $|M|$ on $\overline{X}$ has no base points, and the morphism $\Psi : \overline{X} \to \mathbf{P}^\vee$ induced by $|M|$ is a closed immersion formally at every point of $\overline{X}$. We have $\mathcal{C} = \mathcal{C}^\circ$. For simplicity, we denote by X for $\overline{X}$ or X° and by $\mathcal{C}$ for $\mathcal{C}^\circ$. We also assume that

$$\mathcal{E} \text{ is of codimension 1 in } \mathcal{C}. \tag{7.2}$$

Then $\mathcal{C}$ and $\mathcal{E}$ are closed subschemes of dimensions $m-1$ and $m-2$, respectively, in the smooth projective variety $X \times \mathbf{P}$. The purpose of this section is to calculate

$$\deg \mathcal{C} := \int_{X\times\mathbf{P}} c_1(\mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}}(1))^{m-1} \cap [\mathcal{C}] \quad \text{and} \quad \deg \mathcal{E} := \int_{X\times\mathbf{P}} c_1(\mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}}(1))^{m-2} \cap [\mathcal{E}].$$

For $\alpha \in A^a(X)$ and $\beta \in A^b(\mathbf{P})$, we denote by the same letters $\alpha \in A^a(X \times \mathbf{P})$ and $\beta \in A^b(X \times \mathbf{P})$ the pull-backs of α and β by the projections. We put

$$h := c_1(\mathcal{O}_{\mathbf{P}}(1)) \qquad \text{and} \qquad \lambda := c_1(\mathcal{L}).$$

It is easy to see that, if $\alpha \in A^a(X)$ and $\beta \in A^b(\mathbf{P})$, then

$$\int_{X\times\mathbf{P}} h^{(n+m)-(a+b)} \cap \alpha\beta = \begin{cases} 0 & \text{if } a < n, \\ \left(\int_X \alpha\right) \cdot \left(\int_{\mathbf{P}} h^{m-b} \cap \beta\right) & \text{if } a = n. \end{cases}$$

By the definition of the divisor $\mathcal{D}$ of $X \times \mathbf{P}$, we have

$$\mathcal{O}_{X\times\mathbf{P}}(\mathcal{D}) = \widetilde{\mathcal{L}} = \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}}(1).$$

Therefore

$$[\mathcal{D}] = (\lambda + h) \cap [X \times \mathbf{P}] \qquad \text{in } A_*(X \times \mathbf{P}).$$

By Proposition 3.4, the subscheme $\mathcal{C}$ of $\mathcal{D}$ is defined as the degeneracy subscheme of the homomorphism

$$(d\sigma_X)^\vee : (\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{D}})^\vee \to (p_1^* T(X))^\vee.$$

Using Thom-Porteous formula [3, Chapter 14], we have

$$[\mathcal{C}] = \Delta_n^{(1)}(c(T(X)^\vee - \widetilde{\mathcal{L}}^\vee)) \cap [\mathcal{D}] = \left((\lambda + h) \sum_{i=0}^{n} (-1)^i c_i(X)(\lambda+h)^{n-i} \right) \cap [X \times \mathbf{P}]$$

in $A_*(X \times \mathbf{P})$. In particular, we obtain the following well-known formula ([14, 15]):

$$\deg \mathcal{C} = \sum_{i=0}^{n} \left\{ (-1)^i (n-i+1) \int_X c_i(X) \lambda^{n-i} \cap [X] \right\}.$$

By Proposition 3.14, the divisor $\mathcal{E}$ of $\mathcal{C}$ is defined as the degeneracy subscheme of the symmetric homomorphism

$$\mathcal{H}^{\sim} : \pi_1^* T(X) \to \widetilde{\mathcal{L}} \otimes \pi_1^* T(X)^\vee$$

By Harris-Tu-Pragacz formula ([5, Theorem 10], [20, Theorem 4.1], see also [10]), we have

$$[\mathcal{E}] = 2\, c_1 \left(\pi_1^* T(X)^\vee \otimes \sqrt{\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{C}}} \right) \cap [\mathcal{C}] \in A^1(\mathcal{C}).$$

Hence we obtain the following. (Compare with [2, Formula (2.2)].)

Proposition 7.1. *In $A_*(X \times \mathbf{P})$, we have*

$$\begin{aligned} [\mathcal{E}] &= (-2c_1(X) + n\lambda + nh) \cap [\mathcal{C}] \\ &= \left((-2c_1(X) + n\lambda + nh)(\lambda + h) \sum_{i=0}^{n} (-1)^i c_i(X)(\lambda+h)^{n-i} \right) \cap [X \times \mathbf{P}]. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}\deg \mathcal{E} &= n\sum_{j=0}^{n}\frac{(-1)^{n-j}(j+1)(j+2)}{2}\int_X c_{n-j}(X)\lambda^j \cap [X] - \\ &\qquad \sum_{j=1}^{n}(-1)^{n-j}j(j+1)\int_X c_{n-j}(X)c_1(X)\lambda^{j-1} \cap [X].\end{aligned}$$

Example 7.2. Suppose that $\operatorname{char} k = 3$. Let X be a smooth projective curve of genus g, and let $|M|$ be a 2-dimensional linear system on X without base points such that the induced morphism $\Psi : X \to \mathbf{P}^\vee = \mathbb{P}^2$ is a closed immersion formally at every point of X. Let

$$\gamma : X \to (\mathbb{P}^2)^\vee = \mathbf{P}$$

be the Gauss map of Ψ. For a point $p \in X$, let μ_p denote the multiplicity at p of the divisor $\Psi^*(\gamma(p))$. Suppose that

(i) $\mu_p \le 3$ at every point $p \in X$, and
(ii) there exists $p \in X$ such that $\mu_p = 2$.

Then $\gamma : X \to \mathbf{P}$ is separable onto its image. Hence $\mathcal{E}$ is of dimension 0, and every critical point of γ is of T_l-type by Theorem 6.1. Let t_l be the number of the critical points of T_l-type. Then we have

$$\sum l t_l = \frac{\operatorname{length}\mathcal{O}_{\mathcal{E}}}{3} = \frac{\deg \mathcal{E}}{3} = \int_X (\lambda - c_1(X)) \cap [X] = \deg \Psi^*\mathcal{O}_{\mathbb{P}^2}(1) - 2 + 2g.$$

Therefore the formula (1.1) is proved.

In characteristic 3, the morphism $\mathcal{E}^{\mathrm{sm}} \to \mathbf{P}$ factors through the finite morphism $\mathcal{E}^{\mathrm{sm}} \to (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}}$ of degree 3 by Theorem 4.5. If $\mathcal{E}^{\mathrm{sm}}$ is dense in $\mathcal{E}$, then $\deg\mathcal{E}$ must be divisible by 3. If we take $\mathcal{L}$ to be a cube of a very ample line bundle, then the assumptions (7.1) and (7.2) are satisfied and $\mathcal{E}^{\mathrm{sm}}$ is dense in $\mathcal{E}$ by Corollary 4.10. Therefore we obtain the following non-trivial divisibility relation among the Chern numbers of a smooth projective variety in characteristic 3:

Corollary 7.3. *Let X be a smooth projective variety of dimension n in characteristic 3. Then the integer*

$$\int_X (n\, c_n(X) + 2\, c_{n-1}(X)\, c_1(X)) \cap [X]$$

is divisible by 3.

In fact, this divisibility relation follows from the Hirzebruch-Riemann-Roch theorem by the argument of Libgober and Wood. See [18, Remark 2.4].

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