PRIMITIVITY OF SUBLATTICES GENERATED BY CLASSES OF CURVES ON AN ALGEBRAIC SURFACE

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Abstract. Let X be a smooth projective complex surface. Suppose that a finite set of reduced irreducible curves on X is given. We consider the submodule of the second cohomology group of X with integer coefficients generated by the classes of these curves. We present a method to calculate the primitive closure of this submodule, and apply it to cyclic coverings of the projective plane branching along four lines in general position.

1. Introduction

Let X be a smooth projective complex surface, and let D be an effective divisor on X with the reduced irreducible components $C_1, \dots, C_k$. We regard

$$H^2(X) := H^2(X, \mathbb{Z})/(\text{torsion})$$

as a unimodular lattice by the cup product, and consider the submodule

$$\mathcal{L}(X, D) := \langle [C_1], \dots, [C_k] \rangle \subset H^2(X)$$

generated by the classes $[C_i]$ of the curves C_i. We denote by

$$\overline{\mathcal{L}}(X, D) := (\mathcal{L}(X, D) \otimes \mathbb{Q}) \cap H^2(X) \ \subset \ H^2(X)$$

the primitive closure of $\mathcal{L}(X, D)$ in $H^2(X)$. Then

$$A(X, D) := \overline{\mathcal{L}}(X, D)/\mathcal{L}(X, D)$$

is a finite abelian group. For a submodule $M \subset H^2(X)$, we put

$$\operatorname{disc} M := |\det(S_M)|,$$

where S_M is a symmetric matrix expressing the cup product restricted to M. (If M is of rank 0, then we define $\operatorname{disc} M$ to be 1.) By definition, M is a sublattice of $H^2(X)$ if and only if $\operatorname{disc} M \neq 0$. (See Definition 1.1.) If $\mathcal{L}(X, D)$ is a sublattice, then so is $\overline{\mathcal{L}}(X, D)$ and we have

$$|A(X, D)| = \sqrt{\frac{\operatorname{disc} \mathcal{L}(X, D)}{\operatorname{disc} \overline{\mathcal{L}}(X, D)}}. \tag{1.1}$$

In this paper, we present an algorithm to calculate $\operatorname{disc} \overline{\mathcal{L}}(X, D)$ based on a simple topological observation (Theorem 1.2). Combining this algorithm with an algebro-geometric calculation of $\mathcal{L}(X, D)$, we can calculate the order of $A(X, D)$.

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Definition 1.1. A *quasi-lattice* is a finitely generated $\mathbb{Z}$-module L with a symmetric bilinear form

$$L \times L \to \mathbb{Z} \qquad (x, y) \mapsto x \cdot y.$$

For a quasi-lattice L, we put

$$\ker(L) := \{\, x \in L \mid x \cdot y = 0 \ \text{ for all } \ y \in L \,\},$$

which is the kernel of the natural homomorphism $L \to \mathrm{Hom}(L, \mathbb{Z})$ induced by the symmetric bilinear form. A quasi-lattice is called a *lattice* if $\ker(L) = 0$.

Note that $\ker(L)$ contains the torsion part of L, and that $L/\ker(L)$ has a natural structure of the lattice.

Theorem 1.2. *We put $X^\circ := X \setminus D$. Let $t_1, \dots, t_N$ be topological 2-cycles that generate $H_2(X^\circ, \mathbb{Z})$ modulo torsion. Let T_{X° be the quasi-lattice generated freely by $t_1, \dots, t_N$ with the symmetric bilinear form given by the intersection numbers $t_i \cdot t_j \in \mathbb{Z}$. If $\mathcal{L}(X, D)$ is a sublattice of $H^2(X)$, then $\operatorname{disc} \overline{\mathcal{L}}(X, D)$ is equal to $\operatorname{disc}(T_{X^\circ}/\ker(T_{X^\circ}))$.*

The primary motivation of this article is the following question due to Shioda. Let $X_m \subset \mathbb{P}^3$ be the Fermat surface

$$x_0^m + x_1^m + x_2^m + x_3^m = 0.$$

Then X_m contains $3m^2$ lines. Let $\mathfrak{L}_m$ be the union of these $3m^2$ lines. For simplicity, we assume $m \geq 5$. Aoki and Shioda [1] showed that

$$(m, 6) = 1 \iff \mathrm{NS}(X_m) \otimes \mathbb{Q} = \mathcal{L}(X_m, \mathfrak{L}_m) \otimes \mathbb{Q},$$

where $\mathrm{NS}(X_m) := H^{1,1}(X_m) \cap H^2(X_m)$ is the Néron-Severi lattice of X_m. Shioda then posed the problem whether $\mathrm{NS}(X_m) = \mathcal{L}(X_m, \mathfrak{L}_m)$ holds or not for m prime to 6. In our terminology, this problem is to determine whether $A(X_m, \mathfrak{L}_m)$ is trivial or not for m prime to 6. Recently, Schütt, Shioda and van Luijk [5] showed the following by modulo p reduction technique and computer-aided calculation:

Theorem 1.3 ([5])**.** *Let m be an integer with $5 \leq m \leq 100$. If m is prime to 6, then $\mathrm{NS}(X_m) = \mathcal{L}(X_m, \mathfrak{L}_m)$ holds.*

The Fermat surface X_m is a $(\mathbb{Z}/m\mathbb{Z})^3$-covering of

$$\mathbb{P}^2 := \{y_0 + y_1 + y_2 + y_3 = 0\} \subset \mathbb{P}^3$$

branching along the union of the four lines

$$B := B_0 + B_1 + B_2 + B_3 \subset \mathbb{P}^2, \quad \text{where} \quad B_i := \{y_i = 0\} \cap \mathbb{P}^2,$$

by the morphism

$$\phi_m : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0^m : x_1^m : x_2^m : x_3^m) \in \mathbb{P}^2.$$

For $i = 1, 2, 3$, let Λ_i be the line connecting the intersection point P_{0i} of B_0 and B_i and P_{jk} of B_j and B_k, where $\{0, i, j, k\} = \{0, 1, 2, 3\}$. Then we have

$$\mathfrak{L}_m = \phi_m^*(\Lambda), \quad \text{where} \quad \Lambda := \Lambda_1 + \Lambda_2 + \Lambda_3.$$

We generalize and extend the pair $(X_m, \mathfrak{L}_m)$ as follows. Let

$$\varphi : Y_\varphi \to \mathbb{P}^2$$

be a finite covering branching along B. Since $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}^3$, the covering φ is abelian, and there exists m such that $\phi_m : X_m \to \mathbb{P}^2$ is a composite of a quotient morphism $X_m \to Y_\varphi$ and $\varphi : Y_\varphi \to \mathbb{P}^2$. Note that the singular points of Y_φ are located over the six nodes $\{P_{01}, P_{02}, P_{03}, P_{12}, P_{13}, P_{23}\}$ of B. Let

$$\rho : X_\varphi \to Y_\varphi$$

be a resolution of Y_φ, and put

$$\psi := \varphi \circ \rho \ : \ X_\varphi \to \mathbb{P}^2.$$

We then put

$$D_\varphi := \psi^*(B + \Lambda).$$

Note that $A(X_\varphi, D_\varphi)$ does not depend on the choice of the resolution ρ by Proposition 2.1. Applying our method, we prove the following:

Theorem 1.4. *If φ is cyclic of degree $d \le 50$, then $A(X_\varphi, D_\varphi) = 0$.*

We will see as a corollary of Proposition 3.2 that

$$\mathcal{L}(X_\varphi, \psi^*(\Lambda)) \subset \mathcal{L}(X_\varphi, D_\varphi) \subset \overline{\mathcal{L}}(X_\varphi, \psi^*(\Lambda)) = \overline{\mathcal{L}}(X_\varphi, D_\varphi)$$

for any covering φ of $\mathbb{P}^2$ branching along B. When φ is the covering ϕ_m by the Fermat surface, we have $\mathcal{L}(X_\varphi, \psi^*(\Lambda)) = \mathcal{L}(X_\varphi, D_\varphi)$ by Proposition 3.3. This equality does not hold in general, and we have examples of cyclic coverings φ for which $A(X_\varphi, \psi^*(\Lambda))$ is not trivial. See examples in §5.

The method of this paper can be applied to arbitrary covering of $\mathbb{P}^2$ branching along B, and in particular, to Shioda's original problem. However, even in the case of Fermat surface of degree 6, we have to deal with the covering of mapping degree $6^3 = 216$, and our computer has run out of memory. Thus we restrict ourselves to the cyclic coverings in this article.

Our method was initiated in [2]. This method has been recently applied to extremal elliptic surfaces in a sophisticated way by Degtyarev [4].

In §2, we prove Theorem 1.2. In §3 and §4, we explain in detail how to calculate $A(X_\varphi, \psi^*(\Lambda))$. In §3, we calculate the orthogonal complement $\mathcal{L}(X_\varphi, D_\varphi)^\perp$ by the method of Zariski-van Kampen type. In §4, we calculate the discriminant of the lattices $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ and $\mathcal{L}(X_\varphi, D_\varphi)$. The complete result in the case of cyclic coverings of mapping degree 12 is given in §5. In the last section, we present a couple of related results concerned with certain classes of cyclic coverings of $\mathbb{P}^2$.

When we were finishing this article, Degtyarev proposed an alternative method for the proof of the primitivity of $\mathcal{L}(X_\varphi, D_\varphi)$ using the idea of Alexander modules.

2. The algorithm

Let X be a smooth projective complex surface, and let $D = \sum m_i C_i$ be an effective divisor on X. We put

$$X^\circ := X \setminus D.$$

Proposition 2.1. *If X' is another smooth projective surface containing X° such that $D' := X' \setminus X^\circ$ is a union of curves, then we have*

$$\operatorname{disc} \mathcal{L}(X, D) = \operatorname{disc} \mathcal{L}(X', D'), \quad \operatorname{disc} \overline{\mathcal{L}}(X, D) = \operatorname{disc} \overline{\mathcal{L}}(X', D'),$$

and $A(X, D) \cong A(X', D')$.

Proof. We have a smooth projective surface X'' containing X° with birational morphisms $f : X'' \to X$ and $f' : X'' \to X'$ that are isomorphisms over X°. Then each of f and f' is a composite of blowing-ups at points. If $bl : \tilde{X} \to X$ is a blowing up at a point on D with the exceptional (-1)-curve E, then we have

$$H^2(\tilde{X}) = H^2(X) \oplus \mathbb{Z}[E] \quad \text{and} \quad \mathcal{L}(\tilde{X}, bl^*(D)) = \mathcal{L}(X, D) \oplus \mathbb{Z}[E].$$

Applying these to blowing ups composing f and f', we obtain the proof. □

We define a structure of the quasi-lattice on $H_2(X^\circ, \mathbb{Z})$ by the homomorphism

$$\tilde{j}_* \ : \ H_2(X^\circ, \mathbb{Z}) \xrightarrow{j_*} H_2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \twoheadrightarrow H^2(X)$$

and the cup product on $H^2(X)$, where $j : X^\circ \hookrightarrow X$ is the inclusion and the isomorphism in the middle is the Poincaré duality.

Proof of Theorem 1.2. For a sublattice M of $H^2(X)$, let $M^\perp$ denote the orthogonal complement of M. By the assumption that $\mathcal{L}(X, D)$ be a sublattice, we have

$$\overline{\mathcal{L}}(X, D) = (\mathcal{L}(X, D)^\perp)^\perp.$$

Since $H^2(X)$ is a unimodular lattice and both of the sublattices $\overline{\mathcal{L}}(X, D)$ and $\mathcal{L}(X, D)^\perp$ are primitive, we have

$$\operatorname{disc} \overline{\mathcal{L}}(X, D) = \operatorname{disc} \mathcal{L}(X, D)^\perp. \tag{2.1}$$

(Recall that the discriminant of a lattice of rank 0 is 1.) Note that $\mathcal{L}(X, D)^\perp$ is equal to $\operatorname{Ker} r/(\text{torsion})$, where $r : H^2(X, \mathbb{Z}) \to H^2(D, \mathbb{Z})$ is the restriction homomorphism. Under the Poincaré duality $H_2(X^\circ, \mathbb{Z}) \cong H^2(X, D, \mathbb{Z})$ and $H_2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, we have

$$\operatorname{Im} \tilde{j}_* = \operatorname{Ker} r/(\text{torsion}) = \mathcal{L}(X, D)^\perp.$$

Since $\mathcal{L}(X, D)^\perp$ is a sublattice by the assumption, we have $\ker(H_2(X^\circ, \mathbb{Z})) = \operatorname{Ker} \tilde{j}_*$ and hence $\tilde{j}_*$ induces an isomorphism of lattices

$$H_2(X^\circ, \mathbb{Z})/\ker(H_2(X^\circ, \mathbb{Z})) \ \cong \ \mathcal{L}(X, D)^\perp. \tag{2.2}$$

Since the surjection $T_{X^\circ} \twoheadrightarrow H_2(X^\circ, \mathbb{Z})/(\text{torsion})$ that maps t_i to its homology class is a homomorphism of quasi-lattices, it induces an isomorphism of lattices

$$T_{X^\circ}/\ker(T_{X^\circ}) \ \cong \ H_2(X^\circ, \mathbb{Z})/\ker(H_2(X^\circ, \mathbb{Z})).$$

Combining this with (2.1) and (2.2), we obtain the proof. □

3. The discriminant of $\mathcal{L}(X_\varphi, D_\varphi)^\perp$

Let $\varphi : Y_\varphi \to \mathbb{P}^2$ be a d-fold covering of $\mathbb{P}^2$ (not necessarily cyclic) branching along a union $B := B_0 + \cdots + B_3$ of four lines in general position, and let $\rho : X_\varphi \to Y_\varphi$ be a resolution. We put $\psi := \varphi \circ \rho : X_\varphi \to \mathbb{P}^2$. Let Λ be the union of the three lines such that $\Lambda \cap B$ is the six nodes of B, and put $D_\varphi := \psi^*(B + \Lambda)$. We explain in detail the algorithm to calculate $\mathcal{L}(X_\varphi, D_\varphi)^\perp$.

Let $\tilde{b}$ be a base point of $\mathbb{P}^2 \setminus B$, and put $F := \psi^{-1}(\tilde{b})$. Then the covering φ is determined by the monodromy

$$\mu : \pi_1(\mathbb{P}^2 \setminus B, \tilde{b}) \to \mathfrak{S}(F)$$

to the group $\mathfrak{S}(F)$ of permutations on F. Since $\pi_1(\mathbb{P}^2 \setminus B)$ is abelian, the homotopy class β_i of a simple loop around B_i is well-defined, and we have

$$\pi_1(\mathbb{P}^2 \setminus B) = H_1(\mathbb{P}^2 \setminus B, \mathbb{Z}) = \mathbb{Z}[\beta_0] \oplus \cdots \oplus \mathbb{Z}[\beta_3]/\langle[\beta_0] + \cdots + [\beta_3]\rangle.$$

The input of our algorithm is the permutations

$$\mu(\beta_0), \ldots, \mu(\beta_3)$$

that satisfy $\mu(\beta_0)\mu(\beta_1)\mu(\beta_2)\mu(\beta_3) = 1$ and that generate a commutative transitive subgroup of $\mathfrak{S}(F)$.

Remark 3.1. The submodule $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ contains the classes of the exceptional curves of ρ and the class h of the total transform of a general line on $\mathbb{P}^2$. Since $h^2 > 0$, the Hodge index theorem implies that $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ and $\mathcal{L}(X_\varphi, D_\varphi)$ are sublattices of $H^2(X_\varphi)$.

Proposition 3.2. *Let $\tilde{\Gamma}$ de an irreducible curve on X_φ that is mapped onto a line Γ on $\mathbb{P}^2$. Then $[\tilde{\Gamma}]$ is contained in the primitive closure $\overline{\mathcal{L}}(X_\varphi, \psi^*(\Lambda))$. In particular, we have $\overline{\mathcal{L}}(X_\varphi, D_\varphi) = \overline{\mathcal{L}}(X_\varphi, \psi^*(\Lambda))$.*

Proof. Suppose that $\Gamma = B_i$ for some i, and let r be the ramification index of φ at the generic point of $\tilde{\Gamma}$. Then $\psi^*(\Gamma)$ is the sum of $r\tilde{\Gamma}$ and some exceptional divisors of ρ. Since $[\psi^*(\Gamma)] = h$, we have $r[\tilde{\Gamma}] \in \mathcal{L}(X_\varphi, \psi^*(\Lambda))$. Suppose that $\Gamma \neq B_i$ for any i. We denote by $\mathcal{N} := \{P_{01}, \ldots, P_{23}\}$ the set of nodes of B. Then we have a specialization $\{\Gamma_t\}$ from $\Gamma = \Gamma_1$ to $\Gamma_0 = \Lambda_j$ for some j such that $\Gamma_t \cap \mathcal{N} = \Gamma_1 \cap \mathcal{N}$ for $0 < t \leq 1$. Then $\tilde{\Gamma}$ decomposes into a sum of some irreducible components of the total transform of $\Gamma_0 = \Lambda_j$. Hence $[\tilde{\Gamma}] \in \mathcal{L}(X_\varphi, \psi^*(\Lambda))$. □

Proposition 3.3. *If $\varphi : Y_\varphi \to \mathbb{P}^2$ is the $(\mathbb{Z}/m\mathbb{Z})^3$-covering by the Fermat surface $\phi_m : X_m \to \mathbb{P}^2$ and ρ is the identity, then we have $\mathcal{L}(X_m, \phi_m^*(\Lambda)) = \mathcal{L}(X_m, D_{\phi_m})$.*

Proof. Let R_i be the curve $X_m \cap \{x_i = 0\}$. Since $\phi_m^*(B_i) = mR_i$, it is enough to prove $[R_i] \in \mathcal{L}(X_m, \psi^*(\Lambda))$. Let $\zeta \in \mathbb{C}$ be an m-th root of -1, and let H be the curve $X_m \cap \{x_0 - \zeta x_1 = 0\}$. Then H is a union of m lines. Thus we have $[R_i] = [H] \in \mathcal{L}(X_m, \psi^*(\Lambda))$. □

By Proposition 3.2, we have

$$\mathcal{L}(X_\varphi, \psi^*(\Lambda))^\perp = \mathcal{L}(X_\varphi, D_\varphi)^\perp = \mathcal{L}(X_\varphi, D_\varphi + \psi^*(\Gamma_1 + \cdots + \Gamma_k))^\perp$$

for any lines $\Gamma_1, \ldots, \Gamma_k$ on $\mathbb{P}^2$. Hence, in order to calculate $\mathcal{L}(X_\varphi, D_\varphi)^\perp$, it is enough to take suitable lines $\Gamma_1, \ldots, \Gamma_k$ on $\mathbb{P}^2$, put

$$U \ := \ \mathbb{P}^2 \setminus (B + \Lambda + \sum \Gamma_q),$$

and calculate the intersection pairing of topological 2-cycles on

$$X^U := \psi^{-1}(U).$$

We choose U in such a way that U admits a morphism

$$f : U \to \mathbb{C}^\circ := \mathbb{C} \setminus (\text{a finite set of points})$$

such that the composite

$$f \circ \psi|_{X^U} \ : \ X^U \to U \to \mathbb{C}^\circ$$

is a locally trivial fibration (in the classical topology) with fibers being open Riemann surfaces.

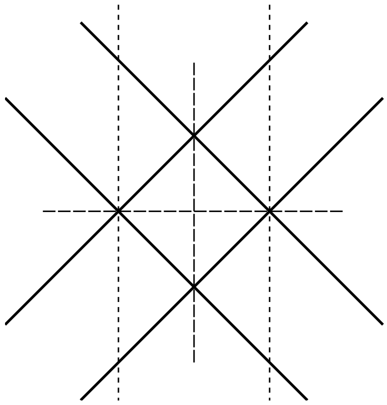

FIGURE 3.1. Lines B_i, Λ_j and Γ_q

Our choice of U and f is as follows. Let (x, y) be affine coordinates on $\mathbb{P}^2$ such that

$$B_0 = \{x - y + 1 = 0\}, \quad B_1 = \{x - y - 1 = 0\},$$
$$B_2 = \{-x - y + 1 = 0\}, \quad B_3 = \{-x - y - 1 = 0\}.$$

Then Λ_1 is the line at infinity, and $\Lambda_2 = \{x = 0\}$, $\Lambda_3 = \{y = 0\}$. We put

$$\Gamma_1 := \{x = 1\}, \quad \Gamma_2 := \{x = -1\}.$$

Let $\bar{f} : \mathbb{C}^2 \to \mathbb{C}$ be the projection $(x, y) \mapsto x$. Then the maps

$$f := \bar{f}|_U \ : \ U \ \to \ \mathbb{C}^\circ := \mathbb{C} \setminus \{-1, 0, 1\} \quad \text{and} \quad f \circ \psi|_{X^U} \ : \ X^U \ \to \ \mathbb{C}^\circ$$

are locally trivial fibrations. See Figure 3.1, in which the thick lines are $B_0, \dots, B_3$. We choose a base point $b \in \mathbb{C}^\circ$ at a large real number, and put

$$\tilde{b} := (b, b') \in f^{-1}(b),$$

where b' is also a large real number such that

$$b' \gg b. \tag{3.1}$$

We then put

$$R := \psi^{-1}(f^{-1}(b)) \subset X^U, \quad F := \psi^{-1}(\tilde{b}) \subset R.$$

Then F is a finite set of d points, and $\psi|_R : R \to f^{-1}(b)$ is an étale covering of the punctured line $f^{-1}(b)$. The five punctured points $\bar{f}^{-1}(b) \setminus f^{-1}(b)$ are located on the real line of $\bar{f}^{-1}(b) = \mathbb{C}$. We index them as Q_1, Q_2, Q_3, Q_4, Q_5 *from left to right*. Let γ_i be the homotopy class of the simple loop on $f^{-1}(b)$ around Q_i with the base point $\tilde{b}$ as in Figure 3.2, in which only three of five simple loops are drawn. These classes γ_i generate the free group $\pi_1(f^{-1}(b), \tilde{b})$, and they are mapped by the natural homomorphism $\pi_1(f^{-1}(b), \tilde{b}) \to \pi_1(\mathbb{P}^2 \setminus B, \tilde{b})$ as follows:

$$\gamma_1 \mapsto \beta_3, \quad \gamma_2 \mapsto \beta_2, \quad \gamma_3 \mapsto 1, \quad \gamma_4 \mapsto \beta_1, \quad \gamma_5 \mapsto \beta_0.$$

Hence, from the input $\mu : \pi_1(\mathbb{P}^2 \setminus B) \to \mathfrak{S}(F)$, we can readily calculate the monodromy $\pi_1(f^{-1}(b), \tilde{b}) \to \mathfrak{S}(F)$. The action of the image of γ_i on $p \in F$ is denoted

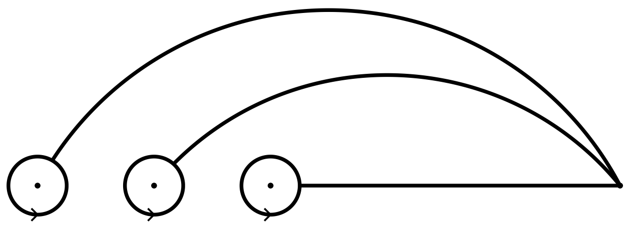

FIGURE 3.2. Simple loops

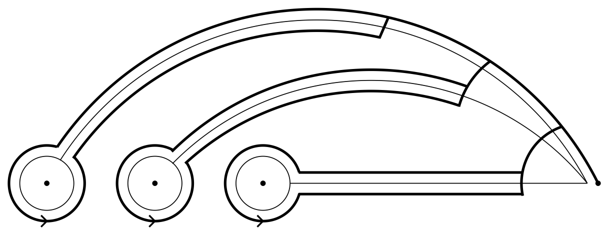

FIGURE 3.3. Shifting of loops

by $p \mapsto \gamma_i(p)$ for simplicity. We denote by $p \otimes \gamma_i$ the path on R that is the lift of γ_i starting from p (and hence ending at $\gamma_i(p)$), and consider the module

$$\mathcal{P} := \bigoplus_{i=1}^{5} \bigoplus_{p \in F} \mathbb{Z}(p \otimes \gamma_i)$$

freely generated by these $p \otimes \gamma_i$. We have a canonical isomorphism

$$\mathcal{P} \cong H_0(F, \mathbb{Z}) \otimes H_1(f^{-1}(b), \mathbb{Z}).$$

Elements of $\mathcal{P}$ are regarded as topological 1-chains on R. Then we have

$$H_1(R, \mathbb{Z}) = \operatorname{Ker}(\ w : \mathcal{P} \to H_0(F, \mathbb{Z})\ ),$$

where w is the homomorphism defined by $w(p \otimes \gamma_i) := (1 - \gamma_i)p$. Thus we have a list of topological 1-cycles on R whose homology classes form a basis of $H_1(R, \mathbb{Z})$. Each member of this list is expressed as an integer vector of length $\operatorname{rank} \mathcal{P} = 5d$. The intersection pairing Q_R among these cycles are calculated from the intersection numbers $Q_{\mathcal{P}}$ of topological chains $p \otimes \gamma_j$ by *shifting* the path γ_j as in Figure 3.3. Namely, we have

$$Q_{\mathcal{P}}(p \otimes \gamma_i, p' \otimes \gamma_j) =$$
$$\begin{cases} \delta(p, p') - \delta(\gamma_i(p), p') - \delta(p, \gamma_j(p')) + \delta(\gamma_i(p), \gamma_j(p')) & \text{if } i < j, \\ -\delta(p, p') + \delta(p, \gamma_i(p')) & \text{if } i = j, \\ -\delta(p, p') + \delta(\gamma_i(p), p') + \delta(p, \gamma_j(p')) - \delta(\gamma_i(p), \gamma_j(p')) & \text{if } i > j, \end{cases}$$

where δ is Kronecker's delta function. (Note that $p' \otimes \gamma_j$ is the shifted cycle.) Restricting $Q_{\mathcal{P}}$ to the submodule $H_1(R, \mathbb{Z}) = \operatorname{Ker} w$, we obtain Q_R.

Next we calculate the monodromy action of $\pi_1(\mathbb{C}^\circ, b)$ on $H_1(R, \mathbb{Z})$ associated with the locally trivial fibration $f \circ \psi|_U : X^U \to \mathbb{C}^\circ$. By (3.1), we have a continuous section

$$s : \mathbb{C}^\circ \to U$$

of $f : U \to \mathbb{C}^\circ$ that satisfies $s(x) = (x, b')$ for x inside a sufficiently large disk on $\mathbb{C}^\circ$. Since $s(b) = \tilde{b}$, $\pi_1(\mathbb{C}^\circ, b)$ acts on $\pi_1(f^{-1}(b), \tilde{b})$ and hence on $H_1(f^{-1}(b), \mathbb{Z})$. Moreover $\pi_1(\mathbb{C}^\circ, b)$ acts on F and hence on $H_0(F, \mathbb{Z})$ by $s_* : \pi_1(\mathbb{C}^\circ, b) \to \pi_1(U, \tilde{b})$. Thus $\pi_1(\mathbb{C}^\circ, b)$ acts on $\mathcal{P} \cong H_0(F, \mathbb{Z}) \otimes H_1(f^{-1}(b), \mathbb{Z})$. This action preserves $\operatorname{Ker} w$, and the restriction to $\operatorname{Ker} w = H_1(R, \mathbb{Z})$ is the desired monodromy.

We write the monodromy on $\mathcal{P}$ explicitly. Let τ_1, τ_2, τ_3 be the homotopy classes of the simple loops on $\mathbb{C}^\circ$ around $-1, 0, 1$, respectively, with the base point b depicted in Figure 3.2. (We use Figure 3.2 twice to illustrate completely different objects; simple loops on the fiber $f^{-1}(b)$ in the previous paragraph and simple loops on the base curve $\mathbb{C}^\circ$ here.) The action of $\pi_1(\mathbb{C}^\circ, b)$ on $H_0(F, \mathbb{Z})$ is trivial, because of the property (3.1). When a point $t \in \mathbb{C}^\circ$ moves along τ_j, the five punctured points $\bar{f}^{-1}(t) \setminus f^{-1}(t)$ undergo the *braid monodromy.* Let $\mathcal{B}_5$ denote the braid group on the strings in $\mathbb{C} \times I$ connecting $(Q_i, 0) \in \mathbb{C} \times I$ and $(Q_i, 1) \in \mathbb{C} \times I$, and let $\sigma_i \in \mathcal{B}_5$ be the simple braid that interchanges Q_i and Q_{i+1} by the positive half-twist and fixes the other $Q_{i'}$. We write the conjunction of braids *from right to left* so that $\mathcal{B}_5$ acts on $\pi_1(f^{-1}(b), \tilde{b})$ from the left:

$$\sigma_i(\gamma_j) = \begin{cases} \gamma_j^{-1}\gamma_{j+1}\gamma_j & \text{if } i = j, \\ \gamma_{j-1} & \text{if } i = j-1, \\ \gamma_j & \text{otherwise.} \end{cases}$$

(The conjunction of loops is also written from right to left.) The braid monodromy $\mathrm{br} : \pi_1(\mathbb{C}^\circ, b) \to \mathcal{B}_5$ is given as follows:

$$\begin{aligned} \mathrm{br}(\tau_1) &= (\sigma_2\sigma_3\sigma_2)^{-1}(\sigma_1\sigma_4)^{-1}(\sigma_2\sigma_3\sigma_2)^2(\sigma_1\sigma_4)(\sigma_2\sigma_3\sigma_2), \\ \mathrm{br}(\tau_2) &= (\sigma_2\sigma_3\sigma_2)^{-1}(\sigma_1\sigma_4)^2(\sigma_2\sigma_3\sigma_2), \\ \mathrm{br}(\tau_3) &= (\sigma_2\sigma_3\sigma_2)^2. \end{aligned}$$

Combining them, we obtain the action of $\tau_j \in \pi_1(\mathbb{C}^\circ, b)$ on $\pi_1(f^{-1}(b), \tilde{b})$, on $H_1(f^{-1}(b), \mathbb{Z})$, and hence on $\mathcal{P}$ and $\operatorname{Ker} w = H_1(R, \mathbb{Z})$, which we write $\alpha \mapsto \tau_j(\alpha)$ for simplicity. For example, we have

$$\tau_1(\gamma_1) = \gamma_1^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_5^{-1}\gamma_4\gamma_1\gamma_4^{-1}\gamma_5\gamma_4\gamma_3\gamma_1,$$

and hence

$$\begin{aligned} &\tau_1(p \otimes \gamma_1) = (p \otimes \gamma_1) + (\gamma_1(p) \otimes \gamma_3) + (\gamma_3\gamma_1(p) \otimes \gamma_4) + \\ &\quad + (\gamma_4\gamma_3\gamma_1(p) \otimes \gamma_5) - (\gamma_4^{-1}\gamma_5\gamma_4\gamma_3\gamma_1(p) \otimes \gamma_4) + (\gamma_4^{-1}\gamma_5\gamma_4\gamma_3\gamma_1(p) \otimes \gamma_1) + \cdots . \end{aligned}$$

Remark that the topological chain of the path γ_i^{-1} on R that starts from $p \in F$ is $-\gamma_i^{-1}(p) \otimes \gamma_i$ in $\mathcal{P}$.

For $\alpha \in \operatorname{Ker} w$ and τ_j, we denote by $\alpha \otimes \tau_j$ the topological 2-chain on X^U that is a tube over the loop τ_j on U whose fiber over $\tau_j(t)$ is a topological 1-cycle on $(f \circ \psi)^{-1}(t)$ that is mapped to α by the inverse of the diffeomorphism $R = (f \circ \psi)^{-1}(0) \cong (f \circ \psi)^{-1}(t)$ along τ_j, and let

$$\mathcal{T} := \operatorname{Ker} w \otimes (\mathbb{Z}[\tau_1] \oplus \mathbb{Z}[\tau_2] \oplus \mathbb{Z}[\tau_3]) \quad \subset \quad \mathcal{P} \oplus \mathcal{P} \oplus \mathcal{P}$$

denote the module of these topological 2-chains. We have a canonical identification

$$\mathcal{T} = H_1(R, \mathbb{Z}) \otimes H_1(\mathbb{C}^\circ, \mathbb{Z}).$$

Since the fiber of $\alpha \otimes \tau_j$ over $\tau_j(1)$ is homologous to $\tau_j(\alpha)$ in R, we have

$$H_2(X^U, \mathbb{Z}) \cong \mathrm{Ker}(W : \mathcal{T} \to \mathrm{Ker}\, w),$$

where W is the homomorphism defined by $W(\sum_{j=1}^{3} \alpha_j \otimes \tau_j) := \sum_{j=1}^{3}(1-\tau_j)\alpha_j$. Therefore we obtain a list of topological 2-cycles whose homology classes form a basis of $H_2(X^U, \mathbb{Z})$. Each member is expressed as an integer vector of length $15d$. The intersection pairing Q_X among these topological 2-cycles are calculated from the intersection numbers $Q_{\mathcal{T}}$ of topological 2-chains $\sum \alpha_j \otimes \tau_j$ by shifting the path τ_j as in Figure 3.3. Namely, we have

$$\begin{aligned} & -Q_{\mathcal{T}}(\sum \alpha_j \otimes \tau_j\, , \sum \alpha'_j \otimes \tau_j\,) \\ = \quad & Q_R((1-\tau_1)(\alpha_1), (1-\tau_2)(\alpha'_2)) + Q_R((1-\tau_1)(\alpha_1), (1-\tau_3)(\alpha'_3)) \\ & + Q_R((1-\tau_2)(\alpha_2), (1-\tau_3)(\alpha'_3)) \\ & + Q_R((1-\tau_1)(\alpha_1), -\tau_1(\alpha'_1)) + Q_R((1-\tau_2)(\alpha_2), -\tau_2(\alpha'_2)) \\ & + Q_R((1-\tau_3)(\alpha_3), -\tau_3(\alpha'_3)). \end{aligned}$$

The six terms correspond to the six intersection points of τ_j and their shifts. Restricting $Q_{\mathcal{T}}$ to $\mathrm{Ker}\, W$, we obtain the intersection pairing Q_X on $H_2(X^U, \mathbb{Z})$.

By Theorem 1.2, the orthogonal complement $\mathcal{L}(X_\varphi, D_\varphi)^\perp$ is then isomorphic to the lattice $\mathrm{Ker}\, W/\ker(\mathrm{Ker}\, W)$ associated to the quasi-lattice $(\mathrm{Ker}\, W, Q_X)$.

Remark 3.4. As the explanation above suggests, the method can be applied inductively to calculate the intersection pairing on the middle homology group of smooth affine varieties with locally trivial fibrations to a product to punctured affine lines.

4. The discriminant of $\mathcal{L}(X_\varphi, D_\varphi)$

Let the notations be as in the previous section. In this section, we explain how to calculate the intersection matrices of $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ and $\mathcal{L}(X_\varphi, D_\varphi)$.

By Remark 3.1, $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ and $\mathcal{L}(X_\varphi, D_\varphi)$ are lattices. Therefore, if we know the intersection matrix of a set of generators C_i, our lattices can be calculated as the quotient of the quasi-lattice $\bigoplus \mathbb{Z}C_i$ by its kernel. We will take the irreducible components of $\psi^*(\Lambda)$ and D_φ as generators.

Recall that $B_i \subset \mathbb{P}^2 = \{y_0 + y_1 + y_2 + y_3 = 0\} \subset \mathbb{P}^3$ is the line $y_i = 0$ for $0 \le i \le 3$, $P_{ij} = B_i \cap B_j$ for $0 \le i < j \le 3$ and Λ_i is the line through P_{0i} and P_{jk} where $\{i, j, k\} = \{1, 2, 3\}$ and $j < k$.

Definition 4.1. In what follows, a *distinguished point* will mean a point in $\varphi^{-1}(P_{ij})$, a *distinguished curve* will mean an irreducible component of $\varphi^{-1}(\Lambda_i)$ or its strict transform on X_φ, and a *boundary curve* will be $\varphi^{-1}(B_i)_{\mathrm{red}}$ or its strict transform on X_φ.

We have to list up the distinguished curves and points, describe the minimal resolutions of singularities at the distinguished points and calculate the intersection numbers of curves involved.

4.1. **Distinguished points and distinguished curves.** First of all, it is necessary to label the distinguished points and distinguished curves in some way. We do this by regarding them as images of points and lines on the Fermat surface X_m.

Definition 4.2. Let $G \subset \mathrm{Gal}(X_m/\mathbb{P}^2) = (\mathbb{Z}/m\mathbb{Z})^3$ be the subgroup corresponding to the intermediate cover $X_m \to Y_\varphi$.

Write $\xi^{(p)}$ for $\exp((2p+1)\pi\sqrt{-1}/m)$.

1. Let i, j, k and l be such that $i < j$, $k < l$ and $\{0,1,2,3\} = \{i,j,k,l\}$.
   - (a) Let $U_{ij} = \{|y_i/y_k| < \epsilon, |y_j/y_k| < \epsilon\}$ be a small polydisc near P_{ij}, $h_{ij} : \pi_1(U_{ij} \setminus B) \to \mathrm{Gal}(X_m/\mathbb{P}^2)$ the natural homomorphism and $H_{ij} := h_{ij}(h_{ij}^{-1}(G))$.
   - (b) For $0 \le p < m$, let P_{ijp} be the point $(x_i = x_j = 0, x_l = \xi^{(p)} x_k)$ in $\mathbb{P}^3$.
2. Let i, j and k be such that $\{1,2,3\} = \{i,j,k\}$ and $j < k$.
   - (a) Let $V_i \subset \Lambda_i$ be a disc near P_{0i}, $k_i : \pi_1(V_i \setminus B) \to \mathrm{Gal}(X_m/\mathbb{P}^2)$ the natural homomorphism and $K_i := k_i(k_i^{-1}(G))$.
   - (b) For $0 \le p, q < m$, let L_{ipq} be the line $(x_i = \xi^{(p)} x_0, x_k = \xi^{(q)} x_j)$ in $\mathbb{P}^3$.

Lemma 4.3. (1) (a) $\phi_m^{-1}(P_{ij}) = \{P_{ijp} | 0 \le p < m\}$.
- (b) *The group G acts on $\{P_{ijp} | 0 \le p < m\}$ with stabilizer subgroup H_{ij}. The distinguished points on Y_φ over P_{ij} can be identified with orbits of the free G/H_{ij}-action on $\{P_{ijp}\}$.*

(2) (a) $\phi_m^* \Lambda_i = \sum_{0 \le p,q < m} L_{ipq}$.
- (b) *The group G acts on $\{L_{ipq} | 0 \le p, q < m\}$ with stabilizer subgroup K_i, and the distinguished curves (on Y_φ or X_φ) over Λ_i can be identified with orbits of the free G/K_i-action on $\{L_{ipq}\}$.*
- (c) *Let $\bar{L}$ be the image of L_{ipq} on Y_φ. Then $\deg(L_{ipq} \to \bar{L}) = |K_i|$ and $\deg(\bar{L} \to \Lambda_i) = m/|K_i|$.*

Proof. (1a), (2a) Straightforward calculations.

(1b) Let W be the connected component of $\phi_m^{-1}(U_{ij})$ containing P_{ijp} and let P be a point of $W \setminus \phi_m^{-1}(B)$. For $g \in G$, we have $g(P_{ijp}) = P_{ijp} \Leftrightarrow g(P) \in W \Leftrightarrow$ g comes from a deck transformation of $W \to U_{ij} \Leftrightarrow g \in H_{ij}$. Thus the stabilizer subgroup is H_{ij}, and the remaining assertions follow.

(2b) is similar.

(2c) Since the group $\mathrm{Gal}(X_m/\mathbb{P}^2)$ acts freely on $X_m \setminus \phi_m^{-1}(B)$, the action of K_i on L_{ipq} is faithful, and $\deg(L_{ipq} \to \bar{L}) = |K_i|$. By the same reason, we have $\deg(L_{ipq} \to \Lambda_i) = (\deg X_m \to \mathbb{P}^2)/(\#\{L_{ipq} | 0 \le p, q < m\}) = m$. □

Definition 4.4. Denote by $\bar{P}_{ijp}$ the image of P_{ijp} on Y_φ (i.e. the distinguished point corresponding to the orbit of P_{ijp}).

Denote by $\bar{L}'_{ipq}$ the image of L_{ipq} on Y_φ and by $\bar{L}_{ipq}$ its strict transform on X_φ (i.e. the distinguished curves corresponding to the orbit of L_{ipq}).

Let us denote by $(C.D)'$ the intersection number outside the inverse image of the distinguished points. For (i, p, q) and (i', p', q') with $\bar{L}_{ipq} \ne \bar{L}_{i'p'q'}$, one can calculate the intersection number $(\bar{L}_{ipq}.\bar{L}_{i'p'q'})'$ as follows:

- If $i = i'$, then it is 0.
- If $i \ne i'$, then it is $\sum_{\bar{g} \in G/K_i} (\bar{g}(L_{ipq}).L_{i'p'q'})'$.

4.2. **Factoring the quotient map and the minimal resolution.** The only singularities of Y_φ are at distinguished points. Let us describe the local situation over P_{ij}.

A linear automorphism g of $\mathbb{C}^n$ is called a reflexion if the set $(\mathbb{C}^n)^g$ of fixed points is a hyperplane, and a finite group with a faithful linear action on $\mathbb{C}^n$ is called small if it has no reflexion. It is known that the quotient of $\mathbb{C}^n$ by any action of a finite group is isomorphic to the quotient by a small group as a singularity. Let us give a detailed description for the case at hand.

Definition 4.5. With the notations of Definition 4.2, write h for h_{ij}.

Identify $\pi_1(U_{ij} \setminus B)$ with $\mathbb{Z} \times \mathbb{Z}$ by sending loops around B_i and B_j to $(1,0)$ and $(0,1)$, and let $\tilde{H} = h^{-1}(G)$.

Let a, b, c and d be determined as follows: a and b are maximal positive integers such that $\tilde{H} \subseteq a\mathbb{Z} \times b\mathbb{Z}$ and c and d are minimal positive integers such that $\tilde{H} \supseteq c\mathbb{Z} \times d\mathbb{Z}$.

Write $H_0 = h(c\mathbb{Z} \times d\mathbb{Z})$, $H_1 = h(\tilde{H}) = H_{ij}$, $H_2 = h(a\mathbb{Z} \times b\mathbb{Z})$ and $H_3 = h(\mathbb{Z} \times \mathbb{Z})$.

Lemma 4.6.
(1) *We have $c = an$ and $d = bn$ for a positive integer n. Mapping by $s : a\mathbb{Z} \times b\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}; (k,l) \mapsto (k/a, l/b)$, we have $\mathbb{Z} \times \mathbb{Z} \supset s(\tilde{H}) \supset s(c\mathbb{Z} \times d\mathbb{Z}) = n\mathbb{Z} \times n\mathbb{Z}$, $s(\tilde{H}) = (n\mathbb{Z} \times n\mathbb{Z}) + \mathbb{Z}(q,1)$ with $0 \leq q < n$ and $\gcd(n,q) = 1$, and $\mathbb{Z} \times \mathbb{Z} = s(\tilde{H}) + \mathbb{Z}(1,0)$. In particular, $(a\mathbb{Z} \times b\mathbb{Z})/\tilde{H}$ and $\tilde{H}/(c\mathbb{Z} \times d\mathbb{Z})$ are cyclic of order n. Furthermore, c and d divide m.*
(2) *Consider the following commutative diagram.*

$$\begin{tikzcd}
& X_m \arrow[dl, "\pi_0"'] \arrow[ddd] \\
X_m/H_0 \arrow[d, "\pi_1"'] & \\
X_m/H_1 \arrow[d, "\pi_2"'] \arrow[dr, "\alpha_1", "\text{étale}"'] & \\
X_m/H_2 \arrow[d, "\pi_3"'] & X_m/G = Y_\varphi \arrow[dd, "\varphi"] \\
X_m/H_3 \arrow[dr, "\alpha_2", "\text{étale}"'] & \\
& \mathbb{P}^2.
\end{tikzcd}$$

Over a neighborhood of P_{ij}, the quotients X_m/H_0, X_m/H_2 and X_m/H_3 are smooth, the quotient maps α_1 and α_2 are étale, and π_0, $\pi_2 \circ \pi_1$ and π_3 are given by $(x,y) \mapsto (x^{m/c}, y^{m/d})$, $(x,y) \mapsto (x^n, y^n)$ and $(x,y) \mapsto (x^a, y^b)$, where $x = 0$ and $y = 0$ are inverse images of B_i and B_j.

The map π_1 is the quotient by the finite cyclic small action $(x,y) \mapsto (\zeta_n^q x, \zeta_n y)$.

Proof. Although most of the statements are proven in [3, Ch. III §5 (i)], we give a proof here for the reader's convenience.

(1) It is obvious that a divides c and b divides d. Write $c = an_1$ and $d = bn_2$. Using the map s in the statement, we have $\mathbb{Z} \times \mathbb{Z} \supset H' := s(\tilde{H}) \supset n_1\mathbb{Z} \times n_2\mathbb{Z}$. The image of H' by the second projection is $\mathbb{Z}$ by the assumption on b. Thus H' contains $(q,1)$ for some q, which can be assumed to satisfy $0 \leq q < n_1$.

For any $y \in \mathbb{Z}$, there can be at most one $x \in \mathbb{Z}$ such that $(x, y) \in H'$ and $0 \leq x < n_1$, by the assumption on c. This shows that $H' = (n_1\mathbb{Z} \times n_2\mathbb{Z}) + \mathbb{Z}(q, 1)$. If $n' := \gcd(n_1, q)$ were greater than 1, then H' would be contained in $n'\mathbb{Z} \times \mathbb{Z}$, contrary to the assumption on a. Therefore q is prime to n_1. From the assumption on d and the fact that $(0, n_1) = n_1(q, 1) - q(n_1, 0)$ is contained in H', n_2 divides n_1. By symmetry, we have $n_1 = n_2$.

Since $c\mathbb{Z} \times d\mathbb{Z} \supset \operatorname{Ker} h = m\mathbb{Z} \times m\mathbb{Z}$, c and d divide m.

(2) By Lemma 4.3(1b), H_1 is the set of elements of $\operatorname{Gal}(X_m/Y_\varphi)$ which fix P_{ijp}. Similarly, H_3 is the set of elements of $\operatorname{Gal}(X_m/\mathbb{P}^2)$ which fix P_{ijp}. Therefore α_1 and α_2 are étale, and the restriction of $X_m \to X_m/H_3$ to the fibers over P_{ij} is one-to-one. Take $(x_i/x_k, x_j/x_k)$ and $(y_i/y_k, y_j/y_k)$ $(k \neq i, j)$ as local coordinate systems on X_m and X_m/H_3, and the map $\pi_3 \circ \pi_2 \circ \pi_1 \circ \pi_0$ is written as $(x, y) \mapsto (x^m, y^m)$. The action of $H_3 \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ on X_m is given by $(\bar{p}, \bar{q}) \cdot (x, y) = (\zeta_m^p x, \zeta_m^q y)$, and our description of quotient maps follows. □

Thus for each point over P_{ij} we are in the following situation: $X = \mathbb{C}^2 \to Z = \mathbb{C}^2$ is the $(\mathbb{Z}/n\mathbb{Z})^2$-quotient given by $(x_1, x_2) \mapsto (z_1, z_2) = (x_1^n, x_2^n)$, and $X \to Y$ is the intermediate quotient by the action $(x_1, x_2) \mapsto (\zeta_n^q x_1, \zeta_n x_2)$. The inverse images of B_i and B_j on Z are given by $z_1 = 0$ and $z_2 = 0$. Let i' be such that $\Lambda_{i'}$ passes through P_{ij}, and $g = \gcd(a, b)$, $a = a'g$ and $b = b'g$. Then the inverse image of $\Lambda_{i'}$ splits into g components of the form $x^{a'} = ty^{b'}$ with different constants t.

The singularity Y can be described as follows.

Proposition 4.7. (1) *The singularity Y is isomorphic to the Hirzebruch-Jung singularity $A_{n,q}$. It is the normalization of $Y' : w^n = z_1 z_2^{n-q}$, and the maps $X \to Y'$ and $Y' \to Z$ are given by $(x_1, x_2) \mapsto (w, z_1, z_2) = (x_1 x_2^{n-q}, x_1^n, x_2^n)$ and $(w, z_1, z_2) \mapsto (z_1, z_2)$.*

(2) *Define integers r and $f_1, \ldots, f_r > 1$ by*

$$n/q = f_1 - \cfrac{1}{f_2 - \cfrac{1}{f_3 - \cdots - \frac{1}{f_r}}}.$$

Then the minimal resolution $\tilde{Y}$ of Y can be described as follows. There are r exceptional curves $C_1, \ldots, C_r$, which are isomorphic to $\mathbb{P}^1$, and which form a chain together with the strict transforms C_0 and C_{r+1} of $z_1 = 0$ and $z_2 = 0$. The self intersections are given by $(C_i^2) = -f_i$.

(3) $\det(C_i.C_j)_{i,j=1}^r = (-1)^r n$.

(4) *In a neighborhood of $C_i \cap C_{i+1}$, $\tilde{Y}$ has an affine open subset U_i with coordinates (u_i, v_i) with the following properties. The curves C_i and C_{i+1} are defined by $u_i = 0$ and $v_i = 0$ respectively. The rational map $X \to \tilde{Y}$ and the morphism $\tilde{Y} \to Y'$ are given by $(u_i, v_i) = (x_1^{\mu_{i+1}} x_2^{-\lambda_{i+1}}, x_1^{-\mu_i} x_2^{\lambda_i})$ and $(w, z_1, z_2) = (u_i^{\frac{\lambda_i + (n-q)\mu_i}{n}} v_i^{\frac{\lambda_{i+1} + (n-q)\mu_{i+1}}{n}}, u_i^{\lambda_i} v_i^{\lambda_{i+1}}, u_i^{\mu_i} v_i^{\mu_{i+1}})$, where λ_i and μ_i are defined as follows. We set $\lambda_{r+1} = 0$, $\lambda_r = 1$, $\mu_0 = 0$ and $\mu_1 = 1$. For other values of i, they are the numerators of*

$$f_{i+1} - \cfrac{1}{f_{i+2} - \cfrac{1}{f_{i+3} - \cdots - \frac{1}{f_r}}} \quad \text{and} \quad f_{i-1} - \cfrac{1}{f_{i-2} - \cfrac{1}{f_{i-3} - \cdots - \frac{1}{f_1}}},$$

respectively.

Proof. (1), (2) This is from [3, Ch. III §5], especially Theorem 5.1 and Proposition 5.3.

(3) can be proven inductively by expanding the determinant along the first column.

(4) One can regard X, Y and Z as affine toric varieties in the following way.

- X is associated to the lattice $\mathbb{Z}^2$ and the cone $\sigma = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1)$.
- Y is associated to the lattice $N = \mathbb{Z}^2 + \mathbb{Z}\frac{1}{n}(q,1)$ and the cone σ.
- Z is associated to the lattice $\left(\frac{1}{n}\mathbb{Z}\right)^2$ and the cone σ.

Vectors $v_k = \frac{1}{n}(\lambda_k, \mu_k)$ decompose σ into a fan Σ. Let V be the toric variety associated to (N, Σ). The equality $\lambda_k\mu_{k+1} - \lambda_{k+1}\mu_k = n$ from [3, Ch. III §5 (5)] shows that V is nonsingular and that, if D_k denotes the toric divisor corresponding to v_k, the curves $D_0, D_1, \dots, D_{r+1}$ form a chain in this order. Since the map $V \to Y$ has r exceptional divisors, V is isomorphic to $\tilde{Y}$, and since D_0 corresponds to C_0, D_k has to correspond to C_k. From this, our local description of $\tilde{Y}$ follows. $\square$

Our intersection numbers can be calculated in the following way.

- Intersection numbers of the exceptional curves, including self intersection numbers, are directly given by the previous proposition.
- Write $(C.D)''$ for the intersection number on the inverse image of a distinguished point. One can write down the local equations for the total and strict transforms of the distinguished or boundary curves, and calculate $(C.D)''$ in the case C is exceptional and D is distinguished or boundary, or C and D are distinct distinguished or boundary curves.
- We are left with self intersection numbers of distinguished or boundary curves. They can be calculated from other intersection numbers. For example, one can write $\psi^*\Lambda_i$ in the form $\bar{L}_{ipq} + \sum \bar{L}_{ip'q'} + \sum m_k C_k$, where (p', q') runs a certain set of indices and C_k are exceptional curves of the resolution. Then we have $(\bar{L}_{ipq}^2) = \deg(\bar{L}_{ipq} \to \Lambda_i) - \sum(\bar{L}_{ipq}.\bar{L}_{ip'q'}) - \sum m_k(\bar{L}_{ipq}.C_k)$

5. Examples

A $\mathbb{Z}/m\mathbb{Z}$-covering of $\mathbb{P}^2$ branching along B is given by a homomorphism

$$\gamma : \pi_1(\mathbb{P}^2 \setminus B, \tilde{b}) \to \mathbb{Z}/m\mathbb{Z}.$$

and hence by

$$a(\gamma) := [\gamma(\beta_0), \gamma(\beta_1), \gamma(\beta_2), \gamma(\beta_3)] \in (\mathbb{Z}/m\mathbb{Z})^4.$$

An element $a = [a_0, \dots, a_3] \in (\mathbb{Z}/m\mathbb{Z})^4$ with $\sum a_i = 0$ corresponds to a cyclic covering of degree m if and only if $a_0, \dots, a_3$ generate $\mathbb{Z}/m\mathbb{Z}$. If such quadruples $a(\gamma)$ and $a(\gamma')$ are contained in the same orbit of $(\mathbb{Z}/m\mathbb{Z})^4$ under the action of the permutation group $\mathfrak{S}_4$ of the components and the diagonal action by $(\mathbb{Z}/m\mathbb{Z})^*$, then the corresponding coverings are topologically equivalent.

Below is the table of the data for all topological equivalence classes of the cyclic coverings of degree $d = 12$, where

$$d^\perp := \operatorname{disc}\mathcal{L}(X_\varphi, D_\varphi)^\perp = \operatorname{disc}\overline{\mathcal{L}}(X_\varphi, D_\varphi),$$

which has turned out to be equal to $\operatorname{disc}\mathcal{L}(X_\varphi, D_\varphi)$, and

$$d_\Lambda := \operatorname{disc}\mathcal{L}(X_\varphi, \psi^*(\Lambda)).$$

The column $\mathrm{rk}^\perp$ denotes the rank of $\mathcal{L}(X_\varphi, D_\varphi)^\perp$, and p_g denotes the geometric genus of X_φ. Note that $\mathrm{rk}^\perp$ and p_g does not depend on the choice of the resolution ρ, and that the signature of $\mathcal{L}(X_\varphi, D_\varphi)^\perp$ is equal to $(p_g, \mathrm{rk}^\perp - p_g)$.

Example 5.1.

No.	a	$d^\perp$	d_Λ	$\mathrm{rk}^\perp$	p_g
1	$[0,0,1,11]$	1	1	0	0
2	$[0,1,1,10]$	1	$(2)^4(3)^4$	0	0
3	$[0,1,2,9]$	1	$(2)^4$	0	0
4	$[0,1,3,8]$	1	1	0	0
5	$[0,1,4,7]$	1	$(3)^4$	0	0
6	$[0,1,5,6]$	1	$(2)^4$	0	0
7	$[1,1,1,9]$	$(2)^2(3)$	$(2)^{10}(3)^5$	8	3
8	$[1,1,2,8]$	$(2)^4(3)$	$(2)^8(3)^5$	6	2
9	$[1,1,3,7]$	$(3)^3$	$(2)^6(3)^7$	6	2
10	$[1,1,4,6]$	1	$(2)^4(3)^4$	4	1
11	$[1,1,5,5]$	$(2)^6$	$(2)^{14}(3)^4$	6	3
12	$[1,1,11,11]$	1	$(2)^6(3)^4$	0	0
13	$[1,2,2,7]$	$(2)^4(3)^3$	$(2)^{10}(3)^7$	6	3
14	$[1,2,3,6]$	$(2)^2(3)^2$	$(2)^8(3)^2$	4	1
15	$[1,2,4,5]$	$(2)^4$	$(2)^8(3)^4$	4	1
16	$[1,2,10,11]$	1	$(2)^6(3)^4$	0	0
17	$[1,3,3,5]$	$(2)^4(3)^2$	$(2)^{10}(3)^2$	4	2
18	$[1,3,4,4]$	$(3)^3$	$(3)^7$	6	1
19	$[1,3,9,11]$	1	$(2)^6$	0	0
20	$[1,3,10,10]$	$(3)^3$	$(2)^6(3)^7$	6	1
21	$[1,4,8,11]$	1	$(3)^4$	0	0
22	$[1,4,9,10]$	(3)	$(2)^4(3)^5$	6	0
23	$[1,5,7,11]$	1	$(2)^6(3)^4$	0	0
24	$[1,5,9,9]$	$(2)^6$	$(2)^{14}$	6	1
25	$[1,6,6,11]$	1	$(2)^6$	0	0
26	$[1,6,7,10]$	$(3)^2$	$(2)^6(3)^6$	4	0
27	$[1,6,8,9]$	$(2)^2$	$(2)^6$	4	0
28	$[1,7,8,8]$	$(2)^4(3)^3$	$(2)^4(3)^7$	6	1
29	$[2,3,3,4]$	$(2)^4$	$(2)^8$	4	1
30	$[2,3,9,10]$	1	$(2)^6$	0	0
31	$[3,4,8,9]$	1	1	0	0

6. Miscellaneous facts

In this section, we will show the primitivity of $\mathcal{L}(X_\varphi, D_\varphi)$ in two special cases. Let $a_0, \dots, a_3$ be as in the previous section.

6.1. Case $a_i \equiv 0$ or $a_i + a_j \equiv 0$.

Proposition 6.1. (1) *If $a_i \equiv 0 \mod m$, then X_φ can be seen as a toric surface in such a way that the support of $\psi^*(B - B_i)$ is the complement of the big orbit.*

(2) *If $a_i + a_j \equiv 0 \mod m$ ($i < j$), then X_φ can be seen as a toric surface in such a way that the support of $\psi^*(B + \Lambda_{i'})$ contains the complement of the big orbit, where i' is chosen so that $\Lambda_{i'}$ passes through P_{ij}.*

Consequently, we have $\mathcal{L}(X_\varphi, D_\varphi) = \mathrm{NS}(X_\varphi) = H^2(X_\varphi)$ in these cases.

Proof. (1) We may assume $i = 3$, and then the cover is in fact branched only along B_0, B_1 and B_2. The projective plane can be seen as the toric surface associated to the lattice $N = \mathbb{Z}^2$ and the fan Σ whose 1-skeletons are $\mathbb{R}_{\geq 0}(1, 0)$, $\mathbb{R}_{\geq 0}(0, 1)$ and $\mathbb{R}_{\geq 0}(-1, -1)$, and Y_φ is the toric surface associated to the sublattice $\{(k, l) \in N | a_0 k + a_1 l \equiv 0 \mod m\}$ and the same fan Σ.

(2) We may assume $i = 0$, $j = 1$ and hence $i' = 1$. Let $S = \mathrm{Bl}_{P_{01}, P_{23}} \mathbb{P}^2$ and $f = \mathrm{bl}_{P_{01}, P_{23}} : S \to \mathbb{P}^2$. Then $(f^{-1})_* \Lambda_1$ is a (-1)-curve, so let $g : S \to T$ be the blowdown and $B_T = g(f^{-1}(B + \Lambda_1))$. We observe the following.

- The surface T is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The curves $g((f^{-1})_* B_0), g((f^{-1})_* B_1)$ and $g(f^{-1}(P_{23}))$ belong to one ruling, and $g((f^{-1})_* B_2), g((f^{-1})_* B_3)$ and $g(f^{-1}(P_{01}))$ to the other ruling.
- $\mathbb{P}^2 \setminus (B + \Lambda_1)$ is isomorphic to $T \setminus B_T$ via f and g, and the covering $Y_\varphi \setminus \varphi^{-1}(B + \Lambda_1) \to T \setminus B_T$ is unramified along $g(f^{-1}(P_{01}))$ and $g(f^{-1}(P_{23}))$.

Therefore $Y_\varphi \setminus \varphi^{-1}(B + \Lambda_1)$ is contained in a cover of $(\mathbb{P}^1 \setminus (2 \text{ points})) \times (\mathbb{P}^1 \setminus (2 \text{ points}))$, and we have the assertion. $\square$

6.2. **Case m is a prime number.**

Proposition 6.2. *Assume the following.*

(a) *$m = p$ is a prime number,*
(b) *$a_i \not\equiv 0 \mod p$ for any i, and*
(c) *$a_i + a_j \not\equiv 0 \mod p$ for any $i \neq j$.*

Then the discriminant of $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ is p^7 and the discriminant of $\mathcal{L}(X_\varphi, D_\varphi)$ is p. Hence $\mathcal{L}(X_\varphi, D_\varphi)$ is primitive when m is a prime number.

Proof. Let i, j and k satisfy $\{i, j, k\} = \{1, 2, 3\}$ and $j < k$, and consider the homomorphism k_i in Definition 4.2. The composite $\gamma \circ k_i$ is given by $n \mapsto n(a_0 + a_i) \mod p$, and by assumption (c), the group K_i in Definition 4.2 is trivial. Thus there is only one distinguished curve $\bar{L}_{i00}$ over Λ_i, with mapping degree p, hence there is only one distinguished point $\bar{P}_{0i0}$ (resp. $\bar{P}_{jk0}$) over P_{0i} (resp. P_{jk}). Denoting the exceptional curves of the minimal resolutions at $\bar{P}_{0i0}$ and $\bar{P}_{jk0}$ by $C_{0i\alpha}$ and $C_{jk\alpha}$, we have $\psi^* \Lambda_i = \bar{L}_{i00} + \sum m_\alpha C_{0i\alpha} + \sum m'_\alpha C_{jk\alpha}$ for some integers m_α and m'_α. It follows that $\mathcal{L}(X_\varphi, \psi^*(\Lambda)) = \mathbb{Z}L \oplus \bigoplus_{0 \leq i < j \leq 3} (\bigoplus_\alpha \mathbb{Z}C_{ij\alpha})$, where L denotes the pullback of a line on $\mathbb{P}^2$. Since $\mathbb{Z}L$ and $\bigoplus_\alpha \mathbb{Z}C_{ij\alpha}$ for different (i, j) are orthogonal to each other, we have $\operatorname{disc} \mathcal{L}(X_\varphi, \psi^*(\Lambda)) = \operatorname{disc}(\mathbb{Z}L) \prod_{0 \leq i < j \leq 3} \operatorname{disc}(\bigoplus_\alpha \mathbb{Z}C_{ij\alpha})$. Let us look at the resolution at P_{ij}. Let h_{ij} be as in Definition 4.2, and then $\gamma \circ h_{ij}$ is given by $(k, l) \mapsto a_i k + a_j l \mod p$. The group $\tilde{H}$ in Definition 4.5 is equal to $\operatorname{Ker} \gamma \circ h_{ij}$, and our assumptions imply $\tilde{H} = (p\mathbb{Z} \times p\mathbb{Z}) + \mathbb{Z}(q, 1)$ for some q with $0 < q < p$. Thus the singularity P_{ij} is of type $A_{p,q}$, and we have $\operatorname{disc} \bigoplus_\alpha \mathbb{Z}C_{ij\alpha} = p$ by Proposition 4.7(3). Hence $\operatorname{disc} \mathcal{L}(X_\varphi, \psi^*(\Lambda)) = p^7$.

By assumption (b), we have $\psi^* B_0 = pR_0 + \sum_{i=1}^3 D_i$, where R_0 is mapped isomorphically onto B_0 and the support of D_i is $\psi^{-1}(P_{0i})$. Define $w : \mathcal{L}(X_\varphi, D_\varphi) \to (\mathbb{Z}/p\mathbb{Z})^3$ by $C \mapsto ((C.D_i) \mod p)_{i=1}^3$.

Since L is the pullback of a line in $\mathbb{P}^2$, $w(L)$ is obviously 0. For an exceptional curve C, we have $(C.D_i) = 0$ if $\psi(C) \neq P_{0i}$. If $\psi(C) = P_{0i}$, then $(C.D_i) =$

$(\psi^* B_0.C) - (pR_0.C) - \sum_{j\neq i}(D_j.C) = -p(R_0.C) \equiv 0 \mod p$. Therefore the sublattice $\mathcal{L}(X_\varphi, \psi^*(\Lambda))$ is contained in $\operatorname{Ker} w$. On the other hand, since the ramification curve R_i over B_i is mapped isomorphically onto B_i, we have $(R_i.D_i) \equiv (R_i.\psi^* B_0) = 1 \mod p$. It is obvious that $(R_i.D_j)$ is 0 if $i \neq j$. Thus there is a surjective homomorphism $\mathcal{L}(X_\varphi, D_\varphi)/\mathcal{L}(X_\varphi, \psi^*(\Lambda)) \to (\mathbb{Z}/p\mathbb{Z})^3$, and it follows that $\operatorname{disc}\mathcal{L}(X_\varphi, \psi^*(\Lambda))/\operatorname{disc}\mathcal{L}(X_\varphi, D_\varphi)$ is a square number which is a multiple of p^6. Since $\operatorname{disc}\mathcal{L}(X_\varphi, \psi^*(\Lambda)) = p^7$, we have $\operatorname{disc}\mathcal{L}(X_\varphi, D_\varphi) = p$. □

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